Complex Algebraic Geometry
MSRI Publications
Volume **28**, 1995

The Schottky Problem: An Update

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Abstract. The aim of these lecture notes is to update Beauville's beautiful 1987 Séminaire Bourbaki talk on the same subject. The Schottky problem is the problem of finding characterizations of Jacobians among all principally polarized abelian varieties. We review the numerous approaches to this problem. In the "analytical approach", one tries to find polynomials in the thetaconstants that define the Jacobian locus in the moduli space of principally polarized abelian varieties. We review van Geemen's (1984) and Donagi's (1987) work in that direction. The loci they get contain the Jacobian locus as an irreducible component. In the "geometrical approach", one tries to give geometric properties that are satisfied only by Jacobians. We review the following: singularities of the theta divisor (Andreotti and Mayer 1967); reducibility of intersections of a theta divisor with a translate (Welters 1984) and trisecants to the Kummer variety (Welters 1983, Beauville and Debarre 1986, Debarre 1992); the K–P equation and Novikov's conjecture (Shiota 1986, Arbarello and De Concini 1984); double translation hypersurfaces (Little 1989); the van Geemen–van der Geer conjectures on the base locus of the set of second order theta functions that vanish with multiplicity ≥ 4 at the origin (Beauville and Debarre 1989, Izadi 1993); subvarieties with minimal class (Ran 1981, Debarre 1992); the Buser–Sarnak approach (1993).

Introduction

For $g \geq 2$, the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g has dimension $\frac{1}{2}g(g+1)$ and the moduli space $\mathcal{M}_g$ of smooth connected projective algebraic curves of genus g has dimension $3g-3$. To any such curve C is associated a principally polarized abelian variety of dimension g, its Jacobian (JC, θ). This defines a map

$$\mathcal{M}_g \longrightarrow \mathcal{A}_g,$$

Partially supported by NSF Grant DMS 9203919 and the European Science Project "Geometry of Algebraic Varieties", Contract no. SCI-0398-C (A).

which is injective (Torelli theorem). The closure $\mathcal{J}_g$ of its image is equal to $\mathcal{A}_g$ only for $g = 2$ or 3. For $g \geq 4$, it is a proper closed subset. The Schottky problem is the problem of finding characterizations of Jacobians among all principally polarized abelian varieties (for $g \geq 4$). For the sake of simplicity, we will say that a property is a *weak characterization* of Jacobians if $\mathcal{J}_g$ is an irreducible component of the set of principally polarized abelian varieties with this property.

1. The analytical approach

Certain modular forms, called "thetaconstants", define an embedding of a finite cover of $\mathcal{A}_g$ into some projective space [I]. The idea is to use the corresponding "coordinates" to give equations of $\mathcal{J}_g$ in $\mathcal{A}_g$. A brief history of this approach goes as follows:

Schottky [S, 1888] found a polynomial of degree 16 in the thetaconstants that vanishes on $\mathcal{J}_4$ but not on $\mathcal{A}_4$.

Schottky and Jung [SJ, 1909], starting from polynomials in the thetaconstants that vanish on $\mathcal{A}_{g-1}$, constructed polynomials that vanish on $\mathcal{J}_g$.

Igusa [I, 1981] and **Freitag** [F, 1983] proved that the divisor defined by the Schottky polynomial is *irreducible*, hence equal to $\mathcal{J}_4$.

Van Geemen [vG, 1984] proved that $\mathcal{J}_g$ is an irreducible component of the locus $\mathcal{S}_g$ defined by the Schottky–Jung polynomials.

Donagi [Do1, 1984] remarked that the Schottky–Jung polynomials depend on the choice of a point of order two on the abelian variety, hence that they define in fact two loci $\mathcal{S}_g$ and $\mathcal{S}_g^{\text{big}}$, depending on whether one considers all or just one single such point. They satisfy

$$\mathcal{J}_g \subset \mathcal{S}_g \subset \mathcal{S}_g^{\text{big}}.$$

While van Geemen proved that $\mathcal{J}_g$ is a component of $\mathcal{S}_g$, Donagi proved that it is also a component of $\mathcal{S}_g^{\text{big}}$, and that $\mathcal{J}_g \neq \mathcal{S}_g^{\text{big}}$ for $g \geq 5$ [Do2]. For example, intermediate Jacobians of cubic threefolds are in $\mathcal{S}_5^{\text{big}}$, but not in $\mathcal{S}_5$. Donagi has a conjecture on the structure of $\mathcal{S}_g^{\text{big}}$ for any g, which would imply $\mathcal{S}_g = \mathcal{J}_g$. He has announced in [Do3] a proof of this conjecture for $g = 5$.

2. The geometrical approach

The idea here is to give geometric properties of a principally polarized abelian variety that are satisfied only by Jacobians.

a) Singularities of the theta divisor. If Θ is a theta divisor on a g-dimensional Jacobian, $\dim \operatorname{Sing} \Theta \geq g - 4$. Andreotti and Mayer proved [AM, 1967] that this property is a *weak characterization* of Jacobians. However, the locus in $\mathcal{A}_g$ defined by this property always has components other than $\mathcal{J}_g$ for $g \geq 4$, although $\mathcal{J}_g$ is the only known component that is not contained in θ_{null} (the divisor in $\mathcal{A}_g$ of principally polarized abelian varieties for which a theta-constant vanishes) [D1].

b) Reducibility of $\Theta \cap \Theta_a$ and trisecants. Weil observed in [W] that if Θ is a theta divisor on the Jacobian JC of a smooth curve C, then, for any points p, q, r and s of C, one has the inclusion

$$\Theta \cap \Theta_{p-q} \subset \Theta_{p-r} \cup \Theta_{s-q}, \tag{$*$}$$

where Θ_x stands for the translate $\Theta + x$. Now let (A, θ) be an indecomposable principally polarized abelian variety and let Θ be a symmetric theta divisor. Assume that

$$\Theta \cap \Theta_a \subset \Theta_x \cup \Theta_y$$

for some distinct non-zero points a, x and y of A (in particular, $\Theta \cap \Theta_a$ is reducible). This inclusion has a nice geometric interpretation in terms of the *Kummer map* $K : A \to |2\Theta|^*$ associated with the linear system $|2\Theta|$: for any point ζ of A such that $2\zeta = x + y$, it is equivalent to the fact that $K(\zeta)$, $K(\zeta - a)$ and $K(\zeta - x)$ are *collinear*. In other words, the Kummer variety $K(JC)$ of a Jacobian has a family of dimension 4 of *trisecant lines*. Mumford suggested that this property should characterize Jacobians. The most ambitious version of this conjecture is due to Welters [We1]:

CONJECTURE. *If the Kummer variety of an indecomposable principally polarized abelian variety (A, θ) has one trisecant line, then (A, θ) is a Jacobian.*

The following is known:

Welters [We2, 1983], using Gunning's criterion [Gu], showed that if $K(A)$ has a one-dimensional family of trisecants and $\dim \operatorname{Sing} \Theta \leq \dim A - 4$, then (A, θ) is a Jacobian.

Beauville and Debarre [BD1, 1986] showed that if $K(A)$ has one trisecant, then $\dim \operatorname{Sing} \Theta \geq \dim A - 4$. This result, combined with the result of Andreotti–Mayer mentioned above, implies that the existence of one trisecant is a weak characterization of Jacobians.

Debarre [D2, 1989] proved that Welters' conjecture holds for $g \leq 5$. Moreover, if (A, θ) is indecomposable, if $K(a)$, $K(b)$ and $K(c)$ are collinear and if the subgroup of A generated by $a - b$ and $a - c$ is dense in A, then (A, θ) is a Jacobian [D3, D4].

c) The K–P equation. If one lets p, q, r and s go to the same point of C in $(*)$, one gets that a theta function of a Jacobian satisfies the K–P equation, a non-linear partial differential equation that depends on three constant vector fields.

Shiota [Sh, 1986] proved that the K–P equation characterizes Jacobians among all indecomposable principally polarized abelian varieties.

Shiota's proof is analytical. It was later partially algebraized and simplified by Arbarello and De Concini [AD], but they could not bypass a crucial point in Shiota's proof. At this point, all existing algebraic proofs need extra hypotheses.

d) Double translation type. If Θ is a symmetric theta divisor on the Jacobian of a curve C of genus g, the existence of an Abel–Jacobi map $C^{g-1} \to \Theta$ implies that Θ can be locally parametrized by

$$(t_1, \ldots, t_{g-1}) \mapsto \alpha_1(t_1) + \cdots + \alpha_{g-1}(t_{g-1}),$$

where $\alpha_i : \mathbf{C} \to \mathbf{C}^g$ are curves. By symmetry of Θ, there are actually two such parametrizations if C is not hyperelliptic, and the theta divisor of a Jacobian is said to be a *double translation hypersurface*. The following is known:

Lie [L1, L2, 1935] and **Wirtinger** [Wi] proved that a non-developable (having a generically finite Gauss map) double translation hypersurface in $\mathbf{C}^g$ is a piece of the theta divisor of a Gorenstein curve of arithmetic genus g.

Little [Li1, 1989] later showed that if the theta divisor of a principally polarized abelian variety is locally a *generalized* double translation hypersurface, that is, has two local parametrizations of the type

$$(t_1, \ldots, t_{g-1}) \mapsto \alpha(t_1) + A(t_2, \ldots, t_{g-1}),$$

the principally polarized abelian variety is a Jacobian. He recently made the connection with other approaches by showing that if the Kummer variety of an indecomposable complex principally polarized abelian variety has a "curve of flexes" (that is, satisfies a one-dimensional family of K–P equations), the theta divisor is a generalized translation hypersurface [Li2].

e) Second order theta functions. Let (A, θ) be a principally polarized abelian variety of dimension g. The linear system $|2\Theta|$ is independent of the choice of a *symmetric* theta divisor Θ. It has dimension $2^g - 1$. When (A, θ) is indecomposable, the subsystem $|2\Theta|_{00}$ of divisors that have multiplicity ≥ 4 at 0 has codimension $\frac{1}{2}g(g+1)$. When (A, θ) is the Jacobian of a smooth curve C, Welters showed [We3] that the base locus of $|2\Theta|_{00}$ is (as a set) the surface $C - C$ in JC (except possibly when $g = 4$, when there might be two extra isolated points). One might hope to use this property to characterize Jacobians:

CONJECTURE [vGvG]. *If the indecomposable principally polarized abelian variety (A, θ) is not a Jacobian, the base locus of $|2\Theta|_{00}$ is $\{0\}$ as a set.*

As explained in [BD2], this conjecture is connected to the trisecant conjecture in the following way: the linear system $|2\Theta|_{00}$ corresponds to hyperplanes in $|2\Theta|^*$ that contain the point $K(0)$ and the Zariski tangent space $T_{K(0)}K(A)$. Its base locus is therefore the inverse image by the Kummer map of $K(A) \cap T_{K(0)}K(A)$. The conjecture says that if (A, θ) is not a Jacobian, there are no lines through $K(0)$ and another point of $K(A)$ that are contained in $T_{K(0)}K(A)$. Note that such a line meets $K(A)$ with multiplicity ≥ 3. What is known is the following:

Beauville and Debarre [BD2, 1989] proved that if (A, θ) is generic of dimension ≥ 4, then the base locus of $|2\Theta|_{00}$ is finite. Also, the conjecture holds for the intermediate Jacobian of a cubic threefold and various other examples.

Izadi's [Iz, 1993] work on $\mathcal{A}_4$ implies the conjecture for $g = 4$.

This conjecture has an infinitesimal analog. By associating to an element of $|2\Theta|_{00}$ the fourth-order term of the Taylor expansion of a local equation at 0, we get a linear system of quartics in $\mathbf{P}T_0A$. Using a result of Green [G], Beauville showed that, for the Jacobian of a smooth curve C, the base locus of this linear system is equal to the canonical curve of C (except possibly when $g = 4$, when there might be an extra isolated point) [BD2]. Again, one might hope to use this property to characterize Jacobians:

Conjecture [vGvG]. *If the indecomposable principally polarized abelian variety (A, θ) is not a Jacobian, this linear system of quartics is base-point-free.*

Donagi explains in [Do3] the relationship between this conjecture and the K–P equation: if D_1 is a base point, the second-order theta functions θ_n satisfy an equation of the type

$$(D_1^4 + \text{ lower order terms }) \, \theta_n(z, \tau) = 0,$$

whereas the K–P equation is equivalent to

$$(D_1^4 - D_1 D_3 + D_2^2 + d) \, \theta_n(z, \tau) = 0.$$

The following is known:

Beauville–Debarre [BD2, 1989] proved the conjecture for a generic principally polarized abelian variety of dimension ≥ 4, for the intermediate Jacobian of a cubic threefold and for various other examples.

Izadi's [Iz, 1993] work on $\mathcal{A}_4$ implies the conjecture for $g = 4$.

The difficulty with these conjectures is that the only known efficient way to construct elements of $|2\Theta|_{00}$ is to use singular points of the theta divisor (if a is singular on Θ, then $\Theta_a + \Theta_{-a}$ is in $|2\Theta|_{00}$). But in general Θ is smooth!

f) Subvarieties with minimal classes. If (JC, θ) is the Jacobian of a smooth curve C of genus g, the curve C embeds (non-canonically) into JC by the Abel–Jacobi map. The image has cohomology class θ_{g-1} (for any integer d, we write θ_d for the *minimal* (that is, non-divisible) cohomology class $\theta^d/d!$). The existence of such a curve characterizes Jacobians (Matsusaka's criterion, [M]). This result has been improved on:

Ran [R, 1981] (see also [C]), proved that if an irreducible curve C generates a g–dimensional principally polarized abelian variety (A, θ) and satisfies $C \cdot \theta = g$, then C is smooth and (A, θ) is isomorphic to its Jacobian.

More generally, for any $d \leq g$, the symmetric product $C^{(d)}$ maps onto a subvariety $W_d(C)$ of JC with cohomology class θ_{g-d}. However, the existence of a subvariety with minimal cohomology class does *not* characterize Jacobians: the Fano surface of lines on a cubic threefold maps onto a surface with class θ_3 in the intermediate Jacobian [CG, B2]. Nonetheless, the following is known:

Ran [R, 1981] proved that if a principally polarized abelian fourfold contains a surface with class θ_2, it is the Jacobian of a curve C of genus 4 and the surface is a translate of $\pm W_2(C)$.

Debarre [D5, 1992] proved that the existence of a subvariety of codimension ≥ 2 with minimal class is a weak characterization of Jacobians. Moreover, the only subvarieties of a Jacobian JC with minimal classes are translates of $\pm W_d(C)$.

A natural extension of Ran's result would be:

CONJECTURE. *If the g-dimensional principally polarized abelian variety (A, θ) contains subvarieties V and W with respective cohomology classes θ_d and θ_{g-d}, then (A, θ) is a Jacobian.*

One can be even more ambitious. Let V be a subvariety of dimension d of a principally polarized abelian variety (A, θ). Call V non-degenerate if the restriction map:

$$H^0(A, \Omega_A^d) \longrightarrow H^0(V_{\mathrm{reg}}, \Omega^d_{V_{\mathrm{reg}}})$$

is injective. For example, a curve is non-degenerate if and only if it generates A. A divisor is non-degenerate if and only if it is ample. Moreover, any subvariety with class a multiple of a minimal class is non-degenerate. Ran's above characterization of Jacobian fourfolds actually holds under the weaker hypotheses that the surface is non-degenerate and of self-intersection 6. In general, since $\theta_{g-d} \cdot \theta_d = \binom{g}{d}$, a nice generalization of Ran's results would be:

CONJECTURE. *If the g-dimensional principally polarized abelian variety (A, θ) contains non-degenerate subvarieties V and W of respective dimensions d and $g - d$ such that $V \cdot W = \binom{g}{d}$, then (A, θ) is a Jacobian.*

g) Buser–Sarnak's approach. A complex abelian variety (A, θ) can be written as the quotient of its universal cover $V \simeq \mathbf{C}^g$ by a lattice L. A polarization induces a positive definite Hermitian form H on V, whose real part B is symmetric positive definite. Set

$$\delta(A) = \min_{x \in L, x \neq 0} B(x, x).$$

Buser and Sarnak show in [BS] that for any g–dimensional Jacobian JC, one has

$$\delta(JC) \leq \frac{3}{\pi} \log(4g + 3).$$

On the other hand,

$$\max_{A \in \mathcal{A}_g} \delta(A) \geq \left(\frac{\pi^g}{2g!} \right)^{-1/g} \simeq \frac{g}{\pi e}.$$

In other words, the maximum of δ on $\mathcal{A}_g$ is much larger than its maximum on $\mathcal{M}_g$. This leads to an effective criterion for determining if a given lattice is not a Jacobian.

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