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# Adjoint Linear Systems

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Abstract. This note describes some of the recent methods and effective results in the study of pluricanonical and adjoint linear systems on higher-dimensional varieties. We describe an algebraic construction for the multiplier ideals and we use it to give a simple proof for the Reider theorem.

§**1.** The purpose of this note is to survey some of the recent results on pluricanonical and adjoint linear systems on algebraic varieties. Let $A$ be a nef and big divisor on a smooth projective variety $X$. We would like to study the linear system $|K_X + A|$. For $X$ a curve, it is well known that if $\deg A \geq 2$, the linear system $|K_X + A|$ is free, and if $\deg A \geq 3$, then $K_X + A$ is very ample. The canonical linear system $|K_X|$ is very ample if and only if $X$ is not a hyperelliptic curve. From the theory of curves, we know that the properties of these linear systems are closely related to the geometry of the curve. It is natural that one would like to obtain similar numerical criteria for freeness and very ampleness for adjoint linear systems on a higher-dimensional variety, and study their geometric properties.

Many results and ideas in this note are based on my joint work with R. Lazarsfeld. I would like to thank him for sharing with me many of his ideas.

For surfaces of general type, Kodaira [Kod], Bombieri [Bmb] and many others have studied the behavior of the pluricanonical maps. A few years ago, Reider obtained the following elegant unifying result for adjoint linear systems on surfaces, by using the famous Bogomolov's instability criterion for rank-two bundles [Rdr].

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Theorem 1 (Reider). *Let $A$ be a nef and big divisor on a smooth projective surface $X$, and let $p$ be a given point in $X$.*

(a) *Assume that $A^2 \geq 5$ and $A\cdot C \geq 2$ for all curves $C$ in $X$ through $p$. Then the linear system $|K_X + A|$ is free at $p$.*
(b) *Assume that $A^2 \geq 10$ and $A\cdot C \geq 3$ for all curves $C$ in $X$. Then $|K_X+A|$ is very ample.*

Naturally one knows a lot less about general adjoint linear systems on higher-dimensional varieties. Using the Kawamata–Viehweg vanishing theorem and an ingenious use of singular hypersurfaces, Kawamata found the following basic base-point free result that is one of the key steps in the higher-dimensional minimal model program.

Theorem 2 (Kawamata [KMM]). *Let $A$ be a nef and big divisor on a projective variety with only canonical singularities. Assume that $K_X+A$ is nef and Cartier. Then $\left|m(K_X + A)\right|$ is free for all $m \gg 0$.*

Remark. Kawamata has first proved this in the case that $X$ is a threefold. He also observed that the proof will work in higher dimensions, if one can show certain linear systems are non-empty. Then Shokurov [S] proved the necessary non-vanishing theorem.

Let $A$ be an ample line bundle on a smooth projective $n$-fold. Mori proved that $K_X + mA$ is nef if $m \geq n + 1$. As for the behavior of the adjoint linear systems, Fujita made the following famous conjectures.

Fujita's conjectures [Fuj 1].

(a) *If $m \geq n+1$, the linear system $|K_X + mA|$ is free.*
(b) *If $m \geq n+2$, the linear system $|K_X + mA|$ is very ample.*
(c) *If $K_X$ is nef and big and $m \geq n+2$, then $|mK_X|$ is free.*

Remarks. (a) By considering the case that $X$ is the projective space and $A$ is the tautological line bundle on the projective space, one sees that the conjectures (a) and (b) are optimal.
(b) For surfaces, one can show that Fujita's conjectures follow from Rieder's theorem easily.

In a real breakthrough, Demailly [Dem] obtained the following effective results in higher dimensions by using powerful analytic tools such as Calabi–Yau's theorem and Nadel's vanishing theorem.

Theorem 3 (Demailly). *Let $B$ be a nef and big divisor on a smooth projective $n$-fold $X$ and let $p$ be a point in $X$. Let $T_X$ denote the tangent sheaf of $X$. Assume that there is a given nonnegative number $m$ such that*

$$T_X \otimes \mathcal{O}_X(mB)$$

*is nef. Then there are $n$ positive constants, $c_1, c_2, \dots, c_n$, expressible explicitly as functions of $n$ and $m$ alone, with the following property. Suppose that the divisor $B$ satisfies the following numerical conditions. For $k = 1, 2, \dots, n$,*

$$B^{k} \cdot Y > c_k \quad \text{for all } k\text{-dimensional subvarieties } Y \quad \text{such that } p \in Y.$$

*Then the linear system $|K_X + B|$ is free at $p$.*

REMARK. The explicit constants in the above theorem are quite large and are unlikely to be optimal. Demailly has also found results for separating points and separating higher-order jets. Using an iteration argument, he obtains the following explicit result.

COROLLARY 4. *Let $A$ be an ample divisor on a smooth projective $n$-fold $X$. Then $2K_X + 12n^n A$ is very ample.*

Together with Lazarsfeld, we begin our study of these questions by observing that one can use the cohomological techniques developed by Kawamata, Reid, Shokurov and others to give a fairly simple proof for Reider's theorem. Then Kollár showed that these techniques give an algebraic proof for the following effective results [Kol1].

THEOREM 5 (Kollár). (a) *Let $A$ be a nef and big divisor on a smooth projective variety $X$. Assume that $K_X + A$ is nef. Then the linear system*

$$\left| m(K_X + A) \right|$$

*is free for $m > 2\,(n+2)!\,(n+1)$.*

(b) $h^0(K_X + mA) \neq 0$ *if* $m > \binom{n+1}{2}$.

Although these effective results are very interesting, the constants involved are very large. It is natural to ask whether one can find results similar to Reider's theorem for higher-dimensional varieties. For threefolds, Lazarsfeld and I were able to find the following results [EL1].

THEOREM 6 (Ein and Lazarsfeld). *Let $X$ be a smooth projective threefold.*

(a) *Let $A$ be an ample divisor on $X$. Then $|K_X + 4A|$ is free.*
(b) *If $X$ is a smooth minimal threefold of general type, $|7K_X|$ is free.*
(c) *Let $B$ be a nef and big divisor on $X$ and $p$ be a given point on $X$. Assume that*

$$B^3 \geq 92,$$
$$B^2 \cdot S \geq 7 \quad \text{for all surfaces } S \text{ in } X \text{ such that } p \in S,$$
$$B \cdot C \geq 3 \quad \text{for all curves in } X \text{ such that } p \in C.$$

*Then $|K_X + B|$ is free at $p$.*

Remark. Theorem 4(a) is the special case of Fujita's conjecture for threefolds. Also, Benveniste had already shown earlier [Ben] that $|mK_X|$ is free when $K_X$ is nef and big and $m \geq 34$.

§**2.** In this section, we will try to demonstrate the cohomological techniques used in the above theorem by giving a proof for the Reider theorem. We begin by considering the following general set-up.

Let $X$ be a smooth projective $n$-fold and let $p$ be a point in $X$. Let $D$ be an effective $\mathbf{Q}$-divisor on $X$ that contains $p$. We will assume that

$$\mathrm{Mult}_p(D) = q \geq n.$$

Definition. We say that $p$ is an almost isolated singularity of $D$, if

$$\mathrm{Mult}_x(D) < 1$$

for all points $x$ in a punctured neighborhood of $p$.

The following result relates singular hypersurfaces to the freeness of the adjoint linear systems. We have learned the result from a lecture of Siu. The algebraic proof given here is based on the construction and properties of multiplier ideals given by Esnault and Viehweg in their lectures on vanishing theorems [EV, §7].

Lemma 7. *Let $X$ be a smooth projective variety and $p$ be a point in $X$. Suppose that there is an effective $\mathbf{Q}$-divisor $D$ such that $\mathrm{Mult}_p(D) \geq n$ and $p$ is an almost isolated singular point of $D$. Let $B$ be a divisor such that $B - D$ is nef and big. Then the linear system $|K_X + B|$ is free at $p$.*

Sketch of the Proof. Let $f : Y \longrightarrow X$ be an embedded resolution for the divisor $D$. We consider the $\mathbf{Q}$-divisor

$$f^*D = [f^*D] + \Delta.$$

We also write

$$K_{Y/X} - [f^*D] = P - N,$$

where $P$ and $N$ are effective divisors with no common components. Observe that all the components of $P$ are $f$-exceptional. It follows that

$$f_*\big(\mathcal{O}_Y(P - N)\big) = f_*\big(\mathcal{O}_Y(-N)\big) = J_Z,$$

where $J_Z$ is the ideal sheaf of a closed subscheme $Z$. This is the multiplier ideal associated to the $\mathbf{Q}$-divisor $D$ as defined in [EV, §7.4]. By our assumption, $p$ is an almost isolated singular point of $D$. It follows from Proposition 7.7 in [EV] that in a neighborhood of $p$ the ideal $J_Z$ is a nontrivial ideal of a zero-dimensional closed subscheme supported at $p$. Now

$$f^*(K_X + B) + P - N \equiv K_Y + f^*(B - D) + \Delta.$$

By the Kawamata–Viehweg vanishing theorem and the Leray spectral sequence, we conclude that

(a) $R^i f_*\big(\mathcal{O}_Y(P-N)\big) = 0 \quad$ for $i > 0$;
(b) $f_*\mathcal{O}_Y(P-N) = J_z$;
(c) $H^i\big(\mathcal{O}_X(K_X+B)\otimes J_z\big) = 0 \quad$ for $i > 0$.

Therefore the restriction map

$$H^0\big(\mathcal{O}_X(K_X+B)\big) \longrightarrow H^0\big(\mathcal{O}_X(K_X+B) \mid Z\big)$$

is surjective. Since $Z$ is zero-dimensional at $p$, this implies that the linear system $|K_X+B|$ is free at $p$. □

Now we'll apply Lemma 7 to give a proof for Reider's theorem (Theorem 1).

Sketch of the Proof of Theorem 1. By the Riemann–Roch theorem, there is a divisor $D \in |kB|$, for $k \gg 0$, such that

$$q = \mathrm{Mult}_p(D) \geq 2k+1.$$

If $p$ is an almost isolated singularity of $2D/q$, the linear system $|K_X+B|$ is free at $p$ by Lemma 7. So we may assume that $p$ is not an almost isolated singular point of $2D/q$. Denote by $F$ the irreducible curve that has the highest multiplicity among all the components of $D$ that go through the given point $p$. We may express $D$ in the form

$$D = rF + G + R,$$

where $r$ is the multiplicity of $D$ along $F$. Since $p$ is not an almost isolated singular point of $2D/q$, we note that $r > q/2$. This inequality implies that $F$ is smooth at $p$. Denote by $G$ the sum of the other components of $D$ that pass through the given point $p$. Denote by $R$ the sum of the other remaining components of $D$. We consider the $\mathbf{Q}$-divisor

$$D/r = F + N + \Delta,$$

where $\Delta$ is the fractional part of $D/r$ and $N = [R/r]$ is an effective divisor that does not contain $p$. Now the vanishing theorem says that the cohomology group

$$H^1\big(\mathcal{O}_X(K_X+B-N-F)\big) = 0.$$

We conclude that the restriction map

$$H^0\big(\mathcal{O}_X(K_X+B-N)\big) \longrightarrow H^0\big(\mathcal{O}_X(K_X+B-N)\big|_F\big)$$

is surjective. We note that

$$K_X+B-N\big|_F \equiv K_F + \big(1-\tfrac{k}{r}\big)\cdot B + \Delta\big|_F.$$

We also observe that $\mathrm{Mult}_p(\Delta) = \big(\frac{1}{r} \cdot q - 1\big)$ and $F \cdot B \geq 2$. We conclude that

$$\deg\big(\big(1 - \tfrac{k}{r}\big) \cdot B + \Delta\big)\big|_F > 1.$$

This implies that $K_X + B - N\big|_F$ is equivalent to a divisor of the form $K_F + C$, where $\deg(C) \geq 2$. Hence we can find a section of $\mathcal{O}_X(K_X + B - N\big|_F)$ that does not vanish at $p$. Since this section can be lifted to a section of $\mathcal{O}_X(K_X + B - N)$, we conclude that the linear system $|K_X + B|$ is free at $p$. □

§**3.** In this last section, we would like to report on some of the recent results on adjoint linear systems. In [M], Mumford gave a proof for the Kodaira vanishing theorem by using Bogomolov's instability criterion. Recently, Fernandez del Busto has found a very elegant proof of the famous inequality of Bogomolov by using the Kawamata–Viehweg vanishing theorem and techniques similar to our proof of Reider's theorem [Fer]. It would be desirable to have a better understanding of the relations among these different results. In another development, using Damailly's theorem Siu obtained the following effective version of the Matsusaka theorem.

Theorem 8 (Siu). *Let $A$ be an ample divisor on a smooth projective $n$-fold $X$. Consider the intersection numbers $a = (A)^n$, $b = K_X \cdot (A)^{n-1}$ and $c = (n+2)a + b$. Assume that*

$$m > 24^n c\big((1+c)^n\big)^{n(6n^3)^n}.$$

*Then the divisor $mA$ is very ample.*

Recently, Lazarsfeld, Nakamaye, and I have found an algebraic proof for the following result that replaces the hypothesis on the tangent sheaf in Demailly's theorem by a simpler assumption on $-K_X$.

Theorem 9 (Ein, Nakamaye, and Lazarsfeld). *Let $B$ be a nef and big divisor on a smooth projective $n$-fold $X$ and let $p$ be point in $X$. Assume that there is a nonnegative number $m$ such that $-K_X + mB$ is ample. Then there exist $n$ positive constants, $c_1, c_2, \dots, c_n$ that can be expressed explicitly as functions of $n$ and $m$ alone with the following property. Suppose that the divisor $B$ satisfies the following numerical conditions. For $k = 1, 2, \dots, n$,*

$$B^k \cdot Y > c_k \quad \text{for all } k\text{-dimensional subvarieties } Y \text{ such that } \; p \in Y.$$

*Then the linear system $|K_X + B|$ is free and separates tangents at $p$.*

Remark. We also obtained similar results for very ampleness. These results give numerical criteria for freeness and very ampleness for ample linear systems on those varieties where $-K_X$ is nef, such as Fano and Calabi–Yau manifolds.

Using some of the ideas of Demailly and Siu, Fernandez del Busto has found the following simple form of effective Matsusaka theorem for ample line bundles on surfaces.

Theorem 10 (Fernandez del Busto). *Let $A$ be an ample divisor on a smooth surface $X$. Set $a = A \cdot A$ and $b = A \cdot K_X$. Then the linear system $|mA|$ is free for $m > (3a + b + 1)^2/(2a + 2)$.*

Let $B$ be an ample divisor on a smooth projective $n$-fold and let $p$ be a given point of $X$. One would like to measure the local positivity of $B$ at $p$. Let $f : Y \longrightarrow X$ be the blowing up of $X$ at $p$ and let $E$ be the exceptional divisor. Following Demailly, we define the Seshadri constant of $B$ at $p$ to be the real number

$$s(B, p) = \sup\{t \in \mathbf{Q} \mid \text{the } \mathbf{Q}\text{-divisor } f^*B - tE \text{ is ample}\}.$$

Suppose that $m$ is a positive integer such that $m \cdot s(B, p) > n$. Then $f^*mB - nE$ is ample. Observe that

$$f^*(K_X + mB) - E = K_Y + f^*mB - nE.$$

Using the Kodaira vanishing theorem, we conclude that $|K_X + mB|$ is free at $p$. Similarly, one can obtain results for separating points and higher-order jets. To apply these observations, one would need to give a lower estimate on $s(B, p)$. In [EL2] we show that for $X$ a surface we have $S(B, p) \geq 1$ for a general point $p$ in $X$. Recently, together with Kuechle and Lazarsfeld, we found the following result for higher-dimensional varieties.

Theorem 11. *Let $B$ be an ample divisor on a smooth projective $n$-fold $X$. If $p$ is a very general point of $X$, then $s(B, p) \geq \frac{1}{n}$.*

Remark. Let $r$ be a positive integer. If $m > n(n+r)$, Theorem 11 implies that the linear system $|K_X + mB|$ separates $r$ jets at a general point $p$ of $X$.

Together with Lazarsfeld and Masek, we have generalized Theorem 6 to threefolds with logterminal singularities. In a recent preprint, Fujita has found an improvement for Theorem 6. His result implies the following theorem [Fuj2].

Theorem 12 (Fujita). *Let $A$ be an ample divisor on a smooth projective threefold $X$. Assume that $A^3 \geq 2$. Then $|K_X + 3A|$ is free.*

In his study of varieties with nontrivial fundamental group, Kollár finds the following interesting result [Kol2].

Theorem 13 (Kollár). *Let $X$ be a smooth projective variety and let $x$ be a general point in $X$. Assume that every irreducible positive-dimensional subvariety $Z$ containing $x$ has the following property. If $Z' \to Z$ is a resolution of singularities of $Z$, then $\pi_1(Z')$ is an infinite group. Then, for every nef and big divisor $B$ on $X$, the linear system $|K_X + B| \neq \varnothing$.*

Despite all this recent progress, many fundamental problems in this area remain unsolved. We conclude with a short list of open problems.

1. Can one generalize Theorem 6 to higher dimension and prove Fujita's conjectures?
2. For threefolds, one may ask what is the optimal result and can one also prove a similar theorem for very ampleness.
3. What is the analog of Bogomolov's theorem for linear systems on a higher-dimensional variety?
4. For surfaces, what is the optimal effective Matsusaka's theorem? Can one describe the properties of the extremal examples?
5. Is the analog of the theorem in [EL2] true in higher dimension? One would like to show that for a nef and big divisor $B$ on a smooth $n$-fold, $s(B,p) \geq 1$ for a general point $p$ in $X$.

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