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Scenic Trails Ascending from Sea-Level Nim to Alpine Chess

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Abstract. *Aim*: To present a systematic development of part of the theory of combinatorial games from the ground up. *Approach*: Computational complexity. Combinatorial games are completely determined; the questions of interest are efficiencies of strategies. *Methodology*: Divide and conquer. Ascend from Nim to chess in small strides at a gradient that's not too steep. *Presentation*: Informal; examples of games sampled from various strategic viewing points along scenic mountain trails illustrate the theory.

1. Introduction

All our games are two-player perfect-information games (no hidden information) without chance moves (no dice). Outcome is (lose, win) or (draw, draw) for the two players, who play alternately. We assume throughout *normal* play, i.e., the player making the last move wins and his opponent loses, unless *misère* play is specified, where the outcome is reversed. A draw is a dynamic tie, that is, a position from which neither player can force a win, but each has a nonlosing next move.

As we progress from the easy games to the more complex ones, we will develop some understanding of the *poset* of tractabilities and efficiencies of game strategies: whereas, in the realm of existential questions, tractabilities and efficiencies are by and large linearly ordered, from polynomial to exponential, for problems with an unbounded number of alternating quantifiers, such as games, the notion of a "tractable" or "efficient" computation is much more complex. (Which is more tractable: a game that ends after four moves, but it's undecidable who

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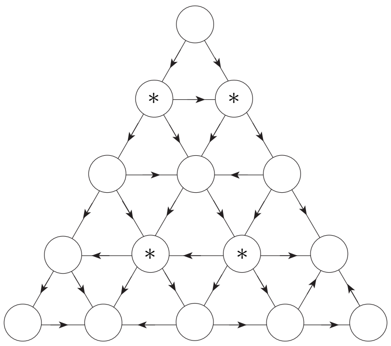

Figure 1. Beat Doug on this DAG (directed acyclic graph).

wins [Rabin 1957], or a game that takes an Ackermann number of moves to finish but the winner can play randomly, having to pay attention only near the end [Fraenkel, Loebl and Nešetřil 1988]?)

When we say that a computation or strategy is polynomial or exponential, we mean that the time it takes to evaluate the strategy is a polynomial or exponential function in a most succinct form of the input size.

In Section 2 we review the classical theory (impartial games without draws, no interaction between tokens). On our controlled ascent to chess we introduce in Section 3 draws, on top of which we then add interaction between tokens in Section 4. In Section 5 we review briefly partizan games. In Section 6 we show how the approach and methodology outlined in the abstract can help to understand some of the many difficulties remaining in the classical theory (numerous rocks are strewn also along other parts of the trails ascending towards chess). This then leads naturally to the notion of tractable and efficient games, taken up in Section 7, together with some more ways in which a game can become intractable or inefficient.

This paper is largely expository, yet it contains material, mainly in parts of Sections 6 and 7, not published before, to the best of our knowledge. The present review is less formal than [Fraenkel 1991]: the emphasis here is on examples that illustrate part of the theory. A fuller and more rigorous treatment is to appear in [Fraenkel $\geq$ 1997].

2. The Classical Theory

Here we will learn to play games such as "Beat Doug" (Figure 1).

Place one token on each of the four starred vertices. A move consists of selecting a token and moving it, along a directed edge, to a neighboring vertex on this acyclic digraph. Tokens can coexist peacefully on the same vertex. For the given position, what's the minimum time to:

(a) compute who can win;

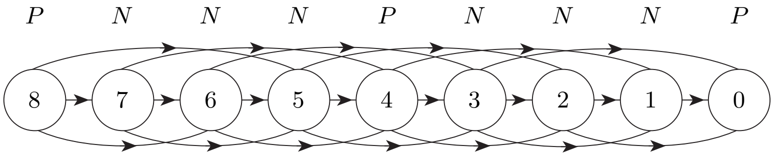

Figure 2. The game graph for Scoring, with initial score $n = 8$ and maximal step $t = 3$. Positions marked N are wins and P are losses for the player about to move.

(b) compute an optimal next move;
(c) consummate the win?

Following our divide and conquer methodology, let's begin with a very easy example, before solving Beat Doug. Given $n \in \mathbb{Z}^+$ (the initial score) and $t \in \mathbb{Z}^+$ (the maximal step size), a move in the game *Scoring* consists of selecting $i \in \{1, \ldots, t\}$ and subtracting i from the current score, initially n, to generate the new score. Play ends when the score 0 is reached.

The *game graph* $G = (V, E)$ for Scoring is shown in Figure 2, for $n = 8$ and $t = 3$. A position (vertex) $u \in V$ is labeled N if the player about to move from it can win; otherwise it's a P-position. Denoting by $\mathcal{P}$ the set of all P-positions, by $\mathcal{N}$ the set of all N-positions, and by $F(u)$ the set of all (direct) *followers* or *options* of any vertex u, we have, for any acyclic game,

$$u \in \mathcal{P} \quad \text{if and only if} \quad F(u) \subseteq \mathcal{N}, \tag{2.1}$$

$$u \in \mathcal{N} \quad \text{if and only if} \quad F(u) \cap \mathcal{P} \neq \varnothing. \tag{2.2}$$

As suggested by Figure 2, we have $\mathcal{P} = \{k(t + 1) : k \in \mathbb{Z}^0\}$ and $\mathcal{N} = \{\{0, \ldots, n\} - \mathcal{P}\}$. The winning strategy consists of dividing n by $t + 1$. Then $n \in \mathcal{P}$ if and only if the remainder r is zero. If $r > 0$, the unique winning move is from n to $n - r$. Is this a "good" strategy?

Input size: $\Theta(\log n)$ (*succinct* input).
Strategy computation: $O(\log n)$ (linear scan of the $\lceil \log n \rceil$ digits of n).
Length of play: $\lceil n/(t + 1) \rceil$.

Thus the computation time is linear in the input size, but the length of play is exponential. This does not prevent the strategy from being good: whereas we dislike computing in more than polynomial time, the human race relishes to see some of its members being tortured for an exponential length of time, from before the era of the Spanish matadors, through soccer and tennis, to chess and Go! But there are other requirements for making a strategy tractable, so at present let's say that the strategy is *reasonable*.

Now take k scores $n_1, \ldots, n_k \in \mathbb{Z}^+$ and $t \in \mathbb{Z}^+$, where $n_j \leq n$ for each j. A move consists of selecting one of the current scores and subtracting from it some $i \in \{1, \ldots, t\}$. Play ends when all the scores are zero. Figure 3 shows an example. This is a *sum* of Scoring games, itself also a Scoring game. It's easy to

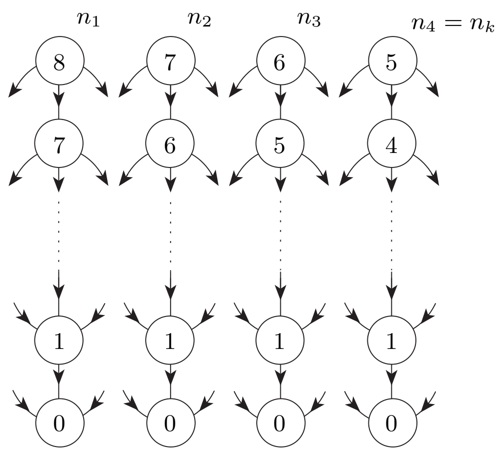

Figure 3. A Scoring game consisting of a sum of four Scoring games. Here $k = 4$, $n_1 = 8$, $n_2 = 7$, $n_3 = 6$, $n_4 = 5$, and $t = 3$.

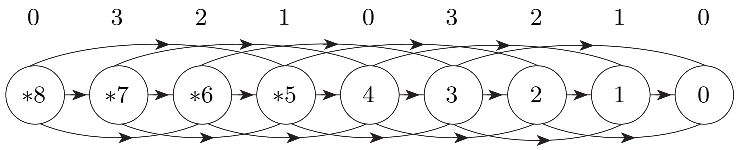

Figure 4. A game on a graph, but not a game graph.

see that the game of Figure 3 is equivalent to the game played on the digraph of Figure 4, with tokens on vertices 5, 6, 7 and 8. A move consists of selecting a token and moving it right by not more than $t = 3$ places. Tokens can coexist on the same vertex. Play ends when all tokens reside on 0. What's a winning strategy?

We hit two snags when trying to answer this question:

(i) The sum of N-positions is in $\mathcal{P} \cup \mathcal{N}$. Thus a token on each of 5 and 7 is seen to be an N-position (the move $7 \to 5$ clearly results in a P-position), whereas a token on each of 3 and on 7 is a P-position. So the simple P-, N-strategy breaks down for sums, which arise frequently in combinatorial game theory.

(ii) The game graph has exponential size in the input size $\Omega(k + \log n)$ of the "regular" digraph $G = (V, E)$ (with $|V| = n + 1$) on which the game is played with k tokens. However, this is not the game graph of the game: each tuple of k tokens on G corresponds to a single vertex of the game graph, whose vertex-set thus has size $\binom{k+n}{n}$—the number of k-combinations of $n+1$ distinct objects with at least k repetitions. For $k = n$ this gives $\binom{2n}{n} = \Theta(4^n/\sqrt{n})$.

The main contribution of the classical theory is to provide a polynomial strategy despite the exponential size of the game graph. On G, label each vertex u with the least nonnegative integer not among the labels of the followers of u (see top of Figure 4). These labels are called the *Sprague–Grundy* function of G, or the g-function for short [Sprague 1935–36; Grundy 1939]. It exists uniquely on every finite acyclic digraph. Then for $\boldsymbol{u} = (u_1, \ldots, u_k)$, a vertex of the game graph (whose very construction entails exponential effort), we have

$$g(\boldsymbol{u}) = g(u_1) \oplus \cdots \oplus g(u_k), \qquad \mathcal{P} = \{\boldsymbol{u} : g(\boldsymbol{u}) = 0\}, \qquad \mathcal{N} = \{\boldsymbol{u} : g(\boldsymbol{u}) > 0\},$$

where $\oplus$ denotes *Nim-sum* (summation over GF(2), also known as exclusive or). To compute a winning move from an N-position, note that there is some i for which $g(u_i)$ has a 1-bit at the binary position where $g(\boldsymbol{u})$ has its leftmost 1-bit. Reducing $g(u_i)$ appropriately makes the Nim-sum 0, and there's a corresponding move with the i-th token. For the example of Figure 4 we have

$$g(5) \oplus g(6) \oplus g(7) \oplus g(8) = 1 \oplus 2 \oplus 3 \oplus 0 = 0,$$

a P-position, so every move is losing.

Is the strategy polynomial? For Scoring, the remainders $r_1, \ldots, r_k$ of dividing $n_1, \ldots, n_k$ by $t+1$ are the g-values, as suggested by Figure 4. The computation of each r_j has size $O(\log n)$. Since $k \log n < (k + \log n)^2$, the strategy computation (items (a) and (b) at the beginning of this section) is polynomial in the input size. The length of play remains exponential.

Now consider a general nonsuccinct digraph $G = (V, E)$, by which we mean that the input size is not logarithmic. If the graph has $|V| = n$ vertices and $|E| = m$ edges, the input size is $\Theta((m + n) \log n)$ (each vertex is represented by its index of size $\log n$, and each edge by a pair of indices), and g can be computed in $O((m + n) \log n)$ steps (by a "depth-first" search; each g-value is at most n, of size at most $\log n$). For a sum of k tokens on the input digraph, the input size is $\Theta((k + m + n) \log n)$, and the strategy computation for the sum can be carried out in $O((k + m + n) \log n)$ steps (Nim-summing k summands of g-values). Note also that for a general digraph the length of play is only linear rather than exponential, as on a succinct (logarithmic input size) digraph.

Since the strategy for Scoring is polynomial for a single game as well as for a sum, we may say, informally, that it's a *tractable* strategy. (We'll see in Section 7 that there are further requirements for a strategy to be truly "efficient".)

Our original Beat Doug problem is now also solved with a tractable strategy. Figure 5 depicts the original digraph of Figure 1 with the g-values added in. Since $2 \oplus 3 \oplus 3 \oplus 4 = 6$, the given position is in $\mathcal{N}$. Moving $4 \to 2$ is a unique winning move. The winner can consummate his win in polynomial time.

Unfortunately, however, the strategy of classical games is not very robust: slight perturbations in various directions can make the analysis considerably more difficult. We'll return to this subject in Sections 6 and 7.

We point out that there is an important difference between the strategies of Beat Doug and Scoring. In both, the g-function plays a key role. But for the latter, some further property is needed to yield a strategy that's polynomial, since the input graph is (logarithmically) succinct. In this case the extra ingredient is the periodicity modulo $(t + 1)$ of g.

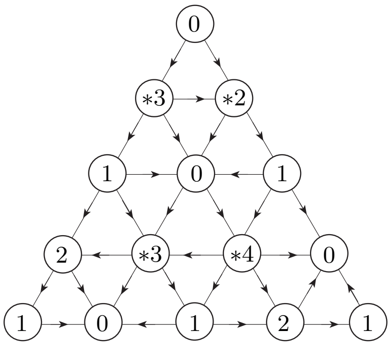

Figure 5. The beaten Doug.

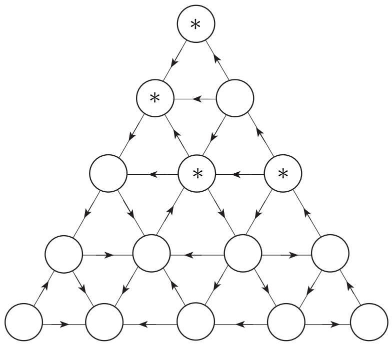

Figure 6. Beat Craig in this cyclic digraph.

3. Introducing Draws

In this section we learn how to beat Craig efficiently. The four starred vertices in Figure 6 contain one token each. The moves are identical to those of Beat Doug, and tokens can coexist peacefully on any vertex. The only difference is that now the digraph $G = (V, E)$ may have cycles, and also loops (which correspond to passing a turn). In addition to the P- and N-positions, which satisfy (2.1) and (2.2), we now may have also Draw-positions D, which satisfy

$$u \in \mathcal{D} \quad \text{if and only if} \quad F(u) \cap \mathcal{P} = \varnothing \text{ and } F(u) \cap \mathcal{D} \neq \varnothing,$$

where $\mathcal{D}$ is the set of all D-positions.

Introducing cycles causes several problems:

- Moving a token from an N-position such as vertex 4 in Figure 7 to a P-position such as vertex 5 is a nonlosing move, but doesn't necessarily lead to a win. A win is achieved only if the token is moved to the leaf 3. The digraph might be embedded inside a large digraph, and it may not be clear to which P-follower to move in order to realize a win.
- The partition of V into $\mathcal{P}, \mathcal{N}$ and $\mathcal{D}$ is not unique, as it is for $\mathcal{P}$ and $\mathcal{N}$ in the classical case. For example, vertices 1 and 2 in Figure 7, if labeled P and N, would still satisfy (2.1) and (2.2), and likewise for vertices 8 and 9 (either can be labeled P and the other N).

Both of these shortcomings can be remedied by introducing a proper counter function attached to all the P-positions [Fraenkel $\geq$ 1977].

For handling sums, we would like to use the g-function, but there are two problems:

- The question of the existence of g on a digraph G with cycles or loops is NP-complete, even if G is planar and its degrees are ≤ 3, with each indegree ≤ 2 and each outdegree ≤ 2 [Fraenkel 1981] (see also [Chvátal 1973; van Leeuwen 1976; Fraenkel and Yesha 1979]).

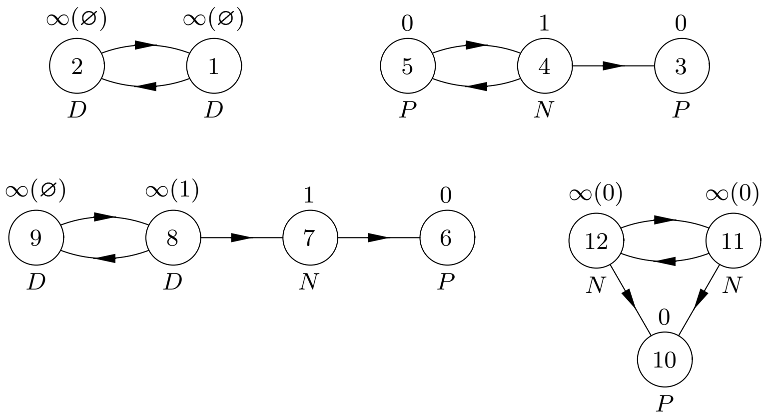

Figure 7. P-, N-, D- and γ-values for simple digraphs.

- The strategy of a cyclic game isn't always determined by the g-function, even if it exists.

This is one of those rare cases where two failures are better than one! The second failure opens up the possibility that perhaps there's another tool that always works, and if we are optimistic, we might even hope that it is also polynomial. There is indeed such a generalized g-function γ. It was introduced in [Smith 1966], and rediscovered in [Fraenkel and Perl 1975]; see also [Conway 1976, Ch. 11]. Of the possible definitions of γ, we give a simple one below; see [Fraenkel and Yesha 1986] for two other definitions and a proof of the equivalence of all three.

The γ-function is defined the same way as the g-function, except that it can assume not only values in $\mathbb{Z}^0 = \{n \in \mathbb{Z} : n \geq 0\}$, but also *infinite* values of the form $\infty(K)$, as we now explain. We have $\gamma(u) = \infty(K)$ if K is the set of finite γ-values of followers of u, and some follower v of u has the following properties: $\gamma(v)$ is also infinite, and v has no follower w with $\gamma(w)$ equal to the least nonnegative integer not in K. Figure 7 shows γ-values for some simple digraphs. Every finite digraph with n vertices and m edges has a unique γ-function that can be computed in $O(mn \log n)$ steps. This is a polynomial-time computation, though bigger than the g-values computation.

We also associate a (nonunique) counter function to every vertex with a finite γ-value, for the reasons explained above. Here is a more formal definition of γ. Given a digraph $G = (V, E)$ and a function $\gamma : V \to \mathbb{Z}^0 \cup \{\infty\}$, set

$$V^f = \bigl\{u \in V : \gamma(u) < \infty\bigr\}$$

and

$$\gamma'(u) = \operatorname{mex} \gamma\bigl(F(u)\bigr),$$

where, for any finite subset $S \subseteq \mathbb{Z}^0$, we define $\operatorname{mex} S = \min(\mathbb{Z}^0 - S)$. The function γ is a *generalized Sprague–Grundy function*, or *γ-function* for short, with *counter function* $c : V^f \to J$, where J is an infinite well-ordered set, if the following conditions hold:

A. If $\gamma(u) < \infty$, then $\gamma(u) = \gamma'(u)$.
B. If there exists $v \in F(u)$ with $\gamma(v) > \gamma(u)$, then there exists $w \in F(v)$ satisfying $\gamma(w) = \gamma(u)$ and $c(w) < c(u)$.
C. If, for every $v \in F(u)$ with $\gamma(v) = \infty$, there is $w \in F(v)$ with $\gamma(w) = \gamma'(u)$, then $\gamma(u) < \infty$.

Remarks. In B we have necessarily $u \in V^f$; and we may have $\gamma(v) = \infty$ as in C. If condition C is satisfied, then $\gamma(u) < \infty$, and so $\gamma(w) = \gamma'(u) = \gamma(u)$ by A. Condition C is equivalent to the following statement:

C′. If $\gamma(u) = \infty$, then there is $v \in F(u)$ with $\gamma(v) = \infty(K)$ such that $\gamma'(u) \notin K$.

To keep the notation simple, we write throughout $\infty(0)$ for $\infty(\{0\})$, $\infty(0, 1)$ for $\infty(\{0, 1\})$, and so on.

To get a strategy for sums, define the *generalized Nim-sum* as the ordinary Nim-sum augmented by:

$$a \oplus \infty(L) = \infty(L) \oplus a = \infty(L \oplus a), \qquad \infty(K) \oplus \infty(L) = \infty(\varnothing),$$

where $a \in \mathbb{Z}^0$ and $L \oplus a = \{l \oplus a : l \in L\}$. For a sum of k tokens on a digraph $G = (V, E)$, let $\boldsymbol{u} = (u_1, \ldots, u_k)$. We then have $\gamma(\boldsymbol{u}) = \gamma(u_1) \oplus \cdots \oplus \gamma(u_k)$, and

$$\left.\begin{aligned} \mathcal{P} &= \big\{\boldsymbol{u} : \gamma(\boldsymbol{u}) = 0\big\}, \\ \mathcal{N} &= \big\{\boldsymbol{u} : 0 < \gamma(\boldsymbol{u}) < \infty\big\} \cup \big\{\boldsymbol{u} : \gamma(\boldsymbol{u}) = \infty(K) \text{ and } 0 \in K\big\}, \\ \mathcal{D} &= \big\{\boldsymbol{u} : \gamma(\boldsymbol{u}) = \infty(K) \text{ and } 0 \notin K\big\}. \end{aligned}\right\} \tag{3.1}$$

Thus a sum consisting of a token on vertex 4 and one on 8 in Figure 7 has γ-value $1 \oplus \infty(1) = \infty(1 \oplus 1) = \infty(0)$, which is an N-position (the move $8 \to 7$ results in a P-position). Also one token on 11 or 12 is an N-position. But a token on both 11 and 12 or on 8 and 12 is a D-position of their sum, with γ-value $\infty(\varnothing)$. A token on 4 and 7 is a P-position of the sum.

With k tokens on a digraph, the strategy for the sum can be computed in $O((k + mn)\log n)$ steps. It is polynomial in the input size $\Theta((k + m + n)\log n)$, since $k + mn < (k + m + n)^2$. Also, for certain succinct "linear" graphs, γ provides a polynomial strategy. See [Fraenkel and Tassa 1975].

Beat Craig is now also solved with a tractable strategy. From the γ-values of Figure 8 we see that the position given in Figure 6 has γ-value $0 \oplus 1 \oplus 2 \oplus \infty(2, 3) = 3 \oplus \infty(2, 3) = \infty(1, 0)$, so by (3.1) it's an N-position, and the unique winning move is $\infty(2, 3) \to 3$. Again the winner can force a win in polynomial time, and can also delay it arbitrarily long, but this latter fact is less interesting.

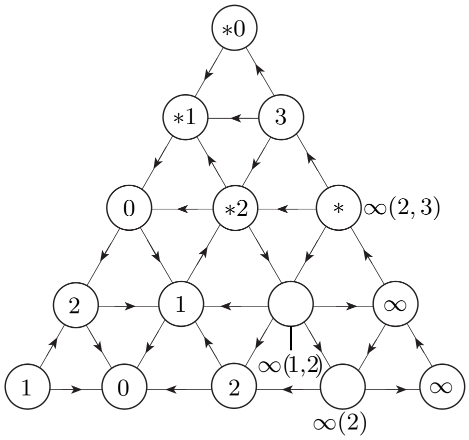

Figure 8. Craig has also been beaten.

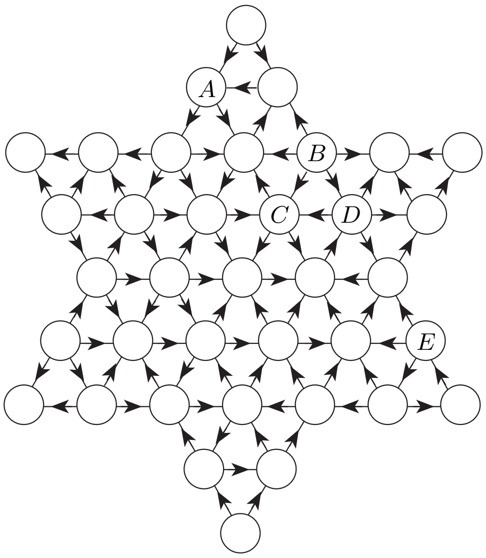

Figure 9. Beat an even bigger Craig.

As a homework problem, beat an even bigger Craig: compute the labels P, N, D for the digraph of Figure 9 with tokens placed as shown, or for various other initial token placements.

4. Adding Interactions between Tokens

Here we learn how to beat Anne. On the five-component digraph depicted in Figure 10, place tokens at arbitrary locations, but at most one token per vertex. A move is defined as in the previous games, but if a token is moved onto an occupied vertex, both tokens are annihilated (removed). The digraph has cycles, and could also have loops (passing positions). Note that the three components

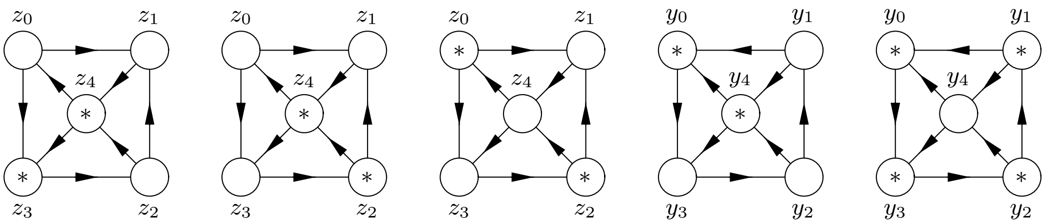

Figure 10. Beat Anne in this Ann-ihilation game.

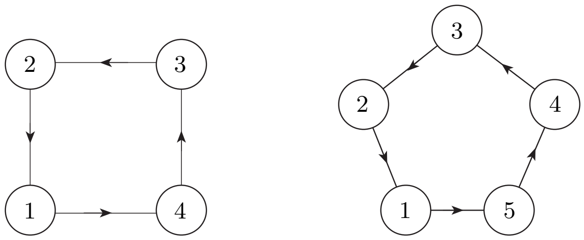

Figure 11. Annihilation on simple cyclic digraphs.

having z-vertices are identical, as are the two y-components. The only difference between a z- and a y- component is in the orientation of the top horizontal edge. With tokens on the twelve starred vertices, can the first player win or at least draw, and if so, what's an optimal move? How "good" is the strategy?

The indicated position may be a bit complicated as a starter. So consider first a position consisting of four tokens only: one on z_3 and the other on z_4 in two of the z-components. Also consider the position consisting of a single token on each of y_3 and y_4 in each y-component. It's clear that in these cases player 2 can at least draw, simply by imitating on one component what player 1 does on the other. Can player 2 actually win in one or both of these games?

Annihilation games were proposed by Conway. It's easy to see that on a finite acyclic digraph, annihilation can affect the length of play, but the strategy is the same as for the classical games. Since $g(u) \oplus g(u) = 0$, the winner doesn't need to use annihilation, and the loser cannot be helped by it. But the situation is quite different in the presence of cycles. In Figure 11 (left), a token on each of vertices 1 and 3 is clearly a D-position for the nonannihilation case, but it's a P-position when played with annihilation (the second move is a winning annihilation move). In Figure 11 (right), with annihilation, a token on each of 1 and 2 is an N-position, whereas a token on each of 1 and 3 is a D-position. The theory of annihilation games is discussed in depth in [Fraenkel and Yesha 1982]; see also [Fraenkel 1974; Fraenkel and Yesha 1976; 1979; Fraenkel, Tassa and Yesha 1978]. Ferguson [1984] considered misère annihilation play.

The *annihilation graph* is a certain game graph of an annihilation game. The annihilation graph of the annihilation game played on the digraph of Figure 11 (left) consists of two components. One is depicted in Figure 12, namely, the com-

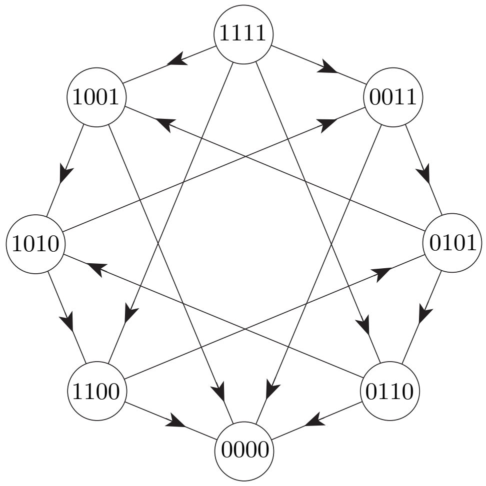

Figure 12. The "even" component $\boldsymbol{G}^0$ of the annihilation graph $\boldsymbol{G}$ of the digraph of Figure 11 (left).

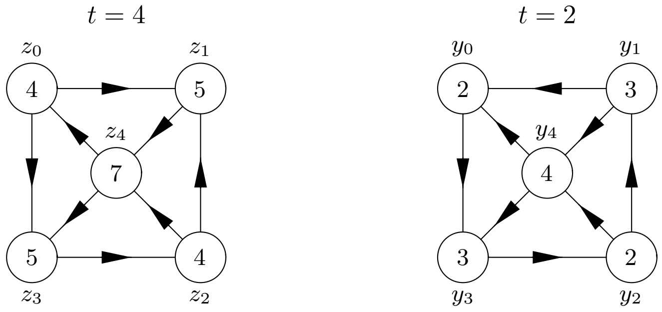

Figure 13. The $\underline{\gamma}$-function.

ponent $\boldsymbol{G}^0 = (\boldsymbol{V}^0, \boldsymbol{E}^0)$ with an even number of tokens. The "odd" component $\boldsymbol{G}^1$ also has 8 vertices. In general, a digraph $G = (V, E)$ with $|V| = n$ vertices has an annihilation graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ with $|\boldsymbol{V}| = 2^n$ vertices, namely all n-dimensional binary vectors. The γ-function on $\boldsymbol{G}$ determines whether any given position is in $\mathcal{P}$, $\mathcal{N}$ or $\mathcal{D}$, according to (3.1); and γ, together with its associated counter function, determines an optimal next move from an N- or D-position.

The only problem is the exponential size of $\boldsymbol{G}$. We can recover an $O(n^6)$ strategy by computing an *extended γ-function* $\underline{\gamma}$ on an induced subgraph of $\boldsymbol{G}$ of size $O(n^4)$, namely, on all vectors of weight ≤ 4 (at most four 1-bits). In Figure 13, the numbers inside the vertices are the $\underline{\gamma}$-values, computed by Gaussian elimination over GF(2) of an $n \times O(n^4)$ matrix. This computation also yields the values $t = 4$ for Figure 13 (left) and $t = 2$ for Figure 13 (right): If $\underline{\gamma}(\boldsymbol{u}) \ge t$, then $\gamma(\boldsymbol{u}) = \infty$, whereas $\underline{\gamma}(\boldsymbol{u}) < t$ implies $\gamma(\boldsymbol{u}) = \underline{\gamma}(\boldsymbol{u})$.

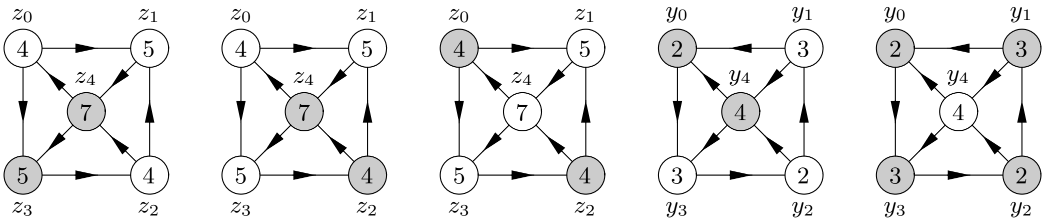

Figure 14. Poor beaten Anne. (Gray circles show initial token positions.)

Thus for Figure 13 (left), we have $\underline{\gamma}(z_3, z_4) = 5 \oplus 7 = 2 < 4$, so $\gamma(z_3, z_4) = 2$. Hence two such copies constitute a P-position ($2 \oplus 2 = 0$). (How can player 2 consummate a win?) In Figure 13 (right) we have $\underline{\gamma}(y_3, y_4) = 3 \oplus 4 = 7 > 2$, so $\gamma(y_3, y_4) = \infty$, in fact, $\infty(0, 1)$, so two such copies constitute a D-position. The position given in Figure 10 is repeated in Figure 14, together with the $\underline{\gamma}$-values. From left to right we have: for the z-components, $\gamma = 2 \oplus 3 \oplus 0 = 1$; and for the y-components, $\infty(0, 1) + 0 = \infty(0, 1)$, so the γ-value is $\infty(0, 1) \oplus 1 = \infty(0, 1)$. Hence the position is an N-position by (3.1). There is, in fact, a unique winning move, namely $y_4 \to y_3$ in the second component from the right. Any other move leads to drawing or losing.

For small digraphs, a counter function c is not necessary, but for larger ones it's needed for consummating a win. The trouble is that we computed γ and c only for an $O(n^4)$ portion of $\boldsymbol{G}$. Whereas γ can then be extended easily to all of $\boldsymbol{G}$, this does not seem to be the case for c. We illustrate a way out on the digraph shown in Figure 13 (right). Suppose that the beginning position is $\boldsymbol{u} = (y_0, y_1, y_2, y_3)$, which has γ-value 0, as can be seen from Figure 13.

With $\boldsymbol{u}$ we associate a *representation* of vectors of weight ≤ 4, each with γ-value 0, in this case, $\tilde{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$, where $\boldsymbol{u}_1 = (y_0, y_2)$ and $\boldsymbol{u}_2 = (y_1, y_3)$, with $\boldsymbol{u} = \boldsymbol{u}_1 \oplus \boldsymbol{u}_2$. Representations can be computed in polynomial time, and a counter function c can be defined on them. Suppose player 1 moves from $\boldsymbol{u}$ to $\boldsymbol{v} = (y_1, y_2)$. The representation $\tilde{v}$ is obtained by carrying out the move on the representation $\tilde{u}$: $F_{0,3}(\boldsymbol{u}_1)$ (i.e., moving from y_0 to y_3) gives $\boldsymbol{u}_3 = (y_2, y_3)$, so $\tilde{v} = (\boldsymbol{u}_3, \boldsymbol{u}_2)$, with $\boldsymbol{u}_3 \oplus \boldsymbol{u}_2 = \boldsymbol{v}$. Now player 2 would like to move to some $\boldsymbol{w}$ with $\gamma(\boldsymbol{w}) = 0$ and $c(\tilde{w}) < c(\tilde{u})$, namely, $\tilde{w} = (\boldsymbol{u}_2)$. However, we see that $\boldsymbol{u}_2$ is a predecessor (immediate ancestor) of $\boldsymbol{v}$, rather than a follower of $\boldsymbol{v}$. Player 2 now pretends that player 1 began play from $\boldsymbol{u}_2$ rather than from $\boldsymbol{u}$, so arrived at $\boldsymbol{v}$ with representation $F_{3,2}(\boldsymbol{u}_2) = \tilde{v}$. This has the empty representation as follower, so player 2 makes the final annihilation move. Followers of representations can always be chosen with c-value smaller than the c-value of their grandfather, and they always correspond to either a follower or predecessor of a position. Since the initial counter value has value $O(n^5)$, player 2 can win in that many moves, using an $O(n^6)$ computation.

This method can easily be extended to handle sums. Thus, according to our definition, we have a tractable strategy for annihilation games. Yet clearly it would be nice to improve on the $O(n^6)$ and to simplify the construction of the counter function.

Is there a narrow winning strategy that's polynomial? A strategy is *narrow* if it uses only the present position u for deciding whether u is a P-, N-, or D-position, and for computing a next optimal move. It is *broad* [Fraenkel 1991] if the computation involves any of the possible predecessors of u, whether actually encountered or not. It is *wide* if it uses any ancestor that was actually encountered in the play of the game. Kalmár [1928] and Smith [1966] defined wide strategies, but then both immediately reverted back to narrow strategies, since both authors remarked that the former do not seem to have any advantage over the latter. Yet for annihilation games we were able to exhibit only a broad strategy that is polynomial. Is this the alpine wind that's blowing?

Incidentally, for certain (Chinese) variations of Go, for chess and some other games, there are rules that forbid certain repetitions of positions, or modify the outcome in the presence of such repetitions. Now if all the history is included in the definition of a move, then every strategy is narrow. But the way Kalmár and Smith defined a move—much the same as the intuitive meaning—there is a difference between a narrow and wide strategy for these games. We also mention here the notion of "positional strategy" [Ehrenfeucht and Mycielski 1979; Zwick and Paterson $\geq$1996]. See also [Beck 1981; 1982; 1985; Chvátal and Erdös 1978].

As a homework problem, compute the label $\in \{P, N, D\}$ of the stellar configuration marked by letters in "Simulation of the SL comet fragments' encounter with Jupiter" (Figure 15), where J is Jupiter, the other letters are various fragments of the comet, and all the vertices are "space-stations". A move consists of selecting Jupiter or a fragment, and moving it to a neighboring space-station along a directed trajectory. Any two bodies colliding on a space-station explode and vanish in a cloud of interstellar dust. Note the six space-stations without exit, where a body becomes a "falling star". Is the given position a win for the (vicious) player 1, who aims at making the last move in the destruction of this subsystem of the solar system, or for the (equally vicious) player 2? Or is it a draw, so that a part of this subsystem will exist forever? And if so, can it be arranged for Jupiter to survive as well? (The encounter of the Shoemaker–Levy comet with Jupiter took place during the MSRI workshop.)

Various impartial and partizan variations of annihilation games were shown to be NP-hard, Pspace-complete or Exptime-complete [Fraenkel and Yesha 1979; Fraenkel and Goldschmidt 1987; Goldstein and Reingold 1995]. We mention here only briefly an interaction related to annihilation. Electrons and positrons are positioned on vertices of the game Matter and Antimatter (Figure 16). A move consists of moving a particle along a directed trajectory to an adjacent station—if not occupied by a particle of the same kind, since two electrons (and two positrons) repel each other. If there is a resident particle, and the incoming particle is of opposite type, they annihilate each other, and both disappear from the play. It is not very hard to determine the label of any position on the

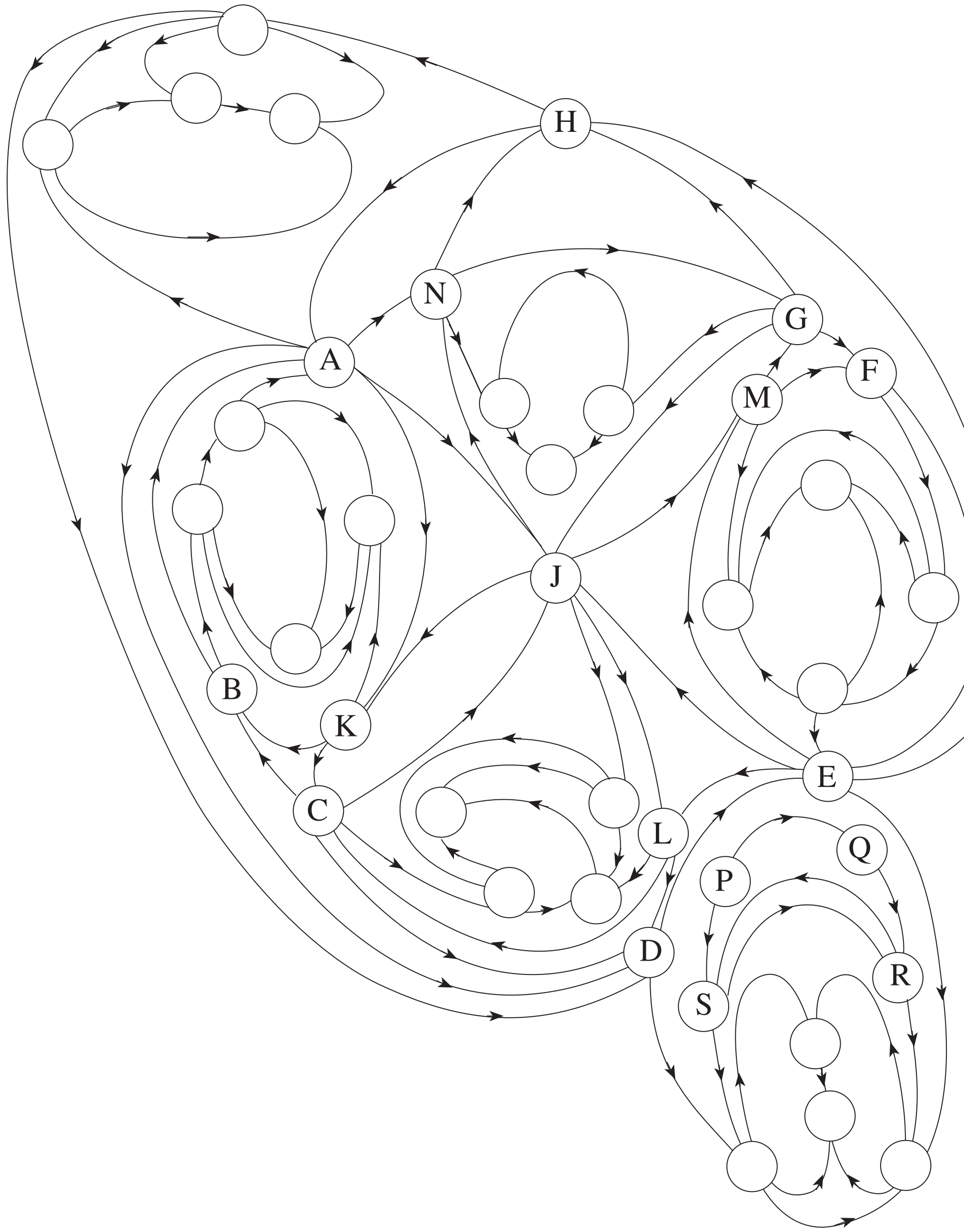

Figure 15. Interstellar encounter with Jupiter.

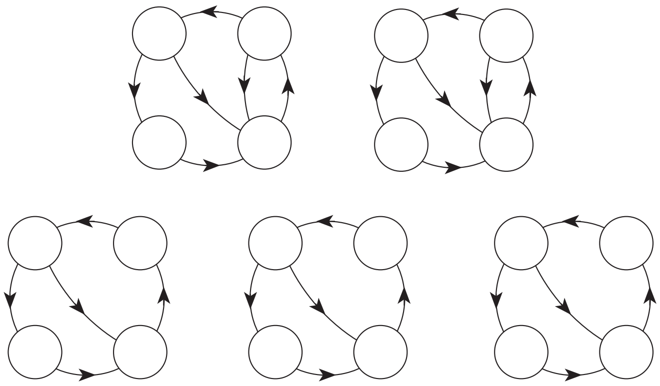

Figure 16. Matter and Antimatter.

given digraph. But what can be said about a general digraph? About succinct digraphs? Note that the special case where all the particles are of the same type, is the generalization of Welter played on the given digraph. Welter [Conway 1976, Ch. 13] is Nim with the restriction that no two piles have the same size. It has a polynomial strategy, but its validity proof is rather intricate. In Nim we are given finitely many piles. A move consists of selecting a pile and removing any positive number of tokens from it. The classical theory (Section 2) shows that the P-positions for Nim are simply those pile collections whose sizes Nim-add to zero.

5. Partizan Games

A game is *impartial* if both players have the same set of moves for all game positions. Otherwise the game is *partizan*. Nim-like games are impartial. Chess-like games are partizan, because if Gill plays the black pieces, Jean will not let him touch the white pieces.

In this section we shall refer to partizan games simply as games. The following two inductive definitions are due to Conway [1976; 1977; 1978a].

(i) If M^L and M^R are any two sets of games, there exists a game $M = \{M^L \mid M^R\}$. All games are constructed in this way.
(ii) If LE and RI are any two sets of numbers and no member of LE is $\geq$ any member of RI, then there exists a number $\{LE \mid RI\}$. All numbers are constructed in this way.

Thus the numbers constitute a subclass of the class of games.

The first games and numbers are created by putting $M^L = M^R = LE = RI = \varnothing$. Some samples are given in Figure 17, where L (Left) plays to the south-west and R (Right) to the south-east. If, as usual, the player first unable to move is the loser and his opponent the winner, then the examples suggest the following statements:

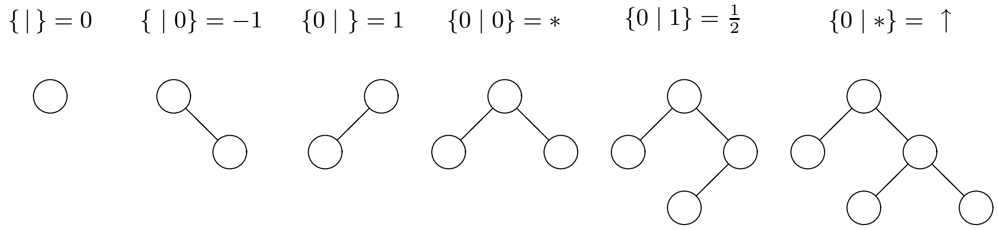

Figure 17. A few early partizan games.

$M > 0$ if L can win,
$M < 0$ if R can win,
$M = 0$ if player 2 can win,
$M \parallel 0$ if player 1 can win (for example in $*$).

We shall in fact define $>$, $<$, $=$, $\parallel$ by these conditions. The relations can be combined as follows:

$M \ge 0$ if L can win as player 2,
$M \le 0$ if R can win as player 2,
$M \mathrel{|\!\rhd} 0$ if L can win as player 1,
$M \mathrel{\lhd\!|} 0$ if R can win as player 1.

Alternative inductive definitions of $\le$ and $\parallel$ can be given. Let $x = \{x^L \mid x^R\}$. Then

$x \le y$ if and only if $x^L \mathrel{\lhd\!|} y$ for all x^L and $x \mathrel{\lhd\!|} y^R$ for all y^R,
$x = y$ if and only if $x \le y$ and $y \le x$,
$x \parallel y$ (x is *fuzzy* with y) if and only if $x \not\le y$ and $y \not\le x$.

One can also provide a consistency proof of both definitions. This enables one to prove many properties of games in a simple manner. For example, define $-M = \{-M^R \mid -M^L\}$. Then $G - G = 0$ (player 2 can win in $G - G$ by imitating the moves of player 1 in the other game). Also:

$$x \le y \mathrel{\lhd\!|} z \qquad \text{or} \qquad x \mathrel{\lhd\!|} y \le z \Rightarrow x \mathrel{\lhd\!|} z$$

(consider the game $z - x = (z - y) + (y - x)$, in which L can win as first player), and

$$x^L \mathrel{\lhd\!|} x \mathrel{\lhd\!|} x^R$$

(clearly L can win as player 1 in $x - x^L$ and in $x^R - x$).

If x is a number, then $x^L < x < x^R$.

Most important is that a sum of games simply becomes a *sum*, defined by:

$$M + H = \{M^L + H, M + H^L \mid M^R + H, M + H^R\}.$$

Consider for example the game of Domineering [Conway 1976, Ch. 10; Berlekamp, Conway and Guy 1982, Ch. 5], played with dominoes each of which covers precisely two squares of an $n \times n$ chessboard. The player L tiles vertically, R

$\{\,|\,\} = 0$ $\{\,|\,0\} = -1$ $\{0\,|\,\} = 1$ $\{0\,|\,0\} = *$ $\{1\,|\,-1\} = \pm 1$ $\{0,-1\,|\,1\} = \frac{1}{2}$

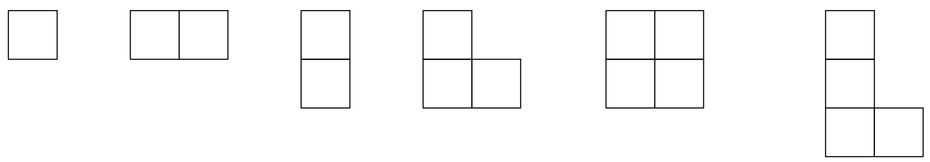

Figure 18. A few values of a Domineering game.

horizontally. After Domineering is played for a while, the board may break up into several parts, whence the game becomes a sum of these parts. The values of several small configurations are given in Figure 18.

Since the relation between the value of a game M and 0 determines the strategy for playing it, computing the value of M is fundamental to the theory. In this direction the *Simplicity Theorem* is helpful: Let $x = \{x^L \mid x^R\}$ be a game. If there exists a *number* z such that

$$x^L \lhd z \rhd x^R$$

and no option of z satisfies this relation, then $x = z$. Thus $\{-1 \mid 99\} = 0$. There also exist algorithms for computing values associated with games that are not numbers. Partizan games with possible draws are discussed in [Li 1976; Conway 1978b; Shaki 1979; Fraenkel and Tassa 1982; Berlekamp, Conway and Guy 1982, Ch. 11].

Sums of partizan games are Pspace-complete [Morris 1981], even if the component games have the form $\{a\,\|\,\{b|c\}\}$ with $a, b, c \in \mathbb{Z}$ [Yedwab 1985; Moews 1994].

6. Sticky Classical Games

As we mentioned in the penultimate paragraph of Section 2, the classical theory is nonrobust (the same holds, even more so, for nonclassical theories). Let's examine a few of the things that can go wrong.

First consider misère play. The *pruned game graph* G, that is, the game graph from which all leaves have been pruned, gives the same full information for misère play as the game graph gives for normal play: its P-, N-positions determine who can win. But in the nontrivial cases, G is exponentially large, so of little help. The game graph G is neither the sum of its components nor of its pruned components, which might explain why nobody seems to have a general theory for misère play, though important advances have been made for many special subclasses and particular games. See, for example, [Berlekamp, Conway and Guy 1982, Ch. 13; Sibert and Conway 1992; Plambeck 1992; Banerji and Dunning 1992].

The classical theory is also very sensitive to interaction between tokens, which again expresses itself as the inability to decompose the game into sums. We mention only two examples: annihilation and Welter. There's an involved special theory for each of these games, but no general unifying theory seems to be known.

Yet another problem is posed by the succinctness of the input size of many interesting games—for example, *octal games*. In an octal game, n tokens are arranged as a linear array (input size: $\Theta(\log n)$), and the array is reduced and/or subdivided according to rules encoded in octal [Guy and Smith 1956; Berlekamp, Conway and Guy 1982, Ch. 4; Conway 1976, Ch. 11]. It's not known whether there's an octal game with finitely many nonzero octal digits that doesn't have a polynomial strategy, though for many such octal games no polynomial strategy has (yet?) been found. Some of them have such a huge period or preperiod that their polynomial nature asserts itself only for impractically high values of n. See [Gangolli and Plambeck 1989].

Of course all these and many more problems bestow upon the class of combinatorial games its distinctive, interesting and challenging flavor.

Let us now examine in some more detail the difficulties presented by a particular classical game, played with just two piles: Wythoff's game [Wythoff 1907; Yaglom and Yaglom 1967]. In this game there's a choice between two types of moves: you can either take any positive number of tokens from a single pile, just as in Nim, or you can remove the same (positive) number of tokens from both piles.

The first few g-values are depicted in Table 1. The structure of the 0's is well-understood, and can be computed in polynomial time [Fraenkel 1982; Fraenkel and Borosh 1973]. But the nonzero values appear at positions that are not yet well-understood. Some structure of the 1-values is portrayed in [Blass and Fraenkel 1990]; see [Pink 1991] for further developments. But no polynomial construction of all the nonzero g-values seems to be known—not even whether such a construction is likely to exist or not.

Why is this the case? The experts say that it's due to the nondisjunctive move of taking from both piles. To test this opinion, let's consider a game, to be called $(k, k+1)$-Nimdi (for reasons to become clear later); see Table 2 for the first few g-values. In this game a move consists of either taking any positive

0	1	2	3	4	5	6	7	8	9	10	11
1	2	0	4	5	3	7	8	6	10	11	9
2	0	1	5	3	4	8	6	7	11	9	10
3	4	5	6	2	0	1	9	10	12	8	7
4	5	3	2	7	6	9	0	1	8	13	14
5	3	4	0	6	8	10	1	2	7	12	15
6	7	8	1	9	10	3	4	5	13	0	2

Table 1. (k, k)-Wythoff.

number from a single pile, or else k from one and $k+1$ from the other, for an arbitrary $k \in \mathbb{Z}^+$.

It won't take long for the reader to see that these values are exactly the same as those for *Nim*. The same holds if "for an arbitrary k" is replaced by "for a fixed k", say $k = 2$. So Table 2 also gives the g-values for $(2,3)$-Nimdi. Let's now consider the same game, but with $(2,3)$ replaced by $(1,3)$. In other words, a move consists of taking any positive number of tokens from a single pile, or else 1 token from one pile and 3 from the other. The first few g-values are listed in Table 3.

The next empty entry, for $(2,3)$, should be $2 \oplus 3 = 1$, according to the Nim-sum rule. However, the true value is 4. The reason is that $(2,3)$ has a follower $(2,3) - (1,3) = (1,0)$, which already has value 1. In other words, the g-value 1 of Nim has been "short-circuited" in $(1,3)$-Nimhoff! Note that in Wythoff, taking (k,k) short-circuits 0-values, but in $(k,\, k+1)$-Nimdi, no g-values have been short-circuited.

More generally, given piles of sizes $(a_1, \ldots, a_n)$ and a move set of the form $S = (b_1, \ldots, b_n)$, where $a_i \in \mathbb{Z}^+$ and $b_i \in \mathbb{Z}^0$ for $i \in \{1, \ldots, n\}$ such that $b_1 + \ldots + b_n > 0$, the moves of the game *Take* are of two types:

(i) Taking any positive number m from a single pile, so

$$(a_1, \ldots, a_n) \to (a_1,\, \ldots,\, a_{i-1},\, a_i - m,\, a_{i+1},\, \ldots,\, a_n).$$

(ii) Taking $b_1, \ldots, b_n$ from all the piles, so

$$(a_1, \ldots, a_n) \to (a_1 - b_1,\, \ldots,\, a_n - b_n).$$

Under what conditions is Take a Nimdi (Nim-in-disguise) game, i.e., a game whose g-values are exactly those of Nim? Blass and Fraenkel (to appear) have proved that Take is a Nimdi-game if and only if S is an odd set in the following sense: let m be the nonnegative integer such that $a_i 2^{-m}$ is an integer for all i but $a_j 2^{-m-1}$ is not an integer for some j. Then S is *odd* if $\sum_{k=1}^{n} a_k 2^{-m}$ is an odd integer. The divide and conquer methodology now suggests that we approach Wythoff gradually, by short-cicuiting only retricted sets of g-values of Nim, thus

0	1	2	3	4	5	6	7	8	9	10	11
1	0	3	2	5	4	7	6	9	8	11	10
2	3	0	1	6	7	4	5	10	11	8	9
3	2	1	0	7	6	5	4	11	10	9	8

Table 2. The first few g-values of $(k, k+1)$-Nimdi.

0	1	2	3	4	5	6	7	8	9	10	11
1	0	3	2	5	4	7	6	9	8	11	10
2	3	0									

Table 3. The first few g-values of (1,3)-Nimhoff.

generating a family of non-Nimdi games. Several such case studies of Nimhoff games are included in [Fraenkel and Lorberbom 1991].

For example, consider *Cyclic Nimhoff*, so named because of the cyclic structure of the g-values, where the restriction is $0 < \sum_{i=1}^{n} b_i < h$, where $h \in \mathbb{Z}^+$ is a fixed parameter. Thus $h = 1$ or $h = 2$ is Nim; $n = 2$, $h = 3$ is the fairy chess king-rook game; and $n = 2$, $h = 4$ is the fairy chess game king-rook-knight game. For a general cyclic Nimhoff game with pile sizes $(a_1, \ldots, a_n)$, we have

$$g(a_1, \ldots, a_n) = h(\lfloor a_1/h \rfloor \oplus \cdots \oplus \lfloor a_n/h \rfloor) + (a_1 + \cdots + a_n) \bmod h.$$

This formula implies that cyclic Nimhoff has a polynomial strategy for every fixed h. Note the combination of Nim-sum and ordinary sum, somewhat reminiscent of the strategy of Welter.

As another example, consider 2^k-*Nimhoff*, where k is any fixed positive integer. In this game we can remove 2^k tokens from two distinct piles, or remove a positive number of tokens from any single pile. Define the k-*Nim-sum* by $a \, ⓚ \, b = a \oplus b \oplus a^k b^k$. In other words, the k-Nim-sum of a and b is $a \oplus b$, unless the k-th bits of a and b are both 1, in which case the least significant bit of $a \oplus b$ is complemented. The k-Nim-sum is not a generalization of Nim-sum, but it is associative. For 2^k-Nimhoff with n piles we then have

$$g(a_1, \ldots, a_n) = a_1 \, ⓚ \cdots ⓚ \, a_n.$$

Now that we have gained some understanding of the true nature of Wythoff's game, we can exploit it in at least two ways:

1. Interesting games seem to be obtained when we adjoin to a game its P-positions as moves! For example, consider W^2, which is Nim with the adjunction of (k, k) and Wythoff's P-positions as moves. We leave it as an exercise to compute the P-positions of W^2, and to iterate other games in the indicated manner.

2. A generalization of Wythoff to more than two piles has long been sought. It's now clear what has to be done: for three-pile Wythoff, the moves are to either take any positive number of tokens from a single pile, or take from all three, say k, l, m (with $k + l + m > 0$), such that $k \oplus l \oplus m = 0$. This is clearly a generalization of the usual two-pile Wythoff's game. Initial values of the P-positions [Chaba and Fraenkel] are listed in Table 4, namely the cases $j = 0$ (one of the three piles is empty—the usual Wythoff game) and $j = 1$ (one of the three piles has size 1). Recall that for two-pile Wythoff the golden section plays an important role. The same holds for three-pile Wythoff, except that there are many "initial disturbances", as in so many other impartial games.

A rather curious variation of the classical theory is *epidemiography*, motivated by the study of long games, especially the Hercules–Hydra game; see the survey article [Nešetřil and Thomas 1987]. Several perverse and maniacal forms of the malady were examined in [Fraenkel and Nešetřil 1985; Fraenkel, Loebl and Nešetřil 1988; Fraenkel and Lorberbom 1989]. The simplest variation is a mild case of *Dancing Mania*, called *Nimania*, sometimes observed in post-*pneumonia* patients.

In Nimania we are given a positive integer n. Player 1 begins by subtracting 1 from n. If $n = 1$, the result is the empty set, and the game ends with player 1 winning. If $n > 1$, one additional copy of the resulting number $n-1$ is adjoined, so at the end of the first move there are two (indistinguishable) copies of $n-1$ (denoted $(n-1)^2$). At the k-th stage, where $k \geq 1$, a move consists of selecting a copy of a positive integer m of the present position, and subtracting 1 from it. If $m = 1$, the copy is deleted. If $m > 1$, then k copies of $m - 1$ are adjoined to the resulting $m - 1$ copies.

D^0	j	B^0	C^0	D^0	j	B^0	C^0	D^1	j	B^1	C^1	D^1	j	B^1	C^1
0	0	0	0	31	0	50	81	0	1	1	1	34	1	50	84
1	0	1	2	32	0	51	83	1	1	3	4	35	1	53	88
2	0	3	5	33	0	53	86	4	1	5	9	36	1	54	90
3	0	4	7	34	0	55	89	6	1	6	12	37	1	55	92
4	0	6	10	35	0	56	91	7	1	7	14	38	1	57	95
5	0	8	13	36	0	58	94	3	1	8	11	39	1	59	98
6	0	9	15	37	0	59	96	8	1	10	18	40	1	61	101
7	0	11	18	38	0	61	99	9	1	13	22	41	1	62	103
8	0	12	20	39	0	63	102	5	1	15	20	42	1	63	105
9	0	14	23	40	0	64	104	12	1	16	28	43	1	65	108
10	0	16	26	41	0	66	107	10	1	17	27	44	1	66	110
11	0	17	28	42	0	67	109	11	1	19	30	45	1	67	112
12	0	19	31	43	0	69	112	15	1	21	36	46	1	69	115
13	0	21	34	44	0	71	115	16	1	23	39	47	1	71	118
14	0	22	36	45	0	72	117	13	1	24	37	48	1	72	120
15	0	24	39	46	0	74	120	18	1	25	43	49	1	74	123
16	0	25	41	47	0	76	123	14	1	26	40	50	1	76	126
17	0	27	44	48	0	77	125	20	1	29	49	51	1	78	129
18	0	29	47	49	0	79	128	21	1	31	52	52	1	80	132
19	0	30	49	50	0	80	130	19	1	32	51	53	1	82	135
20	0	32	52	51	0	82	133	23	1	33	56	54	1	83	137
21	0	33	54	52	0	84	136	24	1	34	58	55	1	85	140
22	0	35	57	53	0	85	138	25	1	35	60	56	1	86	142
23	0	37	60	54	0	87	141	26	1	38	64	57	1	87	144
24	0	38	62	55	0	88	143	27	1	41	68	58	1	89	147
25	0	40	65	56	0	90	146	28	1	42	70	59	1	91	150
26	0	42	68	57	0	92	149	29	1	44	73	60	1	93	153
27	0	43	70	58	0	93	151	30	1	45	75	61	1	94	155
28	0	45	73	59	0	95	154	31	1	46	77	62	1	96	158
29	0	46	75	60	0	97	157	32	1	47	79	63	1	97	160
30	0	48	78	61	0	98	159	33	1	48	81	64	1	99	163

Table 4. Initial P-positions in three-pile Wythoff.

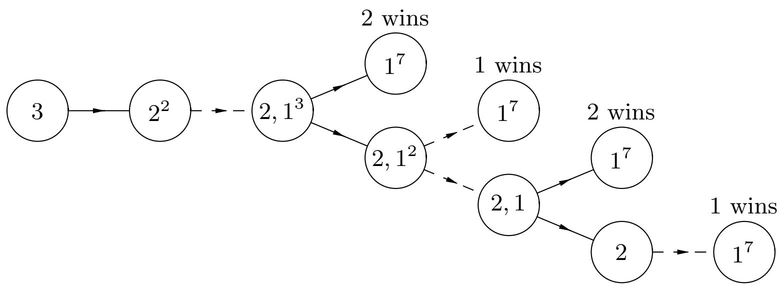

Figure 19. Player 1 can win Nimania for $n = 3$ in 13 moves. Solid arrows indicate Player 1's moves, and dashed arrows those of Player 2.

Since the numbers in successive positions decrease, the game terminates. Who wins? For $n = 1$ we saw above that player 1 wins. For $n = 2$, player 1 moves to 1^2, player 2 to 1, hence player 1 again wins. For $n = 3$, Figure 19 shows that by following the lower path, player 1 can again win. Unlike the cases $n = 1$ and $n = 2$, however, not all moves of player 1 for $n = 3$ are winning.

An attempt to resolve the case $n = 4$ by constructing a diagram similar to Figure 19 is rather frustrating. It turns out that for $n = 4$ the loser can delay the winner so that play lasts over 2^{44} moves! We have proved, however, the following surprising facts:

(i) Player 1 can win for every $n \geq 1$.
(ii) For $n \geq 4$, player 1 cannot hope to see his win being consummated in any reasonable amount of time: the smallest number of moves is $\geq 2^{2^{n-2}}$, and the largest is an Ackermann function.
(iii) For $n \geq 4$, player 1 has a *robust* winning strategy: most of the time player 1 can make random moves; only near the end of play does player 1 have to pay attention (as we saw for the case $n = 3$).

In view of (ii), where we saw that the length of play is at least doubly exponential, it seems reasonable to say that Nimania is not tractable, though the winning strategy is robust.

7. What's a Tractable or Efficient Strategy?

We are not aware that these questions have been addressed before in the literature. Since "Nim-type" games are considered to have good strategies, we now abstract some of their properties in an attempt to define the notions of a "tractable" or "efficient" game, in a slightly more formal way than the way we have used them above.

The subset T of combinatorial games with a *tractable* strategy has the following properties. For normal play of every $G \in T$, and every position u of G:

(a) The P-, N- or D-label of u can be computed in polynomial time.
(b) The next optimal move (from an N- to a P-position; from D- to a D-position) can be computed in polynomial time.
(c) The winner can consummate the win in at most an exponential number of moves.
(d) The set T is closed under summation, i.e., $G_1, G_2 \in T$ implies $G_1 + G_2 \in T$ (so (a), (b), (c) hold for $G_1 + G_2$).

The subset $T' \subseteq T$ for which (a)–(d) hold also for misère play is the subset of *efficient* games.

Remarks. (i) Instead of "polynomial time" in (a) and (b) we could have specified some low polynomial bound, so that some games complete in P (see, for example, [Adachi, Iwata and Kasai 1984]), and possibly annihilation games, would be excluded. But the decision about how low that polynomial should be would be largely arbitrary, and we would lose the closure under composition of polynomials. Hence we preferred not to do this.

(ii) In (b) we could have included also a P-position, i.e., the requirement that the loser can compute in polynomial time a next move that makes play last as long as possible. In a way, this is included in (c). A more explicit enunciation on the speed of losing doesn't seem to be part of the requirements for a tractable strategy.

(iii) In Section 2 we saw that, for Scoring, play lasts for an exponential number of moves. In general, for succinct games, the loser can delay the win for an exponential number of moves. Is there a "more natural" succinct game for which the loser cannot force an exponential delay? There are some succinct games for which the loser cannot force an exponential delay, such as Kotzig's Nim [Berlekamp, Conway and Guy 1982, Ch. 15] of length $4n$ and move set $M = \{n, 2n\}$. This example is somewhat contrived, in that M is not fixed, and the game is not primitive in the sense of [Fraenkel, Jaffray, Kotzig and Sabidussi 1995, Section 3]. Is there a "natural" nonsuccinct game for which the loser can force precisely an exponential delay? Perhaps an epidemiography game with a sufficiently slowly growing function f (where at move k we adjoin $f(k)$ new copies; see [Fraenkel, Loebl and Nešetřil 1988; Fraenkel and Lorberbom 1989]), played on a general digraph, can provide an example.

(iv) There are several ways of compounding a given finite set of games—moving rules and ending rules. See, for example, [Conway 1976, Ch. 14]. Since the sum of games is the most natural, fundamental and important among the various compounds, we only required in (d) closure under game sums.

(v) One might consider a game efficient only if both its succinct and nonsuccinct versions fulfil conditions (a)–(d). But given a succinct game, there are often many different ways of defining a nonsuccinct variation; and given a nonsuccinct game, it is often not so clear what its succinct version is, if any. Hence this requirement was not included in the definition.

A panorama of the poset of strategy efficiencies can be viewed by letting Murphy's law loose on the tractability and efficiency definitions. Just about any perverse game behavior one may think of can be realized by some perturbation of (a)–(d). We have already met misère play, interaction between tokens and succinctness. These tend to affect (d) adversely. Yet we are not aware that misère play has been proven to be NP-hard, nor do we know of any succinct game that has been proven NP-hard; the complexity of so many succinct games is still open! But there are many interesting games involving interaction between tokens that have been proven Pspace-complete, Exptime-complete, or even Expspace-complete.

We have also seen that epidemiography games violate (c), and that Whythoff's game is not known to satisfy (d). The same holds for Moore's Nim [Moore 1909–10; Berlekamp, Conway and Guy 1982, Ch. 15], so for both Wythoff's game and Moore's Nim we only have, at present, a reasonable strategy in the sense of Section 2. For partizan games (d) is violated conditionally, in the sense that sums are Pspace-complete. (It's not more than a curiousity that *certain* sums of impartial Pspace-hard games are polynomial. Thus geography is Pspace-complete [Schaefer 1978]. Yet given two identical geography games, player 2 can win easily by imitating on one copy what the opponent does on the other.)

Incidentally, geography games [Fraenkel and Simonson 1993; Fraenkel, Scheinerman and Ullman 1993; Bodlaender 1993; Fraenkel, Jaffray, Kotzig and Sabidussi 1995] point to another weakness of the classical theory. The input digraph or graph is not succinct, the game does not appear to decompose into a sum, and the game graph is very large. This accounts for the completeness results of many geography games.

While we're at it, we point out another property of geography games. Their *initial position*, as well as an "arbitrary" mid-position, are Pspace-complete, since one is as complex as the other: the initial position has the same form as any mid-position. In fact, this is the case for their prototype, quantified Boolean satisfiability [Stockmeyer and Meyer 1973]. On the other hand, for board games such as checkers, chess, Go, Hex, Shogi or Othello, the completeness result holds for an *arbitrary position*, i.e., a "midgame" or "endgame" position, carefully concocted in the reduction proof. See, for example, [Fraenkel, Garey, Johnson, Schaefer and Yesha 1978; Lichtenstein and Sipser 1978; Fraenkel and Lichtenstein 1981; Robson 1984a; Robson 1984b; Adachi, Kamekawa and Iwata 1987; Iwata and Kasai 1994]. But for the initial position, which is rather simple or symmetric, the decision question who can win might be easier to solve.

For poset games with a largest element and for Hex it's in fact easy to see that player 1 can win, but the proof of this fact is nonconstructive. Yet one of these games, von Neumann's Hackendot, has been given an interesting polynomial strategy by Úlehla [1980]. See also Bushenhack in [Berlekamp, Conway and Guy 1982, Ch. 17]. Still unresolved poset games include chomp [Gale 1974] and a power-set game [Gale and Neyman 1982]; see also [Fraenkel and Scheinerman 1991].

The outcome of certain games can be made to depend on open problems in mathematics. See, for example, [Jones 1982; Jones and Fraenkel $\geq$ 1996]. Lastly, but far from exhaustively, we mention a game in which "the loser wins". Let $Q = x_1 - x_2x_4 - x_2 - x_4 - 1$. Two players alternately assign values to x_1, x_2, x_3, x_4 in this order. Player 1 has to select x_1 as a composite integer > 1 and x_3 as any positive integer, and player 2 selects any positive integers. Player 2 wins if $Q = 0$; otherwise player 1 wins. It should be clear that player 2 can win, since $Q = x_1 - (x_2 + 1)(x_4 + 1)$ and x_1 is composite. But if player 1 picks x_1 as the product of 2 large primes of about the same size, then player 2 can realize his win in practice only if he can crack the RSA public-key cryptosystem [Rivest, Shamir and Adleman 1978]. Thus, in practice, the loser wins! Also note that player 2 has an efficient probabilistic method, such as those of [Solovay and Strassen 1977; Rabin 1976; 1980], to determine with arbitrarily small error that player 1 did not cheat, that is, that he indeed selected a *composite* integer.

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