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Vector Bundles and Brill–Noether Theory

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Abstract. After a quick review of the Picard variety and Brill–Noether theory, we generalize them to holomorphic rank-two vector bundles of canonical determinant over a compact Riemann surface. We propose several problems of Brill–Noether type for such bundles and announce some of our results concerning the Brill–Noether loci and Fano threefolds. For example, the locus of rank-two bundles of canonical determinant with five linearly independent global sections on a non-tetragonal curve of genus 7 is a smooth Fano threefold of genus 7.

As a natural generalization of line bundles, vector bundles have two important roles in algebraic geometry. One is the moduli space. The moduli of vector bundles gives connections among different types of varieties, and sometimes yields new varieties that are difficult to describe by other means. The other is the linear system. In the same way as the classical construction of a map to a projective space, a vector bundle gives rise to a rational map to a Grassmannian if it is generically generated by its global sections. In this article, we shall describe some results for which vector bundles play such roles. They are obtained from an attempt to generalize Brill–Noether theory of special divisors, reviewed in Section 2, to vector bundles. Our main subject is rank-two vector bundles with canonical determinant on a curve C with as many global sections as possible: especially their moduli and the Grassmannian embeddings of C by them (Section 4).

1. Line bundles

Let X be a smooth algebraic variety over the complex number field $\mathbf{C}$. We consider the set of isomorphism classes of line bundles, or invertible sheaves, on X. This set enjoys two good properties, neither of which holds anymore

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for vector bundles of higher rank. One is that it has a natural algebraic structure as a moduli space without any modification. The other is that it becomes a (commutative) group by the tensor product. In fact, the isomorphism classes are parametrized by the first cohomology group $H^1(\mathcal{O}_X^*)$ with coefficient in the (multiplicative) sheaf of nowhere vanishing holomorphic functions. $H^1(\mathcal{O}_X^*)$ endowed with the natural algebraic structure is called the *Picard variety* and denoted by $\operatorname{Pic} X$. Let

$$(1.1)\quad \cdots \longrightarrow H^1(X,\mathbf{Z}) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbf{Z}) \longrightarrow \cdots$$

be the long exact sequence derived from the exponential exact sequence

$$(1.2)\qquad 0 \longrightarrow \mathbf{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$$

of sheaves on X. The connecting homomorphism δ associates the first Chern class $c_1(L)$ to each line bundle $[L] \in H^1(\mathcal{O}_X^*)$. For example, if X is a curve, $\delta(L)$ is the degree of L under the natural identification $H^2(X,\mathbf{Z}) \simeq \mathbf{Z}$. By (1.1), the neutral component $\operatorname{Pic}^0 X$ of $\operatorname{Pic} X$ is isomorphic to the quotient group $H^1(\mathcal{O}_X)/H^1(X,\mathbf{Z})$, which is an abelian variety if X is a projective variety.

Let C be a *curve*, or a compact Riemann surface, of genus g. The Riemann–Roch theorem

$$(1.3)\qquad \begin{cases} \chi(L) := h^0(L) - h^1(L) = \deg L + 1 - g, \\ H^1(L) \simeq H^0(K_C L^{-1})^\vee, \end{cases}$$

is most fundamental for its study. The latter isomorphism is functorial in L and referred to as the *Serre duality*. By (1.1), $\operatorname{Pic} C$ is the disjoint union of $\operatorname{Pic}^d C$, for $d \in \mathbf{Z}$, where $\operatorname{Pic}^d C$ is the set of isomorphism classes of line bundles of degree d. By (1.3), the number $h^0(L)$ of linearly independent global sections is constant on $\operatorname{Pic}^d C$ unless $0 \le d \le 2g-2 = \deg K_C$. Conversely, when $0 \le d \le 2g-2$, the number $h^0(L)$ is equal to $\max\{0,\, d+1-g\}$ on a non-empty Zariski open subset of $\operatorname{Pic}^d C$, but not constant since there exists a *special line bundle*, that is, a line bundle L with $h^0(L)h^1(L) \neq 0$, of degree d. The space $\operatorname{Pic}^d C$ is stratified by $h^0(L)$. For an integer $r \ge \max\{0,\, d+1-g\}$, we set

$$W_d^r(C) = \{[L] \mid h^0(L) \ge r+1\} \subset \operatorname{Pic}^d C,$$

which is closed in the Zariski topology. The case $(d,r) = (g-1,\, 0)$ is most important. W_{g-1}^0 is a divisor and usually denoted by Θ. The self-intersection number (Θ^g) is equal to $g!$, i.e., Θ is a principal polarization of $\operatorname{Pic}^{g-1} C$. This principally polarized abelian variety $(\operatorname{Pic}^{g-1} C, \Theta)$ is called the *Jacobian* of C.

Often the isomorphism class of C is recovered from the variety $W_d^r(C)$ of special line bundles. The case of theta divisor Θ is classical:

THEOREM 1.4 (TORELLI). *Two curves are isomorphic to each other if their Jacobians are so.*

We refer to [13] for various approaches to this important result. Let C be a non-hyperelliptic curve of genus 5. Then $W^1_4(C)$ is a curve of genus 11. (If C is trigonal or $W^1_4(C)$ contains a line bundle with $L^2 \simeq K_C$, then $W^1_4(C)$ is singular. But still the theorem holds true.)

Another example is:

THEOREM 1.5. *The Jacobian of C is isomorphic to the Prym variety of*

$$(W^1_4(C), \sigma),$$

where σ is the involution of $\operatorname{Pic}^4 C$ *defined by* $\sigma[L] = [K_C L^{-1}]$.

See [2] for the proof in the case where C is a complete intersection of three quadrics in $\mathbf{P}^5$.

Another feature of special line bundles is their relation with projective embeddings. If a line bundle L is generated by its global sections, we obtain a morphism

$$\Phi_{|L|} : C \longrightarrow \mathbf{P}^* H^0(L),$$

where $\mathbf{P}^* H^0(L)$ is the projectivization of the dual vector space of $H^0(L)$. The most interesting case is K_C, the canonical line bundle, which appears in (1.3). By the Riemann–Roch theorem, K_C is generated by global sections, and $\Phi_{|K|} : C \longrightarrow \mathbf{P}^* H^0(K_C) = \mathbf{P}^{g-1}$ is an embedding unless C is hyperelliptic. The image $C_{2g-2} \subset \mathbf{P}^{g-1}$ of $\Phi_{|K|}$ is called *the canonical model* of C. Here is a classical example:

THEOREM 1.6 (ENRIQUES–PETRI). *The canonical model $C_{2g-2} \subset \mathbf{P}^{g-1}$ is an intersection of quadrics if and only if $W^1_3(C) = W^2_5(C) = \emptyset$.*

We refer to [1] and [6] for further results of this kind. The latter also discusses an interesting use of vector bundles that we do not treat here.

2. Brill–Noether theory

We study $W^r_d(C)$ more closely. First we note that it is not only a subset but a subscheme of $\operatorname{Pic} C$. Take distinct points $P_1, \dots, P_N$ of C and put $D = \sum_{i=1}^N P_i$. Choose N sufficiently large so that $H^1(L(D))$ vanishes for every $[L] \in \operatorname{Pic}^d C$. The exact sequence

$$0 \longrightarrow L \longrightarrow L(D) \xrightarrow{\text{res}} \bigoplus_{i=1}^{N} L(D)|_{P_i} \longrightarrow 0 \tag{2.1}$$

of sheaves on C induces the exact sequence

$$(2.2)\quad 0 \longrightarrow H^0(L) \longrightarrow H^0(L(D)) \overset{H^0(\mathrm{res})}{\longrightarrow} \bigoplus_{i=1}^{N} H^0(L(D)|_{P_i}) \longrightarrow H^1(L) \longrightarrow 0$$

of vector spaces. There exists a homomorphism $R : \mathcal{E} \longrightarrow \mathcal{F}$ between two vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $\operatorname{Pic}^d C$ whose fibre $R_{[L]}$ at $[L]$ is the above $H^0(\mathrm{res})$ for every $[L] \in \operatorname{Pic}^d C$ (these bundles are the direct images of certain sheaves on $C \times \operatorname{Pic} C$). The difference in rank between $\mathcal{E}$ and $\mathcal{F}$ does not depend on D: we have

$$r(\mathcal{F}) - r(\mathcal{E}) = N - h^0(L(D)) = g - 1 - d$$

by (1.3). The following statement is easy to verify:

LEMMA 2.3. *Let E and F be finite-dimensional vector spaces, let c be a positive integer, and set $W = \{f \in \operatorname{Hom}(E, F) \mid \dim \operatorname{Ker} f \geq c\}$. Then:*

1) *W is a closed subvariety of codimension $\max\{0, c(c+\delta)\}$ in the affine space $\operatorname{Hom}(E, F)$, where $\delta = \dim F - \dim E$, and*
2) *if $\dim \operatorname{Ker} f = c$, then W is smooth at the point f and the normal space $N_{W/\operatorname{Hom},f}$ is isomorphic to $\operatorname{Hom}(\operatorname{Ker} f, \operatorname{Coker} f)$.*

Since $W^r_d(C)$ is

$$\{\alpha \in \operatorname{Pic}^d C \mid \dim \operatorname{Ker} R_\alpha \geq r + 1\},$$

it is a closed subscheme of $\operatorname{Pic}^d C$ and its codimension is at most $(r+1)(g+r-d)$ by the lemma. It follows that

$$(2.4)\qquad \dim W^r_d(C) \geq g - (r + 1)(g + r - d).$$

For a line bundle L on C, we put $\rho(L) = g - h^0(L)h^1(L)$ and call it the *Brill–Noether number*. When $[L] \in W^r_d(C)$ and $h^0(L) = r + 1$, this number is equal to the right-hand side of the above inequality. Since the tangent space of $\operatorname{Pic} C$ is isomorphic to $H^1(\mathcal{O}_C)$ by (1.1), the Zariski tangent space of $W^r_d(C)$ at $[L]$ is the kernel of the tangential map $H^1(\mathcal{O}_C) \longrightarrow \operatorname{Hom}(H^0(L), H^1(L))$ by (2.2) and Lemma 2.3(2).

Now we describe the Zariski tangent space more directly. Let

$$\tau_L : H^1(\mathcal{O}_C) \longrightarrow \operatorname{Hom}(H^0(L), H^1(L))$$

be the linear map induced by the cup product $H^1(\mathcal{O}_C) \times H^0(L) \longrightarrow H^1(L)$. By the Serre duality (1.3), the dual of τ_L is the multiplication map

$$(2.5)\qquad H^0(L) \otimes H^0(K_C L^{-1}) \longrightarrow H^0(K_C),$$

called the *Petri map*.

PROPOSITION 2.6. *Assume that* $[L] \in W^r_d(C)$ *and* $h^0(L) = r+1$. *Then:*
1) *the Zariski tangent space of* $W^r_d(C)$ *at* $[L]$ *is isomorphic to* $\operatorname{Ker}\tau_L$,
2) $\dim \operatorname{Ker}\tau_L \geq \dim_{[L]} W^r_d(C) \geq \rho(L)$, *and*
3) *the following three conditions are equivalent:*
   i) $W^r_d(C)$ *is smooth and of dimension* $\rho(L)$ *at* $[L]$,
   ii) τ_L *is surjective, and*
   iii) *the Petri map* (2.5) *is injective,*

PROOF. Let $\alpha = \{a_{ij}\} \in H^1(\mathcal{O}^*_C)$ be the cohomology class corresponding to L, that is, $a_{ij} \in \mathcal{O}_{U_i\cap U_j}$ are the transition functions of L for a suitable open covering $\{U_i\}$ of C. Let ε be the dual number, i.e., $\varepsilon \neq 0$ but $\varepsilon^2 = 0$. A first-order infinitesimal deformation $\tilde{L}$ of L corresponds to a cohomology class $\tilde{\alpha} = \{\tilde{a}_{ij}\} \in H^1(\mathcal{O}_C[\varepsilon]^*)$ whose reduction modulo ε is α. Also, $\tilde{\alpha}$ is of the form $\{a_{ij}(1 + b_{ij}\varepsilon)\}$ for $\beta = \{b_{ij}\} \in H^1(\mathcal{O}_C)$. Let $h \in H^0(L)$ be a global section of L; then h is a collection $\{h_i\}$ of $h_i \in \mathcal{O}_{U_i}$ such that $h_i = a_{ij}h_j$. The differences $b_{ij}h_j\varepsilon$ of h_i and $\tilde{a}_{ij}h_j$ form a one-cocycle whose cohomology class is the cup product $(\beta^{\cup}h)\varepsilon$. Hence h extends to a global section of $\tilde{L}$ if and only if $\beta^{\cup}h = 0$ in $H^1(L)$. Therefore, all global sections of L extend if and only if the cup product map ${}^{\cup}\beta : H^0(L) \longrightarrow H^1(L)$ is zero, which shows (1). Part (2) follows from (1) and (2.4). Part (3) is straightforward from (2). ☐

Let ρ be the right-hand side of (2.4). We refer to [1] for the following important results:

THEOREM 2.7 (Kempf Kleiman–Laksov; Fulton–Lazarsfeld).

(*Existence*) $W^r_d(C) \neq \emptyset$ *if* $\rho \geq 0$.

(*Connectedness*) $W^r_d(C)$ *is connected if* $\rho > 0$.

Let $\mathcal{M}_g$ be the moduli space of curves of genus g.

THEOREM 2.8 (Gieseker [5], Lazarsfeld [7]). *If* $[C] \in \mathcal{M}_g$ *is general, the Petri map* (2.5) *is injective for every* (*special*) *line bundle* L *on* C.

In particular, $W^r_d(C)$ is of dimension ρ if ρ is nonnegative, and empty otherwise. Thus the estimate (2.4) is best possible for the generic curve. When $\rho = 0$, the number of $W^r_d(C)$ is finite and was first computed by Castelnuovo. Let $G(a,\, a+b) \subset \mathbf{P}_* \bigwedge^a \mathbf{C}^{a+b}$ be the Plücker embedding of the Grassmannian of a-dimensional subspaces of $\mathbf{C}^{a+b}$. The following result is interesting (cf. (4.15)):

THEOREM 2.9. *If* $[C] \in \mathcal{M}_g$ *is general and* $\rho = 0$, *the number of* $W^r_d(C)$ *is equal to the degree of the* g*-dimensional Grassmannian*

$$G(a, a+b) \subset \mathbf{P}_* \bigwedge^a \mathbf{C}^{a+b},$$

where $a = r+1$ *and* $b = g+r-d$.

In fact, both $\#W^r_d(C)$ and $\deg G(a, a+b)$ are equal to

$$g! \prod_{1 \le i \le a < j \le a+b} (j-i)^{-1}.$$

3. Vector bundles on a curve

Let X be a smooth complete algebraic variety (over $\mathbf{C}$). By $\mathcal{O}_X$, we mean either the sheaf of holomorphic functions in the usual topology or the sheaf of regular functions in the Zariski topology. In the study of vector bundles, this is allowed by virtue of the GAGA principle, which says that the two categories of analytic and algebraic coherent sheaves on X are equivalent to each other. By a *vector bundle* E on X we mean a locally free $\mathcal{O}_X$-module. There exists an open covering $\{U_i\}$ of X such that $E\mid_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ for every i. This positive integer r is called the *rank* of E. The highest exterior product $\bigwedge^r E$ is a line bundle on X, denoted by $\det E$.

Assume that E is generated by global sections, that is, the evaluation homomorphism

$$\mathrm{ev}_E : H^0(E) \otimes \mathcal{O}_X \longrightarrow E$$

is surjective. Then every fibre E_x of E is an r-dimensional quotient space of $H^0(E)$. Hence we obtain a map $\Phi_{|E|} : X \longrightarrow G(H^0(E), r)$ to the Grassmannian of r-dimensional *quotient* spaces of $H^0(E)$. This map is holomorphic since E is so. The exterior product

$$\bigwedge^r \mathrm{ev}_E : \bigwedge^r H^0(E) \otimes \mathcal{O}_X \longrightarrow \bigwedge^r E$$

of ev_E induces a linear map

$$\bigwedge^r H^0(E) \longrightarrow H^0(\bigwedge^r E), \tag{3.1}$$

which we denote by λ_E. The exterior product $\bigwedge^r E_x$ of fibres E_x are quotient spaces of both $H^0(\bigwedge^r E)$ and $\bigwedge^r H^0(E)$. The former determines a point in the projective space $\mathbf{P}^*H^0(\bigwedge^r E)$ and the latter the Plücker coordinate of $[E_x] \in G(H^0(E), r)$. Hence we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Phi_{|E|}} & G(H^0(E), r) \\ \Phi_{|\wedge^r E|} \downarrow & & \cap \ \text{Plücker embedding} \\ \mathbf{P}^*H^0(\bigwedge^r E) & \overset{\mathbf{P}^*\lambda_E}{\cdots\to} & \mathbf{P}^* \bigwedge^r H^0(E). \end{array} \tag{3.2}$$

Thus λ_E connects the projective and Grassmannian embeddings. This map is important in other considerations, too. See (3.5), (4.7) and (4.15).

Let $A_{ij} \in \mathrm{GL}(r, \mathcal{O}_{U_i \cap U_j})$ be the transition matrix function of a vector bundle E, that is, the matrix expression of the composite of the two isomorphisms $\mathcal{O}_{U_j}^{\oplus r} \simeq E|_{U_j}$ and $E|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ over $U_i \cap U_j$. The collection $\{A_{ij}\}$ of all such (matrix) functions is a one-cocycle with coefficients in the sheaf $\mathrm{GL}(r, \mathcal{O}_X)$ of noncommutative groups. Hence the rank-r vector bundles on X are parametrized by the first cohomology set $H^1(\mathrm{GL}(r, \mathcal{O}_X))$. The determinant homomorphism $\det : \mathrm{GL}(r, \mathcal{O}_X) \longrightarrow \mathcal{O}_X^*$ induces the map

$$H^1(\det) : H^1(\mathrm{GL}(r, \mathcal{O}_X)) \longrightarrow H^1(\mathcal{O}_X^*) = \operatorname{Pic} X,$$

whose fibre at $[L] \in \operatorname{Pic} X$ is denoted by $B_X(r, L)$.

Moduli Problem. a) Give a natural algebraic structure to a suitable *open* subset of $B_X(r, L)$ and construct its *geometric* compactification.

b) What properties of (X, L) are inherited by the moduli space constructed in (a)?

The fibre $B_X(r, L)$ does not have a nice description such as $\operatorname{Pic} X$ in (1.1). But its Zariski tangent space as a functor is easy to identify. Let E be a vector bundle and $\{A_{ij}\}$ the one-cocycle of transition functions. A one-cochain $\{A_{ij}(I_r + B_{ij}\varepsilon)\}$ with values in $\mathrm{GL}(r, \mathcal{O}_X[\varepsilon])$ is a cocycle if and only if

$$A_{jk}^{-1} B_{ij} A_{jk} + B_{jk} = B_{ik}$$

holds on $U_i \cap U_j \cap U_k$ for every i, j and k, where ε is the dual number. This is the same as saying that $\{B_{ij}\}$ is a one-cocycle with values in the sheaf $\mathcal{E}nd\, E \simeq E^\vee \otimes E$ of (local) endomorphisms of E. By this correspondence, the first-order infinitesimal deformations of E are parametrized by the cohomology group $H^1(\mathcal{E}nd\, E)$. Since

$$\det(A_{ij}(I_r + B_{ij}\varepsilon)) = (\det A_{ij})(I_r + \operatorname{tr} B_{ij}\varepsilon),$$

the Zariski tangent space of $B_X(r, L)$ is isomorphic to the kernel of

$$H^1(\mathrm{tr}) : H^1(\mathcal{E}nd\, E) \longrightarrow H^1(\mathcal{O}_X),$$

which is the tangential map of $H^1(\det)$. Let $\mathrm{sl}(E)$ be the sheaf of traceless endomorphisms of E. Then $\mathcal{E}nd\, E$ is the direct sum of two vector bundles $\mathrm{sl}(E)$ and $\mathcal{O}_X$. Therefore, the Zariski tangent space is isomorphic to $H^1(\mathrm{sl}(E))$.

Let C be a curve of genus g. The Riemann–Roch theorem (1.3) is generalized to the formula

$$\begin{cases} \chi(E) := h^0(E) - h^1(E) = \deg E + r(1-g) \\ H^1(E) \simeq H^0(K_C E^\vee)^\vee \quad \text{(Serre duality)}, \end{cases} \tag{3.3}$$

where $\deg E$ is the degree of $\det E$. For simplicity, we restrict ourselves to the case $r = 2$. The answer to part (a) of the moduli problem is the notion of stability:

DEFINITION 3.4. (Mumford [11]) A rank-two vector bundle E on C is *stable* if $\deg \xi < \frac{1}{2} \deg E$ for every line subbundle ξ of E. It is *semi-stable* if $\deg \xi \leq \frac{1}{2} \deg E$ for every ξ.

Let L be a line bundle on C and E a member of $B_C(2, L)$. Fix an ample line bundle $\mathcal{O}_C(1)$ on C. Then $E(n)$ belongs to $B_C(2, L(2n))$ and we obtain the linear map

$$\lambda_{E(n)} : \bigwedge^2 H^0(E(n)) \longrightarrow H^0(L(2n)) \tag{3.5}$$

as in (3.1). The above condition $\deg \xi < \frac{1}{2} \deg E$ is equivalent to the asymptotic stability of the linear map $\lambda_{E(n)}$ with respect to the action of the special linear group $\mathrm{SL}(H^0(E(n)))$ [4]. By the geometric invariant theory, the (coarse) moduli space $M_C(2, L)$ of stable two-bundles with determinant L exists as a quasi-projective algebraic variety. Moreover, it becomes a projective algebraic variety $\overline{M}_C(2, L)$ by adding certain equivalence classes of semi-stable two-bundles.

Every line bundle M induces an isomorphism $M_C(2, L) \simeq M_C(2, LM^2)$ taking E to $E \otimes M$. Hence there are only two isomorphism classes of the moduli space of two-bundles: $M_C(2, \mathrm{odd})$ and $M_C(2, \mathrm{even})$. Both are smooth and of dimension $3g - 3 = \dim H^1(\mathrm{sl}(E))$. Since every semi-stable two-bundle of odd degree is stable, $M_C(2, \mathrm{odd})$ is a projective variety. Among many known global properties of $M_C(2, \mathrm{odd})$, we state two. One is

THEOREM 3.6 (Ramanan [19]). *$M_C(2, \mathrm{odd})$ is a Fano manifold of index two.*

A smooth projective variety X is called a *Fano manifold* if the anti-canonical line bundle $K_X^{-1} = \det T_X$ is ample. The largest integer that divides $c_1(X)$ in $H^2(X, \mathbf{Z})$ is called the index. In the case of $M_C(2, \mathrm{odd})$, the Picard group is free cyclic and the anti-canonical line bundle is the square of the positive generator.

EXAMPLE. (Desale and Ramanan [21], Newstead [17]) Let C be a curve of genus two defined by the equation $y^2 = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_6)$. Then the moduli space $M_C(2, \mathrm{odd})$ is the complete intersection

$$\sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} \lambda_i x_i^2 = 0$$

in $\mathbf{P}^5$. The anti-canonical line bundle is the square of the restriction of tautological line bundle.

The other result we cite is a Torelli-type theorem:

Theorem 3.7 (Mumford and Newstead [15]). *The intermediate Jacobian*

$$H^2(\Omega^1_M)/H^3(M, \mathbf{Z})$$

of the moduli space $M_C(2, \mathrm{odd})$ *is isomorphic to the Jacobian of* C *as polarized abelian variety.*

4. Toward a Brill–Noether type theory for two-bundles

$M_C(2, \mathcal{O}_C)$ and $M_C(2, K_C)$ are two natural representatives of $M_C(2, \mathrm{even})$. The following is fundamental for the former:

Theorem 4.1 (Narasimhan and Seshadri [16], Donaldson [3]). *There is a natural bijection between the two sets*

1) $M_C(2, \mathcal{O}_C)$*, the set of isomorphism classes of stable two-bundles with trivial determinant, and*
2) $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(C), \mathrm{SU}(2))/\,\mathrm{SU}(2)$*, the set of conjugacy classes of irreducible two-dimensional special unitary representations of the fundamental group of* C.

We take $M_C(2, K)$ to develop a Brill–Noether type theory. See [20] and [18] for another direction of development. The moduli space $M_C(2, K)$ of stable two-bundles with canonical determinant is stratified by the number $h^0(E)$ of linearly independent global sections. By analogy with W^r_d, we set

$$M_C(2, K, n) = \{[E] \mid h^0(E) \geq n + 2\} \subset M_C(2, K).$$

We denote by $\overline{M}_C(2, K, n)$ the union of $M_C(2, K, n)$ and the set of isomorphism classes of semi-stable vector bundles $[\xi \oplus K_C\xi^{-1}] \in \overline{M}_C(2, K)$ with $[\xi] \in W^{n/2}_{g-1}(C)$.

By the same argument as in Section 2, $M_C(2, K, n)$ is a closed subscheme of $M_C(2, K)$. (The universal family does not exist on $C \times M_C(2, K)$, but this does not cause a problem for the study of such local properties of the moduli.) A similar consideration gives the estimate $\dim M_C(2, K, n) \geq 3g - 3 - (n + 2)^2$. But this estimate is not sharp. The proper one is

Theorem 4.2. $\dim M_C(2, K, n) \geq 3g - 3 - \frac{1}{2}(n + 2)(n + 3)$.

Before giving the proof, we recall some notions of symplectic geometry. Let W be a 2ν-dimensional vector space with a non-degenerate skew-symmetric bilinear form $\langle\, ,\rangle : W \times W \longrightarrow \mathbf{C}$. A ν-dimensional subspace V of W is *a Lagrangian* of W if the bilinear form $\langle\, ,\rangle$ is identically zero on $V \times V$. We denote the set of Lagrangians by $\mathcal{L}(W)$, which is a subset of the ν^2-dimensional Grassmannian $G(\nu, W)$. Fix a Lagrangian U_∞ and set $Z = \{[U] \mid U \cap U_\infty = 0\}$ in $G(\nu, W)$. When $[U_0] \in Z$ is fixed, Z is a ν^2-dimensional affine space by the bijection

$$\mathrm{Hom}(U_0, U_\infty) \ni f \mapsto \Gamma_f \in Z, \tag{4.3}$$

where $\Gamma_f \subset U_0 \times U_\infty$ is the graph of f. Assume that U_0 is also a Lagrangian. Then U_0 and U_∞ are dual under the pairing $\langle\, ,\rangle$. Γ_f is a Lagrangian of W if and only if $f \in \mathrm{Hom}(U_0, U_\infty) \simeq U_\infty \otimes U_\infty$ is symmetric. Hence $\mathcal{L}(W)$ is a smooth subvariety of dimension $\frac{1}{2}\nu(\nu+1)$ in the Grassmannian $G(\nu, W)$.

Proposition 4.4. *Fix a Lagrangian $[V_0] \in \mathcal{L}(W)$ and let c be a positive integer. Then the Schubert subvariety*

$$\mathcal{L}(W)_c = \{[V] \in \mathcal{L}(W) \mid \dim V \cap V_0 \geq c\}$$

is of codimension $\frac{1}{2}c(c+1)$ in $\mathcal{L}(W)$.

Proof. For $[V] \in \mathcal{L}(W)$, choose a Lagrangian V_∞ so that $V \cap V_\infty = V_0 \cap V_\infty = 0$. Then V corresponds to a symmetric matrix of size ν via (4.3) and $\dim V \cap V_0$ is equal to the co-rank of the symmetric matrix. Hence we have our assertion.□

Proof of Theorem 4.2. Let E be a two-bundle with canonical determinant and D an effective divisor on C. Since E is self-Serre adjoint, that is, $E^\vee K_C \simeq E$, the two vector spaces $H^0(E)$ and $H^1(E)$ are dual by (3.3). Similarly, $H^0(E(-D))$ and $H^1(E(D))$ are dual. Hence, by the Riemann–Roch theorem (3.3), we have

$$h^0(E(D)) - h^0(E(-D)) = \chi(E(D)) = 2N, \tag{4.5}$$

where $N = \deg D$. Now we denote the quotient sheaf $E(D)/E(-D)$ by A, which is supported by a finite set and has length $4N$. We consider the composite of the pairing $A \times A \longrightarrow K_C(2D)/K_C$, induced by $\bigwedge^2 E \simeq K_C$, and the residue map $r : K_C(2D)/K_C \longrightarrow \mathbf{C}$ given by

$$r(\omega) = \sum_{P \in \mathrm{Supp}\ D} \mathrm{Res}_P\ \omega.$$

This induces a non-degenerate skew-symmetric pairing $\langle\, ,\rangle$ on the vector space $H^0(A)$ of dimension $4N$. Since r is identically zero on the image of $H^0(K_C(2D))$ (by the Residue Theorem), so is $\langle\, ,\rangle$ on the image V of $H^0(E(D)) \longrightarrow H^0(A)$. By the exact sequence

$$0 \longrightarrow E(-D) \longrightarrow E(D) \longrightarrow A \longrightarrow 0,$$

the image V is isomorphic to the quotient space $H^0(E(D))/H^0(E(-D))$. Hence V is a Lagrangian of the symplectic vector space $H^0(A)$ by (4.5). It is obvious that $V_0 = H^0(E/E(-D))$ is also a Lagrangian. Now we choose D so that $H^0(E(-D)) = 0$ for every $[E] \in M_C(2, K)$. Then $H^0(E)$ is the intersection of two Lagrangians $V = H^0(E(D))$ and V_0 of $H^0(A)$. Hence we have our inequality by the above proposition. □

Remark 4.6. This proof works also for a vector bundle E of even rank with a non-degenerate skew-symmetric pairing $E\times E \longrightarrow K_C$. The same trick was used in [12] to show the parity preservation of $h^0(E)$ when E has a non-degenerate quadratic form $q : E \longrightarrow K_C$.

Let E be a rank-two vector bundle on C. For $f \in \mathrm{Hom}(H^0(E), H^1(E))$, let $T(f)$ be the linear map

$$\bigwedge^2 H^0(E) \ni h_1\wedge h_2 \quad\mapsto\quad h_1{}^{\cup}f(h_2) - h_2{}^{\cup}f(h_1) \in H^1\Big(\bigwedge^2 E\Big),$$

where ${}^{\cup} : H^0(E)\times H^1(E) \longrightarrow H^1(\bigwedge^2 E)$ is the cup product. It is easy to check that the following diagram is commutative:

$$\begin{array}{ccc} H^1(\mathcal{E}nd\,E) & \longrightarrow & \mathrm{Hom}(H^0(E), H^1(E)) \\ {\scriptstyle H^1(\mathrm{tr})}\ \downarrow & & \downarrow\ {\scriptstyle T} \\ H^1(\mathcal{O}_C) & \longrightarrow & \mathrm{Hom}(\bigwedge^2 H^0(E), H^1(\bigwedge^2 E)), \end{array} \tag{4.7}$$

where the lower horizontal linear map is the composite of

$$\tau_{\det E} : H^1(\mathcal{O}_C) \longrightarrow \mathrm{Hom}\Big(H^0\Big(\bigwedge^2 E\Big),\ H^1\Big(\bigwedge^2 E\Big)\Big),$$

defined in Section 2, and $\mathrm{Hom}\big(\lambda_E,\ H^1\big(\bigwedge^2 E\big)\big)$.

Theorem 4.8. *Assume that* $[E] \in M_C(2, K, n)$ *and* $h^0(E) = n + 2$. *Then:*

(1) *the Zariski cotangent space of* $M_C(2, K, n)$ *at* $[E]$ *is isomorphic to the cokernel of* $S^2H^0(E) \longrightarrow H^0(S^2E)$,

(2) $\dim \mathrm{Coker}[S^2H^0(E) \longrightarrow H^0(S^2E)] \ge \dim_{[E]} M_C(2, K, n) \ge \sigma(E)$, *and*

(3) $M_C(2, K, n)$ *is smooth and of dimension* $\sigma(E)$ *at* $[E]$ *if and only if the map* $S^2H^0(E) \longrightarrow H^0(S^2E)$ *is injective.*

Proof. As we saw in Section 3, the tangent space of $M_C(2, K)$ at $[E]$ is the kernel $H^1(\mathrm{sl}(E))$ of the trace map $H^1(\mathcal{E}nd\,E) \longrightarrow H^1(\mathcal{O}_C)$. Let $\tilde{E}$ be a first-order infinitesimal deformation of E corresponding to $B = \{B_{ij}\} \in H^1(\mathcal{E}nd\,E)$. By the same argument as in the proof of Proposition 2.6, a global section $h \in H^0(E)$ extends that of $\tilde{E}$ if and only if the cup product $h^{\cup}B \in H^1(E)$ vanishes. Hence, all global sections of E extend if and only if the cup product map ${}^{\cup}B : H^0(E) \longrightarrow H^1(E)$ is zero. Therefore, the Zariski tangent space of $M_C(2, K, n)$ is the kernel of $H^1(\mathrm{sl}(E)) \longrightarrow \mathrm{Ker}\,T \subset \mathrm{Hom}(H^0(E), H^1(E))$. Since $\bigwedge^2 E \simeq K_C$, the dual of (4.7) is reduced to the commutative diagram

$$\begin{array}{ccc} H^0(E\otimes E) & \longleftarrow & H^0(E)\otimes H^0(E) \\ \uparrow & & \uparrow \\ H^0(\bigwedge^2 E) & \xleftarrow{\lambda_E} & \bigwedge^2 H^0(E), \end{array}$$

by Serre duality (3.3), which shows (1). Part (2) follows from (1) and Theorem 4.2. Part (3) is straightforward from (1) and (2). □

REMARK 4.9. The obstructions for $M_C(2,K,n)$ to be smooth at $[E]$ lie in the cokernel of $H^0(S^2E)^\vee \longrightarrow S^2H^0(E)^\vee$. This fact gives another proof of Theorem 4.2 and 4.8.

Let σ be the right-hand side of Theorem 4.2. Theorem 2.7, 2.8 and 2.9 lead us to the following problems:

PROBLEM 4.10.

(*Existence*) Is $\overline{M}_C(2,K,n)$ non-empty when $\sigma \geq 0$?

(*Connectedness*) Is $\overline{M}_C(2,K,n)$ connected when $\sigma > 0$?

PROBLEM 4.11. Assume that $[C] \in \mathcal{M}_g$ is general.

1) Is $S^2H^0(E) \longrightarrow H^0(S^2E)$ injective for every $E \in M_C(2,K)$?
2) Is $M_C(2,K,n)$ of dimension σ when $\sigma \geq 0$?
3) Compute the number of $M_C(2,K,n)$ when $\sigma = 0$. More generally, describe the cohomology class of $\overline{M}_C(2,K,n)$ in $H^*(\overline{M}_C(2,K),\mathbf{Z})$ when $\sigma > 0$.

Another direction is

PROBLEM 4.12. Study the Grassmannian map associated with a member of $M_C(2,K,n)$, and its relation with the canonical model $C_{2g-2} \subset \mathbf{P}^{g-1}$.

We give some sample results in these directions. They are closely related to our classification of Fano threefolds via vector bundles [8]. We first consider the three cases $(g,\, n+2) = (7,5)$, $(9,6)$ and $(11,7)$. The Brill–Noether number σ is equal to 3, 3 and 2, respectively.

THEOREM 4.13. *Let C be a curve of genus 7 with $W^1_4(C) = \emptyset$. Then:*

1) *$M_C(2,K,3)$ is smooth, complete and of dimension 3;*
2) *$M_C(2,K,3)$ is a Fano threefold of genus 7, i.e., $(-K_M)^3 = 12$, and with Picard number one; and*
3) *the intermediate Jacobian $H^2(\Omega^1_M)/H^3(M,\mathbf{Z})$ of $M_C(2,K,3)$ is isomorphic to the Jacobian of C as polarized abelian variety.*

Conversely, every smooth Fano threefold of genus 7 with Picard number one is obtained in this manner from a non-tetragonal curve of genus 7.

Similarly, $\overline{M}_C(2,K,4)$ is a quartic threefold in $\mathbf{P}^4$ with 21 singular points at the boundary if C is a general curve of genus 9, and $M_C(2,K,5)$ is a (polarized) K3 surface of genus 11 if C is a general curve of genus 11.

In the case $(g,\, n+2) = (8,6)$, the number σ is equal to zero.

THEOREM 4.14 (Mukai [9], [10]). *If C is a curve of genus 8 with $W^2_7(C) = \emptyset$, then $M_C(2,K,4)$ consists of the unique isomorphism class of stable two-bundles E. The linear map λ_E in (3.1) is surjective and the following diagram, essentially*

(3.2), *is Cartesian:*

$$\begin{array}{rccl} & C & \xrightarrow{\Phi_{|E|}} & G(H^0(E),2) \\ \text{canonical embedding } \cap & & & \cap \text{ Plücker embedding} \\ & \mathbf{P}^* H^0(K_C) & \xrightarrow{\mathbf{P}^*\lambda_E} & \mathbf{P}^* \bigwedge^2 H^0(E). \end{array}$$

In particular, C is a complete linear section of the 8*-dimensional Grassmannian, that is,*

$$[C \subset \mathbf{P}^7] = [G(6,2) \subset \mathbf{P}^{14}] \cap H_1 \cap \cdots \cap H_7$$

for seven hyperplanes $H_1, \ldots, H_7$.

Let C and E be as in the theorem and consider the intersection of $G(2, H^0(E))$ and $\mathbf{P}_* \operatorname{Ker} \lambda_E$, where $G(2, H^0(E))$ is the Grassmannian of two-dimensional subspaces of $H^0(E)$ embedded into $\mathbf{P}_* \bigwedge^2 H^0(E)$ by the Plücker coordinates. If a subspace $[U] \in G(2, H^0(E))$ belongs to $\mathbf{P}_* \operatorname{Ker} \lambda_E$, the evaluation homomorphism $\mathrm{ev}^U : U \otimes \mathcal{O}_C \longrightarrow E$ is not injective and its kernel is a line bundle. Moreover, the inverse of $\operatorname{Ker} \mathrm{ev}^U$ belongs to $W^1_5(C)$, and if C is general, the map

$$G(2, H^0(E)) \cap \mathbf{P}_* \operatorname{Ker} \lambda_E \longrightarrow W^1_5(C), \quad [U] \mapsto (\operatorname{Ker} \mathrm{ev}^U)^{-1} \tag{4.15}$$

is an isomorphism between two reduced zero-dimensional schemes, which shows Theorem 2.9 in the case $(a,b) = (2,4)$. This idea leads us to a computation-free proof of Theorem 2.9, which we will discuss elsewhere.

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Added in Proof. Our determinantal description of $M_C(2, K, n)$ is similar to that of the loci of special divisors in Prym varieties, for which Problems 4.10 and 4.11 have already been solved. See:

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