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Recent Results in Higher-Dimensional Birational Geometry

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Abstract. This note surveys some recent results on higher-dimensional birational geometry, summarising the views expressed at the conference held at MSRI in November 1992. The topics reviewed include semistable flips, birational theory of Mori fiber spaces, the logarithmic abundance theorem, and effective base point freeness.

Contents

1. Introduction

The purpose of this note is to survey some recent results in higher-dimensional birational geometry. A glance to the table of contents may give the reader some idea of the topics that will be treated. I have attempted to give an informal presentation of the main ideas, emphasizing the common grounds, addressing a general audience. In §3, I could not resist discussing some details that perhaps only the expert will care about, but hopefully will also introduce the non-expert reader to a subtle subject.

Perhaps the most significant trend in Mori theory today is the increasing use, more or less explicit, of the logarithmic theory. Let me take this opportunity

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to advertise the Utah book [Ko], which contains all the recent software on log minimal models. Our notation is taken from there.

I have kept the bibliography to a minimum and made no attempt to give proper credit for many results. The reader who wishes to know more about the results or their history could start from the references listed here and the literature quoted in those references.

The end of a proof or the absence of it will be denoted with a $\square$. Anyway here proof always means "proof": a bare outline will be given at best, usually only a brief account of some of what the author considers to be the main ideas.

In preparing the manuscript, I received considerable help from J. Kollár, J. McKernan and S. Mori. The responsibility for mistakes is of course entirely mine.

2. Notation, Minimal Models, etc.

The aim of this section is to introduce the basic notation and terminology to be used extensively in the rest of the paper, and to give a quick reminder of minimal model theory, including the logarithmic theory.

Unless otherwise explicitly declared, we shall work with complex projective normal varieties.

An *integral Weil divisor* on a variety X is a formal linear combination $B = \sum b_i B_i$ with integer coefficients of irreducible subvarieties $B_i \subset X$ of codimension 1. A $\mathbb{Q}$-*Weil divisor* is a linear combination with rational coefficients. We say B is *effective* if all $b_i \geq 0$. We denote with $\lceil B \rceil = \sum \lceil b_i \rceil B_i$ the round-up and with $\lfloor B \rfloor = \sum \lfloor b_i \rfloor B_i$ the round-down of B.

A *Cartier divisor* is a line bundle together with the divisor of a meromorphic section. A $\mathbb{Q}$-Weil divisor B is $\mathbb{Q}$-*Cartier* if mB is Cartier (i.e., the divisor of the meromorphic section of some line bundle) for some integer $m > 0$. Numerical equivalence of $\mathbb{Q}$-Cartier divisors is defined below and is denoted by $\equiv$, while linear equivalence of Weil divisors is denoted by $\sim$.

A Cartier divisor D on X is *nef* if $D \cdot C \geq 0$ for all algebraic curves $C \subset X$.

A Weil divisor is *qef* (quasieffective) if it is a limit of effective $\mathbb{Q}$-divisors. Clearly a nef divisor is qef.

Here are some elementary examples with nef and qef. Let X be a smooth algebraic surface containing no -1 curves, and let K_X be the the canonical class. If K_X is qef, it is also nef; on the other hand, if K_X is not qef, adjunction terminates (that is, $|D + mK| = \varnothing$ for all D, where m is a sufficiently large positive integer) and X must be uniruled (the reader who does not find these assertions obvious is invited to prove them now as an exercise). It is also true, but it appears to be more delicate to show, that if K_X is qef, (a multiple of) K_X is actually effective (and even free from base points).

A variety X is $\mathbb{Q}$-*factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. This property is local in the Zariski but not in the analytic topology, which makes this notion quite subtle and may lead to confusion. However, to avoid sometimes serious technical problems, when running the minimal model algorithm starting with a variety X, we shall always assume that X is $\mathbb{Q}$-factorial.

Let $f : X \dashrightarrow Y$ be a birational map, $B \subset X$ a Weil divisor. Then $f_*B \subset Y$ denotes the *birational transform*. It is the Weil divisor on Y defined as follows. Let $U \subset X$ be an open subset of X with $\operatorname{codim}_X(X \setminus U) \geq 2$ and $f_U : U \to Y$ a morphism representing f. Then f_*B is the (Zariski) closure in Y of $f_{U*}B$. If $B = \sum b_i B_i$ with B_i prime, $f_{U*}B = \sum b_i f_{U*}B_i$, where by definition $f_{U*}B_i = f_U(B_i)$ if $f_U(B_i)$ is a divisor, and $f_{U*}B_i = 0$ otherwise.

Consistently, if $B \subset Y$ is a Weil divisor on Y, we use $f_*^{-1}B$ for $(f^{-1})_*B$. This way we don't confuse it with the set theoretic preimage $f^{-1}(B)$, defined when f is a morphism, or the pullback as a ($\mathbb{Q}$-)Cartier divisor f^*B, which makes sense when f is a morphism and B happens to be a ($\mathbb{Q}$-)Cartier divisor.

I will now describe the minimal model algorithm. The starting point is always a normal projective $\mathbb{Q}$-factorial variety X, together with a $\mathbb{Q}$-Cartier ($\mathbb{Q}$-)divisor D. All or part of D may only be defined up to linear equivalence of Weil divisors (such is the case in the most important example, where $D = K_X$). In practice, some additional conditions are imposed on the pair (X, D) in order for the program to work: for instance, $D = K_X + B$, where B is an effective $\mathbb{Q}$-Weil divisor and (X, B) is log canonical (see Definition (2.1) below).

In the theory of Zariski decomposition, one tries to remove from D the "negative" part, thus writing $D = D' + D''$, where D'' is nef and $H^0(mD'') = H^0(mD)$ for all positive integers m. Instead, we modify X by a sequence of birational operations, removing all configurations in X, where D is negative. We inductively construct a sequence of birational maps $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X'$, and divisors D_i on X_i, in such a way that $H^0(X_{i-1}, mD_{i-1}) = H^0(X_i, mD_i)$ for all i and all positive integers m, and $D_n = D'$ is nef. After introducing the cone of curves with some motivating remarks, I will describe in more detail how this is done.

Let $NS(X) \otimes \mathbb{R} \subset H^2(X, \mathbb{R})$ be the real Néron–Severi group (it is the subgroup of $H^2(X, \mathbb{R})$ generated by the classes of Cartier divisors). Numerical equivalence on $NS(X) \otimes \mathbb{R}$ is defined by setting $D \equiv D'$ if and only if $D \cdot [C] = D' \cdot [C]$ for all algebraic curves $C \subset X$, and we let $N^1(X)$ be the quotient of $NS(X) \otimes \mathbb{R}$ by $\equiv$ (if X has rational singularities, $\equiv$ is trivial on $NS(X) \otimes \mathbb{Q}$). We also define $N_1(X)$ to be the dual real vector space. By definition $N_1(X)$ can be naturally identified with the free real vector space generated by all algebraic curves $C \subset X$, modulo the obvious notion of numerical equivalence. The *Kleiman–Mori* cone $\overline{NE}(X)$ is by definition the closure (in the natural Euclidean topology) of the convex cone $NE(X) \subset N_1(X)$ generated by the (classes of) algebraic curves.

Let X be a normal projective variety, $f : X \to Y$ a projective morphism. If H is any ample divisor on Y and $C \subset X$ a curve, $f^*H \cdot C = 0$ if and only if C is contained in a fiber of f. By the Kleiman criterion for ampleness, this shows that $\{[C] \mid f(C) = pt\}$ generates a *face* $F \subset \overline{NE}(X)$. A one-dimensional face of $\overline{NE}(X)$ is called an *extremal ray*.

Here is how Mori theory works. Start with $(X_0, D_0) = (X, D)$. Assume that the chain $X_0 \overset{t_0}{\dashrightarrow} \cdots X_{k-1} \overset{t_{k-1}}{\dashrightarrow} X_k$ and divisors D_k on X_k have been constructed. If D_k is nef on X_k, we have reached a *D-minimal model*, and the program stops here at $(X', D') = (X_k, D_k)$. Otherwise if D_k is not nef we need to show that $\overline{NE}(X_k)$ is locally finitely generated in $\{z \mid D_k \cdot z < 0\}$ (*cone theorem*), pick an extremal ray $R \subset \overline{NE}(X_k)$ with $D_k \cdot R < 0$, and construct a morphism $f : X_k \to Y$ to a normal variety Y, with the property that a curve $C \subset X$ is contracted by f if and only if $[C] \in R$ (*contraction theorem*). There are three possibilities for f:

1) $\dim(Y) < \dim(X_k)$. In this case we say that X_k, together with the fibration $f : X_k \to Y$, is a *D-Mori fiber space*. The program stops here, and we are happy because we have a strong structural description of the final product.

2) f is birational and the exceptional set of f contains a divisor (it is easy to see that the exceptional set must then consist of a single prime divisor E). In this case we say that f is a *divisorial contraction* and set $t_k = f$, $X_{k+1} = Y$, $D_{k+1} = f_*D_k$ and proceed inductively.

3) f is birational and *small* (or *flipping*)—that is, the exceptional set of f does not contain a divisor. The problem here is that f_*D_k is not $\mathbb{Q}$-Cartier, so it makes no sense to ask whether it is nef. The appropriate modification $t_k : X_k \dashrightarrow X_{k+1}$ is an entirely new type of birational transformation, called to the *D-opposite* or *D-flip* of f, and $D_{k+1} = t_{k*}D_k$. The flip $f' : X_{k+1} \to Y$ is characterized by the following properties: D_{k+1} is ($\mathbb{Q}$-Cartier and) f'-nef, and f' is small.

If every step of the program (cone theorem, contraction theorem and flip theorem) can be shown to exist, and there is no infinite sequence of flips (i.e., *termination of flips* holds), the minimal model algorithm is established, and terminates in a minimal model or a Mori fibration. Presently, this has only been completed in the following cases:

a) $\dim(X) \leq 3$, $D = K_X$, and X has *terminal* or *canonical* singularities.

b) $\dim(X) \leq 3$, $D = K_X + B$, and the pair (X, B) is *log terminal* or *log canonical*.

c) X is a toric variety and $D = K_X$.

Let me recall the definition:

(2.1) Definition. Let X be a variety, $B = \sum b_i B_i$ a Weil divisor with B_i prime divisors and $0 < b_i \leq 1$ (we allow $B = \varnothing$). We say that the pair (X, B) is terminal (resp. canonical, resp. log terminal, resp. log canonical), or that the divisor $K + B$ is terminal (resp. canonical, etc.) if $K_X + B$ is $\mathbb{Q}$-Cartier and for all normal varieties Y and birational morphism $f : Y \to X$ with exceptional prime divisors E_i we have:

$$K_Y + f_*^{-1} B = f^*(K_X + B) + \sum a_i E_i$$

with all $a_i > 0$ (resp. $a_i \geq 0$, resp. $a_i > -1$, resp. $a_i \geq -1$).

The numbers a_i in definition (2.1) only depend on the valuations ν_i of $\mathbb{C}(X)$ associated to E_i, and can be computed on any normal Z with birational $Z \to X$ such that ν_i is a divisor in Z. This way we may define the discrepancy $a(\nu, K_X + B)$ for any (algebraic) valuation ν with small center on X. In this language $K_X + B$ is log terminal if and only if $a(\nu, K_X + B) > -1$ for all ν with small center on X, and likewise for the other adjectives. The following result is very easy to prove but crucial in all questions concerning termination:

(2.2) Theorem. *Let X be a normal variety, and $D = K_X + B$. Let $t : X \dashrightarrow X'$ be a step (divisorial contraction or flip) in the D-minimal model program. Then $a(\nu, K_{X'} + B') \geq a(\nu, K_X + B)$ for all valuations ν with small center in X. Also, $a(\nu, K_{X'} + B') > a(\nu, K_X + B)$ if and only if t is not an isomorphism at the center of ν in X. In particular, if $K_X + B$ is log terminal (terminal, canonical, log canonical), so is $K_{X'} + B'$.*

Proof. The result is an exercise in the definition if t is a divisorial contraction. If t is a flip, see [Ko, 2.28]. □

I wish to spend a few words on the meaning of the D-minimal model program.

First, when X is smooth and $D = K_X$, this is the genuine minimal model program or Mori program: for surfaces, it consists in locating and contracting -1 curves until none can be found.

When X is smooth, B is a (reduced) divisor on X with global normal crossings, and $D = K_X + B$, the D-minimal model program is a sort of Mori program for the open variety $U = X \setminus B$. The (birational) category of open varieties was introduced by Iitaka, and it may be argued that it too is a primitive God-given entity.

For D general, the D-minimal model program should be considered as a way to obtain some kind of "Zariski decomposition" of D. In particular, it is not clear (certainly not to me) a priori that there should be any interesting reason at all to consider general logarithmic divisors $K_X + B$, where $B = \sum b_i B_i$ is allowed to have rational coefficients b_i, $0 < b_i \leq 1$. Nevertheless, these divisors have been profitably used (especially by Kawamata) since the earlier days of the theory, to direct or construct portions of the (genuine) Mori program, especially in relation to flops and flips. They are crucial in Shokurov's approach to the flipping problem [Ko]. Also, as I will try to show in this survey, the general log category is a key tool in several recent advances in higher-dimensional birational geometry.

3. Semistable Flips

This section summarizes some of the results of [Ka] and [S].

The starting point is a projective semistable family of surfaces $f : X \to T$ over a Dedekind scheme T. This means that f is a projective morphism, X is a regular scheme of dimension 3, and all fibers of f are reduced divisors with global normal crossings. The goal is to establish the minimal model algorithm over T:

$$X/T \dashrightarrow X_1/T \dashrightarrow \cdots \dashrightarrow X_i/T \dashrightarrow X_{i+1}/T \dashrightarrow \cdots \dashrightarrow X_N/T = Y/T.$$

The idea is as follows. It may be assumed that T is the spectrum of a DVR; let $S \subset X$ be the central fiber. By assumption $K_X + S$ is log terminal and $K_X + S \equiv K_X$, since S is a fiber so numerically trivial over T. Every step in the K-minimal model program is then a step in the $(K+S)$-minimal model program. In particular, by (2.2), each $K_{X_i} + S_i$ is log terminal. This implies that K_{S_i} too is log terminal, that is, S_i has quotient singularities. The presence of S_i gives a good control of singularities appearing on X_i, so flips are easier to construct than in the general case. The most important message in [Ka] is that a good understanding of the log surface S is all that is needed to establish the minimal model program in this particular setting. Semistable families of surfaces is the simplest situation where a proof of the existence of flips is known. It is important to revisit this proof in the light of the most recent advances in Mori theory, in the hope to gain a better understanding of flips in general.

[Ka] extends the known results to all Dedekind schemes T (with a new construction of flips), and [S] gives a new proof over $\mathbb{C}$, thereby giving an explicit classification of singularities appearing on each intermediate $S_i \subset X_i$ (and hence on the final product Y/T) which, among other things, allows to explain an old result of Kulikov.

From now on, for simplicity of notation, T will be the spectrum of a DVR $\mathcal{O}$ with parameter τ, $f : X \to T$ a projective morphism from a 3-fold, $S \subset X$ the fiber over the closed point. Starting with (3.4), $T = \Delta \subset \mathbb{C}$ is a small disc centered at the origin.

The approach in [Ka] is based on the following classification.

(3.1) Lemma. *Let T be the spectrum of a complete DVR $\mathcal{O}$ with uniformizing parameter τ and algebraically closed residue field $k = \mathcal{O}/(\tau)$. Let $f : X \to T$ be a* *morphism from a three-dimensional scheme X. Assume that f is smooth outside $S = f^*(0)$, and that $K_X + S$ is log terminal (the conditions imply that X has terminal singularities and S is reduced). If T has positive or mixed characteristic, assume moreover that X is Cohen–Macaulay (this is automatic in characteristic 0). Let $x \in S$ be a point, r the index of K_X at x. One of the following alternatives holds, describing the completion $\hat{\mathcal{O}}_{X,x}$ as a $\mathcal{O}$-algebra:*

(3.1.1) $\hat{\mathcal{O}}_{X,x} \cong \mathcal{O}[[x,y,z]]/(xyz - \tau)$,

(3.1.2) $\hat{\mathcal{O}}_{X,x} \cong \mathcal{O}[[x,y,z]]^{\mathbb{Z}_r}/(xy - \tau)$, *where $\mathbb{Z}_r$ acts with weights $(a,-a,1)$ for some $(a,r)=1$.*

(3.1.3) $r > 1$ *and* $\hat{\mathcal{O}}_{X,x} \cong \mathcal{O}[[x,y,z]]^{\mathbb{Z}_r}/(xy - F(z^r))$, *where $\mathbb{Z}_r$ acts with weights $(a,-a,1)$ for some $(a,r)=1$, or*
$r = 1$ *and* $\hat{\mathcal{O}}_{X,x} \cong \mathcal{O}[[x,y,z]]/(G(x,y,z))$.

Let $\overline{F} \in k[z]$ (resp. $\overline{G} \in k[[x,y,z]]$) be the reduction of F (resp. G) mod τ. Then $xy - \overline{F}$ (resp. $\overline{G}$) defines a rational double point, which must then be of type A_{hr} for some h (resp. could be any rational double point).

Proof. The proof goes by analyzing the canonical cover $\pi : (x \in X') \to (x \in X)$. There are two technical problems if $\mathcal{O}$ has positive or mixed characteristic:

a) It is important to know that $S \subset X$ (hence $S' \subset X'$) is S_2. Here we need the assumption that X is Cohen–Macaulay. In characteristic 0 it is known that log terminal singularities are Cohen–Macaulay, but the proof relies on the Grauert–Riemenschneider vanishing theorem, which in general does not hold in positive or mixed characteristic. It is not known, in positive or mixed characteristic, whether log terminal singularities are Cohen–Macaulay in general.

b) The canonical cover depends on the choice of an isomorphism $\mathcal{O}_X \to \omega_X^{[r]}$. A poor choice may produce a nonnormal X'. $\square$

(3.2) Construction. As a consequence of the classification (3.1), one can (partially) resolve singular points $x \in X$ of index $r > 1$ with weighted blow-ups of the ambient toric variety, in a way that will be now described.

First we introduce some terminology. Let $x \in S \subset X$ be a *higher-index* ($r > 1$) singular point of the type described in (3.1.3). Then $x \in S$ is a rational double point of type A_{hr}, and we call h the *complexity* of x (we also say that x is a *simple* point if $h = 1$, or if we are in case (3.1.2)). Also, the integer $n(x) = \operatorname{ord}_\tau F(0)$ is called the *axial multiplicity* of x ($n(x) = 1$ always in case (3.1.2)).

(3.2.1) With the notation of (3.1), if $r > 1$, let $f : Z \to X$ be induced by the weighted blow-up of $\mathcal{O}[x,y,z]$ with weights of (x,y,z,τ) equal to $\frac{1}{r}(a, r-a, 1, r)$.

Let $S' = f^*S$. The following can be checked directly, by a straightforward, if tedious, calculation in toric geometry:

(3.2.1.1) Z is normal and Cohen–Macaulay. Also, $K_Z = f^*K_X + \frac{1}{r}E$, where E is the exceptional divisor. This means that f "extracts" valuation(s) over x with minimal discrepancy.

(3.2.1.2) Assume that $h(x) = 1$, that is, $x \in X$ is a simple point. Then Z has terminal singularities, $K_Z + S'$ is log terminal, and E is irreducible. More precisely, Z has three singular points $z_i \in Z$, and the completed local rings are as follows:

$\hat{\mathcal{O}}_{Z,z_1} \cong \mathcal{O}[[x,y,z]]^{\mathbb{Z}_a}/(xy-\tau)$, with weights $(r,-r,1)$,

$\hat{\mathcal{O}}_{Z,z_2} \cong \mathcal{O}[[x,y,z]]^{\mathbb{Z}_{r-a}}/(xy-\tau)$, with weights $(r,-r,1)$,

$\hat{\mathcal{O}}_{Z,z_3} \cong \mathcal{O}[[x,y,z]]^{\mathbb{Z}_r}/(xy-\frac{1}{\tau}F(\tau z^r))$, where $\mathbb{Z}_r$ acts with weights $(a,-a,1)$.

In particular z_1 and z_2 have index $< r$, and z_3 has index r but axial multiplicity $n(z) = n(x) - 1$. It can thus be reasonably asserted that Z has simpler singularities than X.

If $h(x) > 1$, various elements change the picture just given for the case $h = 1$, all of which are slightly disturbing for the purpose of constructing flips with the method of [Ka]. Specifically, $K_Z + S'$ is not log terminal (but very mildly so), Z does not always have terminal singularities (it always has a cA_{s-2} curve, for $s = \min\{h(x), n(x)\}$), and the exceptional set E has two irreducible components when $n(x) \geq 2$.

(3.2.2) Assume $h = h(x) \geq 2$; so we are in case (3.1.3). Let $m = \operatorname{ord} F$ (if $F = \sum_{j\geq 0} F_j z^{rj}$; by definition $m = \min\{\operatorname{ord}_\tau(F_j) + j \mid j \geq 0\}$). Let $f: Z \to X$ be induced by the weighted blow-up of $\mathcal{O}[x,y,z]$ with weights of (x,y,z,τ) equal to $(i+\frac{a}{r}, m-i-\frac{a}{r}, \frac{1}{r}, 1)$, for any $0 \leq i < m$. Also write, as above, $S' = f^*S$. Then as above Z is normal, Cohen–Macaulay, and $K_Z = f^*K_X + \frac{1}{r}E$. Again $K_Z + S'$ is not log terminal (very mildly so), but at least now Z has terminal singularities and E is irreducible. More precisely, Z has four singular points $z_i \in Z$, and the completed local rings are as follows:

$\hat{\mathcal{O}}_{Z,z_1} \cong \mathcal{O}[[x,y,z]]^{\mathbb{Z}_{ri+a}}/(xy-\tau)$, with weights $(r,-r,1)$,

$\hat{\mathcal{O}}_{Z,z_2} \cong \mathcal{O}[[x,y,z]]^{\mathbb{Z}_{r(m-i)-a}}/(xy-\tau)$, with weights $(r,-r,1)$,

$\hat{\mathcal{O}}_{Z,z_3} \cong \mathcal{O}[[x,y,z]]^{\mathbb{Z}_r}/(xy-\frac{1}{\tau^m}F(\tau z^r))$, where $\mathbb{Z}_r$ acts with weights $(a,-a,1)$,

$z_4 \in Z$ is a cA_l terminal singular point (for some l). It is at this point that $K_Z + S'$ is not log terminal.

To summarize (3.2), one can use the classification of singularities and weighted blow-ups to construct partial resolutions $f: Z \to X$ (perhaps using (3.2.2) if $h > 1$). The blow-up constructed in (3.2.1), in the case $h = 1$, is useful for constructing flips inductively, because (a) singularities on Z have lower index or axial multiplicity, (b) $K_Z + S'$ is again log terminal. The main disadvantage of

(3.2.2) is that, although the singularities $z_i \in Z$ do appear to be "simpler" than $x \in X$, this is difficult to make precise because $z_1 \in Z$ or $z_2 \in Z$ has index $> r$.

Next I state the main result in [Ka]:

(3.3) THEOREM. *Let X/T be a semistable family of surfaces over a Dedekind scheme T. Assume that the chain*

$$X \overset{t_1}{\dashrightarrow} \cdots \dashrightarrow X_{i-1} \overset{t_i}{\dashrightarrow} X_i$$

has been constructed, where each $t_j : X_{j-1} \dashrightarrow X_j$ is the modification (divisorial contraction or flip) associated to an extremal ray $R_{j-1} \subset \overline{NE}(X_{j-1}/T)$ with $K_{X_{j-1}} \cdot R_{j-1} < 0$. Then the cone theorem, on the structure of $\overline{NE}(X_i/T)$, the contraction theorem and the flip theorem hold for X_i/T in the usual way. In particular, if K_{X_i} is not nef over T, there is an extremal contraction $\varphi : X_i \to Z$. If φ is divisorial, set $X_{i+1} = Z$; if φ is small, let $t_{i+1} : X_i \dashrightarrow X_{i+1}$ be the flip. This inductively establishes the minimal model program for X/T.

PROOF. The cone and contraction theorem for X_i are deduced from the cone and contraction theorem for the log surface S_i. Because of this, and because we need the classification (3.1) to establish the existence of flips, we need at each step to check that X_j is Cohen–Macaulay.

Existence of flips goes as follows. Let $\varphi : X_i \to W$ be a flipping contraction. As usual I may assume that the exceptional set consists of a single irreducible curve $C \cong \mathbb{P}^1 \subset X_i$. It can be seen that there is a point $x \in C \subset X_i$, where K_{X_i} has index $r > 1$. Choose x with maximal r.

If $x \in X_i$ looks like (3.1.3), assume moreover that $h = 1$. Let $f : Z \to X_i$ be the blow-up at $x \in X_i$ described in (3.2.1), and $C' \subset Z$ the proper transform of C. A direct calculation taking place on S_i, and involving the classification of surface quotient singularities, shows that $K_Z \cdot C' \leq 0$. Then C' can be flipped or flopped on Z over W. A sequence of flips on Z/W (possibly preceded by a single flop), followed by a divisorial contraction, gives the original flip of $X_i \to W$. Flips on Z exist because Z is simpler (3.2.1.2). The flop at the beginning sometimes occurs, but in this situation it can be easily constructed directly, so no previous knowledge of (terminal) flops is assumed here.

This point deserves to be emphasized a little more. The reader of [Ka] may notice that, as a byproduct of the above mentioned calculation leading to the inequality $K_Z \cdot C' \leq 0$, it is easy to construct a divisor B in a neighbourhood of $C \subset X$ such that $K_X + B$ is log terminal and numerically trivial on C (we do not even need $h = 1$ for this). Traditionally, one is done at this point via a covering trick: if U is a double covering of a neighbourhood of C with branch divisor B, the flip of X is a quotient of the flop of U. Here, however, U has canonical singularities, so we need to know that canonical flops exist for this approach to work. The whole purpose of using the toric blow-up $f : Z \to X$

(with the complications if $h > 1$ soon to be described) is to avoid using canonical flops at this stage (in fact, any knowledge of flops at all).

If $x \in X_i$ looks like (3.1.3) and $h > 1$, a base change followed by a small analytic $\mathbb{Q}$-factorialization reduces to the case $h = 1$. $\square$

In working with the singularities appearing on X_i/T in (3.3), two attitudes are possible. The maximalistic view of [Ka] studies $S_i \subset X_i$ such that $K_{X_i} + S_i$ is log terminal. In the minimalistic view of [S], one tries to understand exactly which $S_i \subset X_i$ can originate from a semistable X/T. This is based on the following observations.

Assume the germ $(x \in S \subset X)$ is analytically isomorphic to the germ $0 \in (xy + z^2 = 0) \subset \mathbb{C}^3$. Although $K + S$ is log terminal and S is Cartier, this germ does not appear on a semistable family of surfaces. If $f : Y \to X$ is an embedded resolution of S, then $f^*S \subset Y$ is not reduced, nor, consequently, semistable (one has to base extend in order to construct a semistable reduction of $S \subset X$).

Now consider $(x \in S \subset X) \cong (0 \in (z = w) \subset (xy + zw = 0))$. Here $S = f^*(0)$ for $f = z - w$, and this singularity certainly appears on the minimal model of a semistable family of surfaces. However $x \in X$ can be resolved with a single small blow-up, also resolving the double point $x \in S$.

This suggests that Du Val singularities should not appear on special fibers of analytic $\mathbb{Q}$-factorializations of minimal models of semistable families of surfaces. Note that if $X \to \Delta$ is projective and $X' \to X$ is an analytic $\mathbb{Q}$-factorialization of X, the composite $X' \to \Delta$ is often not projective.

(3.4) Definition. Let $f : X \to \Delta$ be a not necessarily projective morphism from a complex threefold X to a small disk $\Delta \subset \mathbb{C}$, and let $S = f^*(0)$ be the central fiber. Then f is S-semistable (Shokurov-semistable) if X has terminal singularities and there is a resolution $g : Y \to X$ such that $(fg)^*(0)$ is a global normal crossing divisor.

The above definition has the obvious disadvantage that the conditions are difficult to check. In particular it is not clear at all (but true) that if $X \to \Delta$ is S-semistable and $X/\Delta \dashrightarrow X'/\Delta$ is a flip, then $X' \to \Delta$ is also S-semistable. The next two statements summarize the main results of [S]:

(3.5) Theorem. *Let $f : X \to \Delta$ be S-semistable and projective, and*

$$X/\Delta \dashrightarrow \cdots \dashrightarrow X_i/\Delta \dashrightarrow X_{i+1}/\Delta \dashrightarrow \cdots \dashrightarrow X_N/\Delta$$

a minimal model program for X over Δ. Then each $X_i \to \Delta$ (and so also the final product $X_N \to \Delta$) is S-semistable. $\square$

The following result should be compared with (3.1) above, and represents the shift in philosophy from [Ka] to [S] (the main difference is (3.6.3)):

(3.6) THEOREM. *Let $X \to \Delta$ be S-semistable, and as usual let $S = f^*(0)$ be the central fiber. Let $x \in S \subset X$ be a point, r the index of K_X at x. One of the following holds*:

(3.6.1) *x belongs to exactly three irreducible components of S and the germ $x \in S \subset X$ is analytically isomorphic to*

$$0 \in (xyz = 0) \subset \mathbb{C}^3.$$

(3.6.2) *x belongs to exactly two irreducible components of S and the germ $x \in S \subset X$ is analytically isomorphic to*

$$0 \in (xy = 0) \subset \mathbb{C}^3/\mathbb{Z}_r,$$

where $\mathbb{Z}_r$ acts with weights $(a, -a, 1)$ and $(a, r) = 1$.

(3.6.3) *x belongs to exactly one irreducible component of S. Let $X' \to X$ be a small analytic $\mathbb{Q}$-factorialization of X, $S' \subset X'$ the proper transform, $x_i \in S'$ a singular point lying over x. The germ $x_i \in S' \subset X'$ is analytically isomorphic to*

$$0 \in (t = 0) \subset (xy + z^r + t^{n_i} = 0 \subset \mathbb{C}^4/\mathbb{Z}_r)$$

where $n_i > 0$ is an integer (*this is the axial multiplicity* (3.2)), *$\mathbb{Z}_r$ acts on $\mathbb{C}^4$ with weights $(a, -a, 1, 0)$, and $(a, r) = 1$.* □

In particular, as a consequence of (3.6), S-semistable analytically $\mathbb{Q}$-factorial singularities have a very simple structure.

In [S], the existence of flips and the above results are proved at the same time, by induction on the depth of S-semistable singularities. The *depth* of a S-semistable singularity, $\operatorname{depth}(x, S, X)$, is by definition the minimum number of g-exceptional divisors in a resolution $g : Y \to X$ as in (3.4). The minimum is to be taken among all g's admitting a factorization $g = g_N \circ g_{N-1} \circ \cdots \circ g_1$ in $g_i : Y_i \to Y_{i+1}$ (here $Y = Y_1$ and $X = Y_{N+1}$) such that g_i is projective *locally analytically over* Y_{i+1}. This approach seems to have two main technical nuisances, which I try to briefly describe. On the one hand the minimal model algorithm only works for projective morphisms, so we are not free to analytically $\mathbb{Q}$-factorialize everything. On the other hand, the inductive approach of Shokurov uses certain divisorial extractions that only exist in the analytic category. As a result, one is led to run the minimal model program on varieties that are not necessarily $\mathbb{Q}$-factorial. A typical paradox arising in this context is that sometimes it is necessary to "flip" divisorial contractions.

As a corollary one immediately obtains an old result of Kulikov:

(3.7) Corollary. *Let $X \to \Delta$ be a semistable degeneration of* K3 *surfaces, $X/\Delta \dashrightarrow Y/\Delta$ the minimal model, $Y' \to Y$ a small analytic $\mathbb{Q}$-factorialization, $Y' \to \Delta$ the induced morphism. Then $Y' \to \Delta$ is semistable* (*in the usual sense*).

Proof. The generic fiber Y_η is a minimal K3 surface, so $K_{Y_\eta} \sim 0$. Because K_X is nef over Δ, we have $K_X \equiv 0$. Since the central fiber is reduced, $K_X \sim 0$, and so K_X has index 1 at all singular points. The result now follows instantly from (3.6). $\square$

Note again that $Y' \to \Delta$ in (3.7) is often not projective.

4. Birational theory of Mori fibrations

This section summarizes the results of [C] on birational maps between Mori fibrations in dimension three.

Recall that X admits a Mori fibration if X has $\mathbb{Q}$-factorial terminal singularities and there is a morphism $\varphi : X \to S$ to a lower-dimensional normal S, with $-K_X$ φ-ample and $\rho(X) - \rho(S) = 1$. The condition $\rho(X) - \rho(S) = 1$ is very important and means that a class D on X is the pull-back of a class on S whenever $D \cdot C = 0$ for some curve C contained in a fiber of φ.

Philosophically, the statements arise from the systematic attempt to consider the morphism φ as a built in structure, not merely as an accessory of the variety X.

We work with the following incarnation of the log category. Let X be a variety with $\mathbb{Q}$-factorial terminal singularities, $\mathcal{H}$ a linear system without base divisors on X. The pair $(X, b\mathcal{H})$ is terminal (canonical, etc.) if and only if for all normal varieties Y and birational morphism $f : Y \to X$ with exceptional divisors E_i one has:

$$K_Y + b f_*^{-1}\mathcal{H} = f^*(K_X + b\mathcal{H}) + \sum a_i E_i$$

with all $a_i > 0$ ($a_i \geq 0$, etc.). Here $f_*^{-1}\mathcal{H}$ denotes the linear system without fixed divisors induced on Y by $\mathcal{H}$, and is called the birational transform.

Throughout this section, a birational map Φ and the following notation will be fixed:

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\ S & S' \end{tikzcd}$$

Choose a sufficiently large (and divisible) positive integer μ', and an extremely ample divisor A' on S'. Then the linear system $\mathcal{H}' = |-\mu' K_{X'} + \varphi'^* A'|$ is very ample on X'.

Let $\mathcal{H} = \Phi_*^{-1}\mathcal{H}'$ be the birational transform on X. For some positive rational number μ and (not necessarily ample) class A on S, we have $\mathcal{H} \equiv -\mu K_X + \varphi^* A$

(recall that $\rho(X) - \rho(S) = 1$!).

Let $p : (Y, \mathcal{H}_Y) \to (X, \mathcal{H})$ be a resolution of X and the base locus of $\mathcal{H}$.

The situation is summarized in the following diagram:

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & & \mathcal{H}_Y = q^*\mathcal{H}' \\ \mathcal{H} \equiv -\mu K_X + \varphi^* A \quad X \arrow[rr, dashed, "\Phi"] \arrow[d, "\varphi"'] & & X' \arrow[d, "\varphi'"] & \mathcal{H}' = \mid -\mu' K_{X'} + \varphi'^* A' \mid \\ S & & S' & \end{tikzcd}$$

The next result is the key to understand the numerical geometry of Φ. Recall that a class in $H^2_{\mathbb{R}}$ is *qef* (quasieffective) if it is a limit of classes of effective $\mathbb{Q}$-divisors.

(4.1) Theorem (Nöther–Fano inequalities).

(4.1.1) *$\mu \geq \mu'$, and equality holds only if Φ induces a rational map $S \dashrightarrow S'$ and Φ^{-1} is contracting.*

(4.1.2) *If $K_X + \frac{1}{\mu}\mathcal{H}$ is canonical and qef, Φ is an isomorphism in codimension one, and it induces an isomorphism $X_\eta \cong X'_{\eta'}$ of generic fibers. In particular, $\mu = \mu'$.*

(4.1.3) *If $K_X + \frac{1}{\mu}\mathcal{H}$ is canonical and nef, Φ is an isomorphism, and it also induces an isomorphism $S \cong S'$. In particular, $\mu = \mu'$.*

Proof. Let's prove (4.1.1) first. We have

$$K_Y + \frac{1}{\mu'}\mathcal{H}_Y = q^*(K_{X'} + \frac{1}{\mu'}\mathcal{H}') + \sum a'_i E_i = q^*\varphi'^* A' + \sum a'_i E_i,$$

with all $a'_i > 0$ because X' has terminal singularities. Also

$$K_Y + \frac{1}{\mu'}\mathcal{H}_Y = p^*(K_X + \frac{1}{\mu'}\mathcal{H}) + \sum a_i E_i,$$

where I know nothing of a_i. Let $C \subset X$ be a general curve contained in a fiber of φ, let $C' \subset X'$ be the transform in X' and C'' the transform in Y. Then

$$(K_X + \frac{1}{\mu'}\mathcal{H}) \cdot C = (K_Y + \frac{1}{\mu'}\mathcal{H}_Y) \cdot C'' = \varphi'^* A' \cdot C' + \sum a'_i E_i \cdot C'' \geq 0.$$

Thus $\mu \geq \mu'$, and $\mu = \mu'$ only if C' is contained in a fiber of φ' and every q-exceptional divisor is also p-exceptional, i.e., Φ^{-1} is contracting.

To prove (4.1.2), reverse the above argument. This gives $\mu = \mu'$. Then (4.1.1) and some extra work imply that Φ is an isomorphism in codimension one.

One has to work a little harder to get (4.1.3). This is standard once the statement of what we wish to prove is known. ☐

The intent is to think of μ as the "degree" of Φ. The plan is to define a class of elementary maps, and use (4.1) to construct an elementary map $\Psi : X/S \dashrightarrow X_1/S_1$ such that $\Phi_1 = \Psi^{-1} \circ \Phi : X_1/S_1 \dashrightarrow X/S$ has degree $\mu_1 < \mu$. Note that the definition itself of μ makes crucial use of the fibration $X \to S$ and the property $\rho(X) - \rho(S) = 1$, while the proof of (4.1) also uses the similar structure of X'. To emphasize even more these concepts, I observe that the elementary $\Psi : X/S \dashrightarrow X_1/S_1$ might very well be the identity map of X, only changing the Mori fibration structure $X \to S$ to $X \to S_1$ (see links of type IV below). Indeed, I win in this game if I can reduce the degree μ. For instance, in the surface case, if $X \to S \cong \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\pi_1} \mathbb{P}^1$ is the first projection, the degree μ equals $\frac{1}{2}\mathcal{H} \cdot \pi_1^{-1}(*)$, and it might well be that the degree is smaller when measured with respect to the second projection $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\pi_2} \mathbb{P}^1$.

The following is the most important statement of [C]. It is due to Sarkisov, as is the philosophy of its proof.

(4.2) Theorem. *Let $\phi : X/S \dashrightarrow X'/S'$ be a birational map between Mori fibrations, with* $\dim(X) \leq 3$. *There is a factorization $\Phi = \Phi_n \cdots \Phi_1$:*

$$\begin{tikzcd}
X \arrow[r, dashed, "\Phi_1"] \arrow[d] & \cdots & X_i \arrow[r, dashed, "\Phi_i"] \arrow[d] & X_{i+1} \arrow[d] & \cdots \arrow[r, dashed, "\Phi_n"] & X' \arrow[d] \\
S & & S_i & S_{i+1} & & S'
\end{tikzcd}$$

where $\Phi_i : X_i/S_i \dashrightarrow X_{i+1}/S_{i+1}$ is one of the following elementary maps:

(4.2.1) Links of type I. *They are commutative*:

$$\begin{tikzcd}
& Z \arrow[r, dashed] \arrow[dl] & X' \arrow[dd] \\
X \arrow[d] & & \\
S & & S' \arrow[ll]
\end{tikzcd}$$

where $Z \to X$ is a Mori extremal divisorial contraction and $Z \dashrightarrow X'$ a sequence of Mori flips, flops or inverse Mori flips. Note that $\rho(S') - \rho(S) = 1$.

(4.2.2) Links of type II. *They are commutative*:

$$\begin{tikzcd}
& Z \arrow[r, dashed] \arrow[dl] & Z' \arrow[dr] & \\
X \arrow[d] & & & X' \arrow[d] \\
S & & & S
\end{tikzcd}$$

where $Z \to X$ and $Z' \to X'$ are Mori extremal divisorial contractions, and $Z \dashrightarrow Z'$ a sequence of Mori flips, flops or inverse Mori flips. The link induces here an isomorphism $S \cong S'$.

(4.2.3) Links of type III. *They are commutative*:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dd] & Z \arrow[d] \\ & X' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

where $X \dashrightarrow Z$ is a sequence of Mori flips, flops or inverse Mori flips, and $Z \to X'$ a Mori extremal divisorial contraction. Note here that $\rho(S) - \rho(S') = 1$.

(4.2.4) Links of type IV. *They are commutative*:

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[d] & & X' \arrow[d] \\ S \arrow[dr] & & S' \arrow[dl] \\ & T & \end{tikzcd}$$

where $X \dashrightarrow X'$ is a sequence of Mori flips, flops or inverse Mori flips. Here $\rho(S) - \rho(T) = \rho(S') - \rho(T) = 1$.

PROOF. There are two cases:

a) If $K_X + \frac{1}{\mu}\mathcal{H}$ does not have canonical singularities, let $c < \frac{1}{\mu}$ be the maximum such that $K_X + c\mathcal{H}$ has canonical singularities. There is an extremal blow-up $f : Z \to X$ with exceptional E such that

$$K_Z + cf_*^{-1}\mathcal{H} = f^*(K_X + c\mathcal{H}).$$

A minimal model program for $K_Z + cf_*^{-1}\mathcal{H}$ over S produces the desired Φ_1 of type I or II.

b) If $K_X + \frac{1}{\mu}\mathcal{H}$ does have canonical singularities, either we are done (4.1.3) or $K_X + \frac{1}{\mu}\mathcal{H}$ is not nef. A minimal model program for $K_X + \frac{1}{\mu}\mathcal{H}$ gives Φ_1 of type III or IV. Care has to be paid not to lose S.

It remains to show that repeated application of steps (a) and (b) above eventually gives a factorization. It is easy to see that μ decreasesafter "untwisting" by Φ_1, and that it strictly decreases unless the base locus of $\mathcal{H}$ has improved by a measurable bit. These bits however may get smaller and smaller. In [C] this is the major technical difficulty. The proof uses results of [Ko, Ch. 18] and Alexeev, showing that certain sets satisfy the ascending chain condition. [Ko, Ch 18] contains some conjectures along these lines, [Ko4] explores a somewhat different direction (see also [A]). $\square$

I will make some comments on the last statement and the future ambitions of this theory.

First, one should ask why the transformations I–IV above should be considered "elementary". Let me once again emphasize that the main philosophical point here is that the Mori fibration morphism is the relevant structure. So whatever an elementary map is, it has to "link" a Mori fibration $X \to S$ to another Mori fibration $X' \to S'$. So, for example, a blow-up $Z \to X$ of the maximal ideal of a smooth point in X should not be considered an elementary map, unless we are in the unlikely case that Z admits a Mori fibration $Z \to S'$ of its own. Elementary transformations of ruled surfaces then show that we need to allow at least two divisorial contractions to form an elementary map. Examples show that in dimension ≥ 3 flops and flips (hence inverse flips) are necessary. This explains that elementary maps satisfy some reasonable economy criterion.

It is to be hoped that the factorization theorem will eventually lead to usable criteria to determine the birational type of a given space admitting a Mori fibration. This definitely requires a better understanding of elementary maps. The idea is that the graph $\Gamma_\Phi \subset X \times X'$ of an elementary map $\Phi : X/S \dashrightarrow X'/S'$ is a relative $\mathbb{Q}$-Fano model with $\rho = 2$ (over S and/or S') and these tend to fit in a finite number of algebraic families. This however is not true because of the possibility of inverse flips, whose role has yet to be clarified. It seems that a close understanding (perhaps a classification) of divisorial contractions would also be necessary.

The following is taken from [I]:

(4.3) Conjecture. *Let $X \to S$ be a standard conic bundle, i.e., X is smooth and $X \to S$ the contraction of an extremal ray. Assume that X is rational. Then the quasieffective threshold $\tau = \tau(S, \Delta)$—that is, the maximum value τ such that $\tau K + \Delta$ is quasieffective—is less than* 2.

The idea is that the quasieffective threshold of the pair (S, Δ) has a strong birational meaning for X.

5. Log abundance

This is the statement of the log abundance theorem for threefolds, proved in [KMM].

(5.1) Theorem. *Let (X, B) be a pair consisting of a threefold X and boundary*

$B \subset X$. If $K+B$ is log canonical and nef, $|m(K+B)|$ is free from base points for some sufficiently large (and divisible) m. $\square$

The genuine (i.e., non logarithmic) abundance theorem states that if X is a three-dimensional minimal model (recall that this means that X has terminal singularities and K_X is nef), $|mK_X|$ is free from base points for some m large and divisible.

Recall that the *Kodaira dimension* $\kappa(D)$ of a divisor D on a variety X is the dimension of $\varphi_{mD}(X)$ for all sufficiently large and divisible integers m (φ_{mD} : $X \dashrightarrow \mathbb{P}H^0(mD)^*$ is the canonical map; by convention $\kappa(D) = -\infty$ if $H^0(mD) = (0)$ for all positive integers m). Also, the *numerical dimension* of a nef $\mathbb{Q}$-Cartier divisor D on a variety X is by definition the largest integer $\nu(D)$ such that $D^\nu \not\equiv 0$. It is easy to show that $\kappa \leq \nu$. It is also easy to see that log abundance is equivalent to $\nu(K+B) = \kappa(K+B)$. Log abundance for $\nu(K+B) = \dim(X)$ is an immediate consequence of the base point free theorem.

The proof of the (genuine) abundance theorem for threefolds (mainly due to Kawamata and Miyaoka) is very long and complicated: it can be found in [Ko], together with proper attributions. Roughly speaking, the proof is divided in two parts, requiring entirely different techniques. Here is a quick summary:

A) First one shows that $|mK| \neq \varnothing$ for some m. This is quite hard, and I refer to [Ko, Ch. 9] for an extremely attractive presentation.

B) We must eventually show that mK_X is free for m large and divisible. By (A) there is a divisor $D \in |mK|$. Note that if $\nu = 0$ we are done already, so we need to discuss $\nu = 1$ and $\nu = 2$. Roughly speaking, the argument proceeds by induction on the dimension: we show that a form of abundance holds on D, then try to use exact sequences to lift it to X. However, D may be too singular to work with. The first step is to modify X to improve the singularities of the support of D, namely to achieve that $K + D_{\mathrm{red}}$ is log canonical. This is achieved by choosing a log resolution X', D' of X, D_{red}, and running a $(K_{X'}+D')$-minimal model program, ending in the new X'', D''. Then $K_{X''} + D''$ is log terminal and D'' is semi log canonical. So, perhaps at the expense of introducing slightly worse singularities on X, we may assume that $K_{D_{\mathrm{red}}}$ is semi log canonical.

If $\nu = 1$ (here we expect the canonical map to give a fibration in surfaces), by a further reduction step, using again log minimal model theory, we may assume that every connected component of D_{red} is irreducible. Then D_{red} is a surface with log terminal singularities and it is not too hard to show that log abundance for D_{red} implies abundance for X. An excellent introduction to the yoga is [Ko, Ch. 11], where log abundance is proved for surfaces with log canonical singularities.

The situation is technically more complicated if $\nu = 2$. Here D_{red} is a surface with semilog canonical singularities. Abundance for D_{red} requires some careful considerations [Ko, Ch. 12]. To show abundance for X, it is enough to show that $H^0(mK_X)$ has at least two sections for m large (i.e., $\kappa \geq 1$). This uses log abundance on D_{red} and some delicate estimates of the relevant terms in the Riemann–Roch formula.

In closing, I wish to emphasize that even to prove genuine abundance one naturally dips in the log category.

The proof of the log abundance theorem for threefolds, part (B), follows closely the proof of genuine abundance, except that it is more difficult at various technical points, especially in the proof that $\nu = 2$ implies $\kappa \geq 1$ in showing the positivity of the relevant terms in the Riemann–Roch formula.

However, showing that $|m(K+B)| \neq \varnothing$ for some m requires some entirely new ideas, which I shall now describe following [KMM].

(5.2) THEOREM. *Let (X, B) be a pair consisting of a threefold X and boundary $B \subset X$. If $K+B$ is log canonical and nef, $|m(K+B)| \neq \varnothing$ for some positive integer m.*

PROOF. a) First construct a terminal modification $f : Z \to X$. This means that Z has terminal singularities,

$$K_Z + f_*^{-1}B + \sum a_i E_i = f^*(K_X + B)$$

with $0 \leq a_i \leq 1$ and $K_Z + B_0$ is log canonical when $B_0 = f_*^{-1}(B) + \sum a_i E_i$.

b) Next run an (ordinary) minimal model program for K_Z. If K_Z is not nef there is a smallest value $0 < \varepsilon \leq 1$ such that $K_Z + \varepsilon B_0$ is nef, and an extremal ray R with $K_Z \cdot R < 0$ and $(K_Z + \varepsilon B_0) \cdot R = 0$. Let $Z \dashrightarrow Z_1$ be the divisorial contraction or flip of R, and $B_1 \subset Z_1$ the proper transform. Then inductively define a chain

$$\cdots (Z_i, \varepsilon_i B_i) \dashrightarrow (Z_{i+1}, \varepsilon_{i+1} B_{i+1}) \dashrightarrow \cdots \dashrightarrow (Z_N, \varepsilon_N B_N) = (X', B'),$$

where $\varepsilon_i \leq \varepsilon_{i-1}$ is the smallest value such that $K_{Z_i} + \varepsilon_i B_i$ is nef, and $Z_i \dashrightarrow Z_{i+1}$ the divisorial contraction or flip associated to an extremal ray $R_i \subset \overline{NE}(Z_i)$ with $K_{Z_i} \cdot R_i < 0$ and $(K_{Z_i} + \varepsilon_i B_i) \cdot R_i = 0$. It is quite clear that $|m(K_Z + B_0)| \neq \varnothing$ for some m if $|m(K_{X'} + B')| \neq \varnothing$ for some m. There are two cases:

b1) X' is a minimal model and $B' = \varnothing$. Here $|mK_{X'}| \neq \varnothing$ by genuine abundance.

b2) X' has a Mori fibration $X' \to S$ with $(K_{X'} + B') \cdot C = 0$ for a curve C contained in a fiber. The result is immediate if X' is a $\mathbb{Q}$-Fano threefold, but the other cases still require considerable work. From now on, I assume (b2) holds.

c) The strategy at this point is to produce sections of $|m(K_{X'/S}+B')|$. It is easier to do this because the relative dualizing sheaf $K_{X'/S}$ behaves well with respect to fiber squares. That is, if $Y' = X' \times_S T$ in

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\varphi'"'] & X' \arrow[d, "\varphi"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

then $K_{Y'/T} = f'^* K_{X'/S}$. By choosing a suitable finite $T \to S$ I can "untwist" X'/S, and going again through (b), I may assume that $X' \to S$ is a $\mathbb{P}^1$-bundle over a surface or a $\mathbb{P}^2$-bundle over a curve, and argue there directly.

d) Now assume $|m(K_{X'/S}+B')| \neq \varnothing$. This means that there is an effective $\mathbb{Q}$-divisor D on S such that $\varphi^*(D) = K_{X'/S} + B'$. Since $K_{X'} = K_{X'/S} + \varphi^* K_S$, this says that $\varphi^*(K_S + D) = K_{X'} + B'$. It can be shown that $K_S + D$ is log terminal, and since it is nef, we are done by log abundance on S. ☐

The log abundance theorem is a very high-brow generalization of the genuine abundance theorem. Its meaning has to be found in the context of log minimal model theory, as a natural extension of minimal model theory. One very interesting feature peculiar to log abundance as opposed to genuine abundance is the analysis of Mori fibrations. The proof just described provides some new tools to study numerical aspects of curves and divisors on Mori fibrations. Let $X \to S$ be a Mori fibration, $\mathcal{H}$ a linear system without base divisors on X, and assume $K_X + \frac{1}{\mu}\mathcal{H}$ to be trivial on fibers. Theorem (5.2) suggests that $K_X + \frac{1}{\mu}\mathcal{H}$ is likely to be quasieffective, which explains the experimental fact that the possibility of X admitting a birationally distinct Mori fibration should be deemed unlikely (compare 4.1.2).

6. Effective base point freeness

A lot of work has been done recently on finding effective nonvanishing and global generation results for pluricanonical (nK) and adjoint ($K+L$) divisors in higher dimension [EL], [Ko2], and [Ko3]. The common ground of most of this work is an attempt to render the "Kawamata technique" effective.

This is a technically simplified version of the main result in [EL]:

(6.1) Theorem. *Let L be a nef and big divisor on a smooth complex projective threefold X, and let $x \in X$ be a given point. Assume*:

(6.1.1) $L^3 > 92$,

(6.1.2) $L^2 \cdot S \geq 7$ *for all surfaces $S \subset X$ with $S \ni x$,*

(6.1.3) $L \cdot C \geq 3$ *for all curves $C \subset X$ with $C \ni x$.*

Then K_X+L is free at x, that is, $\mathcal{O}(K+L)$ has a section that is nonvanishing at x. Moreover, if $L^3 \gg 0$ (e.g., $L^3 \geq 1000$), the same conclusion holds with $L^2 \cdot S \geq 5$. □

The proof of the above result is quite complicated. The Kawamata technique is still in a stage where it is rather difficult to use it to prove things, even when one has a good idea of what should be proved. In the case at hand, the general form of the statement is suggested by earlier results of Reider on algebraic surfaces (proved with Bogomolov's stability of vector bundles). As an introduction to the ideas, following [EL], I will state and completely prove a Reider type statement on surfaces:

(6.2) Theorem. *Let L be a nef and big divisor on a smooth complex projective surface X, and let $x \in X$ be a given point. Assume:*

(6.2.1) $L^2 \geq 5$,

(6.2.2) $L \cdot C \geq 2$ *for all curves* $C \subset X$ *with* $C \ni x$.

Then K_X+L is free at x, that is, $\mathcal{O}(K+L)$ has a section that is nonvanishing at x.

Proof. One starts by writing the divisor we are interested in, $K_X + L$, in the form:

$$K + L = K + B + M$$

where $B \geq 0$ is a nef $\mathbb{Q}$-divisor, M a nef and big $\mathbb{Q}$-divisor. The method works by choosing B very singular at x, and relating the singularity of B at x to sections of $K + L$ at x.

The leading term of Riemann–Roch and (6.2.1) give $D \in |nL|$ with a singular point of multiplicity at least $2n + 1$ at x for some n large. Put $B = \frac{1}{n}D$, and write $B = B' + \sum b_i B_i$, where the B_i's are the irreducible components of B containing x. Let μ_i be the multiplicity of B_i at x. By construction,

$$b = \sum b_i \mu_i > 2 \qquad (*).$$

I discuss two cases:

a) $b > 2b_i$ for all i.

b) There is a value i_0, say $i_0 = 0$, such that $b \leq 2b_0$. In particular, $b_0 > 1$. What happens here is that we chose D to be very singular at x, and unexpectedly got a D that has very high multiplicity along all of B_0. It seems that this occurrence should be considered lucky, but it is the hardest to work with.

a) Let $f : Z \to X$ be the blow-up at x, and E the exceptional curve. Then:

$$f^* K_X = K_Z - E$$

and $f^*B = f_*^{-1}B + bE$, so for all c we have

$$\begin{aligned} f^*(K_X + L) &= f^*(K_X + cB) + (1-c)f^*B \\ &= K_Z + cf_*^{-1}B + (cb-1)E + (1-c)f^*B. \end{aligned}$$

Set $c = 2/b$. Then $M = (1-c)f^*B$ is a nef and big $\mathbb{Q}$-divisor. Also, by assumption, $cb_i < 1$. Then

$$N = f^*(K+L) - \lfloor cf_*^{-1}B \rfloor = K_Z + \lceil M \rceil + E.$$

By Kodaira vanishing $H^0(N) \twoheadrightarrow H^0(N|_E)$. We are done because $N|_E \sim \mathcal{O}_E$ and $H^0(K_X + L) \hookrightarrow H^0(N|_E)$.

b) In this case with $c = 1/b_0$ and $M = (1-c)B$,

$$N = K_X + L - \lfloor cB' \rfloor = K_X + \lceil M \rceil + B_0.$$

By Kodaira vanishing $H^0(N) \twoheadrightarrow H^0(N|_{B_0})$, so we are done as soon as we prove that $H^0(N|_{B_0})$ has a section not vanishing at x. This is the ugly part, where it is very helpful to already know the statement we wish to prove. The observation here is that

$$N|_{B_0} = K_{B_0} + (1-c)B|_{B_0} + \sum_{i>0} cb_iB_i|_{B_0}$$

is an integral divisor, and

$$[(1-c)B + \sum_{i>0} cb_iB_i] \cdot B_0 > 2(1-c) + \sum_{i>0} cb_i\mu_i = 1 + c(\sum_{i\geq 0} b_i\mu_i - 2) > 1$$

by $(*)$ applied twice. The statement then follows from the one-dimensional analogue of (6.2) (more precisely, one needs (6.2) for Gorenstein curves). $\square$

In higher dimensions matters quickly become more complicated, seemingly due to the possibility of case (b) above, especially in conjunction to the more complicated, but necessary in dimension ≥ 3, hypotheses in the Kawamata–Viehweg vanishing theorem. The reader may show as an exercise that case (b) never occurs if $L \cdot C \geq 2$ is replaced with $L \cdot C \geq 3$ in (6.2.2) (I learned this observation from R. Lazarsfeld, who attributes it to Demailly).

Finally, I wish to make a philosophical remark. The log category intervened "behind the scenes" in the proof of (6.2). Indeed, we chose B such that $K + B$ is not log canonical at x, and c the maximum such that $K + cB$ is log canonical at x, in order to isolate the base component with "maximal multiplicity".

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