MSRI Series
Volume **28**, 1995

Torelli Groups and Geometry of Moduli Spaces of Curves

RICHARD M. HAIN

Abstract. The Torelli group T_g is the group of isotopy classes of diffeomorphisms of a compact orientable surface of genus g that act trivially on the homology of the surface. The aim of this paper is to show how facts about the homology of the Torelli group imply interesting results about algebraic curves. We begin with an exposition of some of Dennis Johnson's work on the Torelli groups. We then show how these results imply that the Picard group of the moduli space of curves of genus $g \geq 3$ with a level-l structure is finitely generated. A classification of all "natural" normal functions over the moduli space of curves of genus $g \geq 3$ and a level l structure is obtained by combining Johnson's results with M. Saito's theory of Hodge modules. This is used to prove results that generalize the classical Franchetta Conjecture to the generic curve of genus g with n marked points and a level-l structure. Other applications are given, for example, to computing heights of cycles defined over a moduli space of curves.

1. Introduction

The Torelli group T_g is the kernel of the natural homomorphism $\Gamma_g \to \mathrm{Sp}_g(\mathbb{Z})$ from the mapping class group in genus g to the group of $2g \times 2g$ integral symplectic matrices. It accounts for the difference between the topology of $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties of dimension g, and $\mathcal{M}_g$, the moduli space of smooth projective curves of genus g, and therefore should account for some of the difference between their geometries. For this reason, it is an important problem to understand its structure and its cohomology. To date, little is known about T_g apart from Dennis Johnson's few fundamental results—he has proved that T_g is finitely generated when $g \geq 3$ and has computed $H_1(T_g, \mathbb{Z})$. It is this second result that will concern us in this paper. Crudely stated, it says that there is an $\mathrm{Sp}_g(\mathbb{Z})$-equivariant isomorphism

$$H^1(T_g, \mathbb{Q}) \approx PH^3(\mathrm{Jac}\, C, \mathbb{Q}),$$

Research supported in part by grants from the National Science Foundation and the NWO.

where C is a smooth projective curve of genus g, and P denotes primitive part.

My aim in this paper is to give a detailed exposition of Johnson's homomorphism

$$PH^3(\operatorname{Jac} C, \mathbb{Q}) \to H^1(T_g, \mathbb{Q})$$

and to explain how Johnson's computation, alone and in concert with M. Saito's theory of Hodge modules [43], has some remarkable consequences for the geometry of $\mathcal{M}_g$. It implies quite directly, for example, that for each l, the Picard group of the moduli space $\mathcal{M}_g(l)$ of curves of genus $g \geq 3$ with a level-l structure is finitely generated. Combined with Saito's work, it enables one to completely write down all "natural" generically defined normal functions over $\mathcal{M}_g(l)$ when $g \geq 3$. The result is that, modulo torsion, all are half-integer multiples of the normal function of the cycle $C - C^-$. This is applied to give a new proof of the Harris–Pulte Theorem [27, 41], which relates the mixed Hodge structure on the fundamental group of a curve C to the algebraic cycle $C - C^-$ in its jacobian. Another application is to show that the cycles $C^{(a)} - i_* C^{(a)}$ in $\operatorname{Jac} C$, for $1 \leq a < g-1$, are of infinite order modulo algebraic equivalence for the general curve C. This result is due to Ceresa [10].

Understanding all normal functions over $\mathcal{M}_g(l)$ also allows us to "compute" the archimedean height pairing between any two "natural" cycles in a smooth projective variety defined over the moduli space of curves, provided they are homologically trivial over each curve, disjoint over the generic curve, and satisfy the usual dimension restrictions. The precise statement can be found in Section 14.

Our final application is to the Franchetta conjecture. The classical version of this conjecture asserts that the Picard group of the generic curve is isomorphic to $\mathbb{Z}$ and is generated by the canonical divisor. Beauville (unpublished), and later Arbarello and Cornalba [1], deduced this from Harer's computation of $H^2(\Gamma_g)$. As another application of the classification of normal functions over $\mathcal{M}_g(l)$, we prove a "Franchetta Conjecture" for the generic curve with a level-l structure. The statement is that the Picard group of the generic curve of genus g with a level-l structure is finitely generated of rank 1—the torsion subgroup is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^{2g}$; mod torsion, it is generated by the canonical bundle if l is odd, and by a square root of the canonical bundle if l is even. Our proof is valid only when $g \geq 3$; it does not use the computation of $\operatorname{Pic} \mathcal{M}_g(l)$, which is not known at this time. We also compute the Picard group of the generic genus-g curve with a level-l structure and n marked points.

Our results on normal functions are inspired by those in the last section of Nori's remarkable paper [40], where he studies functions on finite covers of Zariski open subsets of the moduli space of principally polarized abelian varieties. There are analogues of our main results for $\mathcal{A}_g(l)$, the moduli space of principally polarized abelian varieties of dimension g with a level-l structure. These results are similar to Nori's, but differ. The detailed statements, as well as a discussion of the relation between the results, are in Section 15. Our results on abelian varieties are related to some results of Silverberg [45].

Sections 3 and 4 contain an exposition of the three constructions of the Johnson homomorphism that are given in [33]. Since no proof of their equivalence appears in the literature, I have given a detailed exposition, especially since the equivalence of two of these constructions is essential in one of the applications to normal functions.

In Section 13 Johnson's result is used to give an explicit description of the action of the Γ_g on the n-th roots of the canonical bundle. This is a slight refinement of a result of Sipe [46]. A consequence of this computation is that the only roots of the canonical bundle defined over Torelli space are the canonical bundle itself and all theta characteristics.

Acknowledgements. First and foremost, I would like to thank Eduard Looijenga for his hospitality and for stimulating discussions during a visit to the University of Utrecht in the spring of 1992 during which some of the work in this paper was done. I would also like to thank the University of Utrecht and the Dutch NWO for their generous support during that visit. Thanks also to Pietro Pirola and Enrico Arbarello for pointing out to me that the non-existence of sections of the universal jacobian implies the classical Franchetta conjecture. From this it was a short step to the generalizations in Section 12. Thanks also to Arnaud Beauville for useful discussions on roots of the canonical bundle and for bringing Sipe's work to my attention.

2. Mapping class groups and moduli

At this time there is no argument within algebraic geometry to compute the Picard groups of all $\mathcal{M}_g$, and one has to resort to topology to do this computation. Let S be a compact orientable surface of genus g with r boundary components and let P be an ordered set of n distinct marked points of $S - \partial S$. Denote the group of orientation-preserving diffeomorphisms of S that fix $P \cup \partial S$ pointwise by $\mathrm{Diff}^+(S, P \cup \partial S)$. Endowed with the compact-open topology, this is a topological group. The mapping class group $\Gamma_{g,r}^n$ is defined to be its group of path components:

$$\Gamma_{g,r}^n = \pi_0 \, \mathrm{Diff}^+(S, P \cup \partial S).$$

Equivalently, it is the group of isotopy classes of orientation-preserving diffeomorphisms of S that fix $P \cup \partial S$ pointwise. It is conventional to omit the decorations n and r when they are zero. So, for example, $\Gamma_g^n = \Gamma_{g,0}^n$.

The link between moduli spaces and mapping class groups is provided by Teichmüller theory. Denote the moduli space of smooth genus g curves with

n marked points by $\mathcal{M}_g^n$. Teichmüller theory provides a contractible complex manifold $\mathcal{X}_g^n$ on which Γ_g^n acts properly discontinuously—when $2g+n+2>0$, it is the space of all complete hyperbolic metrics on $S-P$ equivalent under diffeomorphisms isotopic to the identity. The quotient $\Gamma_g^n\backslash\mathcal{X}_g^n$ is analytically isomorphic to $\mathcal{M}_g^n$. It is useful to think of Γ_g^n as the orbifold fundamental group of $\mathcal{M}_g^n$. As we shall explain shortly, there is a natural isomorphism

$$H^\bullet(\mathcal{M}_g^n,\mathbb{Q})\approx H^\bullet(\Gamma_g^n,\mathbb{Q}).$$

One can compactify S by filling in the r boundary components of S by attaching disks. Denote the resulting genus-g surface by $\overline{S}$. Elements of $\Gamma_{g,r}^n$ extend canonically to $\overline{S}$ to give a homomorphism $\Gamma_{g,r}^n\to\Gamma_g$. Denote the composite

$$\Gamma_{g,r}^n\to\Gamma_g\to\operatorname{Aut}H_1(\overline{S},\mathbb{Z})$$

by ρ. Since elements of $\Gamma_{g,r}^n$ are represented by orientation-preserving diffeomorphisms, each element of $\Gamma_{g,r}^n$ preserves the intersection pairing

$$q:\Lambda^2H_1(\overline{S},\mathbb{Z})\to\mathbb{Z}.$$

Consequently, we obtain a homomorphism

$$\rho:\Gamma_{g,r}^n\to\operatorname{Aut}(H_1(\overline{S},\mathbb{Z}),q)\approx\operatorname{Sp}_g(\mathbb{Z}).$$

This homomorphism is well-known to be surjective.

Denote the moduli space of principally polarized abelian varieties of dimension g by $\mathcal{A}_g$. Since this is the quotient of the Siegel upper half plane by $\operatorname{Sp}_g(\mathbb{Z})$, it is an orbifold with orbifold fundamental group $\operatorname{Sp}_g(\mathbb{Z})$ and, as in the case of $\mathcal{M}_g^n$, there is a natural isomorphism

$$H^\bullet(\mathcal{A}_g,\mathbb{Q})\approx H^\bullet(\operatorname{Sp}_g(\mathbb{Z}),\mathbb{Q}).$$

The period map $\mathcal{M}_g^n\to\mathcal{A}_g$ is a map of orbifolds and induces ρ on fundamental groups.

The Torelli group $T_{g,r}^n$ is the kernel of the homomorphism

$$\rho:\Gamma_{g,r}^n\to\operatorname{Sp}_g(\mathbb{Z}).$$

Since ρ is surjective, we have an extension

$$1\to T_{g,r}^n\to\Gamma_{g,r}^n\to\operatorname{Sp}_g(\mathbb{Z})\to 1.$$

The Torelli group T_g encodes the differences between the topology of $\mathcal{M}_g$ and that of $\mathcal{A}_g$—between curves and abelian varieties. More formally, we have the Hochschild–Serre spectral sequence

$$H^s(\operatorname{Sp}_g(\mathbb{Z}),H^t(T_{g,r}^n))\implies H^{s+t}(\Gamma_{g,r}^n).$$

Much more (although not enough) is known about the topology of the $\mathcal{A}_g$ than about that of the $\mathcal{M}_g$. For example, the rational cohomology groups of the $\mathcal{A}_g$ stabilize as $g \to \infty$, and this stable cohomology is known by Borel's work [6]: it is a polynomial ring generated by classes $c_1, c_3, c_5, \dots$, where c_k has degree $2k$. As with $\mathcal{A}_g$, the rational cohomology of the $\mathcal{M}_g$ is known to stabilize, as was proved by Harer [22], but the stable cohomology of the $\mathcal{M}_g$ is known only up to dimension 4; the computations are due to Harer [24, 26].

Torelli space $\mathcal{T}_g^n$ is the quotient $T_g^n \backslash \mathcal{X}_g^n$ of Teichmüller space. When $g \geq 3$, it is the moduli space of smooth projective curves C, together with n ordered distinct points and a symplectic basis of $H_1(C, \mathbb{Z})$.

The Torelli group is torsion-free. Perhaps the simplest way to see this is to note that, by standard topology, since $\mathcal{X}_g^n$ is contractible, each element of Γ_g^n of prime order must fix a point of $\mathcal{X}_g^n$. If $\phi \in \Gamma_g^n$ fixes the point corresponding to the marked curve C, there is an automorphism of C that lies in the mapping class ϕ. Since the automorphism group of a compact Riemann surface injects into $\operatorname{Aut} H^0(C, \Omega_C^1)$, and therefore into $H_1(C)$, it follows that T_g^n is torsion-free. Because of this, the Torelli space $\mathcal{T}_g^n$ is the classifying space of T_g^n.

One can view Siegel space $\mathfrak{h}_g$ as the classifying space of principally polarized abelian varieties of dimension g together with a symplectic basis of H^1. The period map therefore induces a map

$$\mathcal{T}_g^n \to \mathfrak{h}_g,$$

which is 2:1 when $g \geq 2$, and ramified along the hyperelliptic locus when $g \geq 3$.

For a finite-index subgroup L of $\mathrm{Sp}_g(\mathbb{Z})$, let $\Gamma_{g,r}^n(L)$ be the inverse image of L in $\Gamma_{g,r}^n$ under the canonical homomorphism $\Gamma_{g,r}^n \to \mathrm{Sp}_g(\mathbb{Z})$. It may be expressed as an extension

$$1 \to T_{g,r}^n \to \Gamma_{g,r}^n(L) \to L \to 1.$$

Set $\mathcal{M}_g^n(L) = \Gamma_g^n(L) \backslash \mathcal{X}_g^n$. We will call $\Gamma_{g,r}^n(L)$ the *level-L* subgroup of $\Gamma_{g,r}^n$, and we will say that points in $\mathcal{M}_g^n(L)$ are curves with a level-L structure and n marked points. The traditional moduli space of curves with a level-l structure, where $l \in \mathbb{N}^+$, is obtained by taking L to be the elements of $\mathrm{Sp}_g(\mathbb{Z})$ that are congruent to the identity mod l.

Since the Torelli groups are torsion-free, $\Gamma_{g,r}^n(L)$ is torsion-free when L is. Note, however, that by the Lefschetz fixed point formula, $\Gamma_{g,r}^n$ is torsion-free when $n + 2r > 2g + 2$, so that $\Gamma_{g,r}^n(L)$ may be torsion-free even when L is not.

Proposition 2.1. *For all $g, n \geq 0$ and for each finite-index subgroup L of $\mathrm{Sp}_g(\mathbb{Z})$, there is a natural homomorphism*

$$H^\bullet(\mathcal{M}_g^n(L), \mathbb{Z}) \to H^\bullet(\Gamma_g^n(L), \mathbb{Z}),$$

which is an isomorphism when $\Gamma_g^n(L)$ is torsion-free, and is an isomorphism after tensoring with $\mathbb{Q}$ for all L.

Proof. Set $\Gamma = \Gamma_g^n$, $\Gamma(L) = \Gamma_g^n(L)$, $\mathcal{M}(L) = \mathcal{M}_g^n(L)$ and $\mathcal{X} = \mathcal{X}_g^n$. Let $E\Gamma$ be any space on which Γ acts freely and properly discontinuously—so $E\Gamma$ is the universal covering space of some model of the classifying space of Γ. Since $\mathcal{X}$ is contractible, the quotient $E\Gamma \times_{\Gamma(L)} \mathcal{X}$ of $E\Gamma \times \mathcal{X}$ by the diagonal action of $\Gamma(L)$ is a model $B\Gamma(L)$ of the classifying space of $\Gamma(L)$. The projection $E\Gamma \times \mathcal{X} \to \mathcal{X}$ induces a map $f : B\Gamma(L) \to \mathcal{M}(L)$, which induces the map of the theorem. If $\Gamma(L)$ is torsion-free, f is a homotopy equivalence. Otherwise, choose a finite-index, torsion-free normal subgroup L' of L. Then $\Gamma(L)$ is torsion-free. Set

$$G = \Gamma(L)/\Gamma(L') \approx L/L'.$$

This is a finite group. We have the commutative diagram of Galois G-coverings

$$\begin{array}{ccc} B\Gamma(L') & \to & \mathcal{M}(L') \\ \downarrow & & \downarrow \\ B\Gamma(L) & \to & \mathcal{M}(L), \end{array}$$

where the top map is a homotopy equivalence. Thus it induces a G-equivariant isomorphism

$$H^\bullet(\mathcal{M}(L')) \to H^\bullet(B\Gamma(L')).$$

The result follows as the vertical projections induce isomorphisms

$$H^\bullet(\mathcal{M}(L), \mathbb{Q}) \xrightarrow{\sim} H^\bullet(\mathcal{M}(L'), \mathbb{Q})^G \text{ and } H^\bullet(\Gamma(L), \mathbb{Q}) \xrightarrow{\sim} H^\bullet(\Gamma(L'), \mathbb{Q})^G. \qquad \square$$

The group $\Gamma_{g,r}^n(L)$ also admits a moduli interpretation when $r > 0$, even though algebraic curves have no boundary components. The idea is that a topological boundary component of a compact orientable surface should correspond to a first-order local holomorphic coordinate about a cusp of a smooth algebraic curve. Denote by $\mathcal{M}_{g,r}^n(L)$ the moduli space of smooth curves of genus g with a level-L structure and with n distinct marked points and r distinct, non-zero cotangent vectors, where the cotangent vectors do not lie over any of the marked points, and where no two of the cotangent vectors are anchored at the same point. This is a $(\mathbb{C}^*)^r$ bundle over $\mathcal{M}_g^{r+n}(L)$.

Proposition 2.2. *For all finite-index subgroups L of $\mathrm{Sp}_g(\mathbb{Z})$ and for all $g, n, r \geq 0$, there is a natural homomorphism*

$$H^\bullet(\mathcal{M}_{g,r}^n(L), \mathbb{Z}) \to H^\bullet(\Gamma_{g,r}^n(L), \mathbb{Z}),$$

which is an isomorphism when $\Gamma_{g,r}^n(L)$ is torsion-free, and is an isomorphism after tensoring with $\mathbb{Q}$ for all L. $\square$

3. The Johnson homomorphism

Dennis Johnson, in a sequence of pioneering papers [30, 31, 32], began a systematic study of the Torelli groups. From the point of view of computing the cohomology of the $\mathcal{M}_g$, the most important of his results is his computation of $H_1(T_g^1)$ [32]. Let S be a compact oriented surface of genus $g \geq 3$ with a distinguished base point x_0.

THEOREM 3.1. *There is an* $\mathrm{Sp}_g(\mathbb{Z})$*-equivariant homomorphism*

$$\tau_g^1 : H_1(T_g^1, \mathbb{Z}) \to \Lambda^3 H_1(S),$$

which is an isomorphism mod 2*-torsion.*

Johnson has also computed $H_1(T_g^1, \mathbb{Z}/2\mathbb{Z})$. It is related to theta characteristics. Bert van Geemen has interesting ideas regarding its relation to the geometry of curves. A proof of Johnson's theorem is beyond our scope, but we will give three constructions of the homomorphism τ_g^1 and establish their equality.

We begin by sketching the first of these constructions. Since the Torelli group is torsion-free, there is a universal curve

$$\mathcal{C} \to \mathcal{T}_g^1$$

over Torelli space. This has a tautological section $\sigma : \mathcal{T}_g^1 \to \mathcal{C}$. There is also the jacobian

$$\mathcal{J} \to \mathcal{T}_g^1$$

of the universal curve. The universal curve can be embedded in its jacobian using the section σ—the restriction of this mapping to the fiber over the point of Torelli space corresponding to (C, x) is the Abel–Jacobi mapping

$$\nu_x : (C, x) \to (\mathrm{Jac}\, C,\ 0)$$

associated to (C, x). Since T_g^1 acts trivially on the first homology of the curve, the local system associated to $H_1(C)$ is framed. There is a corresponding topological trivialization of the jacobian bundle:

$$\mathcal{J} \xrightarrow{\sim} \mathcal{T}_g^1 \times \mathrm{Jac}\, C.$$

Let $p : \mathcal{J} \to \mathrm{Jac}\, C$ be the corresponding projection onto the fiber. Each element ϕ of $H_1(T_g^1, \mathbb{Z})$ can be represented by an embedded circle $\phi : S^1 \to \mathcal{T}_g^1$. Regard the universal curve $\mathcal{C}$ as subvariety of $\mathcal{J}$ via the Abel–Jacobi mapping. Then the part of the universal curve $M(\phi)$ lying over the circle ϕ is a 3-cycle in $\mathcal{J}$. The Johnson homomorphism is defined by

$$\tau_g^1(\phi) = p_*[M(\phi)] \in H_3(\mathrm{Jac}\, C, \mathbb{Z}) \approx \Lambda^3 H_1(C, \mathbb{Z}).$$

This definition is nice and conceptual, but is not so easy to work with. In the remainder of this section, we remake this definition without appealing to Torelli space. In the next section, we will give two more constructions of it, both due to Johnson, and prove that all three constructions agree.

Recall that the *mapping torus* of a diffeomorphism ϕ of a manifold S is the quotient $M(\phi)$ of $S \times [0,1]$ obtained by identifying $(x,1)$ with $(\phi(x),0)$:

$$M(\phi) = S \times [0,1]/\{(x,1) \sim (\phi(x),0)\}.$$

The projection $S \times [0,1] \to [0,1]$ induces a bundle projection

$$M(\phi) \to [0,1]/\{0 \sim 1\} = S^1$$

whose fiber is S and whose geometric monodromy is ϕ.

Now suppose that $\phi : (S,x_0) \to (S,x_0)$ is a diffeomorphism of S that represents an element of T_g^1. The mapping torus bundle

$$M(\phi) \to S^1$$

has a canonical section $\sigma : S^1 \to M(\phi)$, which takes $t \in S^1$ to $(x_0,t) \in M(\phi)$.

Denote $H_\bullet(S,\mathbb{R}/\mathbb{Z})$, the "jacobian" of S, by $\operatorname{Jac} S$. The next task is to embed $M(\phi)$ into $\operatorname{Jac} S$ using the section σ of base points. To this end, choose a basis $\omega_1,\dots,\omega_{2g}$ of $H^1(S,\mathbb{Z})$. This gives an identification of $\operatorname{Jac} S$ with $(\mathbb{R}/\mathbb{Z})^{2g}$. Choose closed, real-valued one-forms $w_1,\dots,w_{2g}$ representing $\omega_1,\dots,\omega_{2g}$. They have integral periods. Since ϕ acts trivially on $H^1(S)$, there are smooth functions $f_j : S \to \mathbb{R}$ such that

$$\phi^* w_j = w_j + df_j.$$

These functions are uniquely determined if we insist, as we shall, that $f_j(x_0) = 0$ for each j. Set $\vec{w} = (w_1,\dots,w_g)$ and $\vec{f} = (f_1,\dots,f_g)$. The map

$$S \times [0,1] \to \operatorname{Jac} S$$

defined by

$$(x,t) \mapsto t\vec{f}(x) + \int_{x_0}^{x} \vec{w}$$

preserves the equivalence relations of the mapping torus $M(\phi)$, and therefore induces a map

$$\nu(\phi) : (M(\phi),\sigma(S^1)) \to (\operatorname{Jac} S,\, 0).$$

Define $\tilde{\tau}(\phi)$ to be the homology class of $M(\phi)$ in $H_3(\operatorname{Jac} S,\, \mathbb{Z})$:

$$\tilde{\tau}(\phi) = \nu(\phi)_*[M(\phi)] \in \Lambda^3 H_1(S,\mathbb{Z}).$$

PROPOSITION 3.2. *Suppose ϕ, ψ are diffeomorphisms of S that act trivially on $H_1(S)$. Then*:

(a) $\tilde{\tau}(\phi)$ *is independent of the choice of representatives* $w_1, \dots, w_g$ *of the basis* $\omega_1, \dots, \omega_{2g}$ *of* $H^1(S, \mathbb{Z})$;
(b) $\tilde{\tau}(\phi)$ *is independent of the choice of basis* $\omega_1, \dots, \omega_{2g}$ *of* $H^1(S, \mathbb{Z})$;
(c) $\tilde{\tau}(\phi)$ *depends only on the isotopy class of* ϕ;
(d) $\tilde{\tau}(\phi\psi) = \tilde{\tau}(\phi) + \tilde{\tau}(\psi)$;
(e) $\tilde{\tau}(g\psi g^{-1}) = g_* \tilde{\tau}(\phi)$ *for all diffeomorphisms* g *of* S, *where* g_* *is the automorphism of* $\Lambda^3 H_1(S)$ *induced by* g.

PROOF. If $w_1', \dots, w_{2g}'$ is another set of representatives of the ω_j, there are functions $g_j : S \to \mathbb{R}$ such that $w_j' = w_j + dg_j$ and $g_j(x_0) = 0$. For each $s \in [0, 1]$, the one-form $w_j(s) = w_j + s dg_j$ is closed on S and represents ω_j. The map

$$\nu_s : M(\phi) \to \operatorname{Jac} S$$

defined using the representatives $w_j(s)$ takes (x, t) to

$$t\big(f_j(x) + s(g_j(\phi(x)) - g_j(x))\big) + s g_j(x) + \int_{x_0}^{x} w_j.$$

Since this depends continuously on s, it follows that ν_0 is homotopic to ν_1, which proves (a).

Assertion (b) follows from linear algebra. The proof of (c) is similar to that of (a).

To prove (d), observe that the quotient of $M(\phi\psi)$ obtained by identifying $(x, 1)$ with $(\psi(x), \frac{1}{2})$ is the union of $M(\phi)$ and $M(\psi)$. The map $\nu(\phi\psi)$ factors through the quotient $M(\phi) \cup M(\psi)$ of $M(\phi\psi)$, and its restrictions to $M(\phi)$ and $M(\psi)$ are $\nu(\phi)$ and $\nu(\psi)$. Additivity follows.

Suppose that $g : (S, x_0) \to (S, x_0)$ is a diffeomorphism. The map $(g, \mathrm{id}) : S \times [0, 1] \to S \times [0, 1]$ induces a diffeomorphism

$$F(g) : M(\phi) \to M(g\phi g^{-1}).$$

To prove (e), it suffices to verify that the diagram

$$\begin{array}{ccc} M(\phi) & \xrightarrow{F(g)} & M(g\phi g^{-1}) \\ {\scriptstyle \nu(\phi)}\downarrow & & \downarrow{\scriptstyle \nu(g\phi g^{-1})} \\ \operatorname{Jac} S & \xrightarrow{g_*} & \operatorname{Jac} S \end{array}$$

commutes up to homotopy. In the proof of (a) we saw that the homotopy class of ν depends only on the basis of $H^1(S, \mathbb{Z})$ and not on the choice of de Rham representatives. Set $w_j' = g^* w_j$. Since the diagram

$$\begin{array}{ccc} H_1(S) & \xrightarrow{g_*} & H_1(S) \\ {\scriptstyle \int w_j'}\downarrow & & \downarrow{\scriptstyle \int w_j} \\ \mathbb{R} & = & \mathbb{R} \end{array}$$

commutes, it suffices to verify that the diagram

$$\begin{array}{ccc} M(\phi) & \xrightarrow{F(g)} & M(g\phi g^{-1}) \\ {\scriptstyle \nu'}\big\downarrow & & \big\downarrow{\scriptstyle \nu} \\ (\mathbb{R}/\mathbb{Z})^{2g} & \xrightarrow{\text{id}} & (\mathbb{R}/\mathbb{Z})^{2g} \end{array}$$

commutes, where ν is defined using $w_1, \dots, w_{2g}$, and ν' is defined using the representatives $w_1', \dots, w_{2g}'$. This is easily done. $\square$

Recall that the homology groups of T_g^1 are $\mathrm{Sp}_g(\mathbb{Z})$-modules; the action on $H_1(T_g)$ is given by

$$g : [\phi] \mapsto [\tilde{g}\phi\tilde{g}^{-1}],$$

where $g \in \mathrm{Sp}_g(\mathbb{Z})$ and $\tilde{g}$ is any element of Γ_g^1 that projects to g under the canonical homomorphism.

Corollary 3.3. *The map $\tilde{\tau}$ induces an $\mathrm{Sp}_g(\mathbb{Z})$-equivariant homomorphism*

$$\tau_g^1 : H_1(T_g^1, \mathbb{Z}) \to \Lambda^3 H_1(S, \mathbb{Z}). \qquad \square$$

From τ_g^1, we can construct a representation τ_g of $H_1(T_g)$. The kernel of the natural surjection $T_g^1 \to T_g$ is isomorphic to $\pi_1(S, x_0)$. The composition of the induced map $H_1(S, \mathbb{Z}) \to H_1(T_g^1, \mathbb{Z})$ with τ_g^1 is easily seen to be the canonical inclusion

$$\_ \times [S] : H_1(S, \mathbb{Z}) \hookrightarrow H_3(\mathrm{Jac}\, S,\ \mathbb{Z})$$

induced by taking Pontrjagin product with $\nu_*[S]$. We therefore have an induced $\mathrm{Sp}_g(\mathbb{Z})$-equivariant map

$$\tau_g : H_1(T_g, \mathbb{Z}) \to \Lambda^3 H_1(S, \mathbb{Z})/H_1(S, \mathbb{Z}).$$

The following result of Johnson is an immediate corollary of Theorem 3.1.

Theorem 3.4. *The homomorphism τ_g is an isomorphism modulo 2-torsion.*

It is not difficult to bootstrap up from Johnson's basic computation to prove the following result.

Theorem 3.5. *There is a natural $\mathrm{Sp}_g(\mathbb{Z})$-equivariant isomorphism*

$$\tau_{g,r}^n : H_1(T_{g,r}^n, \mathbb{Q}) \to H_1(S, \mathbb{Q})^{\oplus(n+r)} \oplus \Lambda^3 H_1(S, \mathbb{Q})/H_1(S, \mathbb{Q}).$$

An important consequence of Johnson's theorem is that the action of $\mathrm{Sp}_g(\mathbb{Z})$ on $H_1(T_{g,r}^n, \mathbb{Q})$ factors through a rational representation of the $\mathbb{Q}$-algebraic group Sp_g. Let $\lambda_1, \dots, \lambda_g$ be a fundamental set of dominant integral weights of Sp_g.

Denote the irreducible Sp_g-module with highest weight λ by $V(\lambda)$. The fundamental representation of Sp_g is $H_1(S)$. It is well-known (and easily verified) that

$$\Lambda^3 H_1(S) \approx V(\lambda_1) \oplus V(\lambda_3).$$

The previous result can be restated by saying that

$$H_1(T^n_{g,r}, \mathbb{Q}) \approx V(\lambda_3) \oplus V(\lambda_1)^{\oplus(n+r)}$$

as Sp_g-modules.

4. A second definition of the Johnson homomorphism

In this section we relate the definition of τ^1_g given in the previous section to Johnson's original definition, which is defined using the action of T^1_g on the lower central series of $\pi_1(S, x_0)$. It is better suited to computations. In order to relate this definition to the one given in the previous section, we need to study the cohomology ring of the mapping torus associated to an element of the Torelli group.

Suppose that the diffeomorphism $\phi : (S, x_0) \to (S, x_0)$ represents an element of T^1_g. As explained in the previous section, the associated mapping torus $M = M(\phi)$ fibers over S^1 and has a canonical section σ. These data guarantee that there is a canonical decomposition of the cohomology of M.

Since ϕ acts trivially on the homology of S, the E_2-term of the Leray–Serre spectral sequence of the fibration $\pi : M \to S^1$ satisfies

$$E_2^{r,s} = H^r(S^1) \otimes H^s(S).$$

This spectral sequence degenerates for trivial reasons. Consequently, there is a short exact sequence

$$0 \to H^1(S^1, \mathbb{Z}) \xrightarrow{\pi^*} H^1(M, \mathbb{Z}) \xrightarrow{i^*} H^1(S, \mathbb{Z}) \to 0,$$

where π is the projection to S^1 and $i : S \hookrightarrow M$ is the inclusion of the fiber over the base point $t = 0$ of S^1. The section σ induces a splitting of this sequence. Denote π^* of the positive generator of $H^1(S^1, \mathbb{Z})$ by θ. Then we have the decomposition

$$H^1(M, \mathbb{Z}) = H^1(S, \mathbb{Z}) \oplus \mathbb{Z}\theta. \tag{1}$$

From the spectral sequence, it follows that we have an exact sequence

$$0 \to \theta \wedge H^1(S, \mathbb{Z}) \to H^2(M, \mathbb{Z}) \xrightarrow{i^*} H^2(S, \mathbb{Z}) \to 0.$$

Denote the Poincaré dual of a homology class u in M by $\mathrm{PD}(u)$. Since

$$\int_S \mathrm{PD}(\sigma) = \sigma \cdot S = 1,$$

it follows that the previous sequence can be split by taking the positive generator of $H^2(S,\mathbb{Z})$ to $\mathrm{PD}(\sigma)$. We therefore have a canonical splitting

$$H^2(M,\mathbb{Z}) = \mathbb{Z}\,\mathrm{PD}(\sigma) \oplus \theta \wedge H^1(S,\mathbb{Z}). \tag{2}$$

The cup product pairing

$$c : H^1(M) \otimes H^2(M) \to H^3(M) \approx \mathbb{Z}$$

induces pairings between the summands of the decompositions (1) and (2).

Proposition 4.1. *The cup product c satisfies*:
(a) $c(\theta \otimes \mathrm{PD}(\sigma)) = 1$;
(b) *the restriction of c to $H^1(S) \otimes \mathrm{PD}(\sigma)$ vanishes*;
(c) *the restriction of c to $\theta \otimes \big(\theta \wedge H^1(S)\big)$ vanishes*;
(d) *the restriction of c to $H^1(S) \otimes \big(\theta \wedge H^1(S)\big)$ takes $u \otimes (\theta \wedge v)$ to $-\int_S u \wedge v$.*

Proof. Since θ is the Poincaré dual of the fiber S, we have

$$\int_M \theta \wedge \mathrm{PD}(\sigma) = \int_M \mathrm{PD}(S) \wedge \mathrm{PD}(\sigma) = S \cdot \sigma = 1.$$

In the decomposition (1), $H^1(S)$ is identified with the kernel of $\sigma^* : H^1(M) \to H^1(S^1)$; that is, with those $u \in H^1(M)$ such that

$$\int_\sigma u = 0.$$

The second assertion now follows as

$$\int_M u \wedge \mathrm{PD}(\sigma) = \int_\sigma u$$

for all $u \in H^1(M)$.

The third and fourth assertions are easily verified. □

To complete our understanding of the cohomology ring of M, we consider the cup product

$$\Lambda^2 H^1(M) \to H^2(M).$$

Since $\theta \wedge \theta = 0$, there is only one interesting part of this mapping, namely, the component

$$\Lambda^2 H^1(S) \to \mathbb{Z}\,\mathrm{PD}(\sigma) \oplus \theta \wedge H^1(S).$$

There is a unique function

$$f_\phi : \Lambda^2 H^1(S,\mathbb{Z}) \to H^1(S,\mathbb{Z})$$

such that

$$u \wedge v \mapsto \left(\int_S u \wedge v, -\theta \wedge f_\phi(u \wedge v) \right) \in H^2(M,\mathbb{Z})$$

with respect to the decomposition (2). We can view f_ϕ as an element of

$$H_1(S,\mathbb{Z})\otimes\Lambda^2 H^1(S,\mathbb{Z}).$$

Using Poincaré duality on the last two factors, we can regard f_ϕ as an element $F(\phi)$ of

$$H_1(S,\mathbb{Z})\otimes\Lambda^2 H_1(S,\mathbb{Z}).$$

There is a canonical embedding of $\Lambda^3 H_1(S,\mathbb{Z})$ into this group. It is defined by

$$a\wedge b\wedge c\mapsto a\otimes(b\wedge c)+b\otimes(c\wedge a)+c\otimes(a\wedge b).$$

THEOREM 4.2. *The invariant $F(\phi)$ of the cohomology ring of $M(\phi)$ is the image of $\tilde{\tau}(\phi)$ under the canonical embedding*

$$\Lambda^3 H_1(S,\mathbb{Z})\hookrightarrow H_1(S,\mathbb{Z})\otimes\Lambda^2 H_1(S,\mathbb{Z}).$$

PROOF. The dual of $\tau_g^1(\phi)$ is the map

$$\Lambda^3 H^1(S)\to\mathbb{Z}$$

defined by

$$u\wedge v\wedge w\mapsto\int_{M(\phi)}u\wedge v\wedge w.$$

Here we have identified $H^1(S)$ with $H^1(\operatorname{Jac}S)$ using the canonical isomorphism

$$\nu^*: H^1(\operatorname{Jac}S)\xrightarrow{\sim}H^1(S).$$

The map $\nu(\phi): M\to\operatorname{Jac}S$ collapses σ to the point 0. It follows that the image of

$$\nu(\phi)^*: H^1(\operatorname{Jac}S)\to H^1(M)$$

lies in the subspace we are identifying with $H^1(S)$ in the decomposition (1) of page 107. Since the restriction of $\nu(\phi)$ to the fiber over the base point $t=0$ of S^1 is the isomorphism ν^*, it follows that the diagram

$$\begin{array}{ccc} H^1(\operatorname{Jac}S) & \xrightarrow{\nu(\phi)^*} & H^1(M)\\ {\scriptstyle\nu^*}\big\downarrow & & \big\| \\ H^1(S) & \xrightarrow{i} & H^1(M)\end{array}$$

commutes, where i is the inclusion given by the splitting (1). That is, all the identifications we have made with $H^1(S)$ are compatible.

We will compute the dual of $\tau_g^1(\phi)$ using $F(\phi)$, which we regard as a homomorphism

$$F(\phi): H^1(S)\otimes\Lambda^2 H^1(S)\to\mathbb{Z}.$$

It follows from Proposition 4.1 that this map takes $u \otimes (v \wedge w)$ to

$$\int_S u \wedge f_\phi(v \wedge w).$$

The assertion that $F(\phi)$ lies in $\Lambda^3 H_1(S)$ is equivalent to the assertion that

$$F(\phi)(u \otimes (v \wedge w)) = F(\phi)(v \otimes (w \wedge u)) = F(\phi)(w \otimes (u \wedge v)),$$

which is easily verified using Proposition 4.1. The equality of $F(\phi)$ and $\tau_g^1(\phi)$ follows as

$$\tau_g^1(\phi)(u \wedge v \wedge w) = \int_M u \wedge v \wedge w = -\int_M u \wedge \theta \wedge f_\phi(v \wedge w) = F(\phi)(u \otimes (v \wedge w)).$$

□

We are now ready to give Johnson's original definition of τ_g^1. Denote the lower central series of a group π by

$$\pi = \pi^{(1)} \supseteq \pi^{(2)} \supseteq \pi^{(3)} \supseteq \cdots$$

We regard the cup product

$$\Lambda^2 H^1(S, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \approx \mathbb{Z}$$

as an element q of $\Lambda^2 H_1(S, \mathbb{Z})$.

Proposition 4.3. *The commutator mapping*

$$[\ ,\ ] : \pi_1(S, x_0) \times \pi_1(S, x_0) \to \pi_1(S, x_0)$$

induces an isomorphism $\Lambda^2 H_1(S, \mathbb{Z})/q \to \pi_1(S, x_0)^{(2)}/\pi_1(S, x_0)^{(3)}$.

Proof. This follows directly from the standard fact (see [44] or [36]) that if F is a free group, the commutator induces an isomorphism

$$\Lambda^2 H_1(F) \xrightarrow{\sim} F^{(2)}/F^{(3)}$$

and from the standard presentation of $\pi_1(S, x_0)$. □

An element ϕ of T_g^1 induces an automorphism of $\pi_1(S, x_0)$. Since it acts trivially on $H_1(S)$,

$$\phi(\gamma)\gamma^{-1} \in \pi_1(S, x_0)^{(2)}$$

for all $\gamma \in \pi_1(S, x_0)$. From Proposition 4.3, it follows that ϕ induces a well defined map

$$\hat{\tau}(\phi) : H_1(S, \mathbb{Z}) \to \Lambda^2 H_1(S, \mathbb{Z})/q$$

Using Poincaré duality, we may view this as an element $L(\phi)$ of

$$H_1(S, \mathbb{Z}) \otimes \left(\Lambda^2 H_1(S, \mathbb{Z})/q\right).$$

Theorem 4.4. *The image of* $F(\phi)$ *in* $H_1(S, \mathbb{Z}) \otimes \left(\Lambda^2 H_1(S, \mathbb{Z})/q\right)$ *is* $L(\phi)$.

Proof. Since $H_1(S,\mathbb{Z})\otimes\big(\Lambda^2 H_1(S,\mathbb{Z})/q\big)$ is torsion-free, it suffices to show that the image of $F(\phi)$ in

$$H_1(S,\mathbb{Q})\otimes\big(\Lambda^2 H_1(S,\mathbb{Q})/q\big)$$

is $L(\phi)$. For the rest of this proof, all (co)homology groups have $\mathbb{Q}$ coefficients.

For all groups π with finite-dimensional $H_1(\ \ ,\mathbb{Q})$, the sequence

$$0\to H^1(\pi)\xrightarrow{h^*}\big(\pi/\pi^{(3)}\big)^*\xrightarrow{[\ ,\ ]^*}\Lambda^2 H^1(\pi)\xrightarrow{\wedge}H^2(\pi) \tag{3}$$

of $\mathbb{Q}$ vector spaces is exact. Here $(\ \ )^*$ denotes the dual vector space, h^* the dual Hurewicz homomorphism, and $[\ ,\ ]$ the map induced by the commutator. This can be proved using results in either [11, §2.1] or [48, §8].

We apply this sequence to the fundamental group of the mapping torus. Choose $m_0=(x_0,0)$ as the base point of M. Since M fibers over the circle with fiber S, we have an exact sequence

$$1\to\pi_1(S,x_0)\to\pi_1(M,m_0)\to\mathbb{Z}\to 0.$$

The section σ induces a splitting $\mathbb{Z}\to\pi_1(M,m_0)$.

Denote the image of 1 by σ. Observe that if $\gamma\in\pi_1(S,x_0)$, then

$$\sigma\gamma\sigma^{-1}=\phi(\gamma).$$

It follows that the inclusion $\pi_1(S,x_0)\hookrightarrow\pi_1(M,m_0)$ induces isomorphisms

$$\pi_1(S,x_0)^{(k)}\approx\pi_1(M,m_0)^{(k)}$$

for all $k>1$ and, as above, that σ induces an isomorphism

$$H_1(M)=H_1(S)\oplus\mathbb{Q}\Sigma,$$

where Σ denotes the homology class of σ. It also follows that for all $a\in H_1(S)$

$$\tilde{\tau}(\phi)(a)=[\Sigma,a]\in\pi_1(M)^{(2)}/\pi_1(M)^{(3)}\approx\pi_1(S)^{(2)}/\pi_1(S)^{(3)}.$$

Using Proposition 4.1 and the exact sequence (3), we see that for all $u\in H_1(S)$, the image of $f_\phi^*(u)$ in

$$\Lambda^2 H_1(S)/q$$

is $[\Sigma,u]$, which is $\hat{\tau}(\phi)(u)$ as we have seen. The result follows. □

The composite of the inclusion

$$\Lambda^3 H_1(S,\mathbb{Z})\hookrightarrow H_1(S,\mathbb{Z})\otimes\Lambda^2 H_1(S,\mathbb{Z})$$

with the quotient mapping

$$H_1(S,\mathbb{Z})\otimes\Lambda^2 H_1(S,\mathbb{Z})\to H_1(S,\mathbb{Z})\otimes\big(\Lambda^2 H_1(S,\mathbb{Z})/q\big)$$

is injective. One way to see this is to tensor with $\mathbb{Q}$ and note that both of these maps are maps of Sp_g-modules. One can then use the fact that $\Lambda^3 H_1(S)$ is the sum of the first and third fundamental representations of Sp_g to check the result. The following result is therefore a restatement of Theorem 4.2.

COROLLARY 4.5. *$L(\phi)$ lies in the image of the canonical injection*

$$\Lambda^3 H_1(S,\mathbb{Z}) \hookrightarrow H_1(S,\mathbb{Z}) \otimes \big(\Lambda^2 H_1(S,\mathbb{Z})/q\big)$$

and the corresponding point of $\Lambda^3 H_1(S)$ is $\tau_g^1(\phi)$.

In his fundamental papers, Johnson defines $\tau_g^1(\phi)$ to be $L(\phi)$. The other two definitions we have given were stated in [33].

5. Picard groups

In [39], Mumford showed that

$$c_1 : \mathrm{Pic}\,\mathcal{M}_g \otimes \mathbb{Q} \to H^2(\mathcal{M}_g, \mathbb{Q})$$

is an isomorphism. Using Johnson's computation of $H_1(T_g, \mathbb{Q})$ and the well-known Theorem 5.3, we will prove the analogous statement for all $\mathcal{M}_{g,r}^n(L)$ when $g \geq 3$. The novelty lies in the variation of the level, and not in the variation of the decorations r and n. The first, and principal, step is to establish the vanishing of the $H^1(\mathcal{M}_{g,r}^n(L))$.

PROPOSITION 5.1. *Suppose that L is a finite-index subgroup of $\mathrm{Sp}_g(\mathbb{Z})$. If $g \geq 3$, then $H^1(\mathcal{M}_{g,r}^n(L), \mathbb{Z}) = 0$.*

Since $H^1(\ \ ,\mathbb{Z})$ is always torsion-free, it suffices to prove that $H^1(\mathcal{M}_g(L), \mathbb{Q})$ vanishes. We will prove a stronger result.

PROPOSITION 5.2. *Suppose that L is a finite-index subgroup of $\mathrm{Sp}_g(\mathbb{Z})$ and that $g \geq 3$. If $V(\lambda)$ is an irreducible representation of Sp_g with highest weight λ, then*

$$H^1(\Gamma_{g,r}^n(L), V(\lambda)) = \begin{cases} \mathbb{Q}^{r+n} & \text{if } \lambda = \lambda_1; \\ \mathbb{Q} & \text{if } \lambda = \lambda_3; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $H^1(\mathcal{M}_{g,r}^n(L), \mathbb{Z})$ vanishes for all r and n when $g \geq 3$.

PROOF. It follows from the Hochschild–Serre spectral sequence

$$H^r(L, H^s(T_{g,r}^n \otimes V(\lambda))) \implies H^{r+s}(\Gamma_{g,r}^n(L), V(\lambda))$$

that there is an exact sequence

$$0 \to H^1(L, V(\lambda)) \to H^1(\Gamma^n_{g,r}(L), V(\lambda)) \\ \to H^0(L, H^1(T^n_{g,r} \otimes V(\lambda))) \xrightarrow{d_2} H^2(L, V(\lambda)).$$

By a result of Ragunathan [42], the first term vanishes when $g \geq 2$. By Theorem 3.5, the third term vanishes except when λ is either λ_1 or λ_3. This proves the result except when λ is either λ_1 or λ_3. In these exceptional cases, the third term has rank $r+n$ or 1, respectively. To complete the proof, we need to show that the differential d_2 is zero.

There are several ways to do this. Perhaps the most straightforward is to use the result, due to Borel [7], that asserts that the last group vanishes when $g \geq 8$. This establishes the result when $g \geq 8$. When $r \geq 1$, the vanishing of d_r for all $g \geq 3$ follows from the fact that the diagram

$$\begin{array}{ccc} H^0(L, H^1(T^n_{g,r} \otimes V(\lambda))) & \xrightarrow{d_2} & H^2(L, V(\lambda)) \\ \uparrow & & \uparrow \\ H^0(L, H^1(T^n_{g+8,r} \otimes V(\lambda))) & \xrightarrow{d_2} & H^2(L_{g+8}, V(\lambda)) \end{array}$$

commutes. Here L_{g+8} is any finite-index subgroup of $\mathrm{Sp}_{g+8}(\mathbb{Z})$ such that

$$L_{g+8} \cap \mathrm{Sp}_g(\mathbb{Z}) \subseteq L$$

and the vertical maps are induced by the "stabilization map"

$$\Gamma^n_{g,r}(L) \to \Gamma^n_{g+8,\,r}(L_{g+8}).$$

When $r=0$ and $\lambda = \lambda_1$, there is nothing to prove. This leaves only the case $r=0$ and $\lambda=\lambda_3$, which follows from the fact that the diagram

$$\begin{array}{ccc} H^0(L, H^1(T^n_{g} \otimes V(\lambda))) & \xrightarrow{d_2} & H^2(L, V(\lambda)) \\ \uparrow & & \uparrow \\ H^0(L, H^1(T^n_{g,1} \otimes V(\lambda))) & \xrightarrow{d_2} & H^2(L, V(\lambda)) \end{array},$$

which arises from the homomorphism $\Gamma^n_{g,1} \to \Gamma^n_g$, commutes. $\square$

Denote the category of $\mathbb{Z}$ mixed Hodge structures by $\mathcal{H}$. Denote the group of "integral $(0,0)$ elements" $\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}, H)$ of a mixed Hodge structure H by ΓH.

Suppose that X is a smooth variety. Since $H^1(X,\mathbb{Z})$ is torsion-free, we can define

$$W_1 H^1(X,\mathbb{Z}) = W_1 H^1(X,\mathbb{Q}) \cap H^1(X,\mathbb{Z}).$$

This is a polarized, torsion-free Hodge structure of weight 1. Set

$$JH^1(X) = \frac{W_1 H^1(X,\mathbb{C})}{W_1 H^1(X,\mathbb{Z}) + F^1 W_1 H^1(X,\mathbb{C})}.$$

This is a polarized Abelian variety.

THEOREM 5.3. *If X is a smooth variety, there is a natural exact sequence*

$$0 \to JH^1(X) \to \operatorname{Pic} X \to \Gamma H^2(X, \mathbb{Z}(1)) \to 0.$$

Alternatively, this theorem may be stated as saying that the cycle map

$$\operatorname{Pic} X \to H^2_{\mathcal{H}}(X, \mathbb{Z}(1))$$

is an isomorphism, where $H^\bullet_{\mathcal{H}}$ denotes Beilinson's absolute Hodge cohomology, the refined version of Deligne cohomology defined in [4].

PROOF. Choose a smooth completion $\overline{X}$ of X for which $\overline{X} - X$ is a normal crossings divisor D in $\overline{X}$ with smooth components. Denote the dimension of X by d. From the usual exponential sequence, we have a short exact sequence

$$0 \to JH^1(\overline{X}) \to \operatorname{Pic} \overline{X} \to \Gamma H^2(X, \mathbb{Z}) \to 0.$$

From [12, (1.8)], we have an exact sequence

$$CH^0(D) \to \operatorname{Pic} \overline{X} \to \operatorname{Pic} X \to 0.$$

The Gysin sequence

$$\begin{aligned} 0 \to H^1(\overline{X}) \to H^1(X) \to H_{2d-2}(D)(-2d) \to H^2(\overline{X}) \to& \\ H^2(X) \to H_{2d-3}(D)(-2d)& \end{aligned}$$

is an exact sequence of $\mathbb{Z}$ Hodge structures. Since $H_{2d-2}(D)(-2d)$ is torsion-free and of weight 2, it follows that

$$W_1 H^1(X, \mathbb{Z}) = H^1(\overline{X}, \mathbb{Z}),$$

and therefore that $JH^1(X) = JH^1(\overline{X})$. Next, since each component D_i of D is smooth, it follows that

$$H_{2d-3}(D)(-2d) = \bigoplus_i H^1(D_i, \mathbb{Z})(-1),$$

and is therefore torsion-free and of weight 3. It follows that the sequence

$$H_{2d-2}(D)(-2d) \to \Gamma H^2(\overline{X}) \to \Gamma H^2(X) \to 0$$

is exact. Since the cycle map

$$CH^0(D) \to H_{2d-2}(D)$$

is an isomorphism [12, (1.5)], the result follows. □

It is now an easy matter to show that the Picard groups of the $\mathcal{M}^n_{g,r}(L)$ are finitely generated.

THEOREM 5.4. *Suppose that L is a finite-index subgroup of $\mathrm{Sp}_g(\mathbb{Z})$. If $g \geq 3$, then for all r, n, the Chern class map*

$$c_1 : \operatorname{Pic} \mathcal{M}^n_{g,r}(L) \to \Gamma H^2(\mathcal{M}^n_{g,r}(L), \mathbb{Z})$$

is an isomorphism when $\Gamma^n_{g,r}(L)$ is torsion-free, and is an isomorphism after tensoring with $\mathbb{Q}$ in general.

PROOF. The case when $\Gamma^n_{g,r}(L)$ is torsion-free follows directly from Proposition 5.1 and Theorem 5.3. To prove the assertion in general, choose a finite-index normal subgroup L' of L such that $\Gamma^n_{g,r}(L')$ is torsion-free. Let

$$G = \Gamma^n_{g,r}(L)/\Gamma^n_{g,r}(L') \approx L/L'.$$

Then it follows from the Teichmüller description of moduli spaces that the projection

$$\pi : \mathcal{M}^n_{g,r}(L') \to \mathcal{M}^n_{g,r}(L)$$

is a Galois covering with Galois group G. It follows from the first case that

$$c_1 : \operatorname{Pic} \mathcal{M}^n_{g,r}(L') \to \Gamma H^2(\mathcal{M}^n_{g,r}(L'), \mathbb{Z})$$

is a G-equivariant isomorphism. The result now follows as the projection π induces isomorphisms

$$\operatorname{Pic} \mathcal{M}^n_{g,r}(L) \otimes \mathbb{Q} \approx H^0(G, \operatorname{Pic} \mathcal{M}^n_{g,r}(L') \otimes \mathbb{Q})$$

and

$$\Gamma H^2(\mathcal{M}^n_{g,r}(L), \mathbb{Q}) \approx \Gamma H^0(G, H^2(\mathcal{M}^n_{g,r}(L'), \mathbb{Q})).$$

$\square$

If we knew that $H^2(T_g, \mathbb{Q})$ were finite-dimensional and a rational representation of Sp_g, we would know from Borel's work [6] that $H^2(\mathcal{M}^n_{g,r}(L), \mathbb{Q})$ would be independent of the level L, once g is sufficiently large; $g \geq 8$ should do it [7]. It would then follow, for sufficiently large g, that the Picard number of $\mathcal{M}^n_{g,r}(L)$ is $n+r+1$. At present it is not even known whether $H^2(T_g, \mathbb{Q})$ is finite-dimensional. The computation of this group, and the related problem of finding a presentation of T_g, appear to be deep and difficult. It should be mentioned that the only evidence for the belief that the Picard number of each $\mathcal{M}_g(L)$ is one comes from Harer's computation [25] of the Picard numbers of the moduli spaces of curves with a distinguished theta characteristic.

6. Normal functions

In this section, we define abstract normal functions that generalize the normal functions of Poincaré and Griffiths. We begin by reviewing how a family of homologically trivial algebraic cycles in a family of smooth projective varieties gives rise to a normal function.

Suppose that X is a smooth variety. A homologically trivial algebraic d-cycle in X canonically determines an element of

$$\operatorname{Ext}^1_{\mathcal{H}}(\mathbb{Z}, H_{2d+1}(X, \mathbb{Z}(-d))).$$

This extension is obtained by pulling back the exact sequence

$$0 \to H_{2d+1}(X, \mathbb{Z}(-d)) \to H_{2d+1}(X, Z, \mathbb{Z}(-d)) \to H_{2d}(Z, \mathbb{Z}(-d)) \to \cdots$$

of mixed Hodge structures along the inclusion

$$\mathbb{Z} \to H_{2d}(|Z|, \mathbb{Z}(-d))$$

that takes 1 to the class of Z.

When H is a mixed Hodge structure all of whose weights are non-positive, there is a natural isomorphism

$$JH \approx \operatorname{Ext}^1_{\mathcal{H}}(\mathbb{Z}, H),$$

where

$$JH = \frac{H_{\mathbb{C}}}{F^0 H_{\mathbb{C}} + H_{\mathbb{Z}}}.$$

(This is well-known; see [9], for example. Our conventions will be taken from [18, (2.2)].)

When X is projective, Poincaré duality provides an isomorphism of the complex torus $JH_{2d+1}(X, \mathbb{Z}(-d))$ with the Griffiths intermediate jacobian

$$\operatorname{Hom}_{\mathbb{C}}(F^d H^{d+1}(X), \mathbb{C})/H_{2d+1}(X, \mathbb{Z}).$$

The point in $JH_{2d+1}(X, Z(d))$ corresponding to the cycle Z under this isomorphism is $\int_\Gamma$, where Γ is a real $2d+1$ chain that satisfies $\partial\Gamma = Z$.

Now suppose that $\mathcal{X} \to T$ is a family of smooth projective varieties over a smooth base T. Suppose that $\mathcal{Z}$ is an algebraic cycle in $\mathcal{X}$, which is proper over T of relative dimension d. Denote the fibers of $\mathcal{X}$ and $\mathcal{Z}$ over $t \in T$ by X_t and Z_t.

The set of $H_{2d+1}(X_t, \mathbb{Z}(-d))$ form a variation of Hodge structure $\mathbb{V}$ over T of weight -1. We can form the relative intermediate jacobian

$$\mathcal{J}_d \to T,$$

which has fiber $JH_{2d+1}(X_t, \mathbb{Z}(-d))$ over $t \in T$. The family of cycles $\mathcal{Z}$ defines a section of this bundle. Such a section is what Griffiths calls *the normal function of the cycle* $\mathcal{Z}$ [14]. Griffiths' normal functions generalize those of Poincaré.

We will generalize this notion further. Before we do, note that the elements of $\mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}, H_{2d+1}(X_t, \mathbb{Z}(-d)))$ defined by the cycles Z_t fit together to form a variation of mixed Hodge structure over T. It follows from the main result of [15] that this variation is good in the sense of [47] along each curve in T, and is therefore good in the sense of Saito [43].

Suppose that T is a smooth variety and that $\mathbb{V} \to T$ is a variation of Hodge structure over T of negative weight. Denote by $J\mathcal{V}$ the bundle over T whose fiber over $t \in T$ is

$$JV_t \approx \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}, V_t).$$

DEFINITION 6.1. A holomorphic section $s : T \to J\mathcal{V}$ of $J\mathcal{V} \to T$ is a *normal function* if it defines an extension

$$0 \to \mathbb{V} \to \mathbb{E} \to \mathbb{Z}_T \to 0$$

in the category $\mathcal{H}(T)$ of good variations of mixed Hodge structure over T.

REMARK 6.2. We know from the preceding discussion that families of homologically trivial cycles in a family $\mathcal{X} \to T$ define normal functions in this sense.

The asymptotic properties of good variations of mixed Hodge structure guarantee that these normal functions have nice properties.

LEMMA 6.3 (RIGIDITY). *If $\mathbb{V} \to T$ and $\mathbb{V}' \to T$ are two good variations of mixed Hodge structure over T with the same fiber V_{t_0} (viewed as a mixed Hodge structure) over some point t_0 of T and with the same monodromy representations*

$$\pi_1(T, t_0) \to \mathrm{Aut}\, V_{t_0},$$

then $\mathbb{V}_1$ and $\mathbb{V}_2$ are isomorphic as variations.

PROOF. The proof is a standard application of the theorem of the fixed part. The local system $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{V}, \mathbb{V}')$ underlies a good variation of mixed Hodge structure. From Saito's work [43], we know that each cohomology group of a variety with coefficients in a good variation of mixed Hodge structure has a natural mixed Hodge structure. So, in particular,

$$H^0(T, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{V}, \mathbb{V}'))$$

has a mixed Hodge structure, and the restriction map

$$H^0(T, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{V}, \mathbb{V}')) \to \mathrm{Hom}_{\mathbb{Z}}(V_{t_0}, V'_{t_0})$$

is a morphism. The result now follows since there are natural isomorphisms

$$H^0(T, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{V}, \mathbb{V}')) \approx \mathrm{Hom}_{\mathbb{Z}\pi_1(T,t_0)}(V_{t_0}, V'_{t_0})$$

and

$$\Gamma H^0(T, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{V}, \mathbb{V}')) \approx \mathrm{Hom}_{\mathcal{H}(T)}(\mathbb{V}, \mathbb{V}'),$$

where $\mathcal{H}(T)$ denotes the category of good variations of mixed Hodge structure over T. □

COROLLARY 6.4. *Two normal functions $s_1, s_2 : T \to J\mathcal{V}$ are equal if and only if there is a point $t_0 \in T$ such that $s_1(t_0) = s_2(t_0)$ and such that the two induced homomorphisms*

$$(s_j)_* : \pi_1(T, t_0) \to \pi_1(J\mathcal{V}, s_1(t_0))$$

are equal. □

7. Extending normal functions

The strong asymptotic properties of variations of mixed Hodge structure imply that almost all normal functions extend across subvarieties where the original variation of Hodge structure is non-singular. Suppose that X is a smooth variety and that $\mathbb{V}$ is a variation of Hodge structure over X of negative weight. Denote the associated intermediate jacobian bundle by $\mathcal{J} \to X$.

THEOREM 7.1. *Suppose that U is a Zariski open subset of X and $s : U \to \mathcal{J}|_U$ is a normal function defined on U. If the weight of $\mathbb{V}$ is not -2, then s extends to a normal function $\tilde{s} : X \to \mathcal{J}$.*

PROOF. Write $U = X - Z$. By Hartog's Theorem, it suffices to show that s extends to a normal function on the complement of the union of the singular locus of Z and the union of the components of Z of codimension ≥ 2 in X. That is, we may assume that Z is a smooth divisor.

The problem of extending s is local. By taking a transverse slice, we can reduce to the case where X is the unit disk Δ and Z is the origin. In this case, we have a variation of Hodge structure over Δ. The normal function $s : \Delta^* \to \mathcal{J}$ corresponds to a good variation of mixed Hodge structure $\mathbb{E}$ over the punctured disk Δ^*, which is an extension

$$0 \to \mathbb{V}|_{\Delta^*} \to \mathbb{E} \to \mathbb{Z}_{\Delta^*}.$$

To prove that the normal function extends, it suffices to show that the monodromy of $\mathbb{E}$ is trivial, for then the local system $\mathbb{E}$ extends uniquely as a flat bundle to Δ and the Hodge filtration extends across the origin as $\mathbb{E}$ is a good variation.

Since $\mathbb{V}$ is defined on the whole disk, it has trivial monodromy. It follows that the local monodromy operator T of $\mathbb{E}$ satisfies

$$(T - I)^2 = 0$$

and that the local monodromy logarithm N is $T - I$. Since E is a good variation, it has a relative weight filtration $M_\bullet$ [47], which is defined over $\mathbb{Q}$ and satisfies

$NM_l \subseteq M_{l-2}$. From the defining properties of $M_\bullet$ ([47, (2.5)]), we have

$$M_0 = \mathbb{E},\ M_m = \mathbb{V},\ \text{and } M_{m-1} = 0,$$

where m is the weight of $\mathbb{V}$.

In the case $m = -1$, the proof that $N = 0$ is simpler. Since this case is the most important (as it is the one that applies to normal functions of cycles), we prove it first. The condition $m = -1$ implies that $M_{-2} = 0$. Since $NM_0 \subseteq M_{-2}$, it follows that $N = 0$ and consequently, that the normal function extends.

In general, we use the defining property [47, (3.13.iii)] of good variations of mixed Hodge structure, which says that

$$(E_t, M_\bullet, F^\bullet_{\mathrm{lim}})$$

is a mixed Hodge structure and N is a morphism of mixed Hodge structures of type $(-1,-1)$, where $F^\bullet_{\mathrm{lim}}$ denotes the limit Hodge filtration. In this case, N induces a morphism

$$\mathbb{Z} \approx \mathrm{Gr}^M_0 \to \mathrm{Gr}^M_{-2},$$

which is zero if $m \neq -2$. Since N is a morphism of mixed Hodge structures, the vanishing of this map implies the vanishing of N. □

When $m = -2$, there are normal functions that don't extend. For example, if we take $\mathbb{V} = \mathbb{Z}(1)$, the bundle of intermediate jacobians is the bundle $X \times \mathbb{C}^*$ and the normal functions are precisely the invertible regular functions $f : X \to \mathbb{C}$. For details see, for example, [20, (9.3)].

8. Normal functions over $\mathcal{M}^n_{g,r}(L)$

Throughout this section, we will assume that $g \geq 3$ and L is a finite-index subgroup of $\mathrm{Sp}_g(\mathbb{Z})$ such that $\Gamma^n_{g,r}(L)$ is torsion-free. With this condition on L, $\mathcal{M}^n_{g,r}(L)$ is smooth. Each irreducible representation of Sp_g defines a polarized $\mathbb{Q}$ variation of Hodge structure over $\mathcal{M}^n_{g,r}(L)$, which is unique up to Tate twist: see Proposition 9.1. It follows that every rational representation of Sp_g underlies a polarized $\mathbb{Z}$ variation of Hodge structure over $\mathcal{M}_g(L)$.

Lemma 8.1. *If $\mathbb{V} \to \mathcal{M}^n_{g,r}(L)$ is a good variation of Hodge structure of negative weight whose monodromy representation*

$$\Gamma^n_{g,r}(L) \to \mathrm{Aut}\, V_\circ \otimes \mathbb{Q}$$

factors through a rational representation of Sp_g and contains no copies of the trivial representation, the group of normal functions $s : \mathcal{M}^n_{g,r}(L) \to J\mathcal{V}$ is finitely generated of rank bounded by

$$\dim H^1(\Gamma^n_{g,r}(L), V_{\mathbb{Z}}).$$

PROOF. A normal function corresponds to a variation of mixed Hodge structure whose underlying local system is an extension

$$0 \to \mathbb{V} \to \mathbb{E} \to \mathbb{Z} \to 0$$

of the trivial local system by $\mathbb{V}$.

One can form the semidirect product $\Gamma^n_{g,r}(L) \ltimes V_{\mathbb{Z}}$, where the mapping class group acts on $V_{\mathbb{Z}}$ via a representation $L \to \operatorname{Aut} V$. The monodromy representation of the local system $\mathbb{E}$ gives a splitting

$$\rho : \Gamma^n_{g,r}(L) \to \Gamma^n_{g,r}(L) \ltimes V_{\mathbb{Z}}$$

of the natural projection

$$\Gamma^n_{g,r}(L) \ltimes V_{\mathbb{Z}} \to \Gamma^n_{g,r}(L). \tag{4}$$

The splitting is well defined up to conjugation by an element of $V_{\mathbb{Z}}$.

The first step in the proof is to show that an extension of $\mathbb{Q}$ by $\mathbb{V}$ in the category of $\mathbb{Q}$ variations of mixed Hodge structure is determined by its monodromy representation. Two such variations can be regarded as elements of the group

$$\operatorname{Ext}^1_{\mathcal{H}(\mathcal{M}^n_{g,r}(L))}(\mathbb{Q}, \mathbb{V}). \tag{5}$$

It is easily seen that their difference is an extension whose monodromy representation factors through the homomorphism $\Gamma^n_{g,r} \to \mathrm{Sp}_g(\mathbb{Q})$. It now follows from Proposition 9.2 and the assumption that $\mathbb{V}$ contain no copies of the trivial representation that this difference is the trivial element of (5). The assertion follows.

From [35, p. 106] it follows that the set of splittings of (4), modulo conjugation by elements of $V_{\mathbb{Z}}$, is isomorphic to

$$H^1(\Gamma^n_{g,r}(L), V_{\mathbb{Z}}).$$

It follows from Proposition 5.2 that this group is finitely generated. Since normal functions are determined by their monodromy, the result follows. $\square$

If $\mathbb{V}$ contains the trivial representation, the group of normal functions is an uncountably generated divisible group. For example, if $\mathbb{V}$ has trivial monodromy, then all such extensions are pulled back from a point. The set of normal functions is then

$$\operatorname{Ext}^1_{\mathcal{H}}(\mathbb{Z}, V_o) \approx JV_o,$$

where V_o denotes the fiber over the base point.

THEOREM 8.2. *If, in addition, the fiber over the base point is an irreducible* Sp_g*-module with highest weight* λ *and Hodge weight* m*, then the group of normal functions* $s : \mathcal{M}_{g,r}^n(L) \to J\mathcal{V}$ *is finitely generated of rank*

$$\dim H^1(\Gamma_{g,r}^n(L), V(\lambda)) = \begin{cases} 1 & \text{if } \lambda = \lambda_3 \text{ and } m = -1; \\ r+n & \text{if } \lambda = \lambda_1 \text{ and } m = -1; \\ 0 & \text{otherwise.} \end{cases}$$

The upper bounds for the rank of the group of normal functions follow from Lemma 8.1, Proposition 5.2, and the fact that the monodromy representation associated to a normal function has to be a morphism of variations of mixed Hodge structure Proposition 9.3. It remains to show that these upper bounds are achieved. We do this by explicitly constructing normal functions.

Multiples of the generators mod torsion of the normal functions associated to $V(\lambda_1)$ can be pulled back from $\mathcal{M}_g^1(L)$ along the $n+r$ forgetful maps $\mathcal{M}_{g,r}^n(L) \to \mathcal{M}_g^1(L)$. There the normal function can be taken to be the one that takes (C,x) to the point $(2g-2)x - \kappa_C$ of $\mathrm{Pic}^0 C$, where κ_C denotes the canonical class of C.

A multiple of the normal function associated to λ_3 can be pulled back from $\mathcal{M}_g(L)$ along the forgetful map $\mathcal{M}_{g,r}^n(L) \to \mathcal{M}_g(L)$. We will describe how this normal function over $\mathcal{M}_g(L)$ arises geometrically. If C is a smooth projective curve of genus g and $x \in C$, we have the Abel–Jacobi mapping

$$\nu_x : C \to \mathrm{Jac}\, C.$$

Denote the algebraic one-cycle $\nu_{x*}C$ in $\mathrm{Jac}\, C$ by C_x. Denote the cycle $i_* C_x$ by C_x^-, where $i : \mathrm{Jac}\, C \to \mathrm{Jac}\, C$ takes u to $-u$. The cycle $C_x - C^-$ is homologous to zero, and therefore defines a point $\tilde{e}(C,x)$ in $JH_3(\mathrm{Jac}\, C, \mathbb{Z}(-1))$. Pontrjagin product with the class of C induces a homomorphism

$$A : \mathrm{Jac}\, C \to JH_3(\mathrm{Jac}\, C, \mathbb{Z}(-1)).$$

Denote the cokernel of A by $JQ(\mathrm{Jac}\, C)$. It is not difficult to show that

$$\tilde{e}(C,x) - \tilde{e}(C,y) = A(x-y).$$

It follows that the image of $\tilde{e}(C,x)$ in $JQ(\mathrm{Jac}\, C)$ is independent of x. The image will be denoted by $e(C)$.

The primitive decomposition

$$H_3(\mathrm{Jac}\, C, \mathbb{Q}) = H_1(\mathrm{Jac}\, C, \mathbb{Q}) \oplus PH_3(\mathrm{Jac}\, C, \mathbb{Q})$$

is the decomposition of $H_3(\mathrm{Jac}\, C)$ into irreducible Sp_g-modules, the highest weights of the pieces being λ_1 and λ_3, respectively.

Fix a level L so that $\mathcal{M}_g(L)$ is smooth. The union of the $JQ(\operatorname{Jac} C)$ forms the bundle $\mathcal{J}_{\lambda_3}$ of intermediate jacobians over $\mathcal{M}_g(L)$ associated to the variation of Hodge structure of weight -1 and monodromy the third fundamental representation $V(\lambda_3)$ of Sp_g.

THEOREM 8.3. *The section e of $\mathcal{J}_{\lambda_3}$ is a normal function. Every other normal function associated to this bundle is, up to torsion, a half-integer multiple of e.*

PROOF. This result is essentially proved in [19]. We give a brief sketch.

To see that e is a normal function, consider the bundle of intermediate jacobians $JH_3(\operatorname{Jac} C, \mathbb{Z}(-1))$ over $\mathcal{M}_g^1(L)$. It follows from Remark 6.2 that $(C,x) \mapsto \tilde{e}(C,x)$ is a normal function. The argument of [19, p. 97] shows that there is a canonical quotient of the variation corresponding to $\tilde{e}$. (It is the extension E in [19, display 10].) This variation does not depend on the base point x, and is therefore constant along the fibers of $\mathcal{M}_g^1(L) \to \mathcal{M}_g(L)$. It follows that this quotient variation is the pullback of a variation on $\mathcal{M}_g(L)$. This quotient variation is classified by e. It follows that e is a normal function.

Each normal function f associated to this bundle of intermediate jacobians induces an L-equivariant homomorphism

$$f_* : H_1(T_g, \mathbb{Z}) \to H_1(JQ, \mathbb{Z}) \approx \Lambda^3 H_1(C, \mathbb{Z})/H_1(C, \mathbb{Z}).$$

It follows from a monodromy computation in [18, (4.3.5)] (see also [19, (6.3)]) that e_* is twice the Johnson homomorphism

$$\tau_g : H_1(T_g, \mathbb{Z}) \to \Lambda^3 H_1(C, \mathbb{Z})/H_1(C, \mathbb{Z}).$$

Since this homomorphism is primitive—that is, not a non-trivial integral multiple of another such normal function—all other normal functions associated to λ_3 must have monodromy representations that are half-integer multiples of that of e. As we have seen in the proof of Lemma 8.1, such normal functions are determined, up to torsion, by their monodromy representation. The result follows. □

I don't know how to realize $e/2$ as a normal function in this sense. But I do know to construct a more general kind of normal function associated to the one-cycle C in $\operatorname{Jac} C$ that does realize $e/2$. It is a section of a bundle whose fiber over C is a principal $JQ(\operatorname{Jac} C)$ bundle. The details may be found in [19, p. 92].

REMARK 8.4. Using the results in Section 9 and Theorem 8.2, one can easily show that the rank of the group of normal functions in the theorem above is

$$\dim \Gamma \operatorname{Hom}_{\mathrm{Sp}_g(\mathbb{Q})}(H_1(T_{g,r}^n, \mathbb{Q}), V_{\mathbb{Q},C}),$$

where $H_1(T_{g,r}^n)$ is given the Hodge structure of weight -1 described in §9.

9. Technical results on variations over $\mathcal{M}_g$

In this section we prove several technical facts about variations of mixed Hodge structure over moduli spaces of curves that were used in Section 8. Throughout we will assume that L has been chosen so that $\Gamma^n_{g,r}(L)$ is torsion-free.

PROPOSITION 9.1. *The local system $\mathbb{V}(\lambda)$ over $\mathcal{M}^n_{g,r}(L)$ associated to the irreducible representation of Sp_g with highest weight λ underlies a good $\mathbb{Q}$ variation of (mixed) Hodge structure, and this variation is unique up to Tate twist.*

PROOF. First observe that the local system $\mathbb{H}$ corresponding to the fundamental representation $V(\lambda_1)$ occurs as a variation of Hodge structure over $\mathcal{M}^n_{g,r}(L)$ of weight 1; it is simply the local system $R^1\pi_*\mathbb{Q}$ associated to the universal curve $\mathcal{C} \to \mathcal{M}^n_{g,r}(L)$. The existence of the structure of a good variation of Hodge structure on the local system corresponding the the Sp_g-module with highest weight λ now follows using Weyl's construction of the irreducible representations of Sp_g—see, for example, [13, §17.3].

To prove uniqueness, suppose that $\mathbb{V}$ and $\mathbb{V}'$ are both good variations of mixed Hodge structure corresponding to the same irreducible Sp_g-module. From Saito [43], we know that

$$\mathrm{Hom}_{\Gamma^n_{g,r}(L)}(\mathbb{V}, \mathbb{V}')$$

has a mixed Hodge structure. By Schur's lemma, this group is one-dimensional. It follows that this group is isomorphic to $\mathbb{Q}(n)$ for some n, and therefore that $\mathbb{V}' = \mathbb{V}(n)$. □

PROPOSITION 9.2. *If $\mathbb{E}$ is a good variation of $\mathbb{Q}$ mixed Hodge structure over $\mathcal{M}_g(L)$ whose monodromy representation factors through a rational representation of the algebraic group Sp_g, then for each dominant integral weight λ of Sp_g, the λ-isotypical part $\mathbb{E}_\lambda$ of $\mathbb{E}$ is a good variation of mixed Hodge structure. Consequently,*

$$\mathbb{E} = \bigoplus_\lambda \mathbb{E}_\lambda$$

in the category of good variations of $\mathbb{Q}$ mixed Hodge structure over $\mathcal{M}_g(L)$. Moreover, for each λ, there is a mixed Hodge structure A_λ such that $\mathbb{E}_\lambda = A_\lambda \otimes \mathbb{V}(\lambda)$.

PROOF. Fix λ, and let $\mathbb{V}(\lambda) \to \mathcal{M}_g(L)$ be a variation of Hodge structure whose fiber over some fixed base point is the irreducible Sp_g-module with highest weight λ. It follows from Saito's work [43] that

$$A_\lambda := \mathrm{Hom}_{\Gamma_g(L)}\big(\mathbb{V}(\lambda), \mathbb{E}\big) = H^0\big(\mathcal{M}_g(L), \mathrm{Hom}_{\mathbb{Q}}(\mathbb{V}(\lambda), \mathbb{E})\big)$$

is a mixed Hodge structure. Let

$$\mathbb{E}' = \bigoplus_\lambda A_\lambda \otimes \mathbb{V}(\lambda).$$

This is a good variation of mixed Hodge structure that is isomorphic to $\mathbb{E}$ as a $\mathbb{Q}$ local system. Now

$$\operatorname{Hom}_{\Gamma_g(L)}(\mathbb{E}', \mathbb{E}) = \bigoplus_\lambda A_\lambda^* \otimes \operatorname{Hom}_{\Gamma_g(L)}(\mathbb{V}(\lambda), \mathbb{E}) = \bigoplus_\lambda \operatorname{Hom}_{\mathbb{Q}}(A_\lambda, A_\lambda).$$

The element of this group that corresponds to $\mathrm{id} : A_\lambda \to A_\lambda$ in each factor is an isomorphism of local systems and an element of

$$\Gamma \operatorname{Hom}_{\Gamma_g(L)}(\mathbb{E}', \mathbb{E}).$$

It is therefore an isomorphism of variations of mixed Hodge structure. $\square$

Now suppose that $g \geq 3$. The local system

$$\{H_1(T_{g,r}^n)\}$$

over $\mathcal{M}_{g,r}^n(L)$ naturally underlies a variation of mixed Hodge structure of weight -1. The λ_1 isotypical component is simply $r + n$ copies of the variation $\mathbb{V}(\lambda_1)$. We shall denote this variation by $\mathbb{H}_1(T_{g,r}^n)$.

Proposition 9.3. *Suppose that $\mathbb{V}$ is a variation of mixed Hodge structure over $\mathcal{M}_{g,r}^n(L)$ whose monodromy representation factors through a rational representation of Sp_g. If $\mathbb{E}$ is an extension of $\mathbb{Q}$ by $\mathbb{V}$ in the category of variations of mixed Hodge structure over $\mathcal{M}_{g,r}^n(L)$, then the restriction of the monodromy representation to $H_1(T_{g,r}^n)$,*

$$\mathbb{H}_1(T_{g,r}^n) \to \mathbb{V},$$

is a morphism of variations of mixed Hodge structure.

Proof. It suffices to prove the assertion for $\mathbb{Q}$ variations of mixed Hodge structure. We will prove the case $n = r = 0$, the proofs of the other cases being similar.

If the monodromy representation of $\mathbb{E}$ is trivial, the result is trivially true. So we shall assume that the monodromy representation is non-trivial.

Using the previous result, we can write

$$\mathbb{V} = \bigoplus_\lambda \mathbb{V}_\lambda$$

as variations of mixed Hodge structure over $\mathbb{V}$. By pushing out the extension

$$0 \to \mathbb{V} \to \mathbb{E} \to \mathbb{Q} \to 0$$

along the projection $\mathbb{V} \to \mathbb{V}_{\lambda_3}$ onto the λ_3 isotypical component, we obtain an extension

$$0 \to \mathbb{V}_{\lambda_3} \to \mathbb{E}' \to \mathbb{Q} \to 0.$$

It follows from Johnson's computation that the restricted monodromy representation of $\mathbb{E}$ factors through that of $\mathbb{E}'$:

$$\mathbb{H}_1(T_g) \to \mathbb{V}_{\lambda_3} \to \mathbb{V}.$$

We have therefore reduced to the case where $\mathbb{V} = \mathbb{V}_{\lambda_3}$.

Let $\mathbb{V}(\lambda_3)$ be the unique variation of Hodge structure of weight -1 over $\mathcal{M}_g(L)$ with monodromy representation given by λ_3. Let $\mathbb{S}$ be the variation of mixed Hodge structure over $\mathcal{M}_g(L)$ given by the cycle $C - C^-$ that was constructed in Section 8. It is an extension of $\mathbb{Q}$ by $\mathbb{V}(\lambda_3)$.

By [43], the exact sequence

$$0 \to \mathrm{Hom}_{\Gamma_g(L)}(\mathbb{S}, \mathbb{V}_{\lambda_3}) \to \mathrm{Hom}_{\Gamma_g(L)}(\mathbb{S}, \mathbb{E}') \to \mathrm{Hom}_{\Gamma_g(L)}(\mathbb{S}, \mathbb{Q})$$

is a sequence of mixed Hodge structures. The group on the right is easily seen to be isomorphic to $\mathbb{Q}(0)$; it is generated by the projection $\mathbb{S} \to \mathbb{Q}$. The group on the left is easily seen to be zero. It follows that

$$\mathrm{Hom}_{\Gamma_g(L)}(\mathbb{S}, \mathbb{E}') \approx \mathbb{Q}(0).$$

Since the monodromy representation of $\mathbb{S}$ is a morphism, so are those of $\mathbb{E}'$ and $\mathbb{E}$. □

10. Normal functions and cycles mod algebraic equivalence

As our first application of the classification of normal functions, we show that certain homologically trivial cycles defined over $\mathcal{M}_{g,r}^n(L)$ are of infinite order modulo algebraic equivalence for the general curve. We first recall a basic result, which follows from the fact that algebraic equivalences are parameterized by curves, and because the correspondence corresponding to a cycle algebraically equivalent to zero induces a map from the jacobian of the base curve to the appropriate intermediate jacobian of the ambient variety.

Proposition 10.1. *Suppose that X is a smooth projective variety. If Z is a d-cycle that is algebraically equivalent to zero, the corresponding point $\nu(Z)$ of $J_d(X)$ lies in an abelian subvariety of $J_d(X)$.* □

Lemma 10.2. *Suppose that $\mathbb{V} \to \mathcal{M}$ is a polarized variation of Hodge structure of weight -1. If the monodromy representation of $\mathbb{V}$ is irreducible, then either $J\mathcal{V}$ is a family of abelian varieties, or else the set*

$$\{m \in \mathcal{M} : JV_m \text{ contains an abelian variety of positive dimension}\}$$

is a countable union of proper subvarieties of $\mathcal{M}$.

Proof. There are only a countable number of orthogonal decompositions

$$V_{o,\mathbb{Q}} = A \oplus B$$

of the fiber over the base point $o \in \mathcal{M}$. For each such decomposition there is the idempotent $p_A \in \operatorname{End} V_o$, which is orthogonal projection onto A. This is, in general, a multivalued section of the local system $\operatorname{End}_{\mathbb{Q}} \mathbb{V}$. The locus of the $m \in \mathcal{M}$ over which the splitting holds in the category of Hodge structures is the locus over which p_A is a Hodge class. This is an analytic subvariety of $\mathcal{M}$. In the case where A is an abelian variety and this locus is all of $\mathcal{M}$, the irreducibility of the monodromy implies that A and all its parallel translates span V_o. Since A has level 1 as a Hodge structure, this implies that V_o, and therefore $\mathbb{V}$ also, has level 1. That is, $J\mathcal{V}$ is a family of abelian varieties. □

Now suppose that $g \geq 3$. As in the previous section, we shall denote the unique $\mathbb{Q}$-variation of Hodge structure over $\mathcal{M}_{g,r}^n(L)$ of weight -1 associated to $V(\lambda_3)$ by $\mathbb{V}(\lambda_3)$. We shall denote the fiber over $C \in \mathcal{M}_{g,r}^n(L)$ of a family $W \to \mathcal{M}_{g,r}^n(L)$ by W_C.

Theorem 10.3. *Suppose that $\pi_* : X \to \mathcal{M}_{g,r}^n(L)$ is a family of projective varieties, smooth over the generic curve, and that Z is a family of homologically trivial algebraic d-cycles in X defined generically over $\mathcal{M}_{g,r}^n(L)$. If the local system $R^{2d+1}\pi_*\mathbb{Q}_X(d+1)$ contains a copy of the variation $\mathbb{V}(\lambda_3)$, and if the component of the normal function of Z in the corresponding bundle of intermediate jacobians $\mathcal{J}_{\lambda_3}$ is of infinite order, then, for the general curve C, the cycle Z_C has infinite order modulo algebraic equivalence.*

Proof. By Theorem 7.1, the normal function of Z is defined over all of the moduli space. Since the λ_3-component ν of this normal function has infinite order, and since $\mathcal{J}_{\lambda_3}$ is not a family of abelian varieties, it follows from Lemma 10.2 that, for the general curve, $\nu(C)$ is of infinite order modulo the maximal abelian subvariety of $JV_{\lambda_3,C}$. The result follows. □

Now take X to be the Jacobian of the universal curve over $\mathcal{M}_g^1(L)$, and Z to be the cycle whose fiber over $(C,x) \in \mathcal{M}_g^1$ is $C_x^{(a)} - i_*C_x^{(a)}$. Here

$$C_x^{(a)} := \{x_1 + \cdots + x_a - ax : x_j \in C\} \subseteq \operatorname{Jac} C$$

and i is the involution $D \mapsto -D$ of the jacobian. Applying the previous result and [19, (8.8)] we obtain the following result of Ceresa [10].

Corollary 10.4. *For the general curve C of genus g and $g \geq 3$, the cycle $C_x^{(a)} - i_*C_x^{(a)}$ is of infinite order modulo algebraic equivalence when $1 \leq a < g-1$.* □

When $g = 2$, the cycle $C_x - i_* C_x$ is algebraically equivalent to zero because, mod algebraic equivalence, we may assume x to be a Weierstrass point. In this case, the cycle is actually zero.

11. The Harris–Pulte theorem

As an application of the classification of normal functions above, we give a new proof of the Harris–Pulte theorem, which relates the mixed Hodge structure on $\pi_1(C, x)$ to the normal function of the cycle $C_x - C_x^-$ when $g \geq 3$. The result we obtain is slightly stronger.

Fix a level so that $\Gamma_g^1(L)$ is torsion-free. Denote by $\mathbb{L}$ the $\mathbb{Z}$ variation of Hodge structure of weight -1 over $\mathcal{M}_g^1(L)$ whose fiber over the pointed curve (C, x) is $H_1(C)$. Denote the corresponding holomorphic vector bundle by $\mathcal{L}$. The cycle $C_x - C_x^-$ defines a normal function ζ, which is a section of

$$J\Lambda^3\mathcal{L} \to \mathcal{M}_g^1(L).$$

Denote the integral group ring of $\pi_1(C, x)$ by $\mathbb{Z}\pi_1(C, x)$, and its augmentation ideal by $I(C, x)$, or I when there is no possibility of confusion. There is a canonical mixed Hodge structure on the truncated augmentation ideal

$$I(C, x)/I^3.$$

(See, for example, [16].) It is an extension

$$0 \to H_1(C)^{\otimes 2}/q \to I(C, x)/I^3 \to H_1(C) \to 0,$$

where q denotes the symplectic form. Tensoring with $H_1(C)$ and pulling back the resulting extension along the map $\mathbb{Z} \to H_1(C)^{\otimes 2}$, we obtain an extension

$$0 \to H_1(C) \otimes \big(H_1(C)^{\otimes 2}/q\big) \to E(C, x) \to \mathbb{Z} \to 0.$$

Since the set of $I(C, x)$ form a good variation of mixed Hodge structure over $\mathcal{M}_g^1(L)$ [17], the set of $E(C, x)$ form a good variation of mixed Hodge structure $\mathbb{E}$ over $\mathcal{M}_g^1(L)$. It therefore determines a normal function ρ, which is a section of

$$J\mathcal{L} \otimes \big(\mathcal{L}^{\otimes 2}/q\big) \to \mathcal{M}_g^1(L).$$

Define the map

$$\Phi : J\Lambda^3\mathcal{L} \to J\mathcal{L} \otimes (\mathcal{L}^{\otimes 2}/q)$$

to be the one induced by the map

$$\Lambda^3\mathbb{L} \to \mathbb{L}^{\otimes 3} \to \mathbb{L} \to \mathbb{L} \otimes (\mathbb{L}^{\otimes 2}/q);$$

the first map is defined by

$$x_1 \wedge x_2 \wedge x_3 \mapsto \sum_\sigma \operatorname{sgn}(\sigma) x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}$$

where σ ranges over all permutations of $\{1, 2, 3\}$.

Our version of the Harris–Pulte Theorem is:

THEOREM 11.1. *The image of ζ under Φ is 2ρ.*

PROOF. The proof uses Corollary 6.4. It is a straightforward consequence of Corollary 4.5 that the monodromy representations of $\Phi(\zeta)$ and 2ρ are equal. It is also a straightforward matter to use functoriality to show that both $\Phi(\zeta)$ and 2ρ vanish at (C, x) when C is hyperelliptic and x is a Weierstrass point [16, (7.5)]. $\square$

12. The Franchetta conjecture for curves with a level

Suppose that L is a finite-index subgroup of $\mathrm{Sp}_g(\mathbb{Z})$, not necessarily torsion-free. Denote the generic point of $\mathcal{M}_g(L)$ by η. There is a universal curve defined generically over $\mathcal{M}_g(L)$. Denote its fiber over η by $\mathcal{C}_g(L)_\eta$. In the statement below, S denotes a compact oriented surface of genus g.

THEOREM 12.1. *For all $g \geq 3$ and all finite-index subgroups L of $\mathrm{Sp}_g(\mathbb{Z})$, the group $\mathrm{Pic}\,\mathcal{C}_g(L)_\eta$ is finitely generated of rank one. The torsion subgroup is isomorphic to $H^0(L, H_1(S, \mathbb{Q}/\mathbb{Z}))$. Modulo torsion, either it is generated by the canonical bundle, or by a divisor of degree $g-1$.*

This has a concrete statement when $L = \mathrm{Sp}_g(\mathbb{Z})(l)$, the congruence subgroup of level l of $\mathrm{Sp}_g(\mathbb{Z})$. It is not difficult to show that the only torsion points of $\mathrm{Jac}\, S$ invariant under L are the points of order l. That is,

$$H^0(L, H_1(S, \mathbb{Q}/\mathbb{Z})) \approx H_1(S, \mathbb{Z}/l\mathbb{Z}).$$

In this case we shall denote $\mathcal{C}_g(L)_\eta$ by $\mathcal{C}_g(l)_\eta$. During the proof of the theorem, we will show that, mod torsion, $\mathrm{Pic}\,\mathcal{C}_g(l)_\eta$ is generated by a theta characteristic when l is even. Combining this with the theorem, we have:

COROLLARY 12.2. *If $g \geq 3$, then for all $l \geq 0$, $\mathrm{Pic}\,\mathcal{C}_g(l)_\eta$ is a finitely generated group of rank one with torsion subgroup isomorphic to $H_1(S, \mathbb{Z}/l\mathbb{Z})$. Modulo torsion, $\mathrm{Pic}\,\mathcal{C}_g(l)_\eta$ is generated by a theta characteristic when l is even, and by the canonical bundle when l is odd.* $\square$

The case $g = 2$, if true, should follow from Mess's computation of $H_1(T_2)$ [37]. One should note that Mess proved that T_2 is a countably generated free group.

SKETCH OF PROOF OF THEOREM 12.1. We first suppose that L is torsion-free. In this case, the universal curve is defined over all of $\mathcal{M}_g(L)$. Denote the restriction of it to a Zariski open subset U of $\mathcal{M}_g(L)$ by $\mathcal{C}_g(L)_U$. Set

$$\mathrm{Pic}_{\mathcal{C}_g/U}\, \mathcal{C}_g(L) = \mathrm{coker}\{\mathrm{Pic}\, U \to \mathrm{Pic}\, \mathcal{C}_g(U)\}.$$

Then

$$\operatorname{Pic} \mathcal{C}_g(L)_\eta = \varinjlim_{U} \operatorname{Pic}_{\mathcal{C}_g/U} \mathcal{C}_g(L),$$

where U ranges over all Zariski open subsets of $\mathcal{M}_g(L)$. There is a natural homomorphism

$$\deg : \operatorname{Pic}_{\mathcal{C}_g/U} \mathcal{C}_g \to \mathbb{Z}$$

given by taking the degree on a fiber. Denote $\deg^{-1}(d)$ by $\operatorname{Pic}^d_{\mathcal{C}_g/U} \mathcal{C}_g(L)$.

We first compute $\operatorname{Pic}^0 \mathcal{C}_g(L)_\eta$. Each element of this group can be represented by a line bundle over $\mathcal{C}_g(L)_U$ whose restriction to each fiber of $\pi : \mathcal{C}_g(L)_U \to U$ is topologically trivial. This line bundle has a section. By tensoring it with the pullback of a line bundle on U, if necessary, we may assume that the divisor of this section intersects each fiber of π in only a finite number of points. We therefore obtain a normal function

$$s : U \to \operatorname{Pic}^0_{\mathcal{C}_g/U} \mathcal{C}_g(L).$$

Since the associated variation of Hodge structure is the unique one of weight -1 associated to $V(\lambda_1)$, it follows from Theorem 8.2 and Theorem 7.1 that this normal function is torsion. It follows that

$$\operatorname{Pic}^0 \mathcal{C}_g(L)_\eta = \operatorname{Pic}^0_{\mathcal{C}_g/U} \mathcal{C}_g(L) = H^0(L, H_1(S, \mathbb{Q}/\mathbb{Z})).$$

Since this group is isomorphic to $H_1(S, \mathbb{Z}/l\mathbb{Z})$ when L is the congruence l subgroup of $\operatorname{Sp}_g(\mathbb{Z})$, and since every finite-index subgroup of $\operatorname{Sp}_g(\mathbb{Z})$ contains a congruence subgroup by [3], it follows that $\operatorname{Pic}^0 \mathcal{C}_g(L)_\eta$ is finite for all L.

The relative dualizing sheaf ω of $\mathcal{C}_g(L)_U$ gives an element of $\operatorname{Pic}^{2g-2} \mathcal{C}_g(L)_\eta$. Denote the greatest common divisor of the degrees of elements of $\operatorname{Pic} \mathcal{C}_g(L)_\eta$ by d. Observe that d divides $2g-2$. Let $m = (2g-2)/d$. We will show that $m = 1$ or 2.

Choose an element δ of $\operatorname{Pic}^d \mathcal{C}_g(L)_\eta$. Then

$$\omega - m\delta \in \operatorname{Pic}^0 \mathcal{C}_g(L)_\eta$$

and is therefore a torsion element of order k, say. Replace L by

$$L' = L \cap \operatorname{Sp}_g(\mathbb{Z})(km).$$

Observe that the natural map

$$\operatorname{Pic}^0 \mathcal{C}_g(L)_\eta \to \operatorname{Pic}^0 \mathcal{C}_g(L')_\eta$$

is injective. We can find

$$\mu \in \operatorname{Pic}^0 \mathcal{C}_g(L')_\eta$$

such that $m\mu = \omega - m\delta$. Then $\delta + \mu$ is an m-th root of the canonical bundle ω. It follows from a result of Sipe [46] that the only non-trivial roots of the canonical bundle that can be defined over $\mathcal{M}_g(L)$ are square roots: see Theorem 13.3. This implies that m divides 2, as claimed.

It also follows from Theorem 13.3 that square roots of the canonical bundle are defined over $\mathcal{M}_g(l)$ if and only if l is even. Combined with the argument above, this shows that, mod torsion, $\operatorname{Pic}^0 \mathcal{C}_g(l)_\eta$ is generated by ω if l is odd, and by a square root of ω if l is even.

Our final task is to reduce the general case to that where L is torsion-free. For arbitrary L, we have

$$\operatorname{Pic} \mathcal{C}_g(L)_\eta = \varinjlim_U \operatorname{Pic}_{\mathcal{C}_g/U} \mathcal{C}_g(L),$$

where U ranges over all smooth Zariski open subsets of $\mathcal{M}_g(L)$. Choose a torsion-free finite-index normal subgroup L' of L and a smooth Zariski open subset U of $\mathcal{M}_g(L)$. Denote the inverse image of U in $\mathcal{M}_g(L')$ by U'. Then the projection $U' \to U$ is a Galois cover with Galois group $G = L/L'$. It follows that

$$\operatorname{Pic}_{\mathcal{C}_g/U} \mathcal{C}_g(L) = \operatorname{Pic}_{\mathcal{C}_g/U'} \mathcal{C}_g(L')^G.$$

Since $\pi_1(U)$ surjects onto $\Gamma_g(L)$, and therefore onto L, the result follows. □

Denote the universal curve over the generic point η of $\mathcal{M}^n_{g,r}(l)$ by $\mathcal{C}^n_{g,r}(l)_\eta$. The proof of the following more general result is similar to that of Theorem 12.1.

THEOREM 12.3. *If $g \geq 3$, then for all $l \geq 0$, $\operatorname{Pic} \mathcal{C}^n_{g,r}(l)_\eta$ is a finitely generated group of rank $r+n+1$ whose torsion subgroup is isomorphic to $H_1(S, \mathbb{Z}/l\mathbb{Z})$. Each of the n marked points and the anchor point of each of the r marked cotangent vectors gives an element of $\operatorname{Pic}^1 \mathcal{C}^n_{g,r}(l)_\eta$. The pairwise differences of these points generate a subgroup of $\operatorname{Pic}^0 \mathcal{C}^n_{g,r}(l)_\eta$ of rank $r+n-1$. Moreover, $\operatorname{Pic}^0 \mathcal{C}^n_{g,r}(l)_\eta$ is generated by these differences modulo torsion, and $\operatorname{Pic} \mathcal{C}^n_{g,r}(l)_\eta$ is generated modulo $\operatorname{Pic}^0 \mathcal{C}^n_{g,r}(l)_\eta$ by the class of one of the distinguished points together with a theta characteristic when l is even, and by the canonical divisor when l is odd.* □

Note that the independence of the pairwise difference of the points follows from the discussion following Theorem 8.2.

13. The monodromy of roots of the canonical bundle

In this section we compute the action of Γ_g on the set of n-th roots of the canonical bundle of a curve of genus $g \geq 3$. This computation is a slight refinement of a result of P. Sipe [46].

If L is an n-th root of the tangent bundle of a smooth projective curve C, its dual is an n-th root of the canonical bundle. That is, there is a one-one correspondence between n-th roots of the canonical bundle and n-th roots of the tangent bundle of a curve. For convenience, we shall work with roots of the tangent bundle.

The first point is that roots of the tangent bundle are determined topologically (see [2, §3] and [46]): denote the $\mathbb{C}^*$ bundle associated to the holomorphic tangent bundle TC of C by T^*. Indeed, an n-th root of TC is a cyclic covering of T^* of degree n, which has degree n on each fiber. The complex structure on such a covering is uniquely determined by that on T^*.

The first Chern class of TC is $2-2g$. So if R is an n-th root of K, the integer n divides $2g-2$. Since the Euler class of T^* is $2-2g$, it follows from the Gysin sequence that there is a short exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to H_1(T^*, \mathbb{Z}/n\mathbb{Z}) \to H_1(C, \mathbb{Z}/n\mathbb{Z}) \to 0. \tag{6}$$

By covering space theory, an n-th root of TC is determined by a homomorphism

$$H_1(T^*, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$$

whose composition with the inclusion $\mathbb{Z}/n\mathbb{Z} \hookrightarrow H_1(T^*, \mathbb{Z}/n\mathbb{Z})$ is the identity. That is, we have the following result:

PROPOSITION 13.1. *There is a natural one-to-one correspondence between n-th roots of the the canonical bundle of C and splittings of the sequence* (6).

□

Throughout this section, we will assume $g \geq 3$. Denote the set of n-th roots of TC by Θ_n. This is a principal $H_1(C, \mathbb{Z}/n\mathbb{Z})$ space. The automorphism group of this affine space is an extension

$$0 \to H_1(C, \mathbb{Z}/n\mathbb{Z}) \to \operatorname{Aut}\Theta_n \xrightarrow{\pi} GL_{2g}(\mathbb{Z}/n\mathbb{Z}) \to 1;$$

the kernel being the group of translations by elements of $H_1(C, \mathbb{Z}/n\mathbb{Z})$. The mapping class group acts on Θ_n, so we have a homomorphism

$$\Gamma_g \to \operatorname{Aut}\Theta_n.$$

The composite of this homomorphism with π is the reduction mod n

$$\rho_n : \Gamma_g \to \mathrm{Sp}_g(\mathbb{Z}/n\mathbb{Z})$$

of the natural homomorphism. Denote the subgroup $\pi^{-1}(\mathrm{Sp}_g(\mathbb{Z}/n\mathbb{Z}))$ of $\operatorname{Aut}\Theta_n$ by $\mathcal{K}_n$. It follows that the action of Γ_g on Θ_n factors through a homomorphism

$$\theta_n : \Gamma_g \to \mathcal{K}_n$$

whose composition with the natural projection $\mathcal{K}_n \to \mathrm{Sp}_g(\mathbb{Z}/n\mathbb{Z})$ is ρ_n. In order to determine θ_n, we will need to compute its restriction

$$\theta_n : H_1(T_g) \to H_1(C, \mathbb{Z}/n\mathbb{Z})$$

to the Torelli group. First some algebra.

Proposition 13.2. *There is a natural homomorphism*

$$\psi_g : H_1(T_g, \mathbb{Z}) \to H_1(C,\, \mathbb{Z}/(g-1)\mathbb{Z}).$$

Proof. By Theorem 3.4, there is a natural homomorphism

$$\tau_g : H_1(T_g, \mathbb{Z}) \to \Lambda^3 H_1(C, \mathbb{Z})/\big([C] \times H_1(C, \mathbb{Z})\big).$$

Here we view $\Lambda^\bullet H_1(C)$ as the homology of $\mathrm{Jac}\, C$ and $[C]$ denotes the homology class of the image of C under the Abel–Jacobi map. There is also a natural homomorphism

$$p : \Lambda^3 H_1(C, \mathbb{Z}) \to H_1(C, \mathbb{Z})$$

defined by

$$p : x \wedge y \wedge z \mapsto (x \cdot y)\, z + (y \cdot z)\, x + (z \cdot x)\, y.$$

It is easy to see that the composite

$$H_1(C, \mathbb{Z}) \xrightarrow{[C]\times} \Lambda^3 H_1(C, \mathbb{Z}) \xrightarrow{p} H_1(C, \mathbb{Z})$$

is multiplication by $g-1$. It follows that p induces a homomorphism

$$\overline{p} : \Lambda^3 H_1(C, \mathbb{Z}) \to H_1(C,\, \mathbb{Z}/(g-1)\mathbb{Z}).$$

The homomorphism ψ_g is the composite $\overline{p} \circ \tau_g$. □

Call a translation of Θ_n *even* if it is translation by an element of

$$2H^1(C, \mathbb{Z}/n\mathbb{Z}).$$

If n is odd, this is the set of all translations. If $n = 2m$, this is the proper subgroup of $H^1(C, \mathbb{Z}/n\mathbb{Z})$ isomorphic to $H^1(C, \mathbb{Z}/m\mathbb{Z})$.

Theorem 13.3. *The image of the natural homomorphism $\theta_n : \Gamma_g \to \mathcal{K}_n$ is a subgroup $\mathcal{K}_n^{(2)}$ of $\mathcal{K}_n$, which is an extension*

$$0 \to 2\, H_1(C, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Aut}\, \Theta_n \xrightarrow{\pi} \mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z}) \to 1$$

of $\mathrm{Sp}_g(\mathbb{Z}/n\mathbb{Z})$ by the even translations. The restriction of θ_n to T_g is the composite of ψ_g with the homomorphism

$$H_1(C,\, \mathbb{Z}/(g-1)\mathbb{Z}) \xrightarrow{r} H_1(C, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\mathrm{PD}} H^1(C, \mathbb{Z}/n\mathbb{Z}),$$

where $r(k)$ equals $2k$ mod n, and PD *denotes Poincaré duality. In particular, the Torelli group acts trivially on Θ_n if and only if n divides 2.*

PROOF. First, Johnson proved in [31] that the kernel of the composite

$$T_g \to H_1(T_g) \xrightarrow{\tau_g} \Lambda^3 H_1(C,\mathbb{Z})/\big([C]\times H_1(C,\mathbb{Z})\big)$$

is generated by Dehn twists on separating simple closed curves. Using this, it is easy to check that the restriction of θ_n to T_g factors through τ_g. In [32], Johnson shows that T_g is generated by Dehn twists on a bounding pair of disjoint simple closed curves. (Actually, all we need is that $\Lambda^3 H_1(C,\mathbb{Z})/([C]\times H_1(C,\mathbb{Z}))$ be generated by the images under τ_g by such bounding pair maps. This is easily checked directly.)

Now suppose that φ is such a bounding pair map. There are two disjoint embedded circles A and B such that φ equals a positive Dehn twist about A and a negative one about B. When we cut C along $A\cup B$, we obtain two surfaces, of genera g' and g'', say. Choose one of these components, and let a be the cycle obtained by orienting A so that it is a boundary component of this component. It is not difficult to show that the image of φ under ψ_g equals

$$-g'\,[a] \in H_1(C,\,\mathbb{Z}/(g-1)\mathbb{Z}),$$

where g' is the genus of the chosen component. Since $g'+g''=g-1$, this is well defined. Next, one can use Morse theory to show that the image of this same bounding pair map in $H^1(C,\mathbb{Z}/n\mathbb{Z})$ is $-2g'\,\mathrm{PD}(a)$. The result follows. □

COROLLARY 13.4. *The only roots of the canonical bundle defined over Torelli space are the canonical bundle itself and its* 2^{2g} *square roots.* □

REMARK 13.5. The homomorphism $\theta_{2g-2} : \Gamma_g \to \mathcal{K}_{2g-2}$ appears in Morita's work [38, §4.A].

14. Heights of Cycles defined over $\mathcal{M}_g(L)$

Suppose that X is a compact Kähler manifold of dimension n and that Z and W are two homologically trivial algebraic cycles in X of dimensions d and e, respectively. Suppose that $d+e=n-1$ and that Z and W have disjoint supports. Denote the current associated to W by δ_W. It follows from the $\partial\overline{\partial}$-lemma that there is a current η_W of type (d,d) that is smooth away from the support of Z and satisfies

$$\partial\overline{\partial}\eta_W = \pi i \delta_W.$$

The (archimedean) height pairing between Z and W is defined by

$$\langle Z, W\rangle = -\int_Z \eta_W.$$

This is a real-valued, symmetric bilinear pairing on such disjoint homologically trivial cycles. It is important in number theory [5].

Now suppose that

$$X \to \mathcal{M}_g(L)$$

is a family of smooth projective varieties of relative dimension n. Suppose that $Z \to \mathcal{M}_g(L)$ and $W \to \mathcal{M}_g(L)$ are families of algebraic cycles in X of relative dimensions d and e, respectively, where $d + e = n - 1$. Denote the fiber of X, Z and W over $C \in \mathcal{M}_g(L)$ by X_C, Z_C and W_C, respectively. Suppose that Z_C and W_C are homologically trivial in X_C and that they have disjoint supports for generic $C \in \mathcal{M}_g(L)$.

We shall suppose that L has been chosen so that every curve has two distinguished theta characteristics α and $\alpha + \delta$, where δ is a non-zero point of order two in $\operatorname{Jac} C$. We shall also suppose that g is odd and ≥ 3. Write g in the form $g = 2d + 1$.

Denote the difference divisor

$$\{x_1 + \cdots + x_d - y_1 - \cdots - y_d : x_j, y_j \in C\}$$

in $\operatorname{Jac} C$ by Δ, and the theta divisor

$$\{x_1 + \cdots + x_{2d} - \alpha : x_j \in C\}$$

in $\operatorname{Jac} C$ by Θ_α. By [18, (4,1.2)], there is a rational function f_C on $\operatorname{Jac} C$ whose divisor is

$$\Delta - \binom{2d}{d}\Theta_\alpha.$$

Denote the unique invariant measure of total mass one on $\operatorname{Jac} C$ by μ.

Theorem 14.1. *Suppose that g is odd and ≥ 3. Suppose that Z and W are families of homologically trivial cycles over $\mathcal{M}_g(L)$ in a family of smooth projective varieties $p : X \to \mathcal{M}_g(L)$, as above. If the monodromy of the local system $R^{2d+1}p_*\mathbb{Q}_X$ factors through a rational representation of Sp_g, there is a rational function h on $\mathcal{M}_g(L)$ and rational numbers a and b such that*

$$\langle Z_C, W_C\rangle = a\bigg(\log|h(C)| + 2b\bigg(\log|f_C(\delta)| - \int_{\operatorname{Jac} C}\log|f_C(x)|d\mu(x)\bigg)\bigg).$$

The numbers a and b are topologically determined, as will become apparent in the proof. The divisor of h is computable when one has a good understanding of how the cycles Z and W intersect. One should be able to derive a similar formula for even g using Bost's general computation of the height in [8] and results from [19].

The proof of Theorem 14.1 occupies the remainder of this section. We only give a sketch. We commence by defining two algebraic cycles in $\operatorname{Pic}^d C$. For

$D \in \operatorname{Jac} C$, let $C_D^{(d)}$ be the d-cycle in $\operatorname{Pic}^d C$ obtained by pushing forward the fundamental class of the d-th symmetric power of C along the map

$$\{x_1, \ldots, x_d\} \mapsto x_1 + \cdots + x_d + D.$$

Let i be the automorphism of $\operatorname{Pic}^d C$ defined by $i : x \mapsto \alpha - x$. Define

$$Z_D = C_D^{(d)} - i_* C_D^{(d)}.$$

This is a homologically trivial d-cycle in $\operatorname{Pic}^d C$.

From [8] and [18], we know that

$$\langle Z_0, Z_\delta \rangle = 2 \log |f_C(\delta)| - 2 \int_{\operatorname{Jac} C} \log |f_C(x)| \, d\mu(x).$$

So the content of the theorem is that there is a rational function h on $\mathcal{M}_g(L)$ and rational numbers a and b such that

$$\langle Z, W \rangle = a \left(\log |h(C)| + b \langle Z_0, Z_\delta \rangle \right).$$

The basic point, as we shall see, is that, up to torsion, all normal functions over $\mathcal{M}_g(L)$ are half-integer multiples of that of $C - C^-$, as was proved in Section 8.

We will henceforth assume that the reader is familiar with the content of [18, §3]. We will briefly review the most relevant points of that section.

A *biextension* is a mixed Hodge structure B with only three non-trivial weight graded quotients: $\mathbb{Z}$, H, and $\mathbb{Z}(1)$, where H is a Hodge structure of weight -1. The isomorphisms

$$\operatorname{Gr}^W_{-2} B \approx \mathbb{Z}(1) \quad \text{and} \quad \operatorname{Gr}^W_0 B \approx \mathbb{Z}$$

are considered to be part of the data of the biextension. If one replaces $\mathbb{Z}$ by $\mathbb{R}$ in this definition, one obtains the definition of a real biextension. To a biextension B one can associate a real number $\nu(B)$, called the *height* of B. It depends only on the associated real biextension $B \otimes \mathbb{R}$.

Associated to a pair of disjoint homologically trivial cycles in a smooth projective variety X satisfying

$$\dim Z + \dim W + 1 = \dim X$$

there is a canonical biextension $B_\mathbb{Z}(Z, W)$, whose weight graded quotients are

$$\mathbb{Z}, \quad H_{2d+1}(X, \mathbb{Z}(-d)), \quad \mathbb{Z}(1),$$

where d is the dimension of Z. The extensions

$$0 \to H_{2d+1}(X, \mathbb{Z}(-d)) \to B_\mathbb{Z}(Z, W)/\mathbb{Z}(1) \to \mathbb{Z} \to 0$$

and

$$0 \to \mathbb{Z}(1) \to W_{-1} B_\mathbb{Z}(Z, W) \to H_{2d+1}(X, \mathbb{Z}(-d)) \to 0$$

are the those determined by Z (directly), and W (via duality) [18, (3.3.2)]. We have

$$\nu(B_{\mathbb{Z}}(Z,W)) = \langle Z,W\rangle.$$

The first step in the proof is to reduce the size of the biextension. Suppose that $\Lambda = \mathbb{Z}$ or $\mathbb{R}$, and that B is a Λ-biextension with weight -1 graded quotient H. Suppose that there is an inclusion $i : A \hookrightarrow H$ of Λ mixed Hodge structures. Pulling back the extension

$$0 \to \Lambda(1) \to W_1 B \to H \to 0$$

along i, we obtain an extension

$$0 \to \Lambda(1) \to E \to C \to 0.$$

If this extension splits, there is a canonical lift $\tilde{\imath} : C \to B$ of i. The quotient B/C is also a Λ biextension.

Proposition 14.2. *The biextensions $B_\Lambda(Z,W)$ and $B_\Lambda(Z,W)/C$ have the same height.*

Proof. This is a special case of [34, (5.3.8)]. It follows directly from [18, (3.2.11)]. □

We will combine this with Theorem 8.2 to prune the biextension $B_{\mathbb{Z}}(Z_C, W_C)$ until its weight -1 graded quotient is either trivial or else one copy of $V(\lambda_3)$.

First observe that if B is a biextension and B' a mixed Hodge substructure of B of finite index, then B' is a biextension and there is a non-zero integer m such that $\nu(B') = m\nu(B)$. This can be proved using [18, (3.2.11)].

To prune the biextension $\mathbb{B}(Z,W)$ over $\mathcal{M}_g(L)$, we consider the portion of the monodromy representation

$$H_1(T_g) \to \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Gr}^W_{-1} B(Z_C, W_C),\, \mathbb{Z}(1))$$

associated to the variation $W_{-1}\mathbb{B}(Z,W)$ over $\mathcal{M}_g(L)$. This map is Sp_g equivariant. Denote $\operatorname{Gr}^W_{-1}\mathbb{B}(Z,W)$ by $\mathbb{H}$. This monodromy representation corresponds to the map

$$\mathbb{H} \to \left\{H^1(T_g, \mathbb{Z}(1))\right\}$$

of local systems over $\mathcal{M}_g(L)$, which takes $h \in H_C$ to the functional $\{\phi \mapsto \phi(h)\}$ on $H_1(T_g)$. For each $C \in \mathcal{M}_g(L)$ this is a morphism of Hodge structures by Proposition 9.3. Denote its kernel by K_C. These form a variation of Hodge structure $\mathbb{K}$ over $\mathcal{M}_g(L)$. If the monodromy representation is trivial on T_g, then $\mathbb{K} = \mathbb{H}$. Otherwise, Schur's lemma implies that $\mathbb{H}/\mathbb{K}$ is isomorphic $\mathbb{V}(\lambda_3)$ placed in weight -1.

We can pull back the extension

$$0 \to \mathbb{Q}(1) \to W_{-1}\mathbb{B}(Z,W) \to \mathbb{H} \to 0$$

along the inclusion $\mathbb{K} \hookrightarrow \mathbb{H}$ to obtain an extension

$$0 \to \mathbb{Q}(1) \to \mathbb{E} \to \mathbb{K} \to 0. \tag{7}$$

If this extension splits over $\mathbb{Q}$, then, by replacing the lattice in $\mathbb{B}_{\mathbb{Z}}(Z,W)$ by a commensurable one, we may assume that the splitting is defined over $\mathbb{Z}$. This has the effect of multiplying the height by a non-zero rational number. Once we have done this, the inclusion $\mathbb{K} \hookrightarrow \mathrm{Gr}^W_{-1}\mathbb{B}(Z,W)$ lifts to an inclusion $\mathbb{K} \hookrightarrow \mathbb{B}(Z,W)$. Using Proposition 14.2, we can replace $B_{\mathbb{Z}}(Z_C,W_C)$ by $\mathbb{B}' = \mathbb{B}(Z,W)/\mathbb{K}$ without changing the height of the biextension. For the time being, we shall assume that (7) splits over $\mathbb{Q}$. This is the case, for example, when $\mathbb{H}$ contains no copies of the trivial representation, as follows from Proposition 9.2 since $\mathbb{K}$ is a trivial T_g-module by construction.

The weight -1 graded quotient of $\mathbb{B}'$ is either trivial or isomorphic to $\mathbb{V}(\lambda_3)$. This biextension is defined over the open subset U of $\mathcal{M}_g(L)$ where Z_C and W_C are disjoint. The related variations $W_{-1}\mathbb{B}'$ and $\mathbb{B}'/\mathbb{Z}(1)$ are defined over all of $\mathcal{M}_g(L)$.

If $\mathbb{K} = \mathbb{H}$, then $\mathbb{B}'$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}(1)$. It therefore corresponds to a rational function h on $\mathcal{M}_g(L)$, which is defined on U [20, (9.3)]. It follows from [18, (3.2.11)] that the height of this biextension $\mathbb{B}'$ is $C \mapsto \log|h(C)|$. This completes the proof of the theorem in this case.

Dually, when the extension

$$0 \to \mathrm{Gr}^W_{-1}\mathbb{B}' \to \mathbb{B}'/\mathbb{Z}(1) \to \mathbb{Z} \to 0$$

has finite monodromy, there is a rational function h on $\mathcal{M}_g(L)$ such that the height of B', and therefore $B(Z_C,W_C)$, is rational multiple of $\log|h(C)|$.

We have therefore reduced to the case where $\mathbb{B}'$ has weight graded -1 quotient $\mathbb{V}(\lambda_3)$ and where neither of the extensions

$$0 \to \mathbb{V}(\lambda_3) \to \mathbb{B}'/\mathbb{Z}(1) \to \mathbb{Z} \to 0$$

or

$$0 \to \mathbb{Z}(1) \to W_{-1}\mathbb{B}' \to \mathbb{V}(\lambda_3) \to 0$$

is torsion. We also have the biextension $\mathbb{B}''$ associated to the cycles Z_0 and Z_δ. It has these same properties. After replacing the lattices in each by lattices of finite index, we may assume that the extensions of variations $W_{-1}\mathbb{B}'$ and $W_{-1}\mathbb{B}''$ are isomorphic, and that $\mathbb{B}'/\mathbb{Z}(1)$ and $\mathbb{B}''/\mathbb{Z}(1)$ are isomorphic. As in [18, (3.4)], the biextensions $\mathbb{B}'$ and $\mathbb{B}''$ each determine a canonically metrized holomorphic line bundle over $\mathcal{M}_g(L)$. These metrized line bundles depend only on the variations $\mathbb{B}/\mathbb{Z}(1)$ and $W_{-1}\mathbb{B}$, and are therefore isomorphic. Denote this common line bundle by $\mathcal{B} \to \mathcal{M}_g(L)$. The biextensions $\mathbb{B}'$ and $\mathbb{B}''$ determine (and are determined by) meromorphic sections s' and s'' of $\mathcal{B}$, respectively. There is therefore a meromorphic function h on $\mathcal{M}_g(L)$ such that $s'' = hs'$. It follows from the main result of [34] that this function is a rational function. (The philosophy is that period maps of variations of mixed Hodge structure behave well at the boundary.) The result follows as

$$\nu(B''_C) = \log \|s''(C)\| = \log |h(C)| + \log \|s'(C)\| = \nu(B'_C) + \log |h(C)|.$$

To conclude the proof, we now explain how to proceed when the extension (7) of page 137 is not split as a $\mathbb{Q}$ variation. Write $\mathbb{K} = \mathbb{T} \oplus \mathbb{T}'$, where $\mathbb{T}$ is the trivial submodule of $\mathbb{K}$ and $\mathbb{T}'$ is its orthogonal complement. This is a splitting in the category of $\mathbb{Q}$ variations by Proposition 9.2. It also follows from Proposition 9.2 that the restriction of (7) to $\mathbb{T}'$ is split. Consequently, there is an inclusion of mixed Hodge structures $\mathbb{T}' \hookrightarrow \mathbb{B}(Z, W)$. As above, we may replace $\mathbb{B}(Z, W)$ by the biextension $\mathbb{B}' = \mathbb{B}(Z, W)/\mathbb{T}'$ after rescaling lattices. This only changes the height by a non-zero rational number. The weight graded -1 quotient of $\mathbb{B}'$ is the sum of at most one copy of $\mathbb{V}(\lambda_3)$ and a trivial variation of weight -1.

Now suppose that B_1 and B_2 are two biextensions. We can construct a new biextension $B_1 \boxplus B_2$ from them as follows: Begin by taking their direct sum. Pull this back along the diagonal inclusion

$$\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} = \operatorname{Gr}^W_0 (B_1 \oplus B_2)$$

to obtain a mixed Hodge structure B whose weight -2 graded quotient is

$$\operatorname{Gr}^W_{-2} (B_1 \oplus B_2) = \mathbb{Z}(1) \oplus \mathbb{Z}(1).$$

Push this out along the addition map

$$\mathbb{Z}(1) \oplus \mathbb{Z}(1) \to \mathbb{Z}(1)$$

to obtain the sought after biextension $B_1 \boxplus B_2$. The following result follows directly from [18, (3.2.11)].

Proposition 14.3. *The height of $B_1 \boxplus B_2$ is the sum of the heights of B_1 and B_2.*

The biextension $\mathbb{B}'$ is easily seen to be the sum, in this sense, of two biextensions. The first is constant with weight -1 quotient equal to the trivial variation $\mathbb{T}$ and the second is a variation with weight -1 quotient equal to $\mathbb{H}/\mathbb{K}$, which is either zero or one copy of $\mathbb{V}(\lambda_3)$. Since the height of a constant biextension is a constant, the result follows from the computation of the height of a biextension with weight -1 quotient $\mathbb{V}(\lambda_3)$ above.

15. Results for Abelian Varieties

Denote by $\mathcal{A}_g(L)$ the quotient of Siegel space $\mathfrak{h}_g$ of rank g by a finite-index subgroup L of $\mathrm{Sp}_g(\mathbb{Z})$. This is the moduli space of abelian varieties with a level-L structure. In this section we state results for $\mathcal{A}_g(L)$ analogous to those in Sections 8 and 14. The proofs are similar, but much simpler, and are left to the reader.

We call a representation of Sp_g *even* if it has a symmetric Sp_g-invariant inner product, and *odd* if it has a skew symmetric Sp_g-invariant inner product. It follows from Schur's Lemma that every irreducible representation of Sp_g is either even or odd. The even ones occur as polarized variations of Hodge structure of even weight over each $\mathcal{A}_g(L)$, while the odd ones occur as polarized variations of Hodge structure only over $\mathcal{A}_g(L)$ of odd weight provided $-I \notin L$. These facts are easily proved by adapting the arguments in Section 9.

The first theorem is the analogue of Lemma 8.1 for abelian varieties. It is similar to the result [45] of Silverberg. The point in our approach is that $H^1(L, V)$ vanishes for all non-trivial irreducible representations of Sp_g by [42].

THEOREM 15.1. *Suppose that $g \geq 2$ and that $L/\pm I$ is torsion-free. If $\mathbb{V} \to \mathcal{A}_g(L)$ is a variation of Hodge structure of negative weight whose monodromy representation is the restriction to L of a rational representation of Sp_g, the group of generically defined normal functions associated to this variation is finite.* □

Since there are no normal functions of infinite order over $\mathcal{A}_g(L)$, we have the following analogue of Theorem 14.1. Suppose that Z and W are families of homologically trivial cycles over $\mathcal{A}_g(L)$ in a family of smooth projective varieties $p : X \to \mathcal{A}_g(L)$. Suppose that they are disjoint over the generic point. Suppose further that $d + e = n - 1$, where d, e and n are the relative dimensions over $\mathcal{A}_g(L)$ of Z, W and X, respectively. Denote the fiber of Z over $A \in \mathcal{A}_g(L)$ by Z_A, etc.

THEOREM 15.2. *If $g \geq 2$ and the monodromy of the local system $R^{2d+1}p_*\mathbb{Q}_X$ is the restriction to L of a rational representation of Sp_g, there is a rational function h on $\mathcal{A}_g(L)$ such that*

$$\langle Z_A, W_A \rangle = \log |h(A)|$$

for all $A \in \mathcal{A}_g(L)$. □

One can formulate and prove analogues of these results for the moduli spaces $\mathcal{A}_g^n(L)$ of abelian varieties of dimension g, n marked points, and a level-L structure.

We conclude with a discussion of Nori's results and their relation to Theorems 8.2 and 15.1. We first recall the main result of the last section of [40].

THEOREM 15.3 (NORI). *Suppose that X is a variety that is an unbranched covering of a Zariski open subset U of $\mathcal{A}_g(L)$, where L is torsion-free. Suppose that $\mathbb{V}$ is a variation of Hodge structure of negative weight over X that is pulled back from the canonical variation over $\mathcal{A}_g(L)$ of the same weight whose monodromy representation is irreducible and has highest weight λ. Then the group of normal functions defined on X associated to this variation is finite unless*

$$\lambda = \begin{cases} 0 & \text{and } g \geq 2, \quad \text{or} \\ \lambda_1 & \text{and } g \geq 3, \quad \text{or} \\ \lambda_3 & \text{and } g = 3, \quad \text{or} \\ m_1\lambda_1 + m_2\lambda_2 & g = 2 \text{ and } m_1 \geq 2. \end{cases}$$

□

This result may seem to contradict Theorem 15.1. The difference can be accounted for by noting that Theorem 15.1 only applies to open subsets of the $\mathcal{A}_g(L)$, whereas Nori's theorem applies to a much more general class of varieties that contains unramified coverings of open subsets of the $\mathcal{A}_g(L)$. One instructive example is $\mathcal{M}_3(l)$, where l is odd and ≥ 3. The map $\mathcal{M}_3(l) \to \mathcal{A}_3(l)$ is branched along the hyperelliptic locus. Theorem 15.1 does not apply. However, Nori's Theorem 15.3 does apply—remember, normal functions in weight -1 extend by Theorem 7.1. In this way we realize the normal function associated to λ_3 in Nori's result. Also, by standard arguments, for each n, there is an open subset U of $M_3(l)$ and an unbranched finite cover V of U over which the natural projection $\mathcal{M}_3^n(l) \to \mathcal{M}_3(l)$ has a section. From this one can construct n linearly independent normal sections of the jacobian bundle defined over V. Note that Nori's result does apply to V, whereas Theorem 15.1 does not.

REFERENCES

1. E. Arbarello and M. Cornalba, *The Picard group of the moduli space of curves*, Topology **26** (1987), 153–171.
2. M. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. **4** (1971), 47–62.
3. H. Bass, J. Milnor, and J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$, Publ. Math. IHES **33** (1967), 59–137.
4. A. Beilinson, *Notes on absolute Hodge cohomology*, Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory (S. J. Bloch et al., eds.), Contemp. Math. **55**, AMS, 1986, part 1, pp. 35–68.
5. ———, *Height pairing between algebraic cycles*, Current Trends in Arithmetical Algebraic Geometry (K. Ribet, ed.), Contemp. Math. **67**, AMS, 1987, pp. 1–24.

6. A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. **7** (1974), 235–272.

7. ______, *Stable real cohomology of arithmetic groups II*, Manifolds and Lie Groups, Papers in Honor of Yozo Matsushima (J. Hano et al., eds.) Prog. in Math. **14**, Birkhäuser, Boston, 1981, pp. 21–55.

8. J.-B. Bost, *Green's currents and height pairing on complex tori*, Duke. Math. J. **61** (1990), 899–912.

9. J. Carlson, *The geometry of the extension class of a mixed Hodge structure*, Algebraic Geometry, Bowdoin, 1985 (S. J. Bloch, ed.), Proc. Sympos. Pure Math. **46**, AMS, 1987, pp. 199–222.

10. G. Ceresa, *C is not algebraically equivalent to C^- in its jacobian*, Ann. of Math. **117** (1983), 285–291.

11. K.-T. Chen, *Extension of C^∞ function algebra and Malcev completion of π_1*, Adv. Math. 23 (1977), 181–210.

12. W. Fulton, *Intersection Theory*, Springer, New York, 1984.

13. W. Fulton, J. Harris, *Representation Theory, a First Course*, Grad. Texts in Math. **129**, Springer, New York, 1991.

14. P. Griffiths, *On the periods of certain rational integrals*, Ann. of Math. **90** (1969), 460–541.

15. F. Guillén,V. Navarro Aznar, P. Pascula-Gainza, F. Puerta, *Hyperrésolutions cubiques et descente cohomologique*, Lecture Notes in Math. **1335**, Springer, Berlin, 1988.

16. R. Hain, *The geometry of the mixed Hodge structure on the fundamental group*, Proc. Sympos. Pure Math. **46**, AMS, 1987, vol. 2, pp. 247–281.

17. ______, *The de Rham homotopy theory of complex algebraic varieties II*, K-Theory **1** (1987), 481–497.

18. ______, *Biextensions and heights associated to curves of odd genus*, Duke. Math. J. **61** (1990), 859–898.

19. ______, *Completions of mapping class groups and the cycle $C - C^-$*, Mapping Class Groups and Moduli Spaces of Riemann Surfaces, (C.-F. Bödigheimer and R. Hain, eds.), Contemp. Math. **150**, AMS, 1993, pp. 75–105.

20. ______, *Classical polylogarithms*, Motives (U. Janssen et al., eds.), Proc. Sympos. Pure Math. **55**, AMS, 1994.

21. J. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. **72** (1983), 221–239.

22. ______, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. Math. **121** (1985), 215–249.

23. ______, *The cohomology of the moduli space of curves*, Theory of Moduli, (E. Sernesi, ed.), Lecture Notes in Math. **1337**, Springer, Berlin, 1988, pp. 139–221.

24. ______, *The third homology group of the moduli space of curves*, Duke. Math. J. **63** (1992), 25–55.

25. ______, *The rational Picard group of the moduli spaces of Riemann surfaces with spin structure*, Mapping Class Groups and Moduli Spaces of Riemann Surfaces (C.-F. Bödigheimer and R. Hain, eds.), Contemp. Math. **150**, AMS, 1993, pp. 107–136.

26. ______, *The fourth homology group of the moduli space of curves*, preprint, 1993.

27. B. Harris, *Harmonic volumes*, Acta Math. **150** (1983), 91–123.

28. N. Ivanov, *Complexes of curves and the Teichmüller modular group*, Uspekhi Mat. Nauk **42** (1987), 49–91; English translation: *Russian Math. Surveys* **42** (1987), 55–107.

29. D. Johnson, *An abelian quotient of the mapping class group $\mathcal{I}_g$*, Math. Ann. **249** (1980), 225–242.

30. ______, *The structure of the Torelli group I: A finite set of generators for $\mathcal{I}$*, Ann. of Math. **118** (1983), 423–442.

31. ______, *The structure of the Torelli group—II: A characterization of the group generated by twists on bounding curves*, Topology **24** (1985), 113–126.

32. ______, *The structure of the Torelli group—III: The abelianization of $\mathcal{I}$*, Topology **24** (1985), 127–144.

33. ______, *A survey of the Torelli group*, Low dimensional topology (S. J. Lomonaco, Jr., ed.), Contemp. Math. **20**, AMS, 1983, pp. 165–179.

34. D. Lear, *Extensions of Normal Functions and Asymptotics of the Height Pairing*, Ph.D. Thesis, University of Washington, 1990.

35. S. Mac Lane, *Homology*, Springer, Berlin, 1963.

36. W. Magnus, A. Karras, and D. Solitar, *Combinatorial Group Theory*, Interscience, 1966.

37. G. Mess, *The Torelli groups for genus 2 and 3 surfaces*, MSRI preprint 05608-90, 1990.

38. S. Morita, *The structure of the mapping class group and characteristic classes of surface bundles*, Mapping Class Groups and Moduli Spaces of Riemann Surfaces (C.-F. Bödigheimer and R. Hain, eds.), Contemp. Math. **150**, AMS, 1993.

39. D. Mumford, *Abelian quotients of the Teichmüller modular group*, J. Anal. Math. **18** (1967), 227–244.

40. M. Nori, *Algebraic cycles and Hodge theoretic connectivity*, Invent. Math. **111** (1993), 349–373.

41. M. Pulte, *The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles*, Duke. Math. J. **57** (1988), 721–760.

42. M. Ragunathan, *Cohomology of arithmetic subgroups of algebraic groups: I*, Ann. of Math. **86** (1967), 409–424.

43. M. Saito, *Mixed Hodge modules and admissible variations*, C. R. Acad. Sci. Paris, **309** (1989), Série I, 351–356.

44. J.-P. Serre, *Lie Algebras and Lie Groups*, Benjamin, 1965.

45. A. Silverberg, *Mordell–Weil groups of generic abelian varieties*, Invent. Math. **81** (1985), 71–106.

46. P. Sipe, *Some finite quotients of the mapping class group of a surface*, Proc. Amer. Math. Soc. **97** (1986), 515–524.

47. J. Steenbrink and S. Zucker, *Variations of mixed Hodge structure I*, Invent. Math. **80** (1985), 489–542.

48. D. Sullivan, *Infinitesimal computations in topology*, Publ. Math. IHES **47** (1977), 269–331.

Richard M. Hain
Department of Mathematics, Duke University
Durham, NC 27708-0320
E-mail address: hain@math.duke.edu