Complex Algebraic Geometry
MSRI Publications
Volume **28**, 1995

Fundamental Groups of Smooth Projective Varieties

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To the memory of Boris Moishezon

Abstract. This article is a brief survey of work related to the structure of topological fundamental groups of complex smooth projective varieties.

These notes, which are based on a talk given at MSRI in April 1993, are intended as a brief guide to some recent work on fundamental groups of varieties. For the most part, I have just tried to explain the results (often in nonoptimal form) and give a few simple examples to illustrate their use. Proofs are either sketched or omitted entirely. The basic question that will concern us is:

Which groups can be fundamental groups of smooth projective varieties?

This is certainly of importance in the topological study of algebraic varieties, but it is also linked to broader issues in algebraic geometry. Let us call the class of such groups $\mathcal{P}$. As an application of the Lefschetz hyperplane theorem [Mi], we can see that any group in $\mathcal{P}$ is the fundamental group of an algebraic surface, and in fact we can even arrange the surface to have general type. Thus failure to answer this question can be viewed as an obstruction to completely classifying algebraic surfaces (even up to homotopy).

I should mention that there is also a nice survey article by Johnson and Rees [JR2] that reviews much of the work done on this problem prior to 1990. My own view of the subject has been shaped, to a large extent, by conversations and correspondence with many people, of whom I would especially like to mention Paul Bressler, Jim Carlson, Dick Hain, János Kollár, Madhav Nori, Mohan Ramachandran and Domingo Toledo. My thanks to F. Campana for catching a silly mistake in an earlier version of these notes.

The terminology used here is fairly standard. The only thing that could cause confusion is that I will say that a group G is an extension of B by A if it fits into an exact sequence:

$$1 \to A \to G \to B \to 1,$$

and not the other way around.

1. Positive results

Since most of the known results are in the negative direction, let us start with some positive ones.

($\mathbf{A}^+$) (Serre) Any finite group lies in $\mathcal{P}$. What Serre [S, Proposition 15] in fact proves is that any finite group acts without fixed points on some smooth complete intersection of any prescribed dimension. Since, by Lefschetz's hyperplane theorem, complete intersections of dimension at least 2 are simply connected, the first statement follows.

($\mathbf{B}^+$) $\mathcal{P}$ is closed under finite products because the class of projective varieties is.

($\mathbf{C}^+$) If $G \in \mathcal{P}$, any subgroup of finite index lies in $\mathcal{P}$, because a finite-sheeted covering of a smooth projective variety can be given the structure of a smooth projective variety.

($\mathbf{D}^+$) As we shall see later, the converse of (C^+) is false; however, a weak form (needed below) does hold. Suppose that X is a simply connected complex manifold on which a group G acts faithfully, biholomorphically and properly discontinuously. Assume furthermore that a finite index subgroup $H \subseteq G$ acts freely on X and that the quotient is a projective variety. Then $G \in \mathcal{P}$.

PROOF (KOLLÁR). We can assume that H is normal, since otherwise we can replace it by a stabilizer of a coset in G/H. Let S be a smooth projective variety with fundamental group G/H, and let $\tilde{S}$ be its universal cover. Then the diagonal action of G on $X \times \tilde{S}$ is free, so the quotient is smooth and has G as its fundamental group. Furthermore $(X \times \tilde{S})/G$ is a projective variety since it possesses a finite holomorphic map to $X/G \times S$. $\square$

($\mathbf{E}^+$) For any positive integer g, the group

$$\langle a_1, a_2, \ldots, a_{2g} \mid [a_1, a_{g+1}], \ldots, [a_g, a_{2g}] = 1\rangle,$$

which is the fundamental group of a curve of genus g, lies in $\mathcal{P}$.

($\mathbf{F}^+$) If G is a semisimple Lie group such that the quotient D of G by a maximal compact subgroup is a Hermitian symmetric space of noncompact type, any cocompact discrete subgroup $\Gamma \subset G$ lies in $\mathcal{P}$. When Γ is torsion-free, this follows from the fact that G acts freely on D and the quotient when endowed with the Bergman metric satisfies the conditions of Kodaira's embedding theorem (see [H, ch. VIII] and [KM, p. 144]), and is consequently a smooth projective variety with fundamental group Γ. In the general case, Γ contains a torsion-free subgroup of finite index [Sel, Lemma 9], so we can appeal to (D^+). Two simple examples to keep in mind are: $G = \mathrm{SU}(n,1)$ the group of unimodular matrices preserving the indefinite form $|z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$, and $G = \mathrm{Sp}(2n, \mathbb{R})$ the group of matrices preserving the standard symplectic form on $\mathbb{R}^{2n}$; maximal compact subgroups are given by

$$K_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix} \mid A \in U(n) \right\}$$

and

$$K_2 = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \mid X + iY \in U(n) \right\},$$

respectively.

$\mathrm{SU}(n,1)/K_1$ can be identified with the unit ball B^n in $\mathbb{C}^n$, and $\mathrm{Sp}(2n,\mathbb{R})/K_2$ can be identified with the Siegel upper half plane $\mathcal{H}_n$ of $n \times n$ symmetric matrices with positive definite imaginary part. In particular, when $n = 1$, we see that all the groups described in (D^+) with $g \geq 2$ arise in this fashion. Although it is difficult to construct cocompact lattices explicitly, they always exist for any G [Bo].

($\mathbf{G}^+$) (Toledo [T1]) In the above situation, suppose that G is the group of real points of an algebraic group defined over $\mathbb{Q}$. Assume that none of the irreducible factors of D is isomorphic to B^1, B^2, or $\mathcal{H}_2$. Then any arithmetic subgroup of G lies in $\mathcal{P}$. In particular, $\mathrm{Sp}(2n,\mathbb{Z}) \in \mathcal{P}$ when $n > 2$. I will describe the idea in this example. As in the proof of (D^+), we can obtain a free action of $\mathrm{Sp}(2n,\mathbb{Z})$ on the product of $X = \mathcal{H}_n$ and a suitably chosen simply connected smooth projective variety $\tilde{S}$. The variety

$$Y = (X \times \tilde{S}) / \mathrm{Sp}(2n, \mathbb{Z})$$

is not projective but only quasiprojective. It has a compactification $\bar{Y}$ obtained by normalizing $\bar{A}_n \times S$ in the function field of Y, where $\bar{A}_n$ is the Satake compactification of $A_n = X/\mathrm{Sp}(2n,\mathbb{Z})$. The variety $\bar{Y}$ is projective and we will fix a projective embedding. The codimension of the complement $\bar{Y} - Y$ is at least 3; therefore we can slice $\bar{Y}$ by hyperplanes, in general position, until we get a smooth projective surface Z contained in Y. A strong form of the Lefschetz theorem, due to Goresky and Macpherson, guarantees that the fundamental group of Z is isomorphic to that of Y, which is of course $\mathrm{Sp}(2n,\mathbb{Z})$.

($\mathbf{H}^+$) Toledo [T2] has solved the long-outstanding problem of showing that $\mathcal{P}$ contains nonresidually finite groups (i.e., groups that don't embed into their profinite completions). Other examples have since been constructed by Catanese, Kollár and Nori (see [CK]). The simplest such example is the preimage of $\mathrm{Sp}(6,\mathbb{Z})$ in the connected 3-fold cyclic cover of $\mathrm{Sp}(6,\mathbb{R})$. The nonresidual finiteness of this and related groups is due to Deligne [De].

(I$^+$) Sommese and Van de Ven [SV], and later Campana [Ca], have shown that $\mathcal{P}$ contains nonabelian torsion-free nilpotent groups, and this contradicts a long-held belief by many workers in the area including this author. (Unfortunately the belief and the counterexample in [SV] existed concurrently for quite some time.) The examples are constructed as follows: Choose an abelian n-fold A and a finite map to $\mathbb{P}P^n$. Let X be the preimage in A of a generic translate of an abelian d-fold in $\mathbb{P}P^n$ with $d \geq 2$. Then a suitable double cover of X has as fundamental group a nonsplit central extension of an abelian group by $\mathbb{Z}$. Sommese and Van de Ven used a specific choice $(n,d) = (4,2)$, and this yields an extension of $\mathbb{Z}^{12}$ by $\mathbb{Z}$ in $\mathcal{P}$.

2. Simple obstructions

Now let us consider the various known obstructions for a group to lie in $\mathcal{P}$. The first obvious constraint, coming from the fact that any variety admits a finite triangulation, is that the groups in $\mathcal{P}$ are finitely presented. A more subtle constraint comes from Hodge theory, which implies that the first Betti number of a smooth projective variety is even (see for example [GH, p. 117] or [KM, p. 115]). Therefore, by Hurewicz's theorem:

(A$^-$) If $\mathrm{rank}(G/[G,G])$ is odd, then $G \notin \mathcal{P}$.

By virtue of (C$^+$) we obtain a strengthening:

(A$'^-$) If G has a subgroup H of finite index with $\mathrm{rank}(H/[H,H])$ odd, then $G \notin \mathcal{P}$.

In particular, *the free group on n generators F_n is not in $\mathcal{P}$.* This is immediate from (A$^-$) if n is odd. If n is even, let $g_1, g_2, \ldots, g_n$ be free generators of F_n. Then the subgroup generated by

$$g_1^2,\, g_2,\, g_3,\, \ldots,\, g_n,\, g_1g_2g_1^{-1},\, g_1g_3g_1^{-1},\, \ldots,\, g_1g_ng_1^{-1}$$

is a subgroup of finite index that is free on an odd number of generators. This can be seen geometrically as follows. Let X be a bouquet of n circles; its fundamental group is F_n, one generator g_i for each circle. Let $\tilde{X} \to X$ be the $\mathbb{Z}/2\mathbb{Z}$-covering corresponding to the homomorphism $F_n \to \mathbb{Z}/2\mathbb{Z}$ defined by sending g_1 to 1 and the other generators to 0. Then the image of $\pi_1(\tilde{X})$ in $\pi_1(X)$ is the subgroup defined above and the essential loops in $\tilde{X}$ correspond to the given generators.

As a second example, let G be the semidirect product of $\mathbb{Z}^2$ with $\mathbb{Z}/2\mathbb{Z}$, where the second group acts on the first through the involution $(x,y) \mapsto (x,-y)$. Then $G \notin \mathcal{P}$ since $\mathrm{rank}(G/[G,G]) = 1$. This example shows that the converse of (C$^+$)

is false. It may also be worth remarking that this example is the fundamental group of the Klein bottle.

At this point one might be tempted to speculate that (A$'^{-}$) is the only obstruction. But now let's consider a subtler example, the 3×3 Heisenberg group H, which is the group of 3×3 upper triangular integer matrices with 1's on the diagonal. The rank of the abelianization of any finite index subgroup is 2. Nevertheless, $H \notin \mathcal{P}$. We will give several proofs of this. The first, due to Johnson and Rees [JR1], is the simplest. Before indicating the argument, we will recall a few facts about group cohomology (details can be found in [Br]). If M is an abelian group upon which a group G acts, then the cohomology group $H^i(G, M)$ can be defined in an entirely algebraic manner, either explicitly in terms of cocycles or more abstractly via derived functors. When $M = \mathbb{R}$ with trivial G action, $H^*(G, \mathbb{R})$ becomes a graded ring under cup product. If X is a connected topological space, there is a natural ring homomorphism

$$H^*(\pi_1(X), \mathbb{R}) \to H^*(X, \mathbb{R}),$$

which is an isomorphism when $* = 1$, and is an isomorphism for all $*$ provided that the universal cover of X is contractible, in which case X is called a $K(\pi_1(X), 1)$.

(B$^-$) (Johnson–Rees) If $H^1(G, \mathbb{R}) \neq 0$ and the map $\mathrm{sq}_G : \wedge^2 H^1(G, \mathbb{R}) \to H^2(G, \mathbb{R})$ induced by cup product vanishes, then $G \notin \mathcal{P}$.

Proof. Suppose $G = \pi_1(X)$, where X is a smooth projective n-dimensional variety. Then there is an isomorphism $H^1(G, \mathbb{R}) \cong H^1(X, \mathbb{R})$ and sq_G factors through $\wedge^2 H^1(X, \mathbb{R}) \to H^2(X, \mathbb{R})$, so it is enough to check that this is nonzero. Given a nonzero class $\alpha \in H^1(X)$, there exists by Poincaré duality a $\beta \in H^{2n-1}(X)$ such that $\alpha \cup \beta \neq 0$. By the Hard Lefschetz theorem [GH, p. 122] we have $\beta = \gamma \cup L^{n-1}$, where $\gamma \in H^1(X)$ and L is the class of a hyperplane section. Therefore $\alpha \cup \gamma \neq 0$. □

The cohomology ring of H can be easily computed topologically once one observes that a $K(H, 1)$ is given by

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in R \right\} \Big/ H.$$

The first and second cohomology groups H are isomorphic to $\mathbb{R}[dx] \oplus \mathbb{R}[dy]$ and $\mathbb{R}[dx \wedge dz] \oplus \mathbb{R}[dy \wedge dz]$; in particular $\mathrm{sq}_H = 0$.

3. Groups with more than one end

The fact that F_n is not in $\mathcal{P}$ is part of a more general phenomenon that we will explain in this section. If X is a topological space, for any subset K let

$E(K)$ be the set of connected components of $X - K$ with noncompact closure. The number of ends of X is defined as

$$\sup\{\#E(K) \mid K \subseteq X \ \text{ compact}\} \in \mathbb{N} \cup \{\infty\}.$$

Exercise: show that $\mathbb{C}$ has one end and $\mathbb{R}$ has two.

The number of ends of a finitely generated group G can be defined in a purely group-theoretic fashion as the dimension of a cohomology group

$$1 + \dim\, H^1(G, \mathbb{Z}/2\mathbb{Z}[G]).$$

However, it has a more geometric interpretation. Suppose that X is a simplicial complex upon which G acts freely and simplicially with compact (i.e., finite) quotient. Then the number of ends of G and X coincide. So, for example, as a corollary of the exercise: $\mathbb{Z}^2$ has one end and $\mathbb{Z}$ has two.

Given a finite set of generators, there is a classical method for building a space upon which G acts, namely the Cayley graph: the vertices are the elements of the group and two vertices are connected by an edge if one vertex can be obtained from the other by multiplication (on the right, say) by a generator or an inverse of one. This has an obvious left action by G with compact quotient; thus the number of ends of G and its graph are the same. Let's consider F_n with its standard generators. Its Cayley graph is an infinite tree where $2n$ branches emanate from any vertex. Clearly this space has infinitely many ends when $n > 1$. More generally, any free product with nontrivial factors other than $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ has infinitely many ends; the exceptional case has two. The converse of the previous sentence is very close to being true, thanks to a deep theorem of Stallings; see [SW, Section 6] for the precise statement.

Building on work of Gromov [G], Bressler, Ramachandran and the author [ABR] have obtained:

($\mathbf{C}^-$) If a group has more than one end, it does not lie in $\mathcal{P}$. An extension of a group with infinitely many ends by a finitely generated group does not lie in $\mathcal{P}$.

The first statement implies Gromov's result [G] (see also [JR1]) that $\mathcal{P}$ contains no nontrivial free products. The Heisenberg group H has one end and does not surject onto a group with infinitely many ends, so it is not covered by the above result. As an application of the last part of the above criterion, let's show that the *braid group* B_n is not in $\mathcal{P}$. This group can be defined as the fundamental group of the space of n distinct unordered points in the plane $\mathbb{R}^2$. For other definitions and basic properties see [Bi]. Since $B_2 = \mathbb{Z}$, it cannot lie in $\mathcal{P}$. The standard presentation for B_3 is

$$\langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2 \rangle.$$

If we set $a = s_1 s_2$ and $b = s_1 s_2 s_1$, we get a new presentation $B_3 = \langle a, b \mid a^3 = b^2 \rangle$. The subgroup $N = \langle a^3 \rangle$ is normal and

$$B_3/N \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

has infinitely many ends; consequently $B_3 \notin \mathcal{P}$. Let P_n be the pure braid group. This is the fundamental group of the space X_n of n ordered points in $\mathbb{R}^2$. It is a subgroup of finite index in B_n, so it suffices to show that $P_n \notin \mathcal{P}$ when $n > 3$. Note that the image of P_3 in B_3/N has infinitely many ends because this property is stable under passage to subgroups of finite index. The projection $X_n \to X_3$ is a fiber bundle and its fiber is homotopic to a finite complex. Thus there is a surjection $p : P_n \to P_3$ with finitely generated kernel. Therefore $p^{-1}(N \cap P_3) \subset P_n$ is a finitely generated normal subgroup such that the quotient has infinitely many ends.

4. Rational homotopy

One of the key insights coming from rational homotopy theory is that the algebra of differential forms on a manifold contains a lot more topological information than just the cohomology ring. One can obtain information about all nilpotent quotients of the fundamental group (not just abelian ones). This information can be systematized by replacing G by the inverse limit of all of its nilpotent quotients:

$$\hat{G} = \lim_{\leftarrow} N,$$

which is called its *nilpotent completion*. We would like to introduce a coarser construction, the Malcev (or rational nilpotent) completion $\mathcal{G}$, which should be thought of as "$\hat{G} \otimes \mathbb{Q}$". Malcev has shown that any finitely generated torsion-free nilpotent group can be embedded as a Zariski dense subgroup of a unipotent linear algebraic group (i.e., an algebraic subgroup of a group of upper triangular matrices) over $\mathbb{Q}$. This leads us to define

$$\mathcal{G} = \lim_{\leftarrow} U,$$

where the limit runs over all representations of G into unipotent algebraic groups U defined over $\mathbb{Q}$. As a trivial but instructive example, let G be abelian. Then $\hat{G} = G$ and $\mathcal{G}$ really is $G \otimes \mathbb{Q}$. The prounipotent group $\mathcal{G}$ is completely determined by its Lie algebra:

$$L(G) = \mathrm{Lie}(\mathcal{G}) = \lim_{\leftarrow} \mathrm{Lie}(U),$$

and this is usually more convenient to work with. $L(G)$ can be topologized by taking the above inverse limit in the category of topological Lie algebras, where each factor $\mathrm{Lie}(U)$ is equipped with the discrete topology. These definitions, while efficient, have the disadvantage of making $\mathcal{G}$ and $L(G)$ seem mysterious;

they aren't. They can be realized quite explicitly as subsets of the completion of the group ring $\mathbb{Q}[G]$ at its augmentation ideal [Q, Appendix A].

Let's work out two examples. First we will compute $L(H)$ for the 3×3 Heisenberg group. We take U to be the unipotent group of upper triangular 3×3 rational matrices. Then the map $H\to U$ is an initial object in the above inverse system; therefore $\mathcal{G}=U$ and $L(H)=\mathrm{Lie}(U)$, the Lie algebra of strictly upper triangular 3×3 rational matrices. Next consider the free group F_n on n generators $X_1,\ldots,X_n$. Let FL_n be the free $\mathbb{Q}$-Lie algebra on n generators $x_1,\ldots,x_n$. Let

$$C^N\,\mathrm{FL}_n=[\mathrm{FL}_n,[\mathrm{FL}_n,\ldots,[\mathrm{FL}_n,\mathrm{FL}_n],\ldots]]\qquad (N+1\ \ \mathrm{FL}_n\text{'s})$$

be the N-th term of the lower central series. Then it can be checked that $L(F_n)$ is the completion

$$\widehat{\mathrm{FL}}_n=\lim_{\substack{\leftarrow\\ N}}\,(\mathrm{FL}_n/C^N\,\mathrm{FL}_n)$$

of the free Lie algebra with respect to the topology determined by the lower central series. The main point is that

$$X_i\mapsto \exp(\mathrm{ad}(x_i))\in \mathrm{GL}(\mathrm{FL}_n/C^N\,\mathrm{FL}_n)$$

determines a cofinal family of unipotent representations of F_n.

We would like to describe the Lie algebra $L(G)$, for arbitrary G, in terms of generators and relations. For generators, choose $x_1,x_2,\ldots,x_n\in L(G)$ such that they determine a basis of

$$L(G)/[L(G),L(G)]\cong (G/[G,G])\otimes\mathbb{Q}.$$

These elements will generate a dense subalgebra of $L(G)$; thus we obtain a continuous surjective homomorphism $\widehat{\mathrm{FL}}_n\to L(G)$. Call the kernel $I(G)$ (we'll suppress the dependence on the x_i). Any element of $\widehat{\mathrm{FL}}_n$ can be expanded as an infinite series

$$\sum a_ix_i+\sum b_{ij}[x_i,x_j]+\sum c_{ijk}[x_i,[x_j,x_k]]+\cdots.$$

The degree of the element is the degree of the smallest term—in other words, the length of the shortest commutator appearing in the series. Let $I_2(G)$ be the closed ideal of $\widehat{\mathrm{FL}}_n$ generated by elements of $I(G)$ of degree 2. The following result of Deligne, Griffiths, Morgan, and Sullivan is the first deep result in this area. (The result as stated here does not appear explicitly in their paper [DGMS], but it is a well known consequence of it. See [CT2] and [M, Section 9, 10] for further discussion.)

(D$^-$) If for some (any) choice of generators, $I(G) \neq I_2(G)$ then $G \notin \mathcal{P}$.

Or, in plain English, the essential relations of $L(G)$, for $G \in \mathcal{P}$, are quadratic. One can even say what they are: they are dual (in a natural sense) to the kernel of the map sq_G of Section 2. For example, let G be the fundamental group of a smooth projective curve of genus g. Then $L(G)$ is isomorphic to the quotient of $\widehat{\mathrm{FL}}_{2g}$ by the quadratic relation

$$\sum_{i=1}^{g} [x_i, x_{i+g}] = 0.$$

Consider the 3×3 Heisenberg group. We can choose two generators x_1, x_2 of $L(H)$ corresponding to the matrices with 1's at $(1,2)$ and $(2,3)$, respectively, and zeros elsewhere. Then clearly there aren't any quadratic relations, although there are cubic ones. Thus again we conclude that $H \notin \mathcal{P}$. This sort of reasoning allows one to eliminate a lot of nilpotent groups from $\mathcal{P}$, although even for this class the method does not yield a definitive answer. Carlson and Toledo have pointed out to me that the Lie algebra of the 5×5 Heisenberg group does in fact have quadratic relations; nevertheless it doesn't lie in $\mathcal{P}$ for other reasons [CT2]. In fact, there are many examples of groups $G \notin \mathcal{P}$ for which $L(G)$ has quadratic relations—for instance, F_n, and less trivially P_n (see [Ko]).

5. Representation varieties

Let Γ be a group generated by finitely many elements $\gamma_1, \ldots, \gamma_n$. Giving a representation of Γ into G is the same thing as choosing n elements $g_i \in G$ satisfying the relations satisfied by γ_i. Thus, if G is a real algebraic group, the set of representations $\mathrm{Hom}(\Gamma, G)$ carries the structure of a (possibly reducible) real algebraic variety. When $\Gamma \in \mathcal{P}$ the local structure of this variety is well understood, thanks to the work of Deligne, Goldman, Milson [GM] and Simpson [S1]:

(E$^-$) If $\Gamma \in \mathcal{P}$, then $X = \mathrm{Hom}(\Gamma, G)$ has quadratic singularities at points corresponding to semisimple representations. More precisely, the completion of the local ring of X at such a point is isomorphic to the quotient of a formal power series ring by an ideal generated by quadratic polynomials.

There is a strong formal similarity between these results and (D$^-$), and in fact the proof uses a generalization of the methods of (D$^-$) to local systems. For the third time, let's look at the Heisenberg group H. Let G be the group of 3×3 upper triangular real matrices. Then the singularity of $\mathrm{Hom}(H, G)$ at the trivial representation can be shown to be a cubic cone. Thus $H \notin \mathcal{P}$.

6. Lattices in Lie groups

While it seems very difficult to characterize all groups in $\mathcal{P}$, a more reasonable problem would be to classify the discrete subgroups of Lie groups that lie in $\mathcal{P}$. In the first section we indicated some positive results in this direction; now we consider some obstructions. But first some terminology. A lattice of a Lie group is a discrete subgroup such that the quotient has finite volume with respect to Haar measure (this is certainly the case when the quotient is compact, for example). $\mathrm{SO}(n,1)$ is the group of unimodular matrices preserving the form $x_1^2 + \cdots + x_n^2 - x_{n+1}^2$. Using techniques from the theory of harmonic maps, Carlson and Toledo [CT] obtain:

($\mathbf{F}^-$) No cocompact discrete subgroup of $\mathrm{SO}(n,1)$, with $n > 2$, lies in $\mathcal{P}$.

Note that the symmetric space associated to $\mathrm{SO}(n,1)$ is not Hermitian when $n > 2$. In fact these authors conjecture that a cocompact lattice in a semisimple group is never in $\mathcal{P}$ unless the associated symmetric space is Hermitian. As further evidence, Carlson and Hernandez [CH] show that a lattice in the automorphism group of the Cayley plane does not lie in $\mathcal{P}$.

The strongest results of this sort have been obtained by Simpson [S1] (see also [C]). To state them, we need some more terminology. Let W be a real algebraic group, G the associated complex group and σ the complex conjugation of G corresponding to W. A Cartan involution is an automorphism C of G such that $C^2 = 1$, $\tau = C\sigma = \sigma C$, and the set of τ-fixed points of G is a compact group that meets every component G. The group W is of Hodge type if there is a γ in the identity component of G such that $x \mapsto \gamma x \gamma^{-1}$ is a Cartan involution (this is equivalent to Simpson's definition [S1, Section 4.42]). For example, $\mathrm{Sp}(2n,\mathbb{R})$ is of Hodge type, for we can take

$$\gamma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The group of τ-fixed points on the associated complex group $\mathrm{Sp}(2n,\mathbb{C})$ is

$$\mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{SU}(2n).$$

A list of simple groups of Hodge type can be found in [S1, pp. 50-51]. $\mathrm{SL}_n(\mathbb{R})$ is not of Hodge type as soon as $n \geq 3$. The most important (and motivating) examples of groups of Hodge type are the Zariski closures of the monodromy groups of complex variations of Hodge structure. This gives another explanation of why $\mathrm{Sp}(2n,\mathbb{R})$ is of Hodge type: namely, it is the (real) Zariski closure of the monodromy group of the variation of Hodge structure associated to the first cohomology of a family of n-dimensional abelian varieties. Simpson shows that any representation of the fundamental group of a smooth projective variety into a reductive group can be deformed into one coming from a variation of

Hodge structure. In particular, if the given representation is rigid, in the sense that every nearby representation is conjugate to it, then it must be conjugate to one coming from a variation of Hodge structure. Thus if the image of the representation is Zariski dense (which would be the case if it were a lattice) the target group must be of Hodge type. Therefore:

($\mathbf{G}^-$) If Γ is a rigid lattice in a reductive real algebraic group that is not of Hodge type, $\Gamma \notin \mathcal{P}$.

To apply this, one needs to be able to check that a given lattice is rigid. Fortunately there are a number of rigidity theorems in existence; they are summarized in [S1, p. 53]. In particular $\mathrm{SL}_n(\mathbb{Z})$ is rigid when $n \geq 3$, so it cannot lie in $\mathcal{P}$. (Neither does $\mathrm{SL}_2(\mathbb{Z})$, but one must appeal to (C^-) in this case.) Note also that $\mathrm{SO}(n,1)$ is of Hodge type when n is even, so that (G^-) does not imply (F^-).

The above results were concerned with lattices in reductive groups. Now let's look at opposite end of the spectrum, namely lattices in nilpotent or, more generally, solvable Lie groups. A lattice in a nilpotent Lie group is finitely generated and nilpotent as a discrete group, and conversely any finitely generated torsion-free nilpotent group is such a lattice by Malcev's theorem. We have already mentioned in (I^+) that it had been expected that such groups cannot lie in $\mathcal{P}$ unless they are abelian. At the moment the situation regarding which nilpotent groups lie in $\mathcal{P}$ is somewhat murky, but hopefully a clearer picture will emerge with time. For the state of the art, see [CT2] and [CT3].

Now let us turn to solvable groups. A lattice in a solvable Lie group is solvable as a discrete group, and moreover satisfies a noetherian condition that all of its subgroups are finitely generated. A group satisfying these conditions is called polycyclic (we're taking this as the definition for simplicity; the usual one is different although equivalent). Nori and the author [AN] have shown that the problem of understanding polycyclic groups in $\mathcal{P}$ reduces to the previous one:

($\mathbf{H}^-$) If a polycyclic group lies in $\mathcal{P}$, it contains a nilpotent group of finite index.

Let's consider the following example. An action of $\mathbb{Z}^2$ on itself is determined by specifying two commuting automorphisms $T_i \in \mathrm{GL}_2(\mathbb{Z})$. For simplicity, let $T_2 = I$ and let G be the semidirect product of $\mathbb{Z}^2$ with itself using this action. Then G is polycyclic. Using (H^-) and (A^-), one can check that $G \notin \mathcal{P}$ unless T_1 has finite order. When T_1 has finite order, it is not too hard to see that G can be realized as the fundamental group of one of the bielliptic surfaces listed in [Be, pp. 113-114]. Thus $G \in \mathcal{P}$.

7. Maps to curves

In recent years a number of remarkable results have been obtained that fit into the following general pattern:

If the fundamental group of smooth projective variety satisfies some suitable hypothesis (say $\diamondsuit$) the variety maps onto curve of genus 2 or more.

In keeping with the aims of these notes, we will concentrate on the group-theoretic consequences. For the sake of expedience let's call a group "curve-dominating" if it possesses a surjective homomorphism onto

$$\Gamma_g = \langle a_1, a_2, \ldots, a_{2g} \mid [a_1, a_{g+1}], \ldots, [a_g, a_{2g}] = 1 \rangle$$

for some $g \geq 2$. Thus any group in $\mathcal{P}$ satisfying ($\diamondsuit$) must be curve-dominating. Now let's look at some specific instances.

In Section 3 we saw that $\mathcal{P}$ does not contain nontrivial free products. However, it does contain amalgamated free products. For example, Γ_{g+h} is a free product of Γ_g and Γ_h amalgamated over $\mathbb{Z}$ (this can seen by decomposing a curve of genus $g+h$ into a connected sum of curves of genus g and h). Call an amalgamated free product $G_1 *_K G_2$ *proper* if the index of K is at least 2 in one of the factors and 3 in the other. Gromov and Schoen [GS, 9.1] have proved:

($\mathbf{I}^-$) Any proper amalgamated free product contained in $\mathcal{P}$ is curve-dominating.

Put another way: a proper amalgamated free product cannot lie in $\mathcal{P}$ unless it's curve-dominating. For example, if G and H are perfect (equal to their commutator subgroups), a proper amalgamated free product $G *_K H$ can never lie in $\mathcal{P}$. Conversely, Mohan Ramachandran has pointed out to me that any curve-dominating group is a proper amalgamated free product. Thus this characterizes curve-dominating elements in $\mathcal{P}$. To see this, let $G \to \Gamma_{g+h}$ be a surjection with $g, h \geq 1$. If we decompose Γ_{g+h} into an amalgamated free product as above, we obtain

$$G \cong G_1 *_K G_2,$$

where G_1, G_2 and K are the preimages of Γ_g, Γ_h and $\mathbb{Z}$ respectively.

The second result of this kind is due, independently, to Green and Lazarsfeld [GL] and Gromov [G]:

($\mathbf{J}^-$) Let $G \in \mathcal{P}$ have a presentation with at least two more generators than relations. Then G is curve-dominating; in fact it surjects onto Γ_g, where g is the rank of the abelianization of G.

This puts a strong restriction on the groups in $\mathcal{P}$ with a small number of relations. We have already seen than that $\mathcal{P}$ contains no groups without relations (free groups), but $\mathcal{P}$ does contain groups with a single defining relation, namely the Γ_g's. Are there any others? I will not venture to guess at this point; however, the previous results yield some nice restrictions.

Lemma. *Suppose that*

$$G = \langle x_1, x_2, \ldots, x_n \mid R(x_1, \ldots, x_n) \rangle \ \in \ \mathcal{P}$$

with $n > 2$. Then

1) *n is even.*
2) *Each x_i occurs at least once in the word R and the number of occurrences of x_i and x_i^{-1} coincide.*
3) *G surjects onto Γ_g with $g = n/2$.*

Proof. (B^-) implies that $H^2(G, \mathbb{R}) \neq 0$. By a theorem of Lyndon [L, 11.4], this cohomology group is nonzero if and only if each exponent $e_i = 0$, where e_i is the difference between the number of occurrences of x_i and x_i^{-1} in R. Therefore the image of R in the abelianization vanishes. Consequently n is the rank of the abelianization, and so (1) and (3) follow from (A^-) and (J^-). Finally, if some x_i did not occur in R, then G could be decomposed into a free product of the groups generated by x_i and by the remaining variables. But this would contradict the results of Section 3. □

To bring this section to a close, we give a necessary and almost sufficient homological condition for a group in $\mathcal{P}$ to be curve-dominating. For any group G, we recall that

$$H_1(G', \mathbb{Q}) = \frac{G'}{G''} \otimes \mathbb{Q},$$

where $G' = [G, G]$.

($\mathbf{K}^-$) If $G \in \mathcal{P}$ is curve-dominating then

$$\dim H_1(G', \mathbb{Q}) = \infty.$$

Conversely, if this space is infinite-dimensional, G contains a curve-dominating subgroup of finite index.

This follows from the corollary in [A, p. 313]. (Actually, the corollary is misstated there. It should read: if X maps onto a curve of genus at least two then $\dim H_1(\pi_1(X)', \mathbb{Q}) = \infty$, and if $\dim H_1(\pi_1(X)', \mathbb{Q}) = \infty$ then there is a finite sheeted abelian cover that maps onto a curve of genus at least two.)

8. Complex manifolds

In this final section we will indicate the situation for general complex manifolds. For arbitrary manifolds there are no restrictions: Taubes [Ta] has shown that any finitely presented group can occur as the fundamental group of some three-dimensional compact complex manifold. The story for Kähler manifolds is almost identical to the algebraic one; in fact all the results stated previously are known to show that a given group is (or is not) the fundamental group of a compact Kähler manifold. Such groups are called *Kähler groups*; Clearly any group in $\mathcal{P}$ is a Kähler group. The converse is unknown, and would be very hard to disprove. If one were to try, it would be a good idea to first discover a property about $\mathcal{P}$ that is proved by methods peculiar to projective algebraic geometry (such as slicing by hyperplanes, or changing the field of definition). We will mention one such property, whose proof will appear in [AN], which gives a strengthening of (H^-). The proof is arithmetic in nature and we don't know whether the result holds for Kähler groups.

($\mathbf{L}^-$) If $G \in \mathcal{P}$ is a solvable group that admits a faithful finite dimensional representation over a field of characteristic 0, then G contains a nilpotent subgroup of finite index.

Let's look at a concrete example. Let $p \in \mathbb{Z}$ be a prime and let G be the group of matrices of the form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the entries lie in $\mathbb{Z}[1/p]$ with units on the diagonal. G turns out to be finitely presented [Ab]; however, it contains no nilpotent groups of finite index, so $G \notin \mathcal{P}$. This can also be deduced from a theorem of Simpson [S2] that implies the validity of a conjecture of Beauville and Catanese for smooth projective varieties. A weak form of this conjecture states that the set of line bundles with nonzero first cohomology in Pic^0 of a variety is either infinite or consists of torsion points. It can be checked that, were a smooth projective variety with fundamental group G to exist, it would be a counterexample to this conjecture. Neither argument applies to nonalgebraic manifolds. Is it possible that G is a Kähler group?

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