Games of No Chance
MSRI Publications
Volume **29**, 1996

New Toads and Frogs Results

JEFF ERICKSON

ABSTRACT. We present a number of new results for the combinatorial game Toads and Frogs. We begin by presenting a set of simplification rules, which allow us to split positions into independent components or replace them with easily computable numerical values. Using these simplication rules, we prove that there are Toads and Frogs positions with arbitrary numerical values and arbitrarily high temperatures, and that any position in which all the pieces are contiguous has an integer value that can be computed quickly. We also give a closed form for the value of any starting position with one frog, and derive some partial results for two-frog positions. Finally, using a computer implementation of the rules, we derive new values for a large number of starting positions.

1. Introduction

Toads and Frogs is a two-player game, played on a one-dimensional board. Left has a number of toads, and Right has a number of frogs, each on its own square of the board. Each player has two types of legal moves: he may either *push* one of his pieces forward into an adjacent empty square, or he may *jump* one of his pieces over an adjacent opposing piece, into an empty square. Jumps are never forced, and jumped-over pieces are not affected in any way. Toads move to the right, frogs to the left. The first player without a legal move loses the game.

Throughout the paper, we represent toads by T, frogs by F, and empty squares by the symbol □.

Here is a typical Toads and Frogs game. Left moves first and wins.

$$\begin{array}{lll}
 & \swarrow & \mathrm{TT\square FF} \\
\mathrm{T\square TFF} & & \\
 & \searrow & \mathrm{TFT\square F} \\
 & \swarrow & \\
\mathrm{TF\square TF} & & \\
 & \searrow & \mathrm{TFFT\square} \\
 & \swarrow & \\
\mathrm{TFF\square T} & &
\end{array}$$

In this particular game, and in almost all previously analyzed positions, all the moves are forced, but this is certainly not typical.

Toads and Frogs was described very early in *Winning Ways* [Berlekamp et al. 1982], in order to introduce some simple concepts such as numbers, fractions, and infinitesimals. *Winning Ways* contains a complete analysis of a few simple types of Toads and Frogs positions.

We assume the reader is familiar with the terminology (value of a game, follower, temperature, atomic weight, ...) and notation ($\{x\,|\,y\}$, $*$, $\uparrow$, $+_x$, ...) presented in *Winning Ways*. Throughout the paper, we will use the notation X^n to denote n contiguous copies of the Toads and Frogs position X. For example, $(\mathrm{TF}\square)^3\mathrm{F}^4$ is shorthand for TF□TF□TF□FFFF.

2. Simplifying Positions

We will employ a number of general rules to simplify the evaluation of board positions. These rules allow us to split positions into independent components, or replace positions with easily derived values.

The simplest rule is to remove dead pieces. A piece is *dead* if it cannot be moved, either in the current position or in any of its followers. For example, in the position TFFT□, the first three pieces are dead, so we can simplify to the position T□. Sometimes removing dead pieces causes the board to be split into multiple independent components. For example, in the position T□TTFF□F, the middle four pieces are dead. Furthermore, the pieces to the left of the dead group can never interfere with the pieces to the right of the dead group. Thus, this position is equivalent to the sum T□ + □F. It's possible that after we split the board into components, some of the components have no pieces; obviously, they can be ignored. For example, □□TTFF□TF = □□ + □TF = □TF.

Fortunately, it's easy to tell what pieces are dead just by looking at the board.

Identifying Dead Pieces: *Any contiguous sequence of toads and frogs beginning with two toads* (*or the left edge of the board*) *and ending with two frogs* (*or the right edge of the board*) *is dead. Any piece that is not in such a sequence is alive.*

In *Winning Ways*, the "Death Leap Principle" was used to help analyze Toads and Frogs positions with only one space. We can generalize it to more complicated positions. We call a space *isolated* if none of its neighboring squares is empty.

The Death Leap Principle: *Any position in which the only legal moves are jumps into isolated spaces has value zero.*

PROOF. Suppose it's Left's turn. If she has no moves, Right wins. Otherwise, she must jump a toad into a single space. Right's response is to push the jumped-over frog forward. Now Left is in the same situation as before—her only moves are jumps into single spaces. (Right may have more moves at this point, but this only makes the situation better for him.) Eventually, Left has no moves, and Right wins. We can argue symmetrically if Right goes first. $\square$

We can also express this principle purely in syntactic terms, which makes it somewhat easier to implement on a computer.

The Death Leap Principle: *Any position that does not contain any of the four subpositions* TF□□, □□TF, T□, *or* □F *has value zero.*

If there are no frogs between a toad and the right edge of the board, we call the toad *terminal*. If there are no toads between a frog and the left edge of the board, we call the frog *finished*. Suppose Left has a terminal toad with three spaces in front of it. Intuitively, this toad is worth exactly three free moves for Left, since none of Right's pieces can ever interfere with it. If we change the position by moving this toad to the right end of the board (where it dies) and crediting Left with three free moves, we expect the value of the game to stay the same. After all, whenever she would have moved her terminal toad, she can just take one of her free moves instead.

This intuition suggests the following simplification principle.

The Terminal Toads Theorem: *Let* X *be any position. Then*

$$\mathrm{XT}\square^n = \mathrm{X}\square^n + \mathrm{T}\square^n = \mathrm{X}\square^n + n.$$

Proof. The second player wins the difference $\mathrm{XT}\square^n - (\mathrm{X}\square^n + \mathrm{T}\square^n)$ by the following mirror strategy. Initially, the last toad in the first component is marked T_*. Any move in either copy of X is answered by the corresponding move in the other copy. Any move in the third component is answered by moving T_*, and vice versa.

This is enough to show that Left loses if she moves first, but there are two special cases to consider when Right moves first. If Right moves a frog in the second component whose twin in the first component is blocked by T_*, then Left moves T_* instead, and then *moves the mark back* to the blocked toad. If Right moves in the third component, but there's a toad in front of T_*, then Left *moves the mark forward* and then pushes the new T_*. Any move by Right is answerable by a corresponding move by Left, so Right also loses going first. $\square$

Naturally, there is a symmetric version of this theorem for removing finished frogs.

The Finished Frogs Formula: *Let* X *be any position. Then*

$$\square^n\mathrm{FX} = \square^n\mathrm{X} + \square^n\mathrm{F} = \square^n\mathrm{X} - n.$$

Consider the position TFTFF□FT□□TTFF□TFF□□TT. At first glance it might look too complicated to evaluate the usual way. Fortunately, our simplification rules let us get a value quickly, without recursively evaluating any of its followers:

$$\begin{aligned}\cancel{\mathrm{TFTFF}}\square\mathrm{FT}\square\square\cancel{\mathrm{TTFF}}\square\mathrm{TFF}\square\square\cancel{\mathrm{TT}} &= \square\cancel{\mathrm{FT}}\square\square + \square\mathrm{TFF}\square\square \\ &= \cancel{\square\square\square} + \square\mathrm{TFF}\square\square + 1 \\ &= \cancel{\square\mathrm{TFF}\square\square} + 1 \\ &= 1.\end{aligned}$$

(i) We remove the three groups of dead pieces, splitting the board into two independent components.
(ii) The first component contains a terminal toad with two remaining moves, and a finished frog with one remaining move. We remove them, crediting Right with one free move and Left with two, for a net gain of one move for Left.
(iii) The first component is now completely empty, so we remove it.
(iv) We remove the remaining component, since it has value zero by the Death Leap Principle.

3. Arbitrary Numbers and Arbitrary Temperatures

We make two somewhat surprising observations. First, for any number, there is a Toads and Frogs position with that number as its value. Second, there are Toads and Frogs positions with arbitrarily high (half-integer) temperatures.

Theorem 3.1. $(\mathrm{TF})^m\mathrm{T}\square(\mathrm{TF})^n = 2^{-n}$ *for all* m *and* n.

Proof. We use induction on n. The base case $n = 0$ follows immediately from the Death Leap Principle. Suppose $n > 0$. Left has only one legal move, to the position $(\mathrm{TF})^m\square\mathrm{T}(\mathrm{TF})^n = (\mathrm{TF})^m\square = 0$ by the Death Leap Principle. Similarly, Right can only move to the position $(\mathrm{TF})^{m+1}\mathrm{T}\square(\mathrm{TF})^{n-1} = 2^{-n+1}$, by the induction hypothesis. □

Corollary 3.2. *For any dyadic rational number* q, *there is a Toads and Frogs position with value* q.

Proof. Write $q = (2k+1)/2^n$, and assume without loss of generality that $k \geq 0$. Then the position $(\mathrm{T}\square(\mathrm{TF})^n\mathrm{TTFF})^{2k+1}$ has value q. □

Theorem 3.3. $\mathrm{TF}\square\mathrm{F}\square^{n+3} = \{n\,|\,0\}$.

Proof. Right has one legal move, to $\mathrm{TFF}\square^{n+4}$, which clearly has value 0. Left has one legal move, to $\square\mathrm{FTF}\square^{n+3}$, which, by the Finished Frogs Formula, is equivalent to $\square\mathrm{TF}\square^{n+3}-1 = \{n\,|\,n+4\}-1 = n$. □

Corollary 3.4. *For any integers* $a \geq b$, *there is a Toads and Frogs position with value* $\{a\,|\,b\}$.

Proof. One of $\mathrm{T}\square^b\mathrm{TTFFTF}\square\mathrm{F}\square^{a-b+3}$ or $\mathrm{TF}\square\mathrm{F}\square^{a-b+3}\mathrm{TTFF}\square^{-b}\mathrm{F}$ has value $\{a\,|\,b\}$, depending on whether b is positive or negative. □

We do not know whether there are positions with arbitrary (dyadic rational) temperatures. The following theorem gives us positions whose temperatures are arbitrary *sufficiently large* multiples of $\frac{1}{4}$. We omit the proof, which follows from exhaustive case analysis. Every position eight moves away is an integer.

THEOREM 3.5. *For any* $n \geq 6$, *we have* $\mathrm{T}\square\mathrm{TF}\square\mathrm{F}\square^n = \{a{+}\frac{1}{2}\,|\,1\}$.

4. Knots Have Integer Values

We now consider positions in which the toads and frogs form a single contiguous group. We call such positions *knots*, the collective term for toads [Lipton 1991]. Somewhat surprisingly, every knot has an integer value, which can be determined without evaluating any of the position's followers. We derive a series of rules that allow us to reduce every such position to one of a few simple cases. These rules are proved using relatively simple counting arguments, but disinterested readers are encouraged to skip the proofs.

Thanks to our simplification rules, we only need to consider positions that start with a toad and end with a frog, but not two toads and two frogs (or similar positions in which the knot is against one edge of the board). In the discussion that follows, unless otherwise stated, all superscripts are positive.

LEMMA 4.1. $\square^a\mathrm{TF}\square^b = \{b{-}a{-}2\,|\,b{-}a{+}2\} = \begin{cases} b-a-1 & \text{if } b-a \geq 2, \\ 0 & \text{if } |b-a| \leq 1, \\ b-a+1 & \text{if } b-a \leq -2. \end{cases}$

PROOF. Immediate. □

LEMMA 4.2. $\square^a\mathrm{TFTF}\square^b = \begin{cases} b-1 & \text{if } 2a \leq b, \\ 2(b-a)-1 & \text{if } a < b \leq 2a, \\ 0 & \text{if } b = a, \\ 2(b-a)+1 & \text{if } b < a \leq 2b, \\ 1-a & \text{if } 2b \leq a. \end{cases}$

PROOF. This follows from Lemma 4.1 and the Terminal Toads Theorem by case analysis. See Figure 1. □

LEMMA 4.3. $\square^a\mathrm{TTF}\square^b = \begin{cases} 2b-a-2 & \text{if } a \leq b-1, \\ b-1 & \text{if } b-1 \leq a \leq b+1, \\ 2b-a & \text{if } b+1 \leq a \leq 2b, \\ 0 & \text{if } 2b \leq a. \end{cases}$

PROOF. This follows from Lemma 4.1 and the Terminal Toads Theorem by simple case analysis. Every position three moves away is an integer. □

LEMMA 4.4. $\square^a\mathrm{T}^b\mathrm{F}\square^c = (b-2)(c-1) + \square^a\mathrm{TTF}\square^c$.

PROOF. We use induction on b. The base case $b = 2$ follows from Lemma 4.3. To complete the proof, we need to show that the second player wins the game $\square^a\mathrm{T}^b\mathrm{F}\square^c - \square^a\mathrm{T}^{b-1}\mathrm{F}\square^c - (c-1)$. This follows from a simple counting argument.

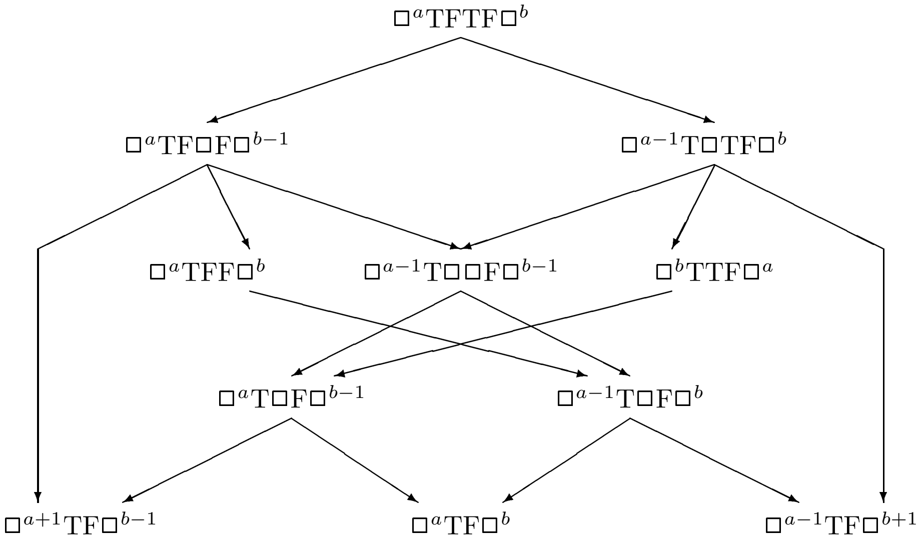

Figure 1. Followers of $\square^a\mathrm{TFTF}\square^b$, with integer components (from the Terminal Toads Theorem) omitted.

Provided neither frog ever jumps, each player will move exactly $bc + a + b - 1$ times before the game ends. Moreover, as long as the second player's frog never jumps, the first player will run out of moves first. The second player can keep his frog from jumping by always pushing it immediately after the opponent's toad jumps over it. $\square$

LEMMA 4.5. *If neither player can move from the position* $\mathrm{XF}\square$, *then*

$$\mathrm{XT}^a\mathrm{F}\square^b = a(b-1).$$

PROOF. The base case $a = 0$ is trivial. Otherwise, the second player wins the game $\mathrm{XT}^a\mathrm{F}\square^b - a(b-1)$ as follows.

Left moves first. Left is forced to jump her rightmost toad. Right responds by moving his frog forward, leaving the position $\mathrm{XT}^{a-1}\mathrm{F}\square^b - a(b-1) + (b-1)$ by the Terminal Toads Theorem. This position has value zero by induction. Thus, Right wins going second.

Right moves first. Left wins by a simple counting argument. Each of her a toads will move or jump at least b times before reaching the edge of the board, for a total of ab moves. Right's frog will move at most a times before hitting X, and he has $ab - a$ moves available in the integer component. If Right never jumps his frog, each player will have the same number of moves, but that's the best Right can do. So when Right uses up all his moves, Left will have at least one move left, to the winning position $\mathrm{XF}\square^b = 0$. $\square$

The previous lemmas let us evaluate any knotted position from which only one player can legally move. Of course, we already knew that such positions had integer values, but we can now easily determine *what* integer. The only positions left to analyze are those in which both sides can legally move.

LEMMA 4.6. *If neither player can move from the position* $\square$TXF$\square$, *then*

$$\square^a\mathrm{TF}^b\mathrm{XT}^c\mathrm{F}\square^d = \square^a\mathrm{TF}^b + \mathrm{T}^c\mathrm{F}\square^d = c(d-1) - b(a-1).$$

PROOF. The second player wins $\square^a\mathrm{TF}^b\mathrm{TXFT}^c\mathrm{F}\square^d + b(a-1) - c(d-1)$ by the same counting argument. Each player can move at most $ab + cd$ times, with equality if neither of the lone pieces ever jumps. Thus, as long as the second player's lone piece never jumps, which is easy to guarantee, the first player will run out of moves first. $\square$

LEMMA 4.7. $\square^a\mathrm{TF}^b\mathrm{T}^c\mathrm{F}\square^d = \square^a\mathrm{TFTF}\square^d + (c-1)(d-1) - (a-1)(b-1)$.

PROOF. The base case $b = c = 0$ follows immediately from Lemma 4.2. To complete the inductive proof, it suffices to prove that

$$\square^a\mathrm{TF}^b\mathrm{T}^c\mathrm{F}\square^d = \square^a\mathrm{TF}^b\mathrm{T}^{c-1}\mathrm{F}\square^d + (d-1).$$

The second player wins the difference game by the usual counting argument. If Right moves first, Left marks the first toad in the first component and the last frog in the second. If Left moves first, right marks the last frog in the first component and the first toad in the second. Either way, the second player moves so that the marked pieces never jump. This guarantees at least $ab+cd+a+b+c+d+1$ moves to the second player, and at most that many to the first player, so the first player will always run out of moves first and lose.

This guarantees that Left will get at least $cd + b + d + 1$ moves in the first component and $ab+a+c$ in the second. Right gets at most $ab+a+c+1$ moves in the first component, $cd-c+b+d+1$ in the second, and $d-1$ in the third. $\square$

THEOREM 4.8. *Every knot has an integer value*, *which can be determined directly*, *without evaluating its followers*. $\square$

5. One-Frog Starting Positions

Starting positions are those in which all toads are against the left edge of the board, and all frogs are against the right edge. In this section, we give a closed form for the value of a starting position in which there is only one frog.

LEMMA 5.1. $\mathrm{T}\square^a\mathrm{F} = a\cdot * = \begin{cases} * & \textit{if } a \textit{ is odd},\\ 0 & \textit{if } a \textit{ is even}.\end{cases}$

PROOF. For any follower of the game $\mathrm{T}\square^a\mathrm{F}$, both players' moves are forced. Now suppose a is even. The second player's $(a/2)$-th move leaves the position $\square^{a/2}\mathrm{TF}\square^{a/2}$, which equals 0 by Lemma 4.1, so the value of the original game is zero.

Suppose a is even. Then the second player wins the game $\mathrm{T}\square^a\mathrm{F} + *$ as follows. Suppose Left moves first. If Left moves in $*$ before the two pieces meet, then Right's $\lceil a/2\rceil$-th moves leaves the winning position $\square^{\lfloor a/2\rfloor}\mathrm{TF}\square^{\lceil a/2\rceil} = 0$ by Lemma 4.1. Otherwise, Right's $\lfloor a/2\rfloor$-th move leaves the winning position $\square^{\lfloor a/2\rfloor}\mathrm{T}\square\mathrm{F}\square^{\lfloor a/2\rfloor} + * = 0$ by Lemma 4.1. $\square$

THEOREM 5.2. $\mathrm{T}^a\square^b\mathrm{F} = \begin{cases} \{\{a{-}2\,|\,1\}\,|\,0\} & \textit{if } b = 1, \\ (a{-}1)(b{-}1)* & \textit{if } b > 1 \textit{ is odd}, \\ (a{-}1)(b{-}1) & \textit{if } b \textit{ is even}. \end{cases}$

PROOF. The base case $a = 1$ follows immediately from the previous lemma, and the case $b = 1$ follows from the analysis in *Winning Ways* (p. 126). Otherwise, it suffices to prove that $\mathrm{T}^a\square^b\mathrm{F} = \mathrm{T}^{a-1}\square^b\mathrm{F} + (b-1)$, since the theorem then follows by induction. The second player wins the difference by the usual counting argument. If the second player never jumps his frog, then he will get at least $ab{+}a{-}b{+}1$ moves, provided that all of his pieces eventually reach the appropriate end of the board. He can guarantee this by moving his frog forward on his first move. The first player gets at most $ab + a - b + 1$ moves, so the second player will win. $\square$

6. Partial Results for Two-Frog Starting Positions

We were unable to derive a closed form for two-frog starting positions, but we do have some interesting partial results.

LEMMA 6.1. $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^b\mathrm{FT}^c\square = c$ *for all* $b \neq 0$.

PROOF. Follows immediately from the Terminal Toads Theorem and the Death Leap Principle. $\square$

LEMMA 6.2. $\mathrm{T}^a\mathrm{F}\square\mathrm{FT}^c\square \leq c$.

PROOF. It suffices to show that Right can win the game $\mathrm{T}^a\mathrm{F}\square\mathrm{F}\square$ if he moves second. Left's first move is forced, and Right responds by moving to $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{TF}\square = 0$ by the Death Leap Principle. $\square$

LEMMA 6.3. $\mathrm{T}^n\square\mathrm{F}\square\mathrm{F} = 0$ *for all* $n \geq 2$.

PROOF. The second player wins as follows.

Left moves first. Right moves his leftmost frog on his first move, and his rightmost frog on his second move. Left's moves are forced. Right's second move leaves the position $\mathrm{T}^{n-1}\mathrm{F}\square\mathrm{TF}\square$, which has value zero by the Death Leap Principle. Thus, Right wins moving second.

Right moves first. Right has two options. If he moves the rightmost frog forward, Left will move her rightmost toad forward three times. This forces Right's next two moves. Left's third move leaves the winning position $\mathrm{T}^{n-1}\mathrm{FF}\square\mathrm{T}\square = 1$.

If Right moves his leftmost Frog forward, Left must respond by jumping it, leaving $\mathrm{T}^{n-1}\square\mathrm{FT}\square\mathrm{F}$. Again, Right has two options. If he moves his rightmost frog, Left responds by jumping it, leaving the positions $\mathrm{T}^{n-1}\square\mathrm{F}\square\mathrm{F}$, which is zero by induction. (One can check the base case $n = 2$ directly, or see *Winning Ways*, p. 135.) Otherwise, Right moves to the position $\mathrm{T}^{n-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F}$, and Left can force the following sequence of moves:

$$\begin{array}{rl}
 & \swarrow\ \mathrm{T}^{n-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F} \\
\mathrm{T}^{n-1}\mathrm{F}\square\square\mathrm{TF} & \\
 & \searrow\ \mathrm{T}^{n-1}\mathrm{F}\square\mathrm{FT}\square \\
\mathrm{T}^{n-2}\square\mathrm{FTFT}\square & \swarrow \\
 & \searrow\ \mathrm{T}^{n-2}\mathrm{F}\square\mathrm{TFT}\square \\
\mathrm{T}^{n-2}\mathrm{F}\square\mathrm{TF}\square & \swarrow
\end{array}$$

The final position is zero by the Death Leap Principle. Thus, Left wins moving second. □

7. Known Values and Open Questions

We list here all known values for positions in which all toads are on the left edge of the board and all frogs are on the right edge of the board. These values, and the values of most of the positions used earlier in the paper, were derived with the help of David Wolfe's `games` package [Wolfe 1996]. I added code (about 600 lines of ANSI C) for evaluating Toads and Frogs positions, using all the simplification rules described in Section 2.

Here are all known values for symmetric games of the form $\mathrm{T}^a\square^b\mathrm{F}^a$, apart from the cases $a = 1$ and $b = 1$, for which all values were previously known (see *Winning Ways*).

	b					
a	1	2	3	4	5	6
1	$*$	0	$*$	0	$*$	0
2	$*$	$*$	$*$	$*$	0	0
3	$*$	$\pm\frac{1}{8}$	0	$*$	0	
4	$*$	$\cdots$	$*$	0		
5	$*$	$\cdots$	$*$			
6	$*$	$\cdots$				

$$\mathrm{T}^4\square\square\mathrm{F}^4 = \pm\big(\{1\,|\,*\},\{\{\tfrac{3}{2}\,|||\,1\,||\,0\,|\,0,{\downarrow}*\}\,|\,0\}\big)$$

$$\mathrm{T}^5\square\square\mathrm{F}^5 = \pm\big(\{2\,|\,*\},\{\tfrac{5}{2}\,|\,\{2\,|||\,0\,||\,\{0\,|||\,{\uparrow}*\,|\,0\,||\,{-1}\},\{0\,|\,\{0\,||\,{-\tfrac{1}{32}}\,|\,{-2}\,|||\,{-\tfrac{1}{2}}*\}\}\,|\,\{{\uparrow}*\,|\,0\,||\,{-\tfrac{1}{2}}\,|||\,{-1}*\}\}$$
$$||\,0\}\big)$$

$$\mathrm{T}^6\square\square\mathrm{F}^6 = \pm\big(\{3\,|\,*\},\{\tfrac{7}{2}\,||\,3\,|\,\{0,G\,|||\,\{0,G\,|\,\{\tfrac{1}{2}{\downarrow}\,|\,m_1\,||\,0\,|||\,{-2}\}\},$$
$$\{0\,|\,\{0\,||\,m_2\,|\,{-3}\,|||\,{-1}{\downarrow}\}\}\,||\,\{\tfrac{1}{2}{\downarrow}\,|\,m_1\,||\,0\,|||\,{-\tfrac{1}{2}}\}\,|\,{-2}*\}$$
$$|||\,0\}\big),$$

where $m_1 = {}_{-2\|0|+\frac{1}{64}}$, $m_2 = {}_{-1\|0|+\uparrow\uparrow\uparrow *}$, and

$$G = \{\tfrac{1}{2}\,|\,0\}, \tfrac{1}{2}\{0\,\|\downarrow|\,{-2}\}\,|\,\{0\,\|\,m_1\,|\,{-2}\}, \{\tfrac{1}{2}{\downarrow}\,|\,m_1\,\|\,{-\tfrac{3}{2}}\,|\,{-2}\}.$$

Except for TTT□□FFF $= \pm\frac{1}{8}$, all values are infinitesimals with atomic weight zero. The values for $b = 2$ are particularly interesting, since they seem to be totally patternless. The other values are certainly much tamer, but there are still no apparent patterns.

The table on the next page lists all known values for positions of the form $\mathrm{T}^a\Box^b\mathrm{F}^c$ for $1 \le c < a \le 7$. The only really nasty value we get is for $\mathrm{T}^7\Box^4\mathrm{F}^3$, which is $(7{*}\,|\,\frac{9}{2}\,\|\,4)$-ish.

These values naturally suggest a number of patterns, some of which we confirmed earlier in the paper. We list here several conjectures that we have been unable to prove, all of which are strongly supported by our experimental results.

Conjecture 7.1. $\mathrm{T}^a\Box\Box\mathrm{F}^b = \{\{a{-}3\,|\,a{-}b\}\,|\,\{{*}\,|\,3{-}b\}\}$ *for all* $a > b \ge 2$.

Conjecture 7.2. $\mathrm{T}^a\Box\Box\Box\mathrm{FF} = (a-2){*}$ *for all* $a \ge 2$.

Conjecture 7.3. $\mathrm{T}^a\Box\Box\Box\mathrm{F}^3 = a - \frac{7}{2}$ *for all* $a \ge 5$.

Conjecture 7.4. $\mathrm{T}^a\Box^a\mathrm{F}^{a-1} = 1$ *or* $1{*}$ *for all* $a \ge 1$.

Conjecture 7.5. $\mathrm{T}^a\Box^b\mathrm{F}^a$ *is an infinitesimal for all* a, b *except* $(a, b) = (3, 2)$.

Many of our results seem to be special cases of a more general principle, which sometimes allows us to split board positions into multiple independent components. An important open problem is to characterize exactly what positions can be split this way.

We finish with a much more ambitious conjecture. Despite the existence of simple rules for special positions, Toads and Frogs positions are in general extremely hard to evaluate. It seems difficult, in general, to even determine the winner of a given Toads and Frogs position. We conjecture that Toads and Frogs is at least as hard as other seemingly richer games such as Red-Blue Hackenbush [Berlekamp et al. 1982].

Conjecture 7.6. *Toads and Frogs is NP-hard.*

To prove this conjecture, it would suffice to prove that for any integers $a > b > c$, there is a Toads and Frogs position whose value is $\{\{a\,|\,b\}\,|\,c\}$. The conjecture would then follow from a theorem of Yedwab and Moews. (See [Yedwab 1985, pp. 29–45] or [Berlekamp and Wolfe 1994, pp. 109–111].) The closest we can get is the following theorem, which can be verified by drawing out the complete game tree.

Theorem 7.7. *For any* $a \ge 0$, $\mathrm{TF}\Box\mathrm{F}\Box\mathrm{F}\Box^a = \{\{a{-}6\,|\,{-1}\}\,|\,{-2}\}$.

	b				
a	1	2	3	4	5
			$c = 1$		
2	$\frac{1}{2}\mid 0$	1	$2*$	3	$4*$
3	$1*\mid 0$	2	$4*$	6	$8*$
4	$2\mid 1\Vert 0$	3	$6*$	9	$12*$
5	$3\mid 1\Vert 0$	4	$8*$	12	$16*$
6	$4\mid 1\Vert 0$	5	$10*$	15	$20*$
7	$5\mid 1\Vert 0$	6	$12*$	18	$24*$
			$c = 2$		
3	$*$	$\frac{1}{2}\mid 0$	$1*$	$2*$	3
4	$*$	$\frac{3}{2}\mid 0$	$2*$	$4*$	$\frac{95}{16}$
5	$*$	$\frac{5}{2}\mid 0$	$3*$	$6\mid\frac{11}{2}$	$\frac{17}{2}\mid 8$
6	$*$	$\frac{7}{2}\mid 0$	$4*$	$\frac{15}{2}\mid 7*$	$\frac{21}{2}$
7	$*$	$\frac{9}{2}\mid 0$	$5*$	9	13
			$c = 3$		
4	$*$	$1*\mid\downarrow$	$1\Vert\frac{1}{2}\mid 0$	$1*$	2
5	$*$	$2*\mid\downarrow$	$\frac{3}{2}$	$3*\mid\frac{5}{2}\Vert 2$	4
6	$*$	$3*\mid\downarrow$	$\frac{5}{2}$	$5*\mid\frac{7}{2}\Vert 3, 3{\uparrow}*$	$\frac{41}{8}$
7	$*$	$4*\mid\downarrow$	$\frac{7}{2}$	see below	
			$c = 4$		
5	$*$	$2\mid 1\Vert *\mid -1$	$1*$	$2*\mid 1*\Vert\frac{1}{2}\mid 0$	1
6	$*$	$3\mid 2\Vert *\mid -1$	$2*$	$4*\mid 2*\Vert\frac{3}{2}\mid 1$	
7	$*$	$4\mid 3\Vert *\mid -1$	$3*$		

$$\mathrm{T}^7\square^4\mathrm{F}^3 = 7*\mid\tfrac{9}{2}\Vert 4, 4\{0\mid\{\{\{2\mid\tfrac{1}{2}*\Vert\{2\mid\tfrac{3}{2}\Vert +_{\frac{1}{2}}\,|||\,0, \{0, *\Vert *\mid -\tfrac{1}{4}\}\}\}, \{\tfrac{3}{2}*\mid +_{\frac{1}{2}}\}\mid 0, \{0, *\Vert *\mid -\tfrac{1}{4}\}\},$$
$$\{\tfrac{3}{2}*\mid +_{\frac{1}{2}}\Vert 0\}\mid 0\}\}$$

Known values for positions $\mathrm{T}^a\square^b\mathrm{F}^c$. See Theorem 5.2 for $c = 1$.

Acknowledgments

The author thanks Dan Calistrate for verifying Theorems 3.5 and 7.7.

References

[Berlekamp et al. 1982] Elwyn Berlekamp, John Conway, and Richard Guy, *Winning Ways for your Mathematical Plays*, Academic Press, New York, 1982.

[Berlekamp and Wolfe 1994] Elwyn Berlekamp and David Wolfe, *Mathematical Go Endgames: Nightmares for the Professional Go Player*, Ishi Press, San Jose, 1994.

[Lipton 1991] James Lipton, *An Exaltation of Larks*, Viking, New York, 1991.

[Wolfe 1996] David Wolfe. See pages 93–98 in this volume.

[Yedwab 1985] Laura Yedwab, "On playing well in a sum of games", Master's Thesis, MIT, 1985. Available as Technical Report MIT/LCS/TR-348.

Jeff Erickson
Computer Science Division
University of California
Berkeley, CA 94720
jeffe@cs.berkeley.edu