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Vector Bundles on Curves and Generalized Theta Functions: Recent Results and Open Problems

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Abstract. The moduli spaces of vector bundles on a compact Riemann surface carry a natural line bundle, the determinant bundle. The sections of this line bundle and its multiples constitute a non-abelian generalization of the classical theta functions. New ideas coming from mathematical physics have shed a new light on these spaces of sections—allowing notably to compute their dimension (Verlinde's formula). This survey paper is devoted to giving an overview of these ideas and of the most important recent results on the subject.

Introduction

It has been known essentially since Riemann that one can associate to any compact Riemann surface X an abelian variety, the Jacobian JX, together with a divisor Θ (well-defined up to translation) that can be defined both in a geometric way and as the zero locus of an explicit function, the Riemann theta function. The geometry of the pair (JX, Θ) is intricately (and beautifully) related to the geometry of X.

The idea that higher-rank vector bundles should provide a non-abelian analogue of the Jacobian appears already in the influential paper [We] of A. Weil (though the notion of vector bundle does not appear as such in that paper!). The construction of the moduli spaces was achieved in the 1960's, mainly by D. Mumford and the mathematicians of the Tata Institute. However it is only recently that the study of the determinant line bundles on these moduli spaces and of their spaces of sections has made clear the analogy with the Jacobian. This is

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largely due to the intrusion of Conformal Field Theory, where these spaces have appeared (quite surprisingly for us!) as fundamental objects.

In these notes (based on a few lectures given in the Fall of 1992 at MSRI, UCLA and University of Utah), I will try to give an overview of these new ideas. I must warn the reader that this is by no means intended to be a complete account. I have mainly focused on the determinant line bundles and their spaces of sections, ignoring deliberately important areas like cohomology of the moduli spaces, moduli of Higgs bundles, relations with integrable systems, Langlands' geometric correspondence ..., simply because I felt it would have taken me too far afield. For the same reason I haven't even tried to explain why the mathematical physicists are so interested in these moduli spaces.

1. The moduli space $\mathcal{SU}_X(r)$

Let X be a compact Riemann surface of genus g. Recall that the Jacobian JX parametrizes line bundles of degree 0 on X. We will also consider the variety $J^{g-1}(X)$ that parametrizes line bundles of degree $g-1$ on X; it carries a *canonical* Theta divisor

$$\Theta = \{M \in J^{g-1}(X) \mid H^0(X, M) \neq 0\} .$$

For each line bundle L on X of degree $g-1$, the map $M \mapsto M \otimes L^{-1}$ induces an isomorphism of $(J^{g-1}(X), \Theta)$ onto (JX, Θ_L), where Θ_L is the divisor on JX defined by

$$\Theta_L = \{E \in JX \mid H^0(X, E \otimes L) \neq 0\} .$$

We know a great deal about the spaces $H^0(JX, \mathcal{O}(k\Theta))$. One of the key points is that the sections of $\mathcal{O}(k\Theta)$ can be identified with certain quasi-periodic functions on the universal cover of JX, the theta functions of order k. In this way one gets for instance that the dimension of $H^0(JX, \mathcal{O}(k\Theta))$ is k^g, that the linear system $|k\Theta|$ is base-point free for $k \geq 2$ and very ample for $k \geq 3$, and so on. One even obtains a rather precise description of the ring

$$\bigoplus_{k \geq 0} H^0(JX, \mathcal{O}(k\Theta)),$$

the graded ring of theta functions.

The character who will play the role of the Jacobian in these lectures is the moduli space $\mathcal{SU}_X(r)$ of (semistable) rank-r vector bundles on X with trivial determinant. It is an irreducible projective variety, whose points are isomorphism classes of vector bundles that are direct sums of stable vector bundles of degree 0 (a degree-0 vector bundle E is said to be *stable* if every proper subbundle of E has degree < 0). By the theorem of Narasimhan and Seshadri, the points of $\mathcal{SU}_X(r)$ are also the isomorphism classes of representations $\pi_1(X) \longrightarrow \mathrm{SU}(r)$ (hence the notation $\mathcal{SU}_X(r)$). The stable bundles form a smooth open subset of $\mathcal{SU}_X(r)$, whose complement—which parametrizes decomposable bundles—is singular (except in the cases $g \leq 1$ and $g = r = 2$, where the moduli space is smooth).

The reason for fixing the determinant is that the moduli space $\mathcal{U}_X(r)$ of vector bundles of rank r and degree 0 is, up to a finite étale covering, the product of $\mathcal{SU}_X(r)$ with JX, so the study of $\mathcal{U}_X(r)$ is essentially reduced to that of $\mathcal{SU}_X(r)$. Of course the moduli spaces $\mathcal{SU}_X(r, L)$ of semistable vector bundles with a fixed determinant $L \in \operatorname{Pic}(X)$ are also of interest; for simplicity, in these lectures I will concentrate on the most central case $L = \mathcal{O}_X$.

Observe that when $g \leq 1$ the spaces $\mathcal{SU}_X(r)$ consist only of direct sums of line bundles. Since these cases are quite easy to deal with directly, I will usually assume implicitly $g \geq 2$ in what follows.

2. The determinant bundle

The geometric definition of the theta divisor extends in a natural way to the higher-rank case. For any line bundle $L \in J^{g-1}(X)$, define

$$\Theta_L = \{E \in \mathcal{SU}_X(r) \mid h^0(X, E \otimes L) \geq 1\} .$$

This turns out to be a Cartier divisor on $\mathcal{SU}_X(r)$ [D-N] (the key point here is that the degrees are chosen so that $\chi(E \otimes L) = 0$). The associated line bundle $\mathcal{L} := \mathcal{O}(\Theta_L)$ does not depend on the choice of L. It is called the determinant bundle, and will play a central role in our story. It is in fact canonical, because of the following result (proved in [B1] for $r = 2$ and in [D-N] in general):

Theorem 1. $\operatorname{Pic} \mathcal{SU}_X(r) = \mathbf{Z}\mathcal{L}$.

By analogy with the rank-one case, the global sections of the line bundles $\mathcal{L}^k$ are sometimes called *generalized theta functions*—we will briefly discuss this terminology in §10.

Can we describe $H^0(\mathcal{SU}_X(r), \mathcal{L})$ and the map $\varphi_{\mathcal{L}} : \mathcal{SU}_X(r) \dashrightarrow |\mathcal{L}|^*$ associated to $\mathcal{L}$? Let us observe that we can define a natural (rational) map of $\mathcal{SU}_X(r)$ to the linear system $|r\Theta|$, where Θ denotes the canonical Theta divisor on $J^{g-1}(X)$: for $E \in \mathcal{SU}_X(r)$, define

$$\theta(E) := \{L \in J^{g-1} \mid h^0(E \otimes L) \geq 1\} .$$

It is easy to see that $\theta(E)$ either is a divisor in $J^{g-1}(X)$ that belongs to the linear system $|r\Theta|$, or is equal to $J^{g-1}(X)$. This last case can unfortunately occur (see §3 below), but only for special E's, so we get a rational map

$$\theta : \mathcal{SU}_X(r) \dashrightarrow |r\Theta|.$$

THEOREM 2. *There is a canonical isomorphism*

$$H^0(\mathcal{SU}_X(r),\mathcal{L}) \xrightarrow{\sim} H^0(J^{g-1}(X),\mathcal{O}(r\Theta))^* \ ,$$

making the following diagram commutative:

$$\begin{tikzcd} & {|\mathcal{L}|^*} \arrow[dd, "\wr"] \\ \mathcal{SU}_X(r) \arrow[ur, dashed, "\varphi_{\mathcal{L}}"] \arrow[dr, dashed, "\theta"'] & \\ & {|r\Theta|} \end{tikzcd}$$

This is proved in [B1] for the rank-two case and in [B-N-R] in general. Let me say a few words about the proof. For L in $J^{g-1}(X)$ denote by H_L the hyperplane in $|r\Theta|$ consisting of divisors passing through L. One has $\theta^*H_L=\Theta_L$, so we get a linear map $\theta^* : H^0(J^{g-1}(X),\mathcal{O}(r\Theta))^* \longrightarrow H^0(\mathcal{SU}_X(r),\mathcal{L})$ whose transpose makes the above diagram commutative. It is easy to show that θ^* is injective, hence *the whole problem is to prove that* $\dim H^0(\mathcal{SU}_X(r),\mathcal{L})=r^g$. This was done by constructing an r-to-one covering $\pi : Y\to X$ such that the pushforward map $\pi_* : JY \dashrightarrow \mathcal{SU}_X(r)$ is dominant, which gives an injective map of $H^0(\mathcal{SU}_X(r),\mathcal{L})$ into $H^0(JY,\mathcal{O}(r\Theta))$. Note that surjectivity of θ^* means that the linear system $|\mathcal{L}|$ is spanned by the divisors Θ_L for L in $J^{g-1}(X)$.

This theorem provides a relatively concrete description of the map $\varphi_{\mathcal{L}}$, and gives one some hope of being able to analyze the nature of this map—whether it is a morphism, an embedding, and so on. As we will see, this is a rather intriguing question, which is far from being completely understood. We first consider whether this map is everywhere defined or not.

3. Base points

It follows from Theorem 2 (more precisely, from the fact that the divisors Θ_L span the linear system $|\mathcal{L}|$) that the base points of $|\mathcal{L}|$ are the elements E of $\mathcal{SU}_X(r)$ such that $\theta(E)=J^{g-1}(X)$, that is, $H^0(E\otimes L)\neq 0$ for *all* line bundles L of degree $g-1$. The existence of such vector bundles was first observed by Raynaud [R]. Let me summarize his results in our language:

THEOREM 3. a) *For $r=2$, the linear system $|\mathcal{L}|$ has no base points.*

b) *For $r=3$, $|\mathcal{L}|$ has no base points if $g=2$, or if $g\geq 3$ and X is generic.*

c) *Let n be an integer ≥ 2 dividing g. For $r=n^g$, the system $|\mathcal{L}|$ has base points.*

In case (c), Raynaud's construction gives only finitely many base points. This leaves open a number of questions, which I will regroup under the same heading:

Question 1. *Can one find more examples (say for other values of r)? Is the base locus of dimension > 0? On the opposite side, can one find a reasonable bound on the dimension of the base locus?*

Since the linear system $|\mathcal{L}|$ has (or may have) base points, we have to turn to its multiples. Here we have the following result of Le Potier [LP], improving an idea of [F1]:

Proposition 1. *For $k > \frac{1}{4}r^3(g-1)$, the linear system $|\mathcal{L}^k|$ is base-point free.*

In fact Le Potier proves a slightly stronger statement: *given $E \in \mathcal{SU}_X(r)$ and $k > \frac{1}{4}r^3(g-1)$, there exists a vector bundle F on X of rank k and degree $k(g-1)$ such that* $H^0(X, E \otimes F) = 0$ (in other words, what may fail with a line bundle always works with a rank-k vector bundle). Then

$$\Theta_F := \{E \in \mathcal{SU}_X(r) \mid H^0(X, E \otimes F) \neq 0\}$$

is a divisor of the linear system $|\mathcal{L}^k|$ which does not pass through E, hence the proposition.

The bound on k is certainly far from optimal; in view of Proposition 1 the most optimistic guess is

Question 2. *Is $|\mathcal{L}^2|$ base-point free?*

Let me also mention the following question of Raynaud [R]:

Question 3. *Given $E \in \mathcal{SU}_X(r)$, does there exist an étale covering $\pi : Y \to X$ such that $\theta(\pi^* E) \neq J^{g-1}(Y)$?*

4. Rank 2

The rank-two case is of course the simplest one; it has two special features. On one hand, by Theorem 3(a) (which is quite easy), we know that in this case $\varphi_{\mathcal{L}}$ is a *morphism*; we also know that this morphism is finite because $\mathcal{L}$ is ample. On the other hand, the linear system $|2\Theta|$ on $J^{g-1}(X)$ is particularly interesting because it contains the *Kummer variety* $\mathcal{K}_X$ of X. Recall that $\mathcal{K}_X$ is the quotient of the Jacobian JX by the involution $a \mapsto -a$, and that the map $a \mapsto \Theta_a + \Theta_{-a}$ of JX to $|2\Theta|$ (where as usual Θ_a denotes the translate of Θ by a) factors through an embedding $\kappa : \mathcal{K}_X \hookrightarrow |2\Theta|$. The non-stable part of $\mathcal{SU}_X(2)$ consists of vector bundles of the form $L \oplus L^{-1}$, for L in JX, and can therefore be identified with $\mathcal{K}_X$; recall that for $g \geq 3$ this is the singular locus of $\mathcal{SU}_X(2)$. Theorem 2 thus gives the following commutative diagram

$$\begin{tikzcd}
 & & {|\mathcal{L}|^*} \arrow[dd, "\wr"] \\
\mathcal{K} \arrow[r, hook] \arrow[drr, "\kappa"'] & \mathcal{SU}_X(2) \arrow[ur, "\varphi_{\mathcal{L}}"] \arrow[dr, "\theta"] & \\
 & & {|2\Theta|}
\end{tikzcd}$$

Let me summarize what is known about the structure of $\varphi_{\mathcal{L}}$ (or, what amounts to the same, of θ). Remember that the dimension of $|2\Theta|$ is $2^g - 1$.

Theorem 4. a) *For $g = 2$, θ is an isomorphism of $\mathcal{SU}_X(2)$ onto $|2\Theta| \cong \mathbf{P}^3$* [N-R1].

b) *For $g \geq 3$, X hyperelliptic, θ is* 2*-to-*1 *onto a subvariety of* $|2\Theta|$ *that can be described in an explicit way* [D-R].

c) *For $g \geq 3$, X not hyperelliptic, θ is of degree one onto its image* [B1]. *Moreover if $g(X) = 3$ or if X is generic, θ is an embedding* ([N-R2]; [L], [B-V]).

The genus 3 (non hyperelliptic) case deserves a special mention: in this case Narasimhan and Ramanan prove that θ is an isomorphism of $\mathcal{SU}_X(2)$ onto a quartic hypersurface $\mathcal{Q}_4$ in $|2\Theta|$ ($\cong \mathbf{P}^7$). By the above remark, this quartic is singular along the Kummer variety $\mathcal{K}_X$. Now it had been observed a long time ago by Coble [C] that there exists a *unique* quartic hypersurface in $|2\Theta|$ passing doubly through $\mathcal{K}_X$! Therefore $\mathcal{Q}_4$ is nothing but Coble's hypersurface.

Part (c) of the theorem leaves open an obvious question:

Question 4. *Is θ always an embedding for X non hyperelliptic?*

The case of a generic curve was proved first by Laszlo [L]; Brivio and Verra have recently developed a more geometric approach [B-V], which might hopefully lead to a complete answer to the question—though some serious technical difficulties remain at this moment.

Laszlo's method is to look at the canonical maps

$$\mu_k : S^k H^0(\mathcal{SU}_X(2), \mathcal{L}) \longrightarrow H^0(\mathcal{SU}_X(2), \mathcal{L}^k) \ .$$

Since we know that $\mathcal{L}^k$ is very ample for some (unknown!) integer k, surjectivity of μ_k for k large enough would imply that $\mathcal{L}$ itself is very ample. We can even dream of getting surjectivity for all k, which would mean that the image of $\mathcal{SU}_X(2)$ in $|2\Theta|$ is projectively normal. In [B2] the situation is completely analyzed for μ_2. Recall that a *vanishing thetanull* on X can be defined as a line bundle L on X with $L^{\otimes 2} \cong \omega_X$ and $h^0(L)$ even ≥ 2—this means that the corresponding theta function on JX vanishes at the origin, hence the name. Such a line bundle exists only on a special curve (more precisely on a divisor in the moduli space of curves). Then:

PROPOSITION 2. *If X has no vanishing thetanull, the map μ_2 is an isomorphism of $S^2H^0(\mathcal{SU}_X(2),\mathcal{L})$ onto $H^0(\mathcal{SU}_X(2),\mathcal{L}^2)$.*

More generally, if X has v vanishing thetanulls, one has

$$\dim \operatorname{Ker} \mu_2 = \dim \operatorname{Coker} \mu_2 = v.$$

This is only half encouraging (it shows that $\mathcal{SU}_X(2)$ is *not* projectively normal for curves with vanishing thetanulls), but note that the case $k = 2$ should be somehow the most difficult. On the positive side we have the following results:

PROPOSITION 3. a) *If X has no vanishing thetanull, the map*

$$\mu_4 : S^4H^0(\mathcal{SU}_X(2),\mathcal{L}) \longrightarrow H^0(\mathcal{SU}_X(2),\mathcal{L}^4)$$

is surjective [vG-P].

b) *If X is* generic*, the map $\mu_k : S^kH^0(\mathcal{SU}_X(2),\mathcal{L}) \longrightarrow H^0(\mathcal{SU}_X(2),\mathcal{L}^k)$ is surjective for k even and $\geq 2g-4$* [L].

As already mentioned, (b) implies that $\varphi_{\mathcal{L}}$ is an embedding for generic X.

5. The Verlinde formula

Trying to understand the maps μ_k raises inevitably the question of the dimension of the spaces $H^0(\mathcal{SU}_X(r),\mathcal{L}^k)$. We have seen that even the case $k = 1$ is far from trivial—this is the essential part of [B-N-R]. So it came as a great surprise when the mathematical physicists claimed to have a general (and remarkable) formula for $\dim H^0(\mathcal{SU}_X(r),\mathcal{L}^k)$, called the Verlinde formula [V] (there is actually a more general formula for the moduli space of principal bundles under a semisimple group, but we will stick to the case of $\mathcal{SU}_X(r)$):

THEOREM 5.

$$\dim H^0(\mathcal{SU}_X(r),\mathcal{L}^k) = \Big(\frac{r}{r+k}\Big)^g \sum_{\substack{S \amalg T = [1,r+k] \\ |S| = r}} \prod_{\substack{s\in S \\ t \in T}} \Big| 2\sin\pi\frac{s-t}{r+k}\Big|^{g-1}.$$

This form of the formula (shown to me by D. Zagier) is the simplest for an arbitrary rank; for small r or k I leave as a pleasant exercise to the reader to simplify it (hint: use $\prod_{p=1}^{n-1}(2\sin(p\pi/n)) = n$). One gets r^g in the case $k = 1$, thus confirming Theorem 2, and in the rank-two case:

Corollary. $\dim H^0(\mathcal{SU}_X(2),\mathcal{L}^k)=(\frac{1}{2}k+1)^{g-1}\sum_{i=1}^{k+1}\frac{1}{(\sin(i\pi/(k+2)))^{2g-2}}\cdot$

Note that the spaces $H^i(\mathcal{SU}_X(r),\mathcal{L}^k)$ vanish for $i>0$, by the Kodaira vanishing theorem (or rather its extension by Grauert and Riemenschneider), since the canonical bundle of $\mathcal{SU}_X(r)$ is equal to $\mathcal{L}^{-2r}$ [D-N]. Hence Theorem 5 gives actually $\chi(\mathcal{L}^k)$. The right-hand side must therefore be a polynomial in k, and take integral values, which is certainly not apparent from the formula! In fact I know no direct proof of these properties, except in the case $r=2$.

The leading coefficient of this polynomial is

$$(c_1(\mathcal{L})^n)/n!,$$

where $n=(r^2-1)(g-1)$ is the dimension of $\mathcal{SU}_X(r)$. This number, which is the volume of $\mathcal{SU}_X(r)$ for any Kähler metric with Kähler class $c_1(\mathcal{L})$, has been computed in a beautiful way by Witten [W1], using the properties of the Reidemeister torsion of a flat connection. The result is

$$\frac{c_1(\mathcal{L})^n}{n!}=r\,(2\pi)^{-2n}\mathrm{Vol}\,(\mathrm{SU}(r))^{2g-2}\sum_V\frac{1}{(\dim V)^{2g-2}}\,,$$

where V runs over all irreducible representations of $\mathrm{SU}(r)$, and the volume of $\mathrm{SU}(r)$ is computed with respect to a suitably normalized Haar measure. One should be able to deduce this formula from Theorem 5, but I don't know how to do that except in rank 2.

6. The Verlinde formula: finite-dimensional proofs

As soon as the Verlinde formula became known to mathematicians, it became a challenge for them to give a rigorous proof, so a wealth of proofs have appeared in the last few years. I will try to describe the ones I am aware of. The basic distinction is between the proofs using standard algebraic geometry, which up to now work only in the case $r=2$, and the proofs that use infinite-dimensional algebraic geometry to mimic the heuristic approach of the physicists—these work for all r. Let me start with the "finite-dimensional" proofs.

The first proof of this kind is due to Bertram and Szenes [B-S], who use the explicit description of the moduli space in the hyperelliptic case (cf. Theorem 4(b)) to compute $\chi(\mathcal{L}^k)$—which is the same for all smooth curves. (Actually they work with the moduli space $\mathcal{SU}_X(2,1)$ of vector bundles of rank 2 and fixed determinant of degree 1, which has the advantage of being smooth; they show that $\chi(\mathcal{L}^k)$ is the Euler–Poincaré characteristic of a certain vector bundle $\mathcal{E}_k$ on $\mathcal{SU}_X(2,1)$.)

A more instructive proof has been obtained by Thaddeus [T], building on ideas of Bertram and Bradlow–Daskalopoulos. The idea is to look at pairs consisting of a rank-two vector bundle E of fixed, sufficiently high degree, say $2d$, together with a nonzero section s of E. There is a notion of stability for these pairs—in fact there are various such notions, depending on an integer i with $0 \leq i \leq d$. For each of these values one gets a moduli space M_i, which is projective and smooth; the key point is that one passes from M_{i-1} to M_i (for $i \geq 2$) by a very simple procedure called a *flip*—blowing up a smooth subvariety and blowing down the exceptional divisor in another direction. Moreover, M_0 is just a projective space, M_1 is obtained by blowing up a smooth subvariety in M_0, while M_{d-1} maps surjectively to $\mathcal{SU}_X(2)$. In short one gets the following diagram

$$\begin{tikzcd}
 & \tilde{M}_2 \arrow[dl] \arrow[dr] & & \tilde{M}_3 \arrow[dl] \arrow[dr] & & \tilde{M}_{d-1} \arrow[dl] \arrow[dr] & \\
M_1 \arrow[d] & & M_2 & & \cdots\cdots & & M_{d-1} \arrow[d] \\
M_0 & & & & & & \mathcal{SU}_X(2)\ ,
\end{tikzcd}$$

from which one deduces (with some highly non-trivial computations) the Verlinde formula.

Another completely different proof has been obtained by Zagier (unpublished). Not surprisingly, it is purely computational. Building on the work of Atiyah–Bott, Mumford and others, Zagier gives a complete description of the cohomology ring of $\mathcal{SU}_X(2,1)$; he is then able to write down explicitly the Riemann–Roch formula for $\chi(\mathcal{E}_k)$ (see above).

Another approach, due, I believe, to Donaldson and Witten, starts from Witten's formula for the volume of $\mathcal{SU}_X(r)$ (§5). More precisely, Witten gives also a formula for the volume of the moduli space of stable *parabolic* bundles; here again the stability depends on certain rational numbers, so we get a collection of volumes indexed by these rational numbers. It turns out that one can recover from these volumes all the coefficients of the polynomial $\chi(\mathcal{L}^k)$.

The last approach of this type I'd like to mention has been developed in [N-Rs] and [D-W1], and carried out successfully in [D-W2]. These authors attempt to prove directly that the spaces $H^0(\mathcal{SU}_X(2), \mathcal{L}^k)$ and the analogous spaces defined using parabolic vector bundles obey the so-called *factorization rules* (see below). Though this approach looks quite promising, the details are unfortunately quite technical, and an extension to higher rank seems out of reach.

7. The Verlinde formula: infinite-dimensional proofs

The idea here is to translate in algebro-geometric terms the methods of the physicists. Actually what the physicists are interested in is a vector space that plays a central role in Conformal Field Theory, the *space of conformal blocks* $B_X^k(r)$. This is defined as follows: let $\mathbf{C}((z))$ be the field of formal Laurent series in one variable. There is a canonical representation V_k of the Lie algebra $\mathfrak{sl}_r\big(\mathbf{C}((z))\big)$ (more precisely, of its universal central extension), called the *basic representation of level* k. Let $p\in X$; the affine algebra $A_X:=\mathcal{O}(X-p)$ embeds into $\mathbf{C}((z))$ (by associating to a function its Laurent expansion at p). Then

$$B_X^k(r):=\{\ell\in V_k^*\mid \ell(Mv)=0 \text{ for all} M\in\mathfrak{sl}_r(A_X) \text{ and } v\in V_k\}\ .$$

THEOREM 6. a) *There is a canonical isomorphism* $H^0(\mathcal{SU}_X(r),\mathcal{L}^k)\xrightarrow{\sim} B_X^k(r)$.

b) *The dimension of both spaces is given by the Verlinde formula* (given in Theorem 5).

There are by now several available proofs of these results. The fact that the dimension of $B_X^k(r)$ is given by the Verlinde formula follows from the work of Tsuchiya, Ueno and Yamada [T-U-Y]. They show that the dimension of $B_X^k(r)$ is independent of the curve X, *even if X is allowed to have double points*. Then it is not too difficult to express $B_X^k(r)$ in terms of analogous spaces for the normalization of X (this is called the *factorization rules* by the physicists). One is thus reduced to the genus-0 case (with marked points), that is, to a problem in the theory of representations of semisimple Lie algebras, which is non-trivial in general (actually I know no proof for the case of an arbitrary semisimple Lie algebra), but rather easy for the case of $\mathfrak{sl}_r(\mathbf{C})$.

Part (a) is proved (independently) in [B-L] and [F2]; actually Faltings proves both (a) and (b). He considers a smooth curve X degenerating to a stable curve X_s. It is not too difficult to show that $H^0(\mathcal{SU}_X(r),\mathcal{L}^k)$ embeds into $B_X^k(r)$, and that the $B_X^k(r)$'s are semicontinuous so that $\dim B_X^k(r)\le\dim B_{X_s}^k(r)$. Therefore the heart of [F2] is the proof of the inequality

$$\dim B_{X_s}^k(r)\le\dim H^0(\mathcal{SU}_X(r),\mathcal{L}^k)\ ,$$

of which I cannot say much, since I don't really understand it (note that the proof, as well as that of [T-U-Y], works in the more general set-up of principal bundles).

I would like to explain in a few words how we construct in [B-L] the isomorphism $H^0(\mathcal{SU}_X(r),\mathcal{L}^k)\xrightarrow{\sim} B_X^k(r)$, because I believe its importance goes far beyond the Verlinde formula. The basic object in the proof is not $\mathcal{SU}_X(r)$, but the *moduli stack* $\mathcal{SL}_X(r)$ parametrizing vector bundles E on X together with a trivialization of $\bigwedge^r E$. Though it appears at first glance as a rather frightening object, it is both more natural and easier to work with than the moduli space: basically, working with the moduli stack eliminates all the artificial problems of non-representability due to the fact that vector bundles have non-trivial automorphisms. The proof (which is entirely algebraic) has three steps:

1) We show that *the moduli stack $\mathcal{SL}_X(r)$ is isomorphic to the quotient stack* $\mathrm{SL}_r(A_X)\backslash \mathrm{SL}_r\big(\mathbf{C}((z))\big)/\,\mathrm{SL}_r(\mathbf{C}[[z]])$. The key point here is that a vector bundle with trivial determinant is *algebraically trivial* over $X-p$. (Hint: show that such a bundle has always a nowhere vanishing section, and use induction on the rank). We choose a small disk $D\subset X$ around p (actually, to avoid convergence problems we take $D=\mathrm{Spec}\,(\mathcal{O})$, where $\mathcal{O}$ is the completed local ring of X at p, but this makes essentially no difference). We then consider triples (E,ρ,σ), where E is a vector bundle on X, ρ an *algebraic* trivialization of E over $X-p$ and σ a trivialization of E over D. Over $D-p$ these two trivializations differ by a holomorphic map $D-p\longrightarrow \mathrm{GL}_r(\mathbf{C})$ that is meromorphic at p, that is, given by a Laurent series $\gamma\in \mathrm{GL}_r\big(\mathbf{C}((z))\big)$. Conversely, given such a matrix γ, one can use it to glue together the trivial bundles on $X-p$ and D and recover the triple (E,ρ,σ). Since we want γ in $\mathrm{SL}_r\big(\mathbf{C}((z))\big)$ we impose moreover that $\bigwedge^r\rho$ and $\bigwedge^r\sigma$ coincide over $D-p$. This gives a bijection of the set of triples (E,ρ,σ) (up to isomorphism) onto $\mathrm{SL}_r\big(\mathbf{C}((z))\big)$.

To get rid of the the trivializations, we have to mod out by the automorphism group of the trivial bundle over D and $X-p$. We get the following diagram:

$$\begin{array}{ccc}
\{E,\,\rho,\,\sigma\} & \longleftrightarrow & \mathrm{SL}_r\big(\mathbf{C}((z))\big) \\
\Big\downarrow & & \Big\downarrow \\
\{E,\rho\} & \longleftrightarrow & \mathcal{Q}:=\mathrm{SL}_r\big(\mathbf{C}((z))\big)/\,\mathrm{SL}_r(\mathbf{C}[[z]]) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\
\{E\} & \longleftrightarrow & \mathrm{SL}_r(A_X)\backslash \mathrm{SL}_r\big(\mathbf{C}((z))\big)/\,\mathrm{SL}_r(\mathbf{C}[[z]])\ .
\end{array}$$

Of course I have only constructed a bijection between the set of isomorphism classes of vector bundles on X with trivial determinant and the set of double classes $\mathrm{SL}_r(A_X)\backslash \mathrm{SL}_r\big(\mathbf{C}((z))\big)/\,\mathrm{SL}_r(\mathbf{C}[[z]])$; with some technical work one shows that the construction actually gives an isomorphism of stacks.

2) Recall that, if $Q=G/H$ is a homogeneous space, one associates to any character $\chi: H\to\mathbf{C}^*$ a line bundle L_χ on Q: it is the quotient of the trivial bundle $G\times\mathbf{C}$ on G by the action of H defined by $h(g,\lambda)=(gh,\chi(h)\lambda)$. We apply this to the homogeneous space $\mathcal{Q}=\mathrm{SL}_r\big(\mathbf{C}((z))\big)/\,\mathrm{SL}_r(\mathbf{C}[[z]])$ (this is actually an

ind-variety, i.e., the direct limit of an increasing sequence of projective varieties). By (1) we have a quotient map $\pi : \mathcal{Q} \longrightarrow \mathcal{SL}_X(r)$. The line bundle $\pi^*\mathcal{L}$ does *not* admit an action of $\mathrm{SL}_r\big(\mathbf{C}((z))\big)$, but of a group $\widehat{\mathrm{SL}}_r\big(\mathbf{C}((z))\big)$ that is a central $\mathbf{C}^*$-extension of $\mathrm{SL}_r\big(\mathbf{C}((z))\big)$. This extension splits over the subgroup $\mathrm{SL}_r(\mathbf{C}[[z]])$, so that $\mathcal{Q}$ is isomorphic to $\widehat{\mathrm{SL}}_r\big(\mathbf{C}((z))\big)/(\mathbf{C}^* \times \mathrm{SL}_r(\mathbf{C}[[z]]))$. Then $\pi^*\mathcal{L}$ is the line bundle L_χ, where $\chi : \mathbf{C}^* \times \mathrm{SL}_r(\mathbf{C}[[z]]) \longrightarrow \mathbf{C}^*$ is the first projection.

3) A theorem of Kumar and Mathieu provides an isomorphism $H^0(\mathcal{Q}, L_\chi^k) \cong V_k^*$. From this and the definition of a quotient stack one can identify

$$H^0(\mathcal{SL}_X(r), \mathcal{L}^k)$$

with the subspace of V_k^* invariant under $\mathrm{SL}_r(A_X)$. This turns out to coincide with the subspace of V_k^* invariant under the Lie algebra $\mathfrak{sl}_r(A_X)$, which is by definition $B_X^k(r)$. Finally a Hartogs type argument gives $H^0(\mathcal{SU}_X(r), \mathcal{L}^k) \cong H^0(\mathcal{SL}_X(r), \mathcal{L}^k)$.

To conclude let me observe that all the proofs I have mentioned are rather indirect, in the sense that they involve either degeneration arguments or sophisticated computations. The simplicity of the formula itself suggests the following question:

Question 5. *Can one find a direct proof of Theorem 5?*

What I have in mind is for instance a computation of $\chi(\mathcal{L}^k)$ by simply applying the Riemann–Roch formula; this requires the knowledge of the Chern numbers of the moduli space. In [W2], Witten proposes some very general conjectures that should give the required Chern numbers for $\mathcal{SU}_X(r)$: the preprint [S] sketches how the Verlinde formula follows from these conjectures. Jeffrey and Kirwan have proved some of the Witten's conjectures, and I understand that they are very close to a proof of the Verlinde formula along these lines.

8. The strange duality

Let me denote by $\mathcal{U}_X^*(k)$ the moduli space of semistable vector bundles of rank k and degree $k(g-1)$ on X; it is isomorphic (non canonically) to $\mathcal{U}_X(k)$. A special feature of this moduli space is that it carries a *canonical* theta divisor Θ_k: set-theoretically one has

$$\Theta_k = \{E \in \mathcal{U}_X^*(k) \mid H^0(X,E) \neq 0\} \ .$$

Put $\mathcal{M} := \mathcal{O}(\Theta_k)$. Consider the morphism $\tau_{k,r} : \mathcal{SU}_X(r) \times \mathcal{U}_X^*(k) \longrightarrow \mathcal{U}_X^*(kr)$ defined by $\tau_{k,r}(E,F) = E \otimes F$. An easy application of the theorem of the square shows that $\tau_{k,r}^*(\mathcal{O}(\Theta_{kr}))$ is isomorphic to $pr_1^*\mathcal{L}^k \otimes pr_2^*\mathcal{M}^r$. Now $\tau_{k,r}^*\Theta_{kr}$ is the divisor of a section of this line bundle, well-defined up to a scalar; by the Künneth theorem we get a linear map $\vartheta_{k,r} : H^0(\mathcal{U}_X^*(k), \mathcal{M}^r)^* \longrightarrow H^0(\mathcal{SU}_X(r), \mathcal{L}^k)$, well-defined up to a scalar. In this section I want to discuss the following conjecture:

QUESTION 6. CONJECTURE: *The map $\vartheta_{k,r}$ is an isomorphism.*

I heard of this statement three or four years ago, as being well-known to the physicists. The conjecture is discussed at length, and extended to vector bundles of arbitrary degree, in [D-T].

Let me discuss a few arguments in favor of the conjecture.

a) The case $k = 1$ is exactly Theorem 2.
b) The two spaces have the same dimension. To prove this one needs to compute the dimension of $H^0(\mathcal{U}_X^*(k), \mathcal{M}^r)$; this is easy (assuming the Verlinde formula!) because the map $\tau_{1,k} : \mathcal{SU}_X(k) \times J^{g-1}(X) \longrightarrow \mathcal{U}_X^*(k)$ is an étale (Galois) covering of degree k^{2g}, and $\tau_{1,k}^*(\mathcal{M}) \cong pr_1^*\mathcal{L} \otimes pr_2^*\mathcal{O}_J(k\Theta)$. Therefore we get
$$\begin{aligned}\dim H^0((\mathcal{U}_X^*(k), \mathcal{M}^r) = \chi(\mathcal{M}^r) &= \frac{1}{k^{2g}}\ \chi(\mathcal{L}^r)\ \chi(\mathcal{O}_J(kr\Theta)) \\ &= \frac{r^g}{k^g}\ \dim H^0((\mathcal{SU}_X(k), \mathcal{L}^r)\ .\end{aligned}$$
Now, Theorem 5 shows that $k^{-g} \dim H^0((\mathcal{SU}_X(k), \mathcal{L}^r)$ is symmetric in k and r, which proves our assertion.
c) Therefore it is enough to prove, for example, the surjectivity of the map $\vartheta_{k,r}$, which has the following geometric meaning:

QUESTION 6$'$. *The linear system $|\mathcal{L}^k|$ in $\mathcal{SU}_X(r)$ is spanned by the divisors Θ_F, for F in $\mathcal{U}_X^*(k)$.*

(Recall from §3 that Θ_F is the locus of vector bundles $E \in \mathcal{SU}_X(r)$ such that $H^0(X, E\otimes F) \neq 0$). As an application, taking vector bundles F of the form $L_1 \oplus \cdots \oplus L_k$ with $L_i \in J^{g-1}(X)$, one deduces from Proposition 3(b) that *the conjecture holds for $r = 2$ and k even and* $\geq 2g-4$ (in this way we get the result for a generic curve only, but using the methods in §9 below I can extend it to every curve).

9. The projective connection

So far we have considered the moduli space $\mathcal{SU}_X(r)$ for a fixed curve X. What can we say about the vector spaces $H^0(\mathcal{SU}_X(r), \mathcal{L}^k)$ when the curve X is allowed to vary? It is again a remarkable discovery of the mathematical physicists that these vector spaces are *essentially independent of the curve*. To explain this in mathematical terms, consider a family of (smooth) curves $(X_t)_{t\in T}$, parametrized by a variety T; for t in T, let us denote by $\mathcal{L}_t$ the determinant line bundle on $\mathcal{SU}_{X_t}(r)$. Then:

Theorem 7. *The linear systems $|\mathcal{L}_t^k|$ define a* **flat** *projective bundle over T.*

Here again we have by now a number of proofs for this result. The first mathematical proof is due to Hitchin [H], following the method used by Welters in the rank-one case; I understand that Beilinson and Kazhdan had a similar proof (unpublished). A different approach, inspired by the work of the physicists, appears in [F1]. Finally, one of the main ingredients in [T-U-Y] is the construction of a flat vector bundle over T whose fibre at $t \in T$ is the space of conformal blocks $B_{X_t}^k(r)$ (the curves of the family are required to have a marked point $p_t \in X_t$, together with a distinguished tangent vector $v_t \in T_{p_t}(X_t)$). Thanks to Theorem 6 this provides still another construction of our flat projective bundle. I have no doubt that all these constructions give the same object, but I must confess that I haven't checked it.

Let us take for T the moduli space $\mathcal{M}_g$ of curves of genus g (here again, the correct object to consider is the moduli stack, but let me ignore this). We get a flat projective bundle over $\mathcal{M}_g$, which corresponds to a projective representation

$$\rho_{r,k} : \Gamma_g \longrightarrow \mathrm{PGL}\big(H^0(\mathcal{SU}_X(r), \mathcal{L}^k)\big)$$

of the fundamental group $\Gamma_g = \pi_1(\mathcal{M}_g, \mathrm{X})$. This group, called the *modular group* by the physicists and the *mapping class group* by the topologists, is a fundamental object: it carries all the topology of $\mathcal{M}_g$. So a natural question is

Question 7. *What is the representation $\rho_{r,k}$?*

This is a rather intriguing question. In the rank-one case, the analogue of $H^0(\mathcal{SU}_X(r), \mathcal{L}^k)$ is the space V_k of k-th order theta functions; the group Γ_g acts on V_k through its quotient $\mathrm{Sp}\,(2g, \mathbf{Z})$ (take this with a grain of salt when k is odd), and this action is explicitly described by the classical "transformation formula" for theta functions—which shows in particular that the action factors through a finite quotient of $\mathrm{Sp}\,(2g, \mathbf{Z})$. Using Theorem 2 we get an analogous description for arbitrary rank in the case $k = 1$. I expect that the general case is far more complicated, and in particular that $\rho_{r,k}$ doesn't factor through $\mathrm{Sp}\,(2g, \mathbf{Z})$, but I know no concrete example where this happens.

Conformal Field Theory predicts that the connection should be (projectively) *unitary*. This is one of the remaining challenges for mathematicians:

Question 8. *Find a flat hermitian metric H_X on* $H^0(\mathcal{SU}_X(r), \mathcal{L}^k)$ (i.e., one *such that the image of $\rho_{r,k}$ is contained in $PU(H_X)$).*

Here again the rank-one case is well-known, and Theorem 2 gives an answer in the case $k = 1$—though not a very explicit one.

10. Are there generalized theta functions?

I believe that the above results give some evidence that the spaces

$$H^0(\mathcal{SU}_X(r), \mathcal{L}^k)$$

are non-abelian analogues of the spaces $H^0(JX, \mathcal{O}(k\Theta))$. There is however one aspect of the picture that is missing so far in higher rank, namely the analytic description of the sections of $\mathcal{O}(k\Theta)$ as holomorphic functions. Clearly the theory of theta functions cannot be extended in a straightforward way, if only because $\mathcal{SU}_X(r)$ is simply connected.

One possible approach is provided by our description of the moduli stack as a double quotient $\mathrm{SL}_r(A_X)\backslash \mathrm{SL}_r\big(\mathbf{C}((z))\big)/\mathrm{SL}_r(\mathbf{C}[[z]])$ (§7). The pullback of the determinant line bundle $\mathcal{L}$ to the group $\widehat{\mathrm{SL}}_r\big(\mathbf{C}((z))\big)$, which is a $\mathbf{C}^*$-extension of $\mathrm{SL}_r\big(\mathbf{C}((z))\big)$, is trivial, so we should be able to express sections of $\mathcal{L}^k$ as functions on $\widehat{\mathrm{SL}}_r\big(\mathbf{C}((z))\big)$. This is done in [B-L] in one particular case: we prove that the pullback of the divisor $\Theta_{(g-1)p}$ is the divisor of a certain algebraic function on $\widehat{\mathrm{SL}}_r\big(\mathbf{C}((z))\big)$ known as the τ function. (This idea appears already in [Br], with a slightly different language.) However, it is not clear how to express other sections of $\mathcal{L}$ (and even less of $\mathcal{L}^k$) in a similar way.

There are other possible ways of describing sections of $\mathcal{L}^k$ by holomorphic functions. One of these, explored by D. Bennequin, is to pull back $\mathcal{L}$ to $\mathrm{SL}_r(\mathbf{C})^g$ by the dominant map $\mathrm{SL}_r(\mathbf{C})^g \dashrightarrow \mathcal{SU}_X(r)$ that maps a g-tuple $(M_1,\dots,M_g)$ to the flat vector bundle $E(\rho_M)\in\mathcal{SU}_X(r)$ associated to the representation $\rho_M : \pi_1(X)\longrightarrow \mathrm{SL}_r(\mathbf{C})$ with $\rho(a_i)=I$, $\rho(b_j)=M_j$ (here $(a_1,\dots,a_g,\, b_1,\dots,b_g)$ are the standard generators of $\pi_1(X)$).

Despite these attempts I am afraid we are still far of having a satisfactory theory of generalized theta functions. So I will end up this survey with a loosely formulated question:

QUESTION 9. *Is there a sufficiently simple and flexible way of expressing the elements of $H^0(\mathcal{SU}_X(r), \mathcal{L}^k)$ as holomorphic functions?*

References

[B1] A. Beauville, *Fibrés de rang 2 sur les courbes, fibré déterminant et fonctions thêta*, Bull. Soc. Math. France **116** (1988), 431–448.

[B2] ______, *Fibrés de rang 2 sur les courbes, fibré déterminant et fonctions thêta II*, Bull. Soc. Math. France **119** (1991), 259–291.

[B-L] A. Beauville and Y. Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), 385–419.

[B-N-R] A. Beauville, M. S. Narasimhan and S. Ramanan, *Spectral curves and the generalized theta divisor*, J. Reine Agnew. Math. **398** (1989), 169–179.

[B-S] A. Bertram and A. Szenes, *Hilbert polynomials of moduli spaces of rank* 2 *vector bundles II*, Topology **32** (1993), 599–609.

[B-V] S. Brivio and A. Verra, *The Theta divisor of* $\mathcal{SU}_C(2)$ *is very ample if* C *is not hyperelliptic and Noether-Lefschetz general*, preprint (1993).

[Br] J. -L. Brylinski, *Loop groups and non-commutative theta-functions*, preprint (1989).

[C] A. Coble, *Algebraic geometry and theta functions*, Amer. Math. Soc. Colloq. Publ., vol. 10, Providence, 1929, third edition, (1969).

[D-W1] G. Daskalopoulos and R. Wentworth, *Local degenerations of the moduli space of vector bundles and factorization of rank* 2 *theta functions* I, Math. Ann. **297** (1993), 417–466.

[D-W2] ———, *Factorization of rank* 2 *theta functions II: proof of the Verlinde formula*, preprint (1994).

[D-R] U. V. Desale and S. Ramanan, *Classification of vector bundles of rank* 2 *on hyperelliptic curves*, Invent. Math. **38** (1976), 161–185.

[D-T] R. Donagi and L. Tu, *Theta functions for* $\mathrm{SL}(n)$ *versus* $\mathrm{GL}(n)$, Math. Res. Let. 1 (1994), 345–357.

[D-N] J. M. Drezet and M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. **97** (1989), 53–94.

[F1] G. Faltings, *Stable G-bundles and projective connections*, J. Algebraic Geom. **2** (1993), 507–568.

[F2] ———, *A proof for the Verlinde formula*, J. Algebraic Geom. **3** (1994), 347–374.

[H] N. Hitchin, *Flat connections and geometric quantization*, Comm. Math. Phys. **131** (1990), 347–380.

[L] Y. Laszlo, *A propos de l'espace des modules des fibrés de rang* 2 *sur une courbe*, Math. Annalen **299** (1994), 597–608.

[N-R1] M. S. Narasimhan and S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. **89** (1969), 19–51.

[N-R2] ———, 2θ*-linear systems on Abelian varieties*, Vector bundles on algebraic varieties, Oxford University Press, 1987, pp. 415–427.

[N-Rs] M. S. Narasimhan and T. R. Ramadas, *Factorization of generalized Theta functions I*, Invent. Math. **114** (1993), 565–623.

[LP] J. Le Potier, *Module des fibrés semi-stables et fonctions thêta*, preprint (1993).

[R] M. Raynaud, *Sections des fibrés vectoriels sur une courbe*, Bull. Soc. Math. France **110** (1982), 103–125.

[S] A. Szenes, *The combinatorics of the Verlinde formula*, preprint (1994).

[T] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. **117** (1994), 317–353.

[T-U-Y] A. Tsuchiya, K. Ueno, and Y. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Adv. Stud. Pure Math. **19** (1989), 459–566.

[V] E. Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nuclear Phys. B **300** (1988), 360-376.

[vG-P] B. van Geemen and E. Previato, *Prym varieties and the Verlinde formula*, Math. Annalen **294** (1992), 741–754.

[We] A. Weil, *Généralisation des fonctions abéliennes*, J. de Math. P. et App. **17** (1938), no. 9$^{\text{ème}}$ sér., 47–87.

[W1] E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys. **141** (1991), 153–209.

[W2] ———, *Two-dimensional gauge theories revisited*, J. Geom. Phys. **9** (1992), 308–368.

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