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Sowing Games

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Abstract. At the Workshop, John Conway and Richard Guy proposed the class of "sowing games", loosely based on the ancient African games Mancala and Wari, as an object of study in combinatorial game theory. This paper presents an initial investigation into two simple sowing games, Sowing and Atomic Wari.

1. Introduction

Most well-studied combinatorial games can be classified into a few broad classes.

Taking and Breaking: Games played with piles of chips, in which the basic move is to take some chips and/or split some piles. They include Nim, Kayles, Dawson's Chess, other octal and hexadecimal games. Higher-dimensional variants include Maundy Cake, Cutcake, Eatcake, and Chomp.

Cutting and Coloring: Games played on (colored) graphs, in which the basic move is to cut out a small piece of the graph of the appropriate color, possibly changing other nearby pieces. They include Hackenbush, Col, Snort, Domineering, and Dots and Boxes.

Sliding and Jumping: Games played with tokens on a grid-like board, in which the basic move is to move a token to a nearby spot, possibly by jumping over opponent's pieces, which may then be (re)moved. Examples include Ski-Jumps, Toads and Frogs, Checkers, and Konane.

At the Workshop, John Conway and Richard Guy suggested that studying entirely new classes of games, fundamentally different from all of these, might lead to a more thorough understanding of combinatorial game theory, and proposed a class of games loosely based on the African games Mancala and Wari, which they called "sowing games". These games are played with a row of pots, each containing some number of seeds. The basic move consists of taking all the seeds from one pot and "sowing" them one at a time into succeeding pots. In this paper, we offer a few introductory results on two simple sowing games.

Almost all the values given in this paper were derived with the help of David Wolfe's `games` package [Wolfe 1996].

2. Sowing

The first game in this family, simply called *Sowing*, was invented by John Conway. The basic setup is a row of pots, each containing some number of uncolored seeds. A legal move by Left consists of taking all the seeds out of any pot and putting them in successive pots to the right, subject to the restriction that *the last seed cannot go into an empty pot*. Right's moves are defined symmetrically.

We represent a Sowing position by a string of boldface digits, where each digit represents the number of seeds in a pot. For example, from the position **312**, Left's only legal move is to move the single seed in the second pot into the third pot, leaving the position **303**. Right has two legal moves, to **402** and **420**. Thus, this position can be evaluated as follows.

$$\mathbf{312} = \{\mathbf{303} \,|\, \mathbf{402}, \mathbf{420}\} = \{0 \,|\, \{\,|\,\mathbf{510}\}, 0\} = \{0 \,|\, \{\,|\,\{\,|\,\mathbf{600}\}\}, 0\} = \{0 \,|\, {-2}\}$$

There is a natural impartial version of this game as well, in which either player can move in either direction. For example:

$$\mathbf{312} = \{\mathbf{303}, \mathbf{402}, \mathbf{420}\} = \{0, \{\mathbf{510}\}, 0\} = \{0, \{\{\mathbf{600}\}\}, 0\} = *$$

Simplifying Positions. There are some obvious ways of simplifying a Sowing position. First, if the first or last pot is empty, we can just ignore it, since it will always be empty.

We will call a pot *full* if the number of seeds is greater than the distance to either the first or last (nonempty) pot. Since neither player can ever move from a full pot, the exact number of seeds it contains is unimportant. Unlike empty pots, however, we can't simply ignore full pots at the ends, since it is still possible to end a turn by dropping a seed into a full pot. In this paper, we represent full pots with the symbol $\bullet$. Thus, for example, $\mathbf{110451000} = \mathbf{110451} = \mathbf{110{\bullet}{\bullet}1}$.

Finally, sometimes Sowing positions can be split into independent components. For example, we can write $\mathbf{1200021} = \mathbf{12}+\mathbf{21}$ and $\mathbf{110{\bullet}{\bullet}1} = \mathbf{110{\bullet}}+\mathbf{{\bullet}1}$. Unfortunately, while in many cases, it is easy to detect such splits by hand, we do not know of a general method that always finds a split whenever one is possible. Clearly, any two positions separated by sufficiently many empty or full pots should be considered independent, but we don't know how many "sufficiently many" is!

The Towers of Hanoi Go to Africa: Sowing is Hard. Suppose we start with n pots, each with one seed, and we want to move all the seeds into the last pot. We can use the following algorithm to accomplish this task.

(i) Recursively move to the position $(\boldsymbol{n-1})\mathbf{0}^{n-2}\mathbf{1}$.
(ii) Sow the contents of the first pot, giving the position $\mathbf{1}^{n-2}\mathbf{2}$.

(iii) Recursively move to the position $\boldsymbol{n}$, by pretending the last pot contains only one seed.

The reader should immediately recognize the recursive algorithm for solving the Towers of Hanoi! See p. 753 of *Winning Ways*. Since this algorithm requires $2^{n-1} - 1$ moves to complete, we conclude that the position $\mathbf{1}^n$ has at least $2^{n-1} - 1$ distinct followers. This immediately implies the following theorem.

Theorem 2.1. *Evaluating Sowing positions* (*by recursively evaluating all their followers*) *requires exponential time in the worst case.*

Despite this result, it is possible that a subexponential algorithm exists for determining the value of a Sowing position, by exploiting higher-level patterns. But this seems quite unlikely.

The algorithm we have just described is not the fastest way to get all the seeds into one pot. Consider the following alternate algorithm, which uses only polynomially many steps.

- If $n = 2m$:
  (i) Move recursively to $\boldsymbol{m}\mathbf{0}^{m-1}\boldsymbol{m}$.
  (ii) Sow the first pot to $\mathbf{1}^{m-1}(\boldsymbol{m}+\mathbf{1})$.
  (iii) Move recursively to $(\mathbf{2}\boldsymbol{m})$.
- If $n = 2m + 1$:
  (i) Move recursively to $\boldsymbol{m}\mathbf{1}\mathbf{0}^{m-1}\boldsymbol{m}$.
  (ii) Move to $(\boldsymbol{m}+\mathbf{1})\mathbf{0}^{m}\boldsymbol{m}$.
  (iii) Sow the first pot to $\mathbf{1}^{m}(\boldsymbol{m}+\mathbf{1})$.
  (iv) Move recursively to $(\mathbf{2}\boldsymbol{m}+\mathbf{1})$.

The number of moves $T(n)$ used by this algorithm obeys the following recurrence:

$$\begin{aligned} T(1) &= 0, \\ T(2m) &= 3T(m) + 1, \\ T(2m+1) &= 2T(m) + T(m+1) + 2. \end{aligned}$$

Asymptotically, $T(n) = O(n^{\log_2 3}) = O(n^{1.5850})$. Restricting our attention to powers of two, we get the exact expression $T(2^k) = \frac{1}{2}(3^k - 1)$.

Some Sowing Patterns. Very little can be said about the types of values that Sowing positions can take. The only general pattern we have found so far is that there are Sowing positions whose values are arbitrary integers and switches with arbitrarily high temperatures.

Theorem 2.2. $(\mathbf{10})^m\mathbf{03}(\mathbf{01})^n = 0$ *for all* m *and* n.

Proof. If Left goes first, she loses immediately, since she has no legal moves. If Right goes first, his only legal move is to the position $(\mathbf{10})^{m-1}\mathbf{211}(\mathbf{01})^n$, from which Left can move to $(\mathbf{10})^{m-1}\mathbf{0220}(\mathbf{01})^n$, which has value zero since there are no more legal moves. Thus, the second player always wins. □

THEOREM 2.3. $(\mathbf{01})^m\mathbf{2}(\mathbf{01})^n = n{+}1$, *for all m and n except $m = n = 0$.*

PROOF. Right has no legal moves. If $n = 0$, Left has only one legal move, to $(\mathbf{10})^{m-1}\mathbf{03} = 0$ by the previous theorem. Otherwise, Left has exactly two legal moves, to $(\mathbf{01})^{m+1}\mathbf{2}(\mathbf{01})^{n-1} = n$ by induction, and to $(\mathbf{10})^{m-1}\mathbf{3}(\mathbf{01})^n = 0$, which is a terminal position. □

THEOREM 2.4. $\mathbf{11}(\mathbf{01})^n = \{n{+}1\,|\,0\}$ *for all positive n.*

PROOF. Left has only one move, to $\mathbf{2}(\mathbf{01})^n = n + 1$ by the previous theorem. Right has only one move, to the terminal position $\mathbf{20}(\mathbf{01})^n$. □

THEOREM 2.5. $(\mathbf{10})^m\mathbf{2}(\mathbf{01})^n = \{n\,|\,{-m}\}$ *for all positive m and n.*

PROOF. Left has only one move, to $(\mathbf{10})^m\mathbf{012}(\mathbf{01})^{n-1}$, which, by a slight generalization of Theorem 3, has value n. Similarly, Right has only one move, to $(\mathbf{10})^{m-1}\mathbf{210}(\mathbf{01})^n = m$. □

We have seen Sowing positions whose values are fractions, ups, tinies, higher-order switches, and even some larger Nim-heaps, but no other general patterns are known. Table 1 lists a few interesting values. Values for some "starting" positions, in which all pots have the same number of seeds, are listed in Table 2.

Even less is known about the impartial version. Table 3 lists the "smallest" known positions with values 0 through $*9$. Table 4 lists values for some impartial starting positions.

Open Questions. We close this section with a few open questions. For what values of n do Sowing positions exist with values 2^{-n}, and if they all exist, can we systematically construct them? What about $n \cdot \uparrow$? $+_n$? $*n$?

Is there a simple algorithm that splits Sowing positions into multiple independent components? Are there any other high-level simplification rules that would allow faster evaluation?

3. Atomic Wari

"Is it ... *atomic*?"
"Yes! *Very* atomic!"
—*The 5000 Fingers of Dr. T*

The second sowing game we consider, called *Atomic Wari*, is my invention. Atomic Wari is loosely based on a different family of African games, variously called wari or oware. The board is the same as in Sowing, but the moves are different. A legal move consists of taking all the seeds from one pot, and sowing them to the left or right, *starting with the original pot.* As in Sowing, Left moves seeds to the right; Right moves them to the left. To avoid trivial infinite play, it is illegal to start a move at a pot that contains only one seed. At the end

$\mathbf{211} = \frac{1}{2}$ $\mathbf{12202} = \frac{1}{4}$ $\mathbf{122011} = \frac{1}{8}$ $\mathbf{2121202} = \frac{1}{16}$	$\mathbf{2121} = 1 \Vert {*} \vert {-1}$ $\mathbf{41122} = 2 \vert 0 \Vert {-2} \vert {-4}$
$\mathbf{2202} = \uparrow$ $\mathbf{31011} = \uparrow{*}$ $\mathbf{201321} = \Uparrow$ $\mathbf{22011} = \Uparrow{*}$ $\mathbf{122112} = \Uparrow\!\uparrow$	$\bullet\mathbf{2011} = +_1$ $\bullet\mathbf{013} = +_2$ $\bullet\mathbf{0114} = +_3$ $\mathbf{332011} = +_{1/2}$
	$\mathbf{11} = {*}$ $\mathbf{31}\bullet\mathbf{13} = {*2}$ $\mathbf{313005} = \uparrow{*3} = {*2} \vert 0$

Table 1. Some interesting partisan Sowing values

of a move, if the last pot in which a seed was dropped contains either two or three seeds, those seeds are *captured*, that is, removed from the game. Multiple captures are possible: after any capture, if the previous pot has two or three seeds, they are also captured. The game ends when there are no more legal moves, or equivalently, when no pot contains more than one seed. As usual, the first player who is unable to move loses.

For example, consider the position **312**. Left can sow the contents of the first pot, then capture the contents of the other two pots, leaving the position $\mathbf{100} = \mathbf{1}$. Left can also sow the contents of the rightmost pot, leaving the position **3111**. Right can move to either **301** or **11112**. Thus, the Atomic Wari position **312** has the following value:

$$\begin{aligned}\mathbf{312} &= \{\mathbf{1}, \mathbf{3111} \,|\, \mathbf{301}, \mathbf{11112}\}\\ &= \{0, \{\mathbf{1001} \,|\, \mathbf{111111}\} \,|\, \{\mathbf{11} \,|\, \mathbf{11101}\}, \{\mathbf{111111} \,|\, \mathbf{11101}\}\}\\ &= \{0, {*} \,|\, {*}, {*}\}\\ &= \uparrow\end{aligned}$$

Clearly, for every move by Left, there is a corresponding move by Right. Thus, Atomic Wari is an "all small" game, in the terminology of *Winning Ways*. All Atomic Wari positions have infinitesimal values, and in the presence of remote stars, correct play in a collection of Atomic Wari positions is completely determined by the position's atomic weights. (Hence, the name.)

There is also a naturally defined impartial version of the game. For example,

$$\begin{aligned}\mathbf{312} &= \{\mathbf{1}, \mathbf{3111}, \mathbf{301}, \mathbf{11112}\}\\ &= \{0, \{\mathbf{1001}, \mathbf{111111}\}, \{\mathbf{11}, \mathbf{11101}\}, \{\mathbf{111111}, \mathbf{11101}\}\}\\ &= \{0, {*}, {*}, {*}\}\\ &= {*2}\end{aligned}$$

$\mathbf{1}^2 = *$	$\bullet\mathbf{1}^1\bullet = *$
$\mathbf{1}^3 = 0$	$\bullet\mathbf{1}^2\bullet = 0$
$\mathbf{1}^4 = \pm\frac{1}{2}$	$\bullet\mathbf{1}^3\bullet = 0$
$\mathbf{1}^5 = 0$	$\bullet\mathbf{1}^4\bullet = \pm\big(0, \{1\,|\,0\}\big)$
$\mathbf{1}^6 = 0$	$\bullet\mathbf{1}^5\bullet = 0$
$\mathbf{1}^7 = *2$	$\bullet\mathbf{1}^6\bullet = *$
$\mathbf{1}^8 = *$	$\bullet\mathbf{1}^7\bullet = \pm\big(\{2\,|\,1\,\|\,{+_1}, {+_{3\|1}}\}, \{1\,|\,\{3\,|\,1\,\|\,0\,\|\|\,{-1}\}\}\big)$
$\mathbf{1}^9 = 0$	$\bullet\mathbf{1}^8\bullet = 0$

${\leftarrow}\!\bullet\mathbf{1}^1\bullet\!{\rightarrow} = *$

${\leftarrow}\!\bullet\mathbf{1}^2\bullet\!{\rightarrow} = 0$

${\leftarrow}\!\bullet\mathbf{1}^3\bullet\!{\rightarrow} = 0$

${\leftarrow}\!\bullet\mathbf{1}^4\bullet\!{\rightarrow} = 0$

${\leftarrow}\!\bullet\mathbf{1}^5\bullet\!{\rightarrow} = \pm\big(4*\,|\,\frac{1}{2}, \{\frac{7}{2}\,|\,0\}\big)$

${\leftarrow}\!\bullet\mathbf{1}^6\bullet\!{\rightarrow} = 0$

${\leftarrow}\!\bullet\mathbf{1}^7\bullet\!{\rightarrow} = \pm\big(\{4\,|\,3\,\|\,0\}, \{\frac{5}{2}*\,|\,\{2\,|\,{-2}\}, \{2\,|\,\frac{1}{2}\,\|\|\,0\,|\,{-1}\,\|\,{-3}\}\}\big)$

${\leftarrow}\!\bullet\mathbf{1}^8\bullet\!{\rightarrow} = \pm\big(\frac{23}{4}\,|\,4, \{\{\{6*\,\|\|\,6\,|\,5\,\|\,{-4}\}\,|\,1\}, 4{+_{(2-_5)}}\,\|\|\,\pm\big(4{+_{(2-_5)}}\,|\,\frac{1}{2}\big)\,\|\,\pm\big(4{+_{(2-_5)}}\,|\,\frac{1}{2}\big),$
$\{4\,|\,3, \{3\,\|\,*\,|\,{-1}\}\,\|\,*, \{1\,\|\,{-2}\,|\,{-5}\}\}\big)$

$\mathbf{2}^3 = \mathbf{2}\bullet\mathbf{2} = *$

$\mathbf{2}^4 = *$

$\mathbf{2}^5 = 0$

$\mathbf{2}^6 = \pm\big(1\,\|\|\,1, \{1\,|\,\frac{7}{8}\}\,|\,\{1\,|\,{-1*}\}, \{*, \{1\,|\,0, *\}, \{1\,|\,{\downarrow}*\}\,|\,0, \{0\,|\,{-1}\}\}\,\|\,0\,|\,{-1}\big)$

$\mathbf{2}^7 = *$

$\bullet\mathbf{2}^2\bullet = *$	${\leftarrow}\!\bullet\mathbf{2}^2\bullet\!{\rightarrow} = 0$
$\bullet\mathbf{2}^3\bullet = *2$	${\leftarrow}\!\bullet\mathbf{2}^3\bullet\!{\rightarrow} = \pm 1$
$\bullet\mathbf{2}^4\bullet = 0$	${\leftarrow}\!\bullet\mathbf{2}^4\bullet\!{\rightarrow} = 0$
$\bullet\mathbf{2}^5\bullet = *$	${\leftarrow}\!\bullet\mathbf{2}^5\bullet\!{\rightarrow} = \pm\big(\frac{3}{4}\,|\,\{\frac{1}{2}\,|\,0\}, \{0\,\|\,0, \{0\,|\,{-1}\}\,|\,{-3}\,\|\|\,0\,|\,{-4}\}\big)$
$\bullet\mathbf{2}^6\bullet = 0$	${\leftarrow}\!\bullet\mathbf{2}^6\bullet\!{\rightarrow} = *$

$\mathbf{3}^4 = \mathbf{3}\bullet\bullet\mathbf{3} = *$	$\bullet\mathbf{3}^3\bullet = \bullet\mathbf{3}\bullet\mathbf{3}\bullet = *$	${\leftarrow}\!\bullet\mathbf{3}^2\bullet\!{\rightarrow} = 0$
$\mathbf{3}^5 = \mathbf{33}\bullet\mathbf{33} = *$	$\bullet\mathbf{3}^4\bullet = \pm(1{+_1})$	${\leftarrow}\!\bullet\mathbf{3}^3\bullet\!{\rightarrow} = \pm 1$
$\mathbf{3}^6 = *$	$\bullet\mathbf{3}^5\bullet = \pm\big(\frac{1}{2}, \{1\,|\,*\}\big)$	${\leftarrow}\!\bullet\mathbf{3}^4\bullet\!{\rightarrow} = *$
$\mathbf{3}^7 = *$	$\bullet\mathbf{3}^6\bullet = *$	${\leftarrow}\!\bullet\mathbf{3}^5\bullet\!{\rightarrow} = \pm\big(0, \{\{1, \{1\,|\,0\}\,|\,0, \{1\,|\,0\}\}, 1{-_2}\,|\,0\}\big)$

Table 2. Values of "starting" positions in partisan Sowing. The symbols ${\leftarrow}\!\bullet$ and $\bullet\!{\rightarrow}$ denote full pots going forever to the left and right.

$\mathbf{102}$	$=$	0	$=$	$\mathbf{102}$
$\mathbf{11}$	$=$	$*$	$=$	$\mathbf{11}$
$\mathbf{111}$	$=$	$*2$	$=$	$\mathbf{111}$
$\mathbf{1112}$	$=$	$*3$	$=$	$\mathbf{1112}$
$\mathbf{110111}$	$=$	$*4$	$=$	$\mathbf{11131}$
$\mathbf{111121}$	$=$	$*5$	$=$	$\mathbf{12113}$
$\mathbf{10111121}$	$=$	$*6$	$=$	$\mathbf{111312}$
$\mathbf{11101112}$	$=$	$*7$	$=$	$\mathbf{1111113}$
$\mathbf{11112111}$	$=$	$*8$	$=$	$\mathbf{11112111}$
$\mathbf{111111122}$	$=$	$*9$	$=$	$\mathbf{11132112}$

Table 3. Simplest impartial Sowing positions with given Nim-values. The left column gives the position with the fewest seeds; the right column gives the position with the fewest pots.

	$\bullet\mathbf{1}^1\bullet = *$	$\leftarrow\!\bullet\mathbf{1}^1\bullet\!\rightarrow = *$
$\mathbf{1}^2 = *$	$\bullet\mathbf{1}^2\bullet = *2$	$\leftarrow\!\bullet\mathbf{1}^2\bullet\!\rightarrow = 0$
$\mathbf{1}^3 = *2$	$\bullet\mathbf{1}^3\bullet = 0$	$\leftarrow\!\bullet\mathbf{1}^3\bullet\!\rightarrow = *$
$\mathbf{1}^4 = 0$	$\bullet\mathbf{1}^4\bullet = 0$	$\leftarrow\!\bullet\mathbf{1}^4\bullet\!\rightarrow = *$
$\mathbf{1}^5 = 0$	$\bullet\mathbf{1}^5\bullet = *$	$\leftarrow\!\bullet\mathbf{1}^5\bullet\!\rightarrow = 0$
$\mathbf{1}^6 = *$	$\bullet\mathbf{1}^6\bullet = 0$	$\leftarrow\!\bullet\mathbf{1}^6\bullet\!\rightarrow = 0$
$\mathbf{1}^7 = *2$	$\bullet\mathbf{1}^7\bullet = *4$	$\leftarrow\!\bullet\mathbf{1}^7\bullet\!\rightarrow = 0$
$\mathbf{1}^8 = 0$	$\bullet\mathbf{1}^8\bullet = 0$	$\leftarrow\!\bullet\mathbf{1}^8\bullet\!\rightarrow = *$
$\mathbf{1}^9 = 0$	$\bullet\mathbf{1}^9\bullet = *2$	$\leftarrow\!\bullet\mathbf{1}^9\bullet\!\rightarrow = *$
$\mathbf{1}^{10} = *$	$\bullet\mathbf{1}^{10}\bullet = 0$	$\leftarrow\!\bullet\mathbf{1}^{10}\bullet\!\rightarrow = 0$
$\mathbf{1}^{11} = 0$		
$\mathbf{1}^{12} = 0$		
$\mathbf{2}\bullet\mathbf{2} = *$	$\bullet\mathbf{2}^2\bullet = *$	$\leftarrow\!\bullet\mathbf{2}^2\bullet\!\rightarrow = 0$
	$\bullet\mathbf{2}^3\bullet = *$	$\leftarrow\!\bullet\mathbf{2}^3\bullet\!\rightarrow = *$
$\mathbf{2}^4 = *2$	$\bullet\mathbf{2}^4\bullet = *$	$\leftarrow\!\bullet\mathbf{2}^4\bullet\!\rightarrow = *$
$\mathbf{2}^5 = *3$	$\bullet\mathbf{2}^5\bullet = 0$	$\leftarrow\!\bullet\mathbf{2}^5\bullet\!\rightarrow = 0$
$\mathbf{2}^6 = 0$	$\bullet\mathbf{2}^6\bullet = 0$	$\leftarrow\!\bullet\mathbf{2}^6\bullet\!\rightarrow = 0$
$\mathbf{2}^7 = 0$	$\bullet\mathbf{2}^7\bullet = *2$	$\leftarrow\!\bullet\mathbf{2}^7\bullet\!\rightarrow = *$
$\mathbf{2}^8 = *2$		
$\mathbf{3}\bullet\bullet\mathbf{3} = *$	$\bullet\mathbf{3}\bullet\mathbf{3}\bullet = *$	$\leftarrow\!\bullet\mathbf{3}^2\bullet\!\rightarrow = 0$
		$\leftarrow\!\bullet\mathbf{3}^3\bullet\!\rightarrow = *$
$\mathbf{33}\bullet\mathbf{33} = *2$	$\bullet\mathbf{3}^4\bullet = *$	$\leftarrow\!\bullet\mathbf{3}^4\bullet\!\rightarrow = 0$
	$\bullet\mathbf{3}^5\bullet = *3$	$\leftarrow\!\bullet\mathbf{3}^5\bullet\!\rightarrow = 0$
$\mathbf{3}^6 = 0$	$\bullet\mathbf{3}^6\bullet = 0$	$\leftarrow\!\bullet\mathbf{3}^6\bullet\!\rightarrow = 0$
$\mathbf{3}^7 = *3$	$\bullet\mathbf{3}^7\bullet = *2$	
$\mathbf{3}^8 = *3$		

Table 4. Values of "starting" positions in impartial Sowing. The symbols $\leftarrow\!\bullet$ and $\bullet\!\rightarrow$ denote full pots going forever to the left and right.

x	$y=1$	2	3	4	5	6	7	8	9	10
1	0	$\ast$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
2	$\ast$	$\ast 2$	$\uparrow(1\|0)$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$
3	$\downarrow$	$\downarrow(1\|0)$	$\ast 2$	$\ast 2$	$\downarrow(1\|0)$	$\downarrow(1\|0)$	$\downarrow(1\|0)$	$\downarrow(1\|0)$	$\downarrow(1\|0)$	$\downarrow(1\|0)$
4	$\downarrow$	$\downarrow_2\ast$	$\ast 2$	$\ast 2$	$\downarrow(1\|0)$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$
5	$\downarrow$	$\downarrow_2\ast$	$\uparrow(1\|0)$	$\uparrow(1\|0)$	$\uparrow(\pm 1)$	$\uparrow(1\|0)$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$
6	$\downarrow$	$\downarrow_2\ast$	$\uparrow(1\|0)$	$\downarrow_2\ast$	$\downarrow(1\|0)$	$\ast 2$	$\ast 2$	$\uparrow^2\ast$	$\uparrow^2\ast$	$\uparrow^2\ast$
7	$\downarrow$	$\downarrow_2\ast$	$\uparrow(1\|0)$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\ast 2$	$\ast 2$	$\ast 2$	$\uparrow^2\ast$	$\uparrow^2\ast$
8	$\downarrow$	$\downarrow_2\ast$	$\uparrow(1\|0)$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\ast 2$	$\ast 2$	$\ast 2$	$\uparrow^2\ast$
9	$\downarrow$	$\downarrow_2\ast$	$\uparrow(1\|0)$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\ast 2$	$\ast 2$	$\ast 2$
10	$\downarrow$	$\downarrow_2\ast$	$\uparrow(1\|0)$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\downarrow_2\ast$	$\ast 2$	$\ast 2$

Table 5. Values of Atomic Wari positions of the form $\boldsymbol{xy}$.

Except for deleting leading and trailing empty pots, there don't seem to be any clear-cut rules for simplifying Atomic Wari positions. The situation is similar to Sowing. Positions can often be split into sums independent components by hand, but no algorithm is known to find such splits in general. For example, **1231110101311** $=$ **123111** $+$ **1311**. Similarly, there are several cases where the first or last pot contains only one seed, where the position's value does not change when this pot is removed, but no algorithm is known for detecting such positions. For example, **1001321** $=$ **1321**.

Simple Values. Table 5 lists the values for all Atomic Wari games with two adjacent nonempty pots, each containing ten or fewer seeds. We use the following notation from [Conway 1976]: For any game $G = \{G^L \mid G^R\}$, we recursively define $\uparrow G = \{\ast, \uparrow G^L \mid \ast, \uparrow G^R\}$ and $\downarrow G = \uparrow(-G) = -(\uparrow G)$. For all positive integers n, we define $\uparrow^n = \uparrow n - \uparrow(n-1)$, and $\downarrow_n = -(\uparrow^n)$.

We note that only ten different games appear in Table 5: 0, $\ast$, $\uparrow$, $\downarrow$, $\ast 2$, and the "exotic" games

$$\uparrow(1\,|\,0) = \{\uparrow, \ast \mid 0, \ast\} \qquad \uparrow(\pm 1) = \{\uparrow, \ast \mid \downarrow, \ast\} \qquad \uparrow^2\ast = \{0, \ast \mid \downarrow\}$$
$$\downarrow(1\,|\,0) = \{0, \ast \mid \downarrow, \ast\} \qquad\qquad\qquad \downarrow_2\ast = \{\uparrow \mid 0, \ast\}$$

The games $\uparrow$ and $\uparrow(1|0)$ have atomic weight 1; $\downarrow$ and $\downarrow(1|0)$ have atomic weight -1; all the other have atomic weight zero. These games are partially ordered as follows:

$$\begin{tikzcd}
 & \uparrow \arrow[dl] \arrow[dr] & & \uparrow(1|0) \arrow[dl] \arrow[dr] & & \uparrow^2\ast \arrow[dl] \arrow[dr] & \\
0 \arrow[dr] & & \ast 2 \arrow[dl] \arrow[dr] & & \uparrow(\pm 1) \arrow[dl] \arrow[dr] & & \ast \arrow[dl] \\
 & \downarrow & & \downarrow(1|0) & & \downarrow_2\ast &
\end{tikzcd}$$

There are Atomic Wari positions with arbitrary integer atomic weights. For example, the position $(\mathbf{01300})^n$ has value $n \cdot \uparrow$ and atomic weight n. Even so, for positions arising in normal play, the atomic weight is almost always 0, 1, or

-1, occasionally 2 or -2, and in extremely rare cases, 3 or -3. We have yet to see even one "natural" position with any other atomic weight. It is an open question whether noninteger or nonnumeric atomic weights are possible.

Partisan Atomic Wari Is Partially Impartial. Even though Atomic Wari is a partisan game, there is a special case that can be analyzed as if it were impartial. We call an Atomic Wari position "sparse" if every pot has two or fewer seeds.

THEOREM 3.1. *Every sparse Atomic Wari position has the same value as the corresponding Impartial Atomic Wari position, and any such position can be split into independent components by removing all pots with fewer than two seeds.*

PROOF. We prove the claim by induction on the number of *deuces* (pots with two seeds). The base case, in which each pot contains either one seed or none, is trivial.

Consider a position X with n deuces, and let X' denote the sum of positions obtained by deleting pots with fewer than two seeds. Each of the options of X is a sparse position with either $n-1$ or $n-2$ deuces. For each move by Left, there is a corresponding move by Right in the same contiguous "string" of deuces that results in exactly the same position, once the inductive hypothesis is applied. For example, given the position **1222201**, the Left move to $\mathbf{12102201} = \mathbf{2} + \mathbf{22}$ is matched by the Right move to $\mathbf{12201201} = \mathbf{22} + \mathbf{2}$. Clearly, X and X' have the same options, once the inductive hypothesis is applied. The theorem follows immediately. □

Sparse Atomic Wari is equivalent to the following take-away game. There are several piles of seeds. Each player can remove one seed from any pile, or remove two seeds from any pile and optionally split the remainder into two piles. In the octal notation of *Winning Ways*, this is the game **·37**. A computer search of the first 200,000 values of this game reveals no periodicity, and finds only thirteen $\mathcal{P}$-positions:

$$\{0, 3, 11, 19, 29, 45, 71, 97, 123, 149, 175, 313, 407\}.$$

It seems quite likely that these are in fact the only $\mathcal{P}$-positions. For a short list of Nim values, see page 102 of *Winning Ways*.

Theorem 3.1 implies that the values we see in Table 5 are the only values that a two-pot Atomic Wari position can have. By straightforward case analysis, we can classify all positions $\boldsymbol{xy}$ with $x \le y$ as follows.

$x = 1$	$1 < x \le y - 2$	$1 < x = y - 1$	$1 < x = y$
0 if $y = 1$ $*$ if $y = 2$ $\uparrow$ otherwise	$\downarrow(1\|0)$ if $\mathbf{2}^x = 0$ $\uparrow^2*$ otherwise	$\uparrow(1\|0)$ if $\mathbf{2}^{x-2} = 0$ $\downarrow(1\|0)$ if $\mathbf{2}^{y-2} = 0$ $*2$ otherwise	$*2$ if $x = 2$ $\uparrow(\pm 1)$ if $\mathbf{2}^{x-2} = 0$ $*2$ otherwise

Open Questions. Are noninteger or nonnumeric atomic weights possible in Atomic Wari? How can we systematically construct Impartial Atomic Wari positions with value $*n$ for any n? Is there a *simple* algorithm that splits Atomic Wari positions into multiple independent components? Are there Atomic Wari positions with exponentially many followers? (The answer is yes if we disallow capturing groups of three seeds.) Finally, are there any other high-level simplification rules like Theorem 3.1 that would allow faster evaluation?

4. Other Variants

An amazingly large number of other variations on these games are possible. Here is a short list of possible games, starting with common versions of the original African games, which might give interesting results. This list is by no means exhaustive!

Mancala is played on a two by six grid of pots, where each player owns one row of six. All moves go counterclockwise, but must begin at one of the moving player's pots. Each player has an extra pot called a store. Each player can drop seeds into his own store as if it were the seventh pot on his side, but not into his opponent's. Seeds never leave the stores. If a move ends by putting the last seed into the store, the same player moves again. If the last stone lands in an empty pot on the moving player's side, both that stone and the stones in the opponent's pot directly opposite are put into the moving player's store. If any player cannot move, the other player collects all the seeds on his side and puts them in his store, and the game ends. The winner is the player who has more seeds in his store at the end of the game. In the starting position, there are three (or sometimes six) seeds in each pot.

Wari (or *oware*) is also played on a two by six grid of pots, similarly to mancala, but with no stores. All moves go counterclockwise around the board, but each move must begin on the moving player's side. Otherwise, the rules are identical to Atomic Wari. In one version of the game, if one player cannot move, his opponent moves again; the game ends only when neither player can move. In other versions, the ending conditions are considerably more complicated. When the game ends, the player with more captured seeds wins. Typically, the game beings with four seeds in each pot.

Sowing can also be played in reverse. In *Reaping*, a legal move consists of picking up one seed from each of a successive string of pots and dropping them into the first available empty pot. One could also play Atomic Wari after reversing the movement rules, but one must be careful not to reverse the capturing rules as well!

Partisan sowing games could be played with either player moving in either direction, but with colored seeds. For example, a legal move by Left might consist of sowing all the blue seeds in any pot, in such a way that the last seed is put into a pot containing at least one other blue seed, and capturing all the red seeds in the last pot.

Finally, consider the following two-dimensional sowing game. The board consists of a two-dimensional grid of pots. Left can sow upwards or downwards; Right can sow to the left or right. Seeds are sown exactly as in Sowing; in particular, the last seed must be put into a nonempty pot. Whenever any seed lands in a nonempty pot, the contents of that pot, including the new seed, are captured. Thus, at least two seeds are captured on every turn. One special case of this game is already quite well-known!

References

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