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On Numbers and Endgames: Combinatorial Game Theory in Chess Endgames

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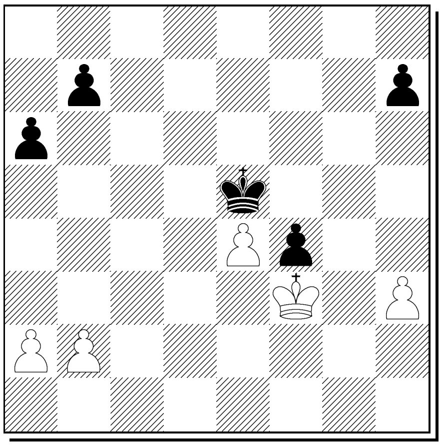
Neither side appears to have any positional advantage in the normal sense. . . . the player with the move is able to arrange the pawn-moves to his own advantage [and win] in each case. It is difficult to say why this should be so, although the option of moving a pawn one or two squares at its first leap is a significant factor.

— *Euwe and Hooper [1960], trying to explain why the position below (Example 87 in their book) should be a first-player win.*

What shall we do with an Up?

— *[Parker and Shaw 1962, 159–160, bass 2]*



ABSTRACT. In an investigation of the applications of CGT to chess, we construct novel mutual Zugzwang positions, explain the pawn endgame above, show positions containing non-integer values (fractions, switches, tinies, and loopy games), and pose open problems concerning the values that may be realized by positions on either standard or nonstandard chessboards.

1. Introduction

It was already noted in *Winning Ways* [Berlekamp et al. 1982, p. 16] that combinatorial game theory (CGT) does not apply directly to chess, because the winner of a chess game is in general not determined by who makes the last move, and indeed a game may be neither won nor lost at all but drawn by infinite play.[1]

Still, CGT has been effectively applied to other games such as Dots-and-Boxes and Go, which are not combinatorial games in the sense of *Winning Ways*. The main difficulty with doing the same for chess is that the 8×8 chessboard is too small to decompose into many independent subgames, or rather that some of the chess pieces are so powerful and influence such a large fraction of the board's area that even a decomposition into two weakly interacting subgames (say a kingside attack and a queenside counteroffensive) generally breaks down in a few moves.

Another problem is that CGT works best with "cold" games, where having the move is a liability or at most an infinitesimal boon, whereas the vast majority of chess positions are "hot": Zugzwang[2] positions (where one side loses or draws but would have done better if allowed to pass the move) are already unusual, and positions of mutual Zugzwang, where neither side has a good or even neutral move, are much rarer.[3]

This too is true of Go, but there the value of being on move, while positive, can be nearly enough constant to be managed by "chilling operators" [Berlekamp and Wolfe 1994], whereas the construction of chess positions with similar behavior seems very difficult, and to date no such position is known.

To find interesting CGT aspects of chess we look to the endgame. With most or all of the long-range pieces no longer on the board, there is enough room for a decomposition into several independent subgames. Also, it is easier to construct mutual Zugzwang positions in the endgame because there are fewer pieces that could make neutral moves; indeed, it is only in the endgame that mutual Zugzwang ever occurs in actual play. If a mutual Zugzwang is sufficiently localized on the chessboard, one may add a configuration of opposing pawns that must eventually block each other, and the first player who has no move on that configuration loses the Zugzwang. Furthermore this configuration may split up as a sum of independent subgames. The possible values of these games are sufficiently varied that one may construct positions that, though perhaps

[1] Of course infinite play does not occur in actual games. Instead the game is drawn when it is apparent that neither side will be able to checkmate against reasonable play. Such a draw is either agreed between both opponents or claimed by one of them using the triple-repetition or the fifty-move rule. These mechanisms together approximate, albeit imperfectly, the principle of draw by infinite play.

[2] This word, literally meaning "compulsion to move" in German, has long been part of the international chess lexicon.

[3] Rare, that is, in practical play; this together with their paradoxical nature is precisely why Zugzwang and mutual Zugzwang are such popular themes in composed chess problems and endgame studies.

not otherwise intractable, illustrate some of the surprising identities from *On Numbers and Games* [Conway 1976]. Occasionally one can even use this theory to illuminate the analysis of a chess endgame occurring in actual play.

We begin by evaluating simple pawn subgames on one file or two adjacent files; this allows us to construct some novel mutual Zugzwang positions and explain the pawn endgame that baffled Euwe. We then show positions containing more exotic values: fractions, switches and tinies, and loopy games. We conclude with specific open problems concerning the values that may be realized by positions either on the 8×8 chessboard or on boards of other sizes.

NOTES. (1) In the vast majority of mutual Zugzwangs occurring in actual play only a half point is at stake: one side to move draws, the other loses. (In tournament chess a win, draw or loss is worth 1, $\frac{1}{2}$, or 0 points, respectively.) We chose to illustrate this article with the more extreme kind of mutual Zugzwang involving the full point: whoever is to move loses. This is mainly because it is easier for the casual player to verify a win than a draw, though as it happens the best example we found in actual play is also a full-point mutual Zugzwang. The CGT part of our analysis applies equally to similar endgames where only half a point hinges on the mutual Zugzwang.

(2) We use "algebraic notation" for chess moves and positions: the ranks of the chessboard are numbered 1 through 8 from bottom to top; the columns ("files") labeled a through h from left to right; and each square is labeled by the file and rank it is on. Thus in the diagram on page 135 the White king is on f3. Pawns, which stay on the same file when not capturing, are named by file alone when this can cause no confusion. Pawn moves are described by the destination square, and other moves by the piece (capital letter, N = knight) followed by the destination square. A capture is indicated with a colon, and the file of departure is used instead of a capital letter in captures by pawns: thus b:a3 means that the b-pawn captures what's on a3. A + indicates a check, and a capital letter after a pawn move indicates a promotion.

(3) All game and subgame values are stated from White's perspective, that is, White is "Left", Black is "Right".

2. Simple Subgames with Simple Values

Integers. Integer values, indicating an advantage in spare tempo moves, are easy to find. (A *tempo move* or *waiting move* is one whose only effect is to give the opponent the turn.) An elementary example is shown in Diagram 1.

The kingside is an instance of the mutual Zugzwang known in the chess literature as the "trébuchet": once either White or Black runs out of pawn moves, he must move his king, losing the g-pawn and the game. Clearly White has one free pawn move on the e-file, and Black has two on the a-file, provided he does

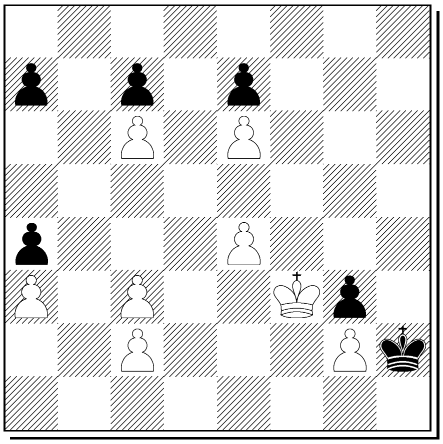


Diagram 1

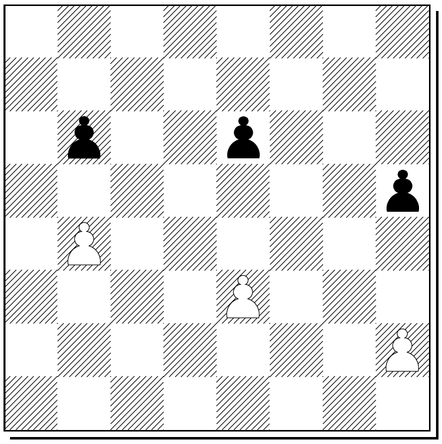


Diagram 2

not rashly push his pawn two squares on the first move. Finally the c-file gives White four free moves (the maximum on a single file), again provided Pc2 moves only one square at a time. Thus the value of Diagram 1 is $1 - 2 + 4 = 3$, and White wins with at least two free moves to spare regardless of who moves first.

Infinitesimals. Simple subgames can also have values that are not numbers, as witness the b- and h-files in Diagram 2. The b-file has value $\{0 \mid 0\} = *$; the same value (indeed an isomorphic game tree) arises if the White pawn is placed on a3 instead of b4. The e-file has value zero, since it is a mutual Zugzwang; this is the identity $\{* \mid *\} = 0$. The h-file, on the other hand, has positive value: White's double-move option gives him the advantage regardless of who has the move. Indeed, since the only Black move produces a $*$ position, while White to move may choose between 0 and $*$, the h-file's value is $\{0, * \mid *\} = \uparrow$. (While $\uparrow$ is usually defined as $\{0 \mid *\}$, White's extra option of moving to $*$ gives him no further advantage; in *Winning Ways* parlance (p. 64), it is *reversible*, and bypassing it gives White no new options. This may be seen on the chessboard by noting that in the position where White has pawns on h2 and f4 and Black has pawns on h5 and f7, Black to move loses even if White is forbidden to play h3 until Black has played h4: 1. ... h4 2. f5, or 1. ... f6 2. f5.)

This accounts for all two-pawn positions with the pawns separated by at most two squares on the same file, or at most three on adjacent files. Putting both pawns on their initial squares of either the same or adjacent files produces a mutual Zugzwang (value zero). This leaves only one two-pawn position to evaluate, represented by the a-file of Diagram 3.

From our analysis thus far we know that the a-file has value $\{0, * \mid \uparrow\}$. Again we bypass the reversible $*$ option, and simplify this value to $\Uparrow * = \{0 \mid \uparrow\}$. Equivalently, Diagram 3 (in which the c-, d- and e-files are $\Downarrow *$, and the kingside is mutual Zugzwang for chess reasons) is a mutual Zugzwang, and remains so if

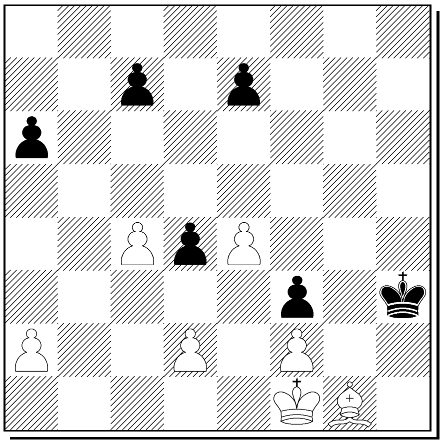


Diagram 3

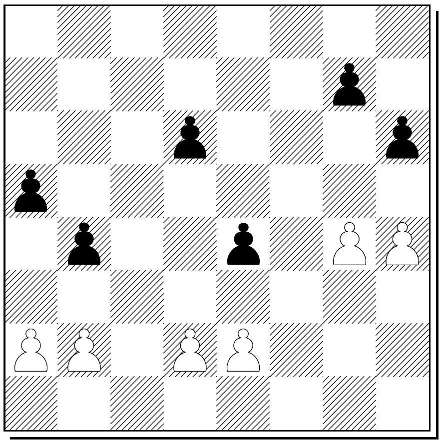


Diagram 4

White is forbidden to play a2-a4 before Black has moved his a6-pawn. This is easily verified: WTM (White to move) 1. c5 a5 and wins by symmetry, or 1. a4 d3 2. a5(c5) e5 or 1. a3 d3 etc.; BTM 1. ... a5 2. c5 wins symmetrically, 1. ... d3 2. c5 e5 (else 3. e5 and 4. a3) 3. c6 a5 4. a4, 1. ... c6 2. e5 and 3. a3, 1. ... c5 2. d3 e5 (e6 3. e5 a5 4. a4) 3. a3 a5 4. a4.

On a longer chessboard we could separate the pawns further. Assuming that a pawn on such a chessboard still advances one square at a time except for an initial option of a double move on its first move, we evaluate such positions thus: a White pawn on a2 against a Black one on a7 has value $\{0,* \mid \Uparrow *\} = \uparrow\uparrow\uparrow$; against a Black Pa8, $\{0,* \mid \uparrow\uparrow\uparrow\} = \uparrow\uparrow\uparrow\uparrow *$; and by induction on n a Black pawn on the $(n+4)$-th rank yields n ups or n ups and a star according to whether n is odd or even, provided the board is at least $n+6$ squares wide so the Black pawn is not on its initial square. With both pawns on their initial squares the file has value zero unless the board has width 5 or 6, when the value is $*$ or $*2$, respectively. Of course if neither pawn is on its starting square the value is 0 or $*$, depending on the parity of the distance between them, as in the b- and e-files of Diagram 2.

Diagram 4 illustrates another family of infinitesimally valued positions. The analysis of such positions is complicated by the possibility of pawn trades that involve entailing moves: an attacked pawn must in general be immediately defended, and a pawn capture parried at once with a recapture. Still we can assign standard CGT values to many positions, including all that we exhibit in diagrams in this article, in which each entailing line of play is dominated by a non-entailing one (*Winning Ways*, p. 64).

Consider first the queenside position in Diagram 4. White to move can choose between 1. b3 (value 0) and 1. a4, which brings about $*$ whether or not Black interpolates the *en passant* trade 1. ... b:a3 2. b:a3. White's remaining choice 1. a3 would produce an inescapably entailing position, but since Black can answer 1. a3 with b:a3 2. b:a3 this choice is dominated by 1. a4 so we may safely ignore it. Black's move a4 produces a mutual Zugzwang, so we have $\{0, * \mid 0\} = \uparrow *$ (*Winning Ways*, p. 68). Our analysis ignored the Black move 1. ... b3?, but 2. a:b3 then produces a position of value 1 (White has the tempo move 3. b4 a:b4 4. b3), so we may disregard this option since $1 > \uparrow *$.

We now know that in the central position of Diagram 4 Black need only consider the move d5, which yields the queenside position of value $\uparrow *$, since after 1. ... e3? 2. d:e3 White has at least a spare tempo. WTM need only consider 1. d4, producing a mutual Zugzwang whether or not Black trades *en passant*, since 1. d3? gives Black the same option and 1. e3?? throws away a spare tempo. Therefore the center position is $\{0 \mid \uparrow *\} = \Uparrow$ (*Winning Ways*, p. 73). Both this and the queenside position turn out to have the same value as they would had the pawns on different files not interacted. This is no longer true if Black's rear pawn is on its starting square: if in the center of Diagram 4 Pd6 is placed on d7, the resulting position is no longer $*$, but mutual Zugzwang (WTM 1. d4 e:d3 2. e:d3 d6; BTM 1. ... d5 2. e3 or d6 2. d4). Shifting the Diagram 4 queenside up one or two squares produces a position (such as the Diagram 4 kingside) of value $\{0 \mid 0\} = *$: either opponent may move to a mutual Zugzwang (1. h5 or 1. ... g6), and neither can do any better, even with Black's double-move option: 1. g5 h5 is again $*$, and 1. ... g5 2. h:g5 h:g5 is equivalent to 1. ... g6, whereas 1. ... h5? is even worse. With the h4-pawn on h3, though, the double-move option becomes crucial, giving Black an advantage (value $\{* \mid 0, *\} = \downarrow$, using the previous analysis to evaluate 1. h4 and 1. ... g6 as $*$).

3. Schweda versus Sika, Brno, 1929

We are now ready to tackle a nontrivial example from actual play—specifically, the subject of our opening quote from [Euwe and Hooper 1960]. We repeat it in Diagram 5 for convenience.

On the e- and f-files the kings and two pawns are locked in a vertical trébuchet; whoever is forced to move there first will lose a pawn, which is known to be decisive in such an endgame. Thus we can ignore the central chunk and regard the rest as a last-mover-wins pawn game.

As noted in the introduction, the 8×8 chessboard is small enough that a competent player can play such positions correctly even without knowing the mathematical theory. Indeed the White player correctly evaluated this as a win when deciding earlier to play for this position, and proceeded to demonstrate this win over the board. Euwe and Hooper [1960, p. 56] also show that Black would win if he had the move from the diagram, but they have a hard time explaining why such a position should a first-player win—this even though Euwe held both the world chess championship from 1935 to 1937 and a doctorate in

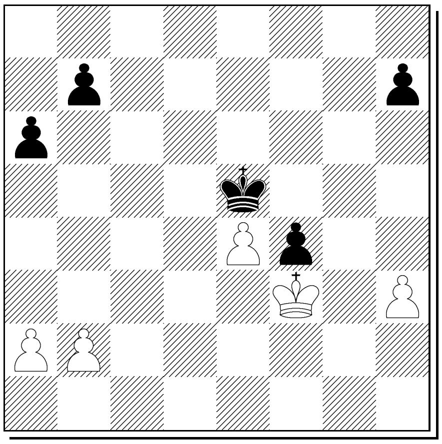


Diagram 5

mathematics [Hooper and Whyld 1992]. (Of course he did not have the benefit of CGT, which had yet to be developed.)

Combinatorial game theory tells us what to do: decompose the position into subgames, compute the value of each subgame, and compare the sum of the values with zero. The central chunk has value zero, being a mutual Zugzwang. The h-file we recognize as $\Downarrow *$. The queenside is more complicated, with a game tree containing hot positions (1. a4 would produce $\{2 \mid 0\}$) and entailing moves (such as after 1. a4 b5); but again it turns out that these are all dominated, and we compute that the queenside simplifies to $\uparrow$. Thus the total value of the position is $\uparrow + \Downarrow * = \downarrow *$. Since this is confused with zero, the diagram is indeed a first-player win. To identify the queenside value as $\uparrow$ we show that White's move 1. h4, converting the kingside to $\downarrow$, produces a mutual Zugzwang, using values obtained in the discussion of Diagram 4 to simplify the queenside computations. For instance, 1. h4 a5 2. h5 a4 3. h6 b6 4. b4 wins, but with WTM again after 1. h4, Black wins after 2. a4 a5, 2. a3 h5, 2. b4 h5 (mutual Zugzwang) 3. a3/a4 b5/b6, 2. b3 a5 (mutual Zugzwang $+ \downarrow$), or 2. h5 a5 and 3. h6 a4 or 3. a4(b3) h6. Black to move from Diagram 7 wins with 1. ... a5, reaching mutual Zugzwang after 2. h4 a4 3. h5 h6 or 2. a4(b3) h6, even without using the ... h5 double-move option (since without it the h-file is $*$ and $\uparrow + * = \uparrow *$ is still confused with zero).

4. More Complicated Values: Fractions, Switches and Tinies

Fractions. Fractional values are harder to come by; Diagram 6 shows two components with value $\frac{1}{2}$. In the queenside component the c2 pawn is needed to assure that Black can never safely play b4; a pawn on d2 would serve the same

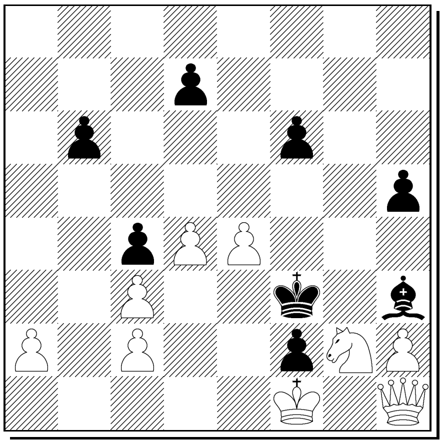


Diagram 6

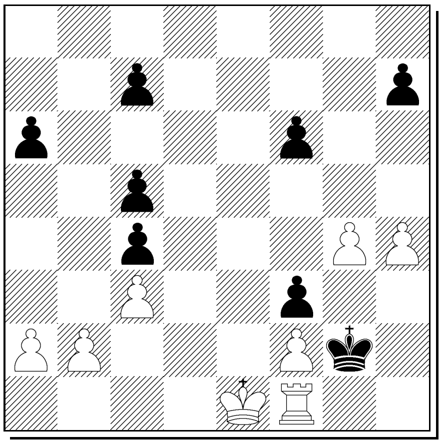


Diagram 7

purpose. In the configuration d4, e4/d7, f6 it is essential that White's e5 forces a pawn trade, i.e., that in the position resulting from 1. e5 f5? 2. d5 White wins, either because the position after mutual promotions favors White or because (as in Diagram 6) the f-pawn is blocked further down the board. Each of these components has the form $\{0, * \mid 1\}$, but (as happened in Diagrams 2 and 3) White's $*$ option gives him no further advantage, and so each component's value simplifies to $\{0 \mid 1\} = \frac{1}{2}$. Since the seven-piece tangle occupying the bottom right corner of Diagram 6 not only blocks the f6-pawn but also constitutes a (rather ostentatious) mutual Zugzwang, and Black's h-pawn provides him a free move, the entire Diagram is itself a mutual Zugzwang illustrating the identity $\frac{1}{2} + \frac{1}{2} - 1 = 0$.

What do we make of Diagram 7, then? Chess theory recognizes the five-man configuration around f2 as a mutual Zugzwang (the critical variation is WTM 1. Rh1 K:h1 2. Kd2 Kg1! 3. Ke3 Kg2, and Black wins the trébuchet; this mutual Zugzwang is akin to the kingside mutual Zugzwang of Diagram 3, but there the d-pawns simplified the analysis). Thus we need to evaluate the three pure-pawn subgames, of which two are familiar: the spare tempo-move of Pc7, and the equivalent of half a spare tempo-move White gets from the upper kingside.

To analyze the queenside position (excluding Pc7), we first consider that position after Black's only move a5. From that position Black can only play a4 (value 1), while White can choose between a4 and a3 (values 0 and $*$), but not 1. b3? c:b3 2. a:b3 c4! and the a-pawn promotes. Thus we find once more the value $\{0, * \mid 1\} = \frac{1}{2}$. Returning to the Diagram 7 queenside, we now know the value $\frac{1}{2}$ after Black's only move a5. White's moves a3 and a4 produce 0 and $*$, and 1. b3 can be ignored because the reply c:b3 2. a:b3 c4 3. b4 shows that this is no better than 1. a3. So we evaluate the queenside of Diagram 7 as $\{0, * \mid \frac{1}{2}\} = \frac{1}{4}$, our first quarter.

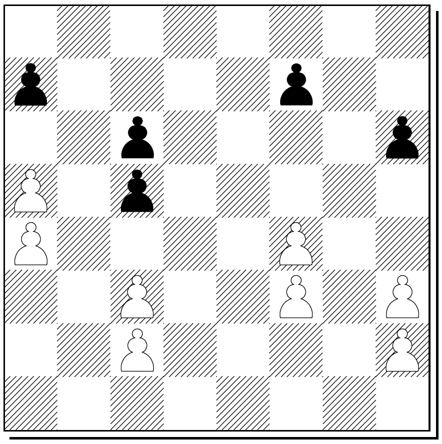


Diagram 8

Thus the whole of Diagram 7 has the negative value $\frac{1}{2}-1+\frac{1}{4}=-\frac{1}{4}$, indicating a Black win regardless of who has the move, though with BTM the only play is 1. ... a5! producing a $\frac{1}{2}+\frac{1}{2}-1$ mutual Zugzwang.

On a longer chessboard we could obtain yet smaller dyadic fractions by moving the b-pawn of Diagram 6 or the Black a-pawn of Diagram 7 further back, as long as this does not put the pawn on its initial square. Each step back halves the value. These constructions yield fractions as small as $1/2^{N-7}$ and $1/2^{N-6}$, respectively, on a board with columns of length $N \geq 8$.

Switches and tinies. We have seen some switches (games $\{m \mid n\}$ with $m > n$) already in our analysis of four-pawn subgames on two files such as occur in Diagrams 4 and 5. We next illustrate a simpler family of switches.

In the a-file of Diagram 8, each side has only the move a6. If Black plays a6 the pawns are blocked, while White gains a tempo move with a6 (compare with the e-file of Diagram 1), so the a-file has value $\{1 \mid 0\}$. On the c-file whoever plays c4 gets a tempo move, so that file gives $\{1 \mid -1\} = \pm 1$. Adding a Black pawn on c7 would produce $\{1 \mid -2\}$; in general, on a board with files of length N, we could get temperatures as high as $(N-5)/2$ by packing as many as $N-3$ pawns on a single file in such a configuration.[1]

The f-file is somewhat more complicated: White's f5 produces the switch $\{2 \mid 1\}$, while Black has a choice between f6 and f5, which yield $\{1 \mid 0\}$ and 0. Bypassing the former option we find that the f-file shows the three-stop game $\{2 \mid 1 \parallel 0\}$. Likewise $\{4 \mid 2 \parallel 0\}$ can be obtained by adding a White pawn on f2, and on a longer board $n+1$ pawns would produce $\{2n \mid n \parallel 0\}$. The h-file shows

[1]For large enough N, it will be impossible to pack that many pawns on a file starting from an initial position such as that of 8×8 chess, because it takes at least $\frac{1}{4}n^2 + O(n)$ captures to get n pawns of the same color on a single file. At any rate one can attain temperatures growing as some multiple of $\sqrt{N}$.

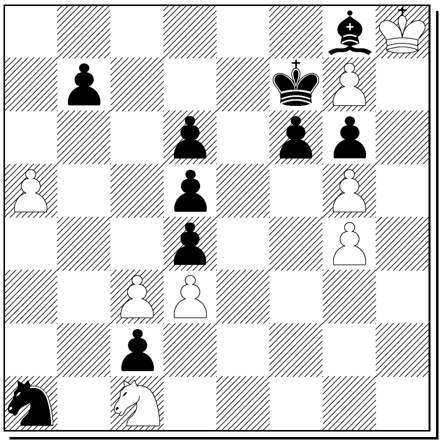


Diagram 9

the same position shifted down one square, with Black no longer able to reach 0 in one step. That file thus has value $\{2 \mid 1 \parallel 1 \mid 0\}$, which simplifies to the number 1, as may be seen either from the CGT formalism or by calculating directly that the addition of a subgame of value -1 to the h-file produces mutual Zugzwang.

Building on this we may construct a few tinies and minies, albeit in more contrived-looking positions than we have seen thus far (though surely no less natural than the positions used in [Fraenkel and Lichtenstein 1981]). Consider Diagram 9. In the queenside (apart from the pawns on a5 and b7, which I put there only to forestall a White defense based on stalemate) the Black pawn on c2 and both knights cannot or dare not move; they serve only to block Black from promoting after ... d:c3. That is Black's only move, and it produces the switch $\{0 \mid -1\}$ as in the a-file of Diagram 8. White's only move is 1. c:d4 (1. c4? d:c4 2. d:c4 d3 3. N:d3 Nb3, or even 2. ... Nb3 3. N:b3 d3), which yields mutual Zugzwang. Thus the queenside evaluates to $\{0 \parallel 0 \mid -1\}$, or tiny-one. Adding a fourth Black d-pawn on d7 would produce tiny-two, and on larger boards we could add more pawns to get tiny-n for arbitrarily large n. In the kingside of Diagram 9 the same pawn-capture mechanism relies on a different configuration of mutually paralyzing pieces, including both kings. With a White pawn on g3 the kingside would thus be essentially the same as the queenside with colors reversed, with value miny-one; but since White lacks that pawn the kingside value is miny-zero, or $\downarrow$ (*Winning Ways*, p. 124). Black therefore wins Diagram 9 regardless of whose turn it is, since his kingside advantage outweighs White's queenside edge.

Some loopy chunks. Since pawns only move in one direction, any subgame in which only pawns are mobile must terminate in a bounded number of moves. Subgames with other mobile pieces may be unbounded, or *loopy* in *Winning Ways*

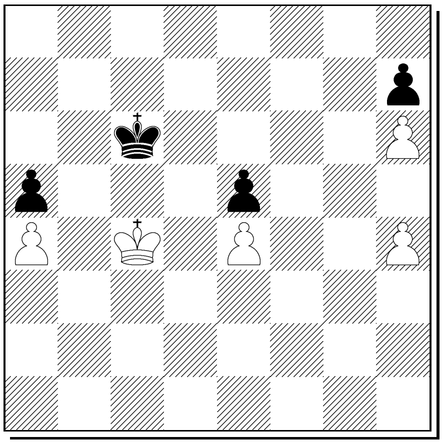


Diagram 10

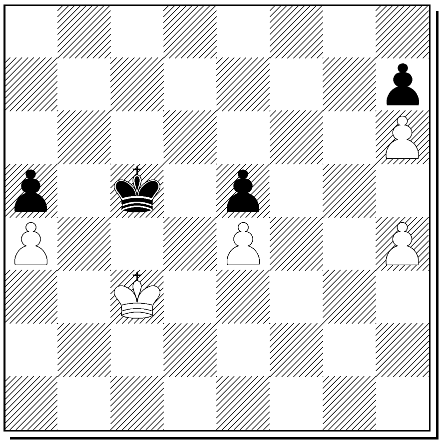


Diagram 10′

terminology (p. 314); indeed, unbounded games must have closed cycles (loops) of legal moves because there are only finitely many distinct chess positions.

Consider for instance the queenside of Diagram 10, where only the kings may move. Black has no reasonable options since any move loses at once to Kb5 or Kd5; thus the queenside's value is at least zero. White could play Kc3, but Black responds Kc5 at once, producing Diagram 10′ with value ≤ 0 because then Black penetrates decisively at b4 or d4 if the White king budges. Thus Kc3 can never be a good move from Diagram 10. White can also play Kb3 or Kd3, though. Black can then respond Kc5, forcing Kc3 producing Diagram 10′. In fact Black might as well do this at once: any other move lets White at least repeat the position with 2. Kc4 Kc6 3. Kb3(d3), and White has no reasonable moves at all from Diagram 10′ so we need not worry about White moves after Kb3(d3). We may thus regard 1. Kb3(d3) Kc5 2. Kc3 as a single move that is White's only option from the Diagram 10 queenside.

By the same argument we regard the Diagram 10′ queenside as a game where White has no moves and Black has only the "move" 1. ... Kb6(d6) 2. Kc4 Kc6, recovering the queenside of Diagram 10. We thus see that these queenside positions are equivalent to the loopy games called **tis** and **tisn** in *Winning Ways*, p. 322 (**istoo** and **isnot** in American English). Since the kingside has value 1, Diagram 10 ($1 + \textbf{tis}$) is won for White, as is Diagram 10′ ($1 + \textbf{tisn} = \textbf{tis}$) with BTM, but with WTM Diagram 10′ is drawn after 1. h5 Kb6(d6) 2. Kc4 Kc6 3. Kb3(d3) Kc5 4. Kc3 etc.

We draw our final examples from the Three Pawns Problem (Diagram 11). See [Hooper and Whyld 1992] for the long history of this position, which was finally solved by Szén around 1836; Staunton [1847] devoted twelve pages (487–500) to its analysis. (Thanks to Jurg Nievergelt for bringing this Staunton reference to my attention.) Each king battles the opposing three pawns. Three pawns on

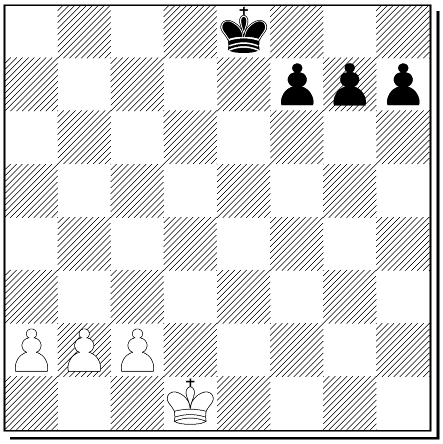


Diagram 11

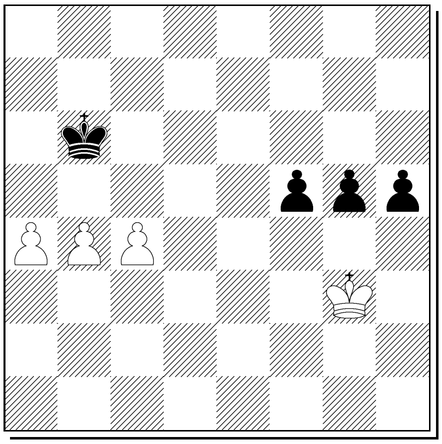


Diagram 12

adjacent files can contain a king but (unless very far advanced) not defeat it. Eventually Zugzwang ensues, and one player must either let the opposing pawns through, or push his own pawns when they can be captured. As with our earlier analysis, we allow only moves that do not lose a pawn or unleash the opposing pawns; thus the last player to make such a move wins. The Three Pawns Problem then in effect splits into two equal and opposite subgames. One might think that this must be a mutual Zugzwang, but in fact Diagram 11 is a first-player win. Diagram 12 shows a crucial point in the analysis, which again is a first-player win despite the symmetry. The reason is that each player has a check (White's a5 or c5, Black's f4 or h4) that entails an immediate king move: Black is not allowed to answer White's 1. a5+ with the Tweedledum move (*Winning Ways*, p. 4) 1. ... h4+, and so must commit his king before White must answer the pawn check. This turns out to be sufficient to make the difference between a win and a loss in Diagrams 11 and 12.

Diagram 13 is a classic endgame study by J. Behting using this material ([Sutherland and Lommer 1938, #61]; originally published in *Deutsche Schachzeitung* 1929). After 1. Kg1! the kingside shows an important mutual Zugzwang: BTM loses all three pawns after 1. ... g3 2. Kg2 or 1. ... f3/h3 2. Kf2/h2 h3/f3 3. Kg3, while WTM loses after 1. Kh2(f1) h3, 1. Kf2(h1) f3, or 1. Kg2 g3, when at least one Black pawn safely promotes to a queen. On other king moves from Diagram 13 Black wins: 1. Kf2(f1) h3 or 1. Kh2(h1) f3 followed by ... g3 and White can no longer hold the pawns, e.g., 1. Kh1 f3 2. Kh2 g3+ 3. Kh3 f2 4. Kg2 h3+ 5. Kf1 h2. Thus we may regard Kg1 as White's only kingside option in Diagram 13. Black can play either ... g3 reaching mutual Zugzwang, or ... f3/h3+ entailing Kf2/h2 and again mutual Zugzwang but BTM; in effect Black can interpret the kingside as either 0 or $*$. In the queenside, White to move can only play a6 reaching mutual Zugzwang. Black to move plays 1. ... Ka7 or

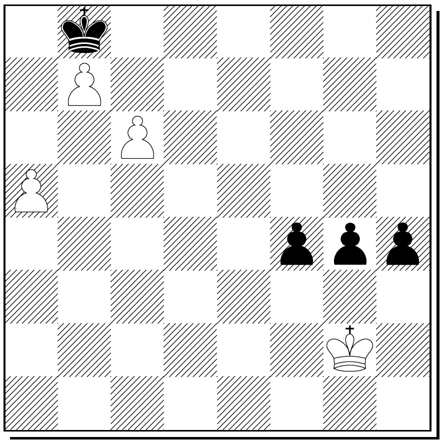


Diagram 13

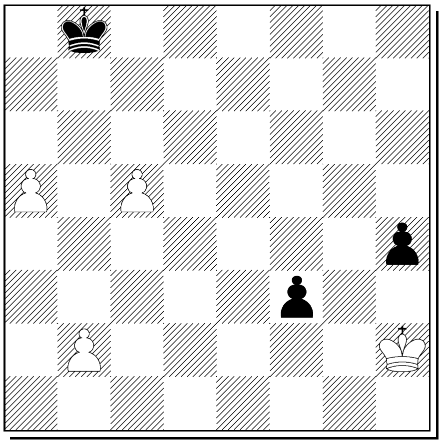


Diagram 14

Kc7, when 2. a6 Kb8 is mutual Zugzwang; but the position after 1. ... Ka7(c7) is not itself a mutual Zugzwang because White to move can improve on 2. a6 with the sacrifice 2. b8Q+! K:b8 3. a6, reaching mutual Zugzwang with BTM. The positions with the Black king on b8 or a7(c7) are then seen to be equivalent: Black can move from one to the other and White can move from either to mutual Zugzwang. Thus the Diagram 13 queenside is tantamount to the loopy game whose infinitesimal but positive value is called **over** $=$ **1/on** in *Winning Ways*, p. 317. White wins Behting's study with 1. Kg1! Ka7(c7) 2. b8Q+! K:b8 3. a6 reaching mutual Zugzwang; all other alternatives (except 2. Kg2 Kb8 repeating the position) lose: 1. a6? g3!, or 2. a6? Kb8. Since **over** exceeds $*$ as well as 0, White wins Diagram 13 even if Black moves first: 1. ... Ka7 2. Kg1! Kb8 3. a6, 1. ... g3 2. a6, or 1. ... f3/h3+ 2. Kf2/h2 Ka7 3. b8Q+ K:b8 4. a6 etc.

The mutual Zugzwang in the analysis of the Diagram 13 queenside after 2. b8Q+ K:b8 3. a6 is the only mutual Zugzwang involving a king and only two pawns. In other positions with a king in front of two pawns either on adjacent files or separated by one file, the king may not be able to capture the pawns, but will at least have an infinite supply of tempo moves. Thus such a position will have value **on** or **off** (*Winning Ways*, p. 317 ff.) according to whether White or Black has the king. For instance, in the kingside of Diagram 14, White must not capture on h4 because then the f-pawn promotes, but the king can shuttle endlessly between h2 and h3 while Black may not move (1. ... f2? 2. Kg2 h3+ 3. K:f2! and the h-pawn falls next). If White didn't have the pawn on b2, the queenside would likewise provide Black infinitely many tempo moves and the entire Diagram would be a draw with value **on** + **off** = **dud** (*Winning Ways*, p. 318). As it is, White naturally wins Diagram 14, since Black will soon run out of queenside moves. We can still ask for the value of the queenside; it turns out to give another realization of **over**. Indeed, the Black king can only shuttle between b7 and b8, since moving to the a- or c-file loses to c6 or a6, respectively; and until the b-pawn reaches b4 White may not move his other pawns since c6/a6 drops a pawn to Kc7/a7. We know from Diagram 13 that if the b-pawn were on b5 the Diagram 14 queenside would be mutual Zugzwang. The same is true with that pawn on b4 and the Black king on b7: WTM 1. b5 Kb8, BTM 1. ... Kb8 2. b5 or 2. c6/a6 Kc7/Ka7 3. b5. Thus pawn on b4 and king on b8 give $*$, as do Pb3/Kb7, while Pb3/Kb8 is again mutual Zugzwang. From b2 the pawn can move to mutual Zugzwang against either Kb7 or Kb8 (moving to $*$ is always worse, as in the Diagram 13 queenside), yielding a value of **over** as claimed. Positions such as this one, which show an advantage of **over** thanks to the double-move option, are again known to chess theory; see for instance [Sutherland and Lommer 1938, endgame #55] by H. Rinck (originally published in *Deutsche Schachzeitung*, 1913), which uses a different pawn trio. Usually, as in that Rinck endgame, the position is designed so that White can only win by moving a pawn to the fourth rank in two steps instead of one.

5. Open Problems

We have seen that pawn endgames can illustrate some of the fundamental ideas of combinatorial game theory in the familiar framework of chess. How much of CGT can be found in such endgames, either on the 8×8 or on larger boards? Of course one could ask for each game value in *Winning Ways* whether it can be shown on a chessboard. But it appears more fruitful to focus on attaining specific values in endgame positions. I offer the following challenges:

Nimbers. Do $*2$, $*4$ and higher nimbers occur on the 8×8 or larger boards? We have seen already that on a file of length 6 the position Pa2 vs. Pa5 gives $*2$, and Pa2 vs. Pb6 does the same for files of length 7. But these constructions extend neither to longer boards nor to nimbers beyond $*2$ and $*3 = * + *2$.

Positive infinitesimals. We have seen how to construct tiny-x for integers $x \geq 0$ (Diagram 9). How about other x such as $\frac{1}{2}$ or $1\uparrow$? Also, do the higher ups $\uparrow^2$, $\uparrow^3$, etc. of *Winning Ways* (pp. 277 and 321) occur?

Fractions. We can construct arbitrary dyadic fractions on a sufficiently large chessboard. Does $\frac{1}{8}$ exist on the standard 8×8 board? Can positions with value $\frac{1}{3}$ or other non-dyadic rationals arise in loopy chess positions? (Thirds do arise as mean values in Go [Berlekamp and Wolfe 1994; Gale 1994], thanks to the Ko rule.)

Chilled chess. Is there a class of chess positions that naturally yields to chilling operators as do the Go endgames of [Berlekamp and Wolfe 1994]?

In other directions, one might also hope for a more systematic CGT-style treatment of *en passant* captures and entailing chess moves such as checks, captures entailing recapture, and threats to capture; and ask for a class of positions on an $N \times N$ board that bears on the computational complexity of pawn endgames as [Fraenkel and Lichtenstein 1981] does for unrestricted $N \times N$ chess positions.

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This paper would never have been written without the prodding, assistance and encouragement of Elwyn Berlekamp. In the Fall of 1992 he gave a memorable expository talk on combinatorial game theory, during which he mentioned that at the time CGT was not known to have anything to do with chess. I wrote a precursor of this paper shortly thereafter in response to that implied (or at least perceived) challenge. The rest of the material was mostly developed in preparation for or during the MSRI workshop on combinatorial games, which was largely Berlekamp's creation. I am also grateful for his comments on the first draft of this paper, particularly concerning [Berlekamp and Wolfe 1994].

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