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Stable Winning Coalitions

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Abstract. We introduce the notion of a stable winning coalition in a multi-player game as a new system of classification of games. An axiomatic refinement of this classification for three-player games is also presented. These classifications are compared in light of a probabilistic model and the existing literature.

1. Introduction

Are multi-player combinatorial games essentially different from two-player combinatorial games? John Nash was recently awarded the Nobel prize in economics in part for his resolution of the analog question about classical or matricial games: matricial games have equilibria regardless of the number of players.

From a classical game-theoretic point of view, combinatorial games are a "trivial" sort of zero-sum game: one of the two players has a forced win in any finite two-player deterministic sequential-move perfect-knowledge winner-take-all game.

When a game has more than two players, it is no longer the case that one always has a forced win. In Section 2, we will study Propp's so-called "queer" three-player games [Propp a], in which no player can force a win, but rather one player chooses which of his two opponents will win.

For example, the game of nim is played with several piles of stones. Players take turns removing stones from a single pile of their choice. At least one stone and up to an entire pile may be taken. The player who makes the last move is the sole winner. With a pile of one stone and a pile of two stones, no player can force a win alone (see Figure 1).

The most obvious classification of two-player combinatorial games is according to who can force a win alone. One begins to understand a game upon discovering who can force a win. (After dividing games into types by "outcome", the next logical step is to refine these types so that the outcome of a sum of games is

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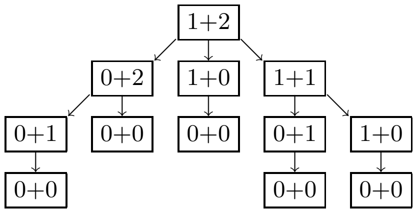

Figure 1. Three-player nim position $1 + 2$ game tree.

unchanged under replacement of one game with another in its class. For two-player impartial games, the two possible results, namely, next player wins and previous player wins, are further refined by Sprague–Grundy numbers [Grundy 1939; Sprague 1935]. All two-player impartial games are equivalent to one-pile nim. Similarly, in three-player nim, all nim positions are equivalent to one member of two countable families of nim positions and three sporadic positions [Propp a].)

However, the classification according to who can force a win is inappropriate to the vast majority of multi-player games, in which no player can force a win. Many authors have proposed various solutions to this problem. In Section 3, we classify multi-player combinatorial games according to which combinations of players can force a win. The set of all winning coalitions is itself a well-known combinatorial object: a maximal intersecting family of sets [Loeb and Meyerowitz]. In essence, an n-player game is reduced by the win classification into its 2^{n-1} two-player quotient games.

According to matricial game theory, the value of a combinatorial game to each player depends only on the win classification of the game [Shapley 1953]. For example, a three-player queer game has value $\frac{1}{3}$ to each player. Such fractional values arise because classical cooperative game theory allows binding side-agreements between players [von Neumann 1928].

However, in some contexts, such side-agreements may be unenforceable, and in other contexts, the players may not be allowed to communicate except through their moves and hence may not make any agreements whatsoever. Is there any hope for cooperative play without the use of side-payments, repeated play or binding agreements?

Just as a three-dimensional object is not completely described by its various two-dimensional projections, the win classification does not reveal all the essential information about a game. According to the probabilistic model introduced in Section 6.1, nearly all three-player games are classified together as queer games. The win classification does not distinguish between these games, and it does not distinguish between winning coalitions. Should players have preferences among queer games? Are all winning coalitions really equally effective?

The notion of a stable coalition is defined in Section 4 to answer these questions. In the example of Figure 1, two players form the only stable coalition.

They have a winning strategy by which either of them can win. The actual winner will be named by the third player. Despite the inability to share winnings, stable coalitions provide each member an incentive to cooperate. We will prove that stable winning coalitions exist in all multi-player combinatorial games.

This classification of multi-player games according to their stable coalitions contrasted with those of [Li 1978; Propp a; Straffin 1985]; however, we refer the reader to the bibliography for the full development of these promising systems for classifying multi-player games.

The fine classification introduced in Section 5 refines all the three-player game classifications above, with the exception of Li's. The fine type of a game is defined axiomatically via a number of tree rewriting rules similar to those of [Straffin 1985]. Thus, Section 5 makes heavy use of the language of graph grammars [Courcelle 1990].

As compared to the case of two-player games, we are confronted here with an embarrassing wealth of conflicting classification schemes. Before spending considerable effort classifying particular multi-player games, we must choose the most appropriate system of classification among the many alternatives. We hope that this paper is a step in that direction.

In order to focus on the general classification problem, we do not attempt an encyclopedic classification of particular games. However, we include examples for pedagogical reasons. To avoid unnecessary distractions, all examples are taken from one of the simplest combinatorial games of all: nim.

Although it is hardly mentioned in this paper, perhaps the most important concept in multi-player games is negotiation; it has close philosophical ties with many of the issues in this paper. Some relevant remarks on negotiation and how these classifications may be used to play a game can be found in [Loeb 1992, § 3].

2. Queer Games

We will consider finite deterministic sequential-move winner-take-all perfect-information games. That is, we make the following "combinatorial" assumptions about all games discussed here:

- They have a finite number of legal positions.
- There is no element of chance involved.
- There is no simultaneous movement.
- The players are at all times completely aware of the state of the game.
- The game necessarily terminates after a finite number of moves with a single unique winner.

We make no assumptions about the preferences of players among possible losses (compare [Li 1978]).

Every such game can be represented by a finite labeled game tree, like the one in Figure 2. All nodes are labeled with the name of a player. The label of

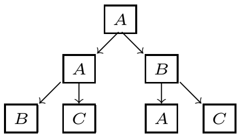

Figure 2. Example game.

an interior node indicates the player whose turn it is. This player selects one of the node's children and play continues at that point. Play terminates at a leaf node. The label of the leaf indicates who wins. Play begins at the root of the tree.

In theory, the minimax algorithm can be used to play any two-player, deterministic, sequential-move game perfectly. (In practice, however, the size of the game tree is too large to analyze directly via the minimax algorithm. The key idea in combinatorial game theory [Berlekamp et al. 1982; Conway 1976] has thus been to express games as a combination of simpler games, and relate them to special families of games such as impartial games and cold games for which rich mathematical theories have been discovered.)

However, what happens when a third player is added?

Let P be a node in a game tree corresponding to player A's turn. Suppose all of P's children $\mathrm{Succ}(P)$ have been evaluated. How do we evaluate P. Obviously, if P has a child whose value is equal to an A-win, then this child is a good strategy for player A, and the value of P will be a win for A. Similarly, if *all* of the children of P are wins for say player B, then P is itself a win for player B. However, what happens if some of the children of P are wins for player B and some are wins for player C? It is then impossible to evaluate P without some psychological information regarding A's attitude to such situations. Given that both choices are losing for A, we have no manner to predict how A will make his choice: randomly or according to some unknown criterion. Nevertheless, A's move is crucial to B and C.

Definition 2.1 [Propp a]. A position in a three-player combinatorial game is called *queer* if no player can force a win.

For example, consider the game of nim from the introduction, played with three players: N next (first), F following (second), and P previous (third). The position 0+0 is a win for P. Thus, N wins 1+0, 0+1, 2+0 and 0+2 by taking all the remaining stones from the remaining pile. Hence, 1+1 is a forced win for F. Therefore, the position 1+2 is queer as it yields a choice by N betweens wins for P and F (Figure 1).

Are queer positions very common? If not, we can hope to frequently apply the traditional techniques of two-player game tree analysis. Unfortunately, we have the following result (see Proposition 6.1 for similar probabilistic arguments):

Proposition 2.2 [Propp a, Claim 11a,e,f]. *If G is an impartial three-player game*, $2+2+G$ *is a queer game*. *In other words*, *any impartial game large enough to contain a* $2+2$ *component is queer*.

Recall that an n-player *impartial game* is an unlabeled tree, or rather a tree whose internal nodes have labels equal to their depth modulo n, and whose leaves have labels one less than their depth modulo n. Thus, players take turns moving. When playing the *sum* of two combinatorial games, players may play in either *component*. The game ends when a player has no available move in either component. The last player to move wins. See [Propp a] for a study of the win types of sums of three-player games.

Proof. First we will show that $G+2$ is not a win for P, and then that $G+2+2$ is not a win for N or F.

Type P: If P can win $G+2$, then F must win the options $G+1$ and G. However, this is a contradiction since G is an option of $G+1$.

Type N: Every move in $G+2+2$ leaves a heap of size 2, so $G+2+2$ has no P options. Thus, it has no winning moves for N.

Type F: Suppose F can win $G+2+2$. Clearly, G is not the trivial game, since $2+2$ is queer. Thus, G has at least one option G'. Now, N must win $G'+2+2$, which contradicts the preceding argument. □

3. The Win Classification

Games with four or more players can also be classified according to which sets of players form winning coalitions. For four-player games, instead of four win types of games A, B, C, Q, twelve are needed (Table 1). In Types 1–4, one player can force a win. In Types 5–8, a coalition of three players is needed to stop a certain player from winning. In Types 9–12, one player plays no significant role, and the three remaining players participate in a "queer" game (in other words, any of the three players can be stopped by the other two).

To generalize this classification to any number of players, we first define the quotient of a game.

Definition 3.1. Given a combinatorial game G with players V and a function $f : V \to W$ of its players, we define the *quotient game* G/f to be a new game with players W, and whose game tree is obtained by taking the game tree for the game G and replacing each label p with the label $f(p)$.

Thus, G/f represents a variant of the game G. Player $f(p)$ moves for the player p, and player $f(p)$ wins if p would have won the original game. If f is not surjective, some players in W will have no effect on the game, and no possibility of winning. If f is bijective, G and G/f are isomorphic as games.

Definition 3.2. The set of players $A \subseteq V$ is said to be a *winning* (or *losing*) coalition in the game G if the quotient game G/χ_A is a win (or loss) for 1, where

win type	$\mathcal{F}(P)$	nim example	probability
1	A	1	0
2	B	$1+1$	
3	C	$1+1+1$	
4	D	$1+1+1+1$	
5	$AB\ AC\ BC$	$2+1$	$(4-\sqrt{10})/8 \sim 10.47\%$
6	$AB\ AD\ BD$	$3+2+1+1$	
7	$AC\ AD\ CD$	$2+1+1$	
8	$BC\ BD\ CD$	$2+2+2+2$	
9	$AB\ AC\ AD\ BCD$	$2+2+1$	$(4+\sqrt{10})/8 \sim 14.53\%$
10	$AB\ BC\ BD\ ACD$	$2+2$	
11	$AC\ BC\ CD\ ABD$	$3+3+3$	
12	$AD\ BD\ CD\ ABC$	$4+4+3+2+1$	

Table 1. Four-player win types, their probabilities in a random game, and examples from nim.

χ_A is the characteristic function of A (that is, $\chi_A(p) = 1$ if $p \in A$ and $\chi_A(p) = 0$ if $p \notin A$). The *win type* of G is given by its set of winning coalitions $\mathcal{F}(G)$.

The quotient defined above is compatible with the quotient in [Loeb a] in the sense that $\mathcal{F}(G/f) = \mathcal{F}(G)/f$.

Win types are in one-to-one correspondence with maximal intersecting families of sets [Shapley 1962] (Table 2).

Definition 3.3 [Loeb and Meyerowitz]. Given a set V of players, a *maximal intersecting family of subsets of* V is a collection $\mathcal{F}$ of winning coalitions subject to the condition that if $V \supseteq B \supseteq A \in \mathcal{F}$ then $B \in \mathcal{F}$, and if $A = V - B$ then exactly one of A and B can be a member of $\mathcal{F}$.

In other words, a maximal intersecting family defines an order-preserving map f from the subsets of V to $\{0,1\}$ such that $f(A) = 1 - f(V - A)$. The number of maximal intersecting families and thus the number of win types grows extremely rapidly as n increases [Korshunov 1991]. For example, there are 1,422,564 seven-player win types [Loeb 1992; Bioch and Ibarki 1994], and 229,809,982,112 eight-player win types [Conway and Loeb 1995]. See also Table 2.

$\mathcal{F}(G)$ is a maximal intersecting family, since adding players to a coalition can only strengthen it, and given a pair of opposing coalitions exactly one can win while the other must lose. It is from this connection that the terminology of strong/simple games is derived [Shapley 1953].

4. Stable Coalitions

All queer games have the same win type. Moreover, under the hypotheses of matricial game theory [von Neumann 1928], every three-player game is either strictly determined (forced win) with pay-offs $(1, 0, 0)$, or symmetric non-strictly determined (queer) with payoffs $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. However, when side-agreements are not allowed, are queer games always "symmetric"? Or are there queer games that favor certain winning coalitions over others?

For example, consider the game in Figure 2, where player A can choose between two queer positions: one where A would name B or C as winner, and one where B would name A or C as winner. Perhaps A would prefer the second choice even though both are queer games, since then he would retain some chance of winning [Straffin 1985, Axiom St2].

In this section, we propose a criteria by which certain "favored" coalitions of players may be deemed *stable*.

Definition 4.1. Let P be a game tree. The set of stable coalitions $\mathrm{St}(P)$ or *stable type* is defined recursively as follows.

Case 0: In a terminal position P, a player p has won the game. Player p himself forms the only stable coalition. $\mathrm{St}(P) = \{\{p\}\}$.

In a non-terminal position P resulting in a set $\mathrm{Succ}(P)$ of choices for the player p, one must consider two cases to determine if a set of players C is stable.

Case 1 ($p \in C$): $C \in \mathrm{St}(P)$ if and only if (1.1) there exists a choice that keeps C stable, and (1.2) there is no stable coalition strictly included in C:

$$\begin{array}{ll} (1.1) & \exists P' \in \mathrm{Succ}(P) : C \in \mathrm{St}(P'), \\ (1.2) & (D \subset C) \Rightarrow (D \notin \mathrm{St}(P)). \end{array}$$

Case 2 ($p \notin C$): $C \in \mathrm{St}(P)$ if and only if (2.1) all choices keep some part of C stable, (2.2) for every player $q \in C$, there is some choice by which some subcoalition containing q is stable, and (2.3) there is no stable coalition strictly included in C:

$$\begin{array}{ll} (2.1) & P' \in \mathrm{Succ}(P) \Rightarrow \exists D \subseteq C : D \in \mathrm{St}(P'). \\ (2.2) & q \in C \Rightarrow \exists P' \in \mathrm{Succ}(P), D \in \mathrm{St}(P) : q \in D \subseteq C. \\ (2.3) & D \subset C \Rightarrow D \notin \mathrm{St}(P). \end{array}$$

In other words, a stable coalition must be a winning coalition; that is to say, it must have a strategy that guarantees a win for one of its members, regardless of the futile resistance of its opposition (Conditions 0, 1.1 and 1.2). However, it is the opposition who, via their choice of resistance, chooses which member of the coalition will actually win the game. In fact, every member of the coalition must be eligible to be elected winner by the opposition (Condition 2.2). Furthermore, we require that no member of the coalition can do better by joining a strictly included stable coalition (Conditions 1.2 and 2.3). (This last requirement seems the least plausible. However, it is not essential for certain of the results that follow; for example, in the proof of Theorem 4.2, nonminimal members of $E(P) \cup F(P)$ would also be considered stable.)

As a game proceeds, stable coalitions will sometimes disappear, as each player is confronted with a choice of conflicting strategies corresponding to different stable coalitions of which he is a member. Stable coalitions will never enlarge in size under ideal play, but they may decrease in size, as the opposition makes an arbitrary resistance. This reduction continues until the stable coalition consists of only a single player, at which point that player can force a win.

The following important result shows that our definition is not devoid of content. In fact, the proof gives a direct method by which $\mathrm{St}(P)$ may be calculated.

THEOREM 4.2. *All games have at least one stable coalition.*

PROOF. Clearly this is true for terminal positions, so proceed by induction. Let P be a position with player p to move. Let $F(P)$ be the set of "friendly coalitions" ($p \in A$) such that $A \in \mathrm{St}(P')$ for some move $P' \in \mathrm{Succ}(P)$. Let $E(P)$ be the set of "enemy coalitions" of the form

$$A = \bigcup_{P' \in \mathrm{Succ}(P)} A_{P'},$$

where $p \notin A_{P'} \in \mathrm{St}(P')$ for each choice $P' \in \mathrm{Succ}(P)$. Note that $\mathrm{St}(P)$ is the set of minimal elements of $E(P) \cup F(P)$.

It suffices to show that $E(P) \cup F(P) \neq \varnothing$. If $E(P)$ is empty, there must be some P' for which p belongs to all stable coalitions. By induction, P' must have at least one stable coalition A. Thus $A \in F(P)$. □

PROPOSITION 4.3. (i) *If the number of players is greater than one, the set of all players is not stable.*

(ii) *No stable coalition can contain another* (*in other words*, $\mathrm{St}(P)$ *is an antichain*).

(iii) *Any pair of stable coalitions has a nonempty intersection.*

The set of all players is winning. Intuitively, however, it is not stable since there is no opposition to choose the winner.

PROOF. (i) By contradiction. Let P be a smallest counterexample. Thus, $V \in \mathrm{St}(P)$. If P is of depth zero, Condition 0 is contradicted. Otherwise, by Condition 1.1, there must be a simpler position P' with $V \in \mathrm{St}(P')$. This contradicts the minimality of P.

(ii) Conditions 1.2 and 2.3.

(iii) Stable coalitions are winning (Conditions 1.1 and 2.1). Subsets of their complements are losing. □

We have the following "converse" to Theorem 4.2 and Proposition 4.3.

PROPOSITION 4.4. *Let $\mathcal{F}$ be a nonempty family of intersecting subsets of X. Suppose that $X \notin \mathcal{F}$, and that $\mathcal{F}$ is an antichain. Then there is some game position P* (*with players X*) *such that* $\mathrm{St}(P) = \mathcal{F}$.

PROOF. The game is played as follows. Players take turns eliminating all but one of the coalitions in $\mathcal{F}$ to which they belong (if any). After each player has

number of players		0	1	2	3	4	5	6	7
win types	(Section 3)	0	1	2	4	12	81	2646	1,422,564
stable types	(Section 4)	0	1	2	10	79	2644	1,422,562	229, 809, 982, 110
Li types	[Li 1978]	0	1	2	3	4	5	6	7
Straffin types	[Straffin 1985]	0	1	2	6				
Fine types	(Section 5)	0	1	2	$3\aleph$				
nim-sum types	[Propp a]	0	1	2	$2\aleph + 3$				

Table 2. Systems of classification of multi-player games.

taken his turn there will be one set left. The first non-member of the set can then choose a member of the set to be the winner. ☐

PROPOSITION 4.5. *The number of n-player stable types* $\mathrm{St}(P)$, *for* $n \geq 2$, *is equal to two less than the number of maximal intersecting families of subsets of an* $(n+1)$*-element set*. (*See Table* 2.)

PROOF. The antichains $\mathcal{F}$ of pairwise intersecting subsets of $\{1, 2, \ldots, n\}$ are in bijection with maximal intersecting families $\mathcal{M}$ on $\{1, 2, \ldots, n+1\}$:

$$\mathcal{F} = \min\{A : A \subseteq \{1, 2, \ldots, n\} \text{ and } A \in \mathcal{M}\}.$$

However, two maximal intersecting families must not be counted: the "dictatorship"

$$\mathcal{M} = \{A : n+1 \in A\}$$

corresponds to $\mathcal{F} = \varnothing$, and the "constitutional monarchy"

$$\mathcal{M} = \{A : n+1 \in A, |A| \geq 2\} \cup \{\{1, 2, \ldots, n\}\}.$$

corresponds to $\mathcal{F} = \{\{1, 2, \ldots, n\}\}$. ☐

We study the stable classification, since a winning coalition itself does not win, but rather only one member of the coalition wins. Thus, in the end, many winning coalitions are "unstable." For example, in Table 2, if player A chooses the winner between players B and C, then the only stable coalition would be BC. A together with any other player wins, but would not be not stable. A would therefore prefer to let B choose between A and C (compare [Straffin 1985]).

Of the ten three-player stable types, three correspond to forced victories for one of the players. The other seven correspond to queer games. (See Table 3.) Nevertheless, the stable classification is not in general finer than the win classification (Table 2). For example, the four-player nim positions $5+1$ and $3+3$ both have stable type $ABD\ ACD$, whereas $5+1$ has win type $AB\ AC\ BC$ and $3+3$ has win type $AB\ BC\ BD\ ABD$. Note in particular that, according to the win classification, player D is a spectator in the game $5+1$; he has no chance to affect the game. However, according to the stable classification, player D is a member of the only two stable coalitions. Depending on the interpretation chosen, player D is either a spectator with no chance of winning, or an active

win type $\mathcal{F}(P)$ (§ 3)	stable type $\mathrm{St}(P)$ (§ 4)	fine type (§ 5)	Straffin type [Straffin 1985]	nim example	prob.	$\mathcal{F}(P) =$ $\mathrm{St}(P)$?
next N	N	n_0	N wins	1	0	yes
following F	F	f_0	F wins	$1+1$	0	yes
previous P	P	p_0	P wins	$1+1+1$	0	yes
queer	FP	n_1	N picks	$2+1$	3/23	no
NF NP FP	NP	f_1	F picks	$3+1$	3/23	no
	NF	f_1	P picks	$3+3$	3/23	no
	NF NP	p_1	N picks	$4+1$	2/23	no
	NF FP	n_2	F picks	$4+4$	2/23	no
	NP FP	f_2	P picks	$4+4+4$	2/23	no
	NF NP FP	p_2	? picks	$6+6$	8/23	yes

Table 3. Classification of three-player games along with probabilities in a random game, and examples from the game of nim.

participant with at least as good a chance of winning the game as any other player.

5. The Fine Classification

Consider three-player games. Although the classification by stable winning coalitions is an improvement over classification by winning coalitions, does the stable classification completely express the preferences of all players? For example, if three two-player coalitions are stable, is the game necessarily symmetrical? In this section we define a finer classification of three-player games. Two positions are in the same fine class if their game trees are identical after pruning dominated options from the game tree, and eliminating moves consisting of a single choice. Using a system of axioms similar to those used in [Straffin 1985], we reduce all game trees to a set of *basic trees*. Interestingly, none of the basic trees will be invariant under permutation of players.

Let n_0, f_0, and p_0 denote the one-node basic game trees $\boxed{N}$, $\boxed{F}$, and $\boxed{P}$. They represent immediate forced wins. Now,

$$n_1 = \begin{tikzcd} & \boxed{N} \arrow[dl] \arrow[dr] & \\ \boxed{F} & & \boxed{P} \end{tikzcd}$$

denotes the basic game tree giving a choice by the next player between f_0 and p_0. The basic trees f_1 and p_1 can be defined similarly. Player N has the following preferences:

$$n_0 > f_1, p_1 > \underbrace{f_0, p_0}_{n_1}.$$

Is he indifferent regarding f_0 and p_0? Can the position n_1 leading to such a choice be simplified by his opponents? Moreover, is N necessarily indifferent between f_1 and p_1? Unfortunately, we can not answer these questions until we clarify how a player makes a choice between symmetrical positions. We will propose a system of preference axioms below. In particular, PR3 states that the preferences of N are invariant under the exchange of F and P.

However, we first review several reasonable alternative hypotheses that have been proposed.

- One assumption is to make a random move in such a situation [V. Lefevre, personal communication]. In this model, the forced wins n_0, f_0 and p_0 are given Lefevre-values $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in $\mathbb{R}^{\nVdash}$. Given a nontrivial game tree

$$\psi = \begin{array}{c} \boxed{i} \\ \swarrow \cdots \searrow \\ \tau_1 \;\; \dots \;\; \tau_n \end{array}$$

in which the i-th player is to move, $v(\psi)$ is calculated by averaging the $v(\tau_j)$ having maximal i-th coordinate $v(\tau_j)_i$. Thus $v(\psi)$ represents the probabilities of victory for each of the three players according to this model.

  However, consider a game whose game tree has redundant subtrees. For example, consider chess but allow rooks to be moved by either hand and all other pieces to be moved only by the left hand. Is it then more likely that the player will move his rook? Although plausible, this is not a very pleasing model, since it is strongly affected by redundant branches from our game tree (compare [Straffin 1985, Axiom Bk1]).
- Another solution [von Neumann 1928; Shapley 1953] is to modify the game being analyzed, and allow *side-payments* by the other players in order to entice player N to agree to a *binding agreement.*
- Straffin [1985, axiom St3] proposed a rule of vengeance (attack the player who eliminated you).
- Li [1978] proposed that the players try to come in "second place". (According to Li's model, there is a forced win for some player assuming the players are ranked in function of position relative to the last player to move. To obtain this result, it is necessary to assume that the ranking of any one player determines the rankings of all players. The actual permutation used generalizes the two-player notation of misère variants of games. Sequential composition [Stromquist and Ullman 1993] of two Li games results in the composition of the misère permutation of the second game with the rankings of the first game.) See Table 2.

Instead, we propose rules by which game trees may be simplified. It will be shown that any position in a three-player game can be reduced to exactly one of the specific *basic games* n_i, p_i, or f_i, where i is an integer, n_{i+1} is recursively defined to be a choice by player N between the games p_i and f_i

$$n_{i+1} = \begin{array}{c} \boxed{N} \\ \swarrow \quad \searrow \\ f_i \qquad p_i \end{array},$$

and f_{i+1} and p_{i+1} are defined similarly:

$$f_{i+1} = \begin{array}{ccc} & \boxed{F} & \\ \swarrow & & \searrow \\ n_i & & p_i \end{array}, \qquad p_{i+1} = \begin{array}{ccc} & \boxed{P} & \\ \swarrow & & \searrow \\ n_i & & f_i \end{array}.$$

For example, n_3 is the choice by player N of who will name the person who will choose the person who selects the winner.

The reduction rules we use are defined in terms of the preferences of players for game subtrees. Preferences are defined only for completely reduced game trees. Note that the preferences defined below are not total. Thus, for certain positions, we do not suppose that the player will make a certain preferred move; he may have a preference, he may move randomly, or he may move based on unknown criteria. We make no assumptions other than the axioms listed below.

Preference Rules

PR1: Let X and Y be distinct players. Player X prefers $x_0 = \boxed{X}$ to $y_0 = \boxed{Y}$. (Players prefer a sure win to a sure loss.)

PR2: Let X and Y be distinct players. Let

$$\sigma = \begin{array}{ccc} & \boxed{Y} & \\ \swarrow & & \searrow \\ \tau_1 & & \tau_2 \end{array}$$

be a completely reduced game tree. If player X prefers τ_1 to τ_2, player X also prefers τ_1 to σ and σ to τ_2. (Players prefer a good result to an uncertain result, and they prefer an uncertain result to a bad result.)

PR3: If player X prefers τ_1 to τ_2, player X also prefers τ_1 to τ_2' and τ_1' to τ_2, where τ_i' is the game subtree obtained by exchanging the two opponents of X wherever they occur in τ_i. (Players are impartial.)

PR4: Let $\psi = \begin{array}{ccc} & \boxed{X} & \\ \swarrow & \cdots & \searrow \\ \tau_1 & \ldots & \tau_n \end{array}$ be a completely reduced game tree.

PR4a: Player X prefers one of $\tau_1, \ldots, \tau_n$ to σ if and only if he prefers ψ to σ.

PR4b: Player X prefers σ to all of $\tau_1, \ldots, \tau_n$ if and only if he prefers σ to ψ.

(A choice is as good as its best option.)

PR5: Let X and Y be distinct players, and let $\psi = \begin{array}{ccc} & \boxed{Y} & \\ \swarrow & \cdots & \searrow \\ \tau_1 & \ldots & \tau_n \end{array}$.

PR5a: If player X prefers all of $\tau_1, \ldots, \tau_n$ to σ, player X prefers ψ to σ. (A tree is good if all its branches are good.)

PR5b: If player X prefers σ to all of $\tau_1, \ldots, \tau_n$, player X prefers σ to ψ. (A tree is bad if all its branches are bad.)

PR6.: We assume no preferences other than those implied by PR1–PR5.

Reduction Rules

RR1: A game tree of the form $\begin{array}{c}\boxed{X}\\ \downarrow\\ \tau\end{array}$ can be reduced to τ. (We may collapse non-decisions [Straffin 1985, Axiom Bk2].)

RR2: A game tree of the form $\begin{array}{c}\boxed{\quad X \quad}\\ \swarrow \cdots \downarrow\downarrow \cdots \searrow\\ \tau_1 \ \ldots \ \tau_1\tau_2 \ \ldots \ \tau_n\end{array}$ may be reduced to $\begin{array}{c}\boxed{X}\\ \swarrow \cdots \searrow\\ \tau_1 \ \ldots \ \tau_n\end{array}$.
(We may remove redundant game subtrees [Straffin 1985, Axiom Bk1].)

RR3: A game tree of the form

$$\begin{array}{ccc} & \boxed{X} & \\ \swarrow & \downarrow \cdots & \searrow \\ \begin{array}{c}\boxed{X}\\ \swarrow \cdots \downarrow\\ \tau_1 \ \cdots \ \tau_n\end{array} & \tau_{n+1} & \cdots \tau_{n+m}\end{array}$$

may be reduced to $\begin{array}{c}\boxed{X}\\ \swarrow \cdots \searrow\\ \tau_1 \ \ldots \ \tau_{n+m}\end{array}$. (We may collapse sequential choices by the same player [Straffin 1985, Axiom Bk3].)

RR4: A game tree of the form $\begin{array}{c}\boxed{X}\\ \swarrow \cdots \searrow\\ \tau_1 \ \ldots \ \tau_n\end{array}$ may be reduced to τ_1 if player X prefers τ_1 to all τ_i with $1 < i \leq n$. (We may assume that players will choose what they prefer [Straffin 1985, Axiom Bk3].)

Contrast PR2 with [Straffin 1985, Axioms St1 and St2]. Note that we do not adopt McCarthy's Revenge Rule [Straffin 1985, Axiom St3].

The reduction rules all strictly decrease the number of nodes in the game tree. Thus, no infinite reductions are possible, and the use of term "completely reduced" is justified. Since preferences are only defined on completely reduced game trees, application of RR4 must be carried out in a bottom-up fashion. Strategies in the reduced trees correspond in an obvious way to strategies in the original game tree.

Lefevre-values are not conserved by RR2 and RR3. However, the criteria of "preference" as defined above will be seen to be related to the simple comparison of Lefevre-values.

THEOREM 5.1. *By using the above reduction rules, all three-player games can be reduced to one and only one of* n_i, f_i, *or* p_i.

The *fine type* of a position P is defined to be the unique game tree n_i, f_i, or p_i to which it can be reduced.

We first establish the preferences (Lemma 5.2) and non-preferences (Lemma 5.3) for a given player, say N. We will then show that n_i, f_i, and p_i can not be further reduced (Lemma 5.4). It will then suffice to show that the set $\{n_i, f_i, p_i : i \geq 0\}$ is closed under tree construction.

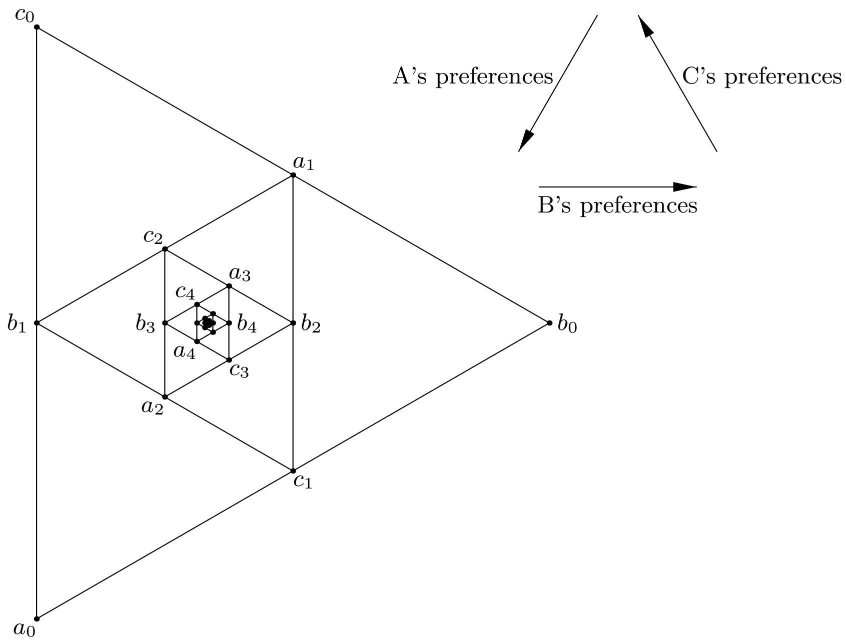

Figure 3. Preferences for basic-game trees.

preferences	$n_0 >$	$\overbrace{f_1, p_1}^{n_2} >$	$\overbrace{f_3, p_3}^{n_4} >$	$\cdots\ .\ \cdots >$	$\overbrace{f_4, p_4}^{n_5} >$	$\overbrace{f_2, p_2}^{n_3} >$	$\overbrace{f_0, p_0}^{n_1}$
probability	1	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{5}{16}$	$\frac{1}{4}$	0

Table 4. Player N's fine type preferences, along with the probability of player N winning given unbiased play by players F and P.

LEMMA 5.2. *The relations shown in the first row of Table* 4 *show preferences by player N for basic-game trees*. (Preferences of other players can be determined *mutatis mutandis*; see Figure 3.)

PROOF. Induction on k. The base case follows immediately from PR1. All of the required preferences for n_{2k+1} follow easily by hypothesis and the "only if" parts of PR4ab. We have $n_{2k} > f_{2k+1} > p_{2k}$ and $n_{2k} > p_{2k+1} > f_{2k}$ by hypothesis and PR2. Now, $f_{2k+1} > f_{2k}$ and $p_{2k+1} > p_{2k}$ by PR5. Thus, $f_{2k+1} > n_{2k+1}$ by PR4b. Furthermore, $f_{2k+1} < f_{2k-1}, p_{2k-1}$ by PR3 and the "if" part of PR4a. The remaining preferences for f_{2k+1} follow easily by hypothesis and PR5ab.

Preferences for $n_{2k+2}, p_{2k+2}, f_{2k+2}$ follows *mutatis mutandis* making use of the "if" part of PR4b to show that $f_{2k+2} > f_{2k}, p_{2k}$. □

LEMMA 5.3. *Let $\tau, \sigma \in \{n_i, f_i, p_i : i \geq 0\}$ be basic game trees. If $v(\tau)_1 \geq v(\sigma)_1$, the next player N does not prefer τ to σ.*

Proof. It suffices by PR6 to suppose that a counterexample was derived from one of the axioms PR1–PR5 from earlier preferences, and deduce that at least one of the earlier preferences is another counterexample.

PR1: Only n_0, f_0, and p_0 apply. PR1 shows that $n_0 > f_0$ and $n_0 > p_0$, neither of which are counterexamples, since $v(n_0)_1 = 1 > 0 = v(f_0)_1 = v(p_0)_1$.

PR2: This rule shows for example that $f_{i+1} = \begin{array}{c} \boxed{F} \\ \swarrow \qquad \searrow \\ n_i \qquad\quad p_i \end{array}$ lies between n_i and p_i in the preferences of player N. However, $v(f_{i+1})_1$ lies strictly between $v(p_i)_1$ and $v(n_i)_1$.

PR3: If $v(\tau) = (a, b, c)$ then $v(\tau') = (a, c, b)$.

PR4ab: In the "only if" direction, the only choice by player N that interest us here is n_{j+1}.

In the "if" direction, by RR4, the player N must have no preferences between the τ_i. The τ_i may not equal n_{j+1} by RR3, and they must be unequal. Thus we have $n = 2$, $\tau_1 = f_j$, and $\tau_2 = p_j$ for some $j \geq 0$, implying $\psi = n_{j+1}$.

However, note that $v(f_j)_1 = v(p_j)_1 = v(n_{j+1})_1$.

PR5ab. Suppose, for example, that $v(\tau_i)_1 > v(\sigma)_1$. Now, $v(\psi)_1$ is the average of certain $v(\tau_i)_1$. Thus $v(\psi)_1 > v(\sigma)_1$. □

Lemma 5.4. *The basic game trees n_i, f_i, and p_i are irreducible.*

Proof. In none of the game trees f_i, n_i, or p_i is there a node with a single child (by RR1), two identical subtrees depending on the same node (by RR2), or consecutive choices by the same player (by RR3). In all cases of a node with two children, we have shown that neither child is preferred by the player to move (by RR4). □

Theorem 5.1. Without loss of generality, we need only show how

$$\sigma = \begin{array}{c} \boxed{N} \\ \swarrow \qquad \searrow \\ \tau_1 \qquad\quad \tau_2 \end{array}$$

can be reduced for all basic game trees $\tau_1, \tau_2 \in \{n_i, f_i, p_i : i \geq 0\}$. If player N prefers one of τ_1 and τ_2 to the other, we are done by RR4. If $\tau_1 = \tau_2$, we are done by RR2 and RR1. The remaining possibilities for $\{\tau_1, \tau_2\}$ are as follows:

$\{f_i, p_i\}$: By definition, $\sigma = n_{i+1}$.
$\{n_i, p_{i-1}\}$: By RR3 and RR2, $\sigma \to n_i$.
$\{n_i, f_{i-1}\}$: Same as above. □

It is clear how moves in the reduced game correspond to strategies in the original game. The analysis in [Propp a] does not assume that the other players will necessarily play well. However, unless some assumption about the intelligence of other players is made, there is no way to discriminate between queer games. Moreover, the mild degree of fairness imposed by PR3 is necessary in order to compare, say, f_0 with f_1.

(i,j)	0	1	2	3	4	5	6	7	8	9	10
0	p_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0
1	n_0	$\boxed{f_0}$	n_1	f_1	n_2	n_2	n_2	n_2	n_2	n_2	n_2
2	n_0	n_1	f_1	n_2	n_2	n_2	n_2	n_2	n_2	n_2	n_2
3	n_0	f_1	n_2	p_1	p_1	p_1	p_1	p_1	p_1	p_1	p_1
4	n_0	n_2	n_2	p_1	f_2	n_3	f_3	n_4	n_4	n_4	n_4
5	n_0	n_2	n_2	p_1	n_3	f_3	n_4	n_4	n_4	n_4	n_4
6	n_0	n_2	n_2	p_1	f_3	n_4	p_3	p_3	p_3	p_3	p_3
7	n_0	n_2	n_2	p_1	n_4	n_4	p_3	f_4	n_5	f_5	n_6
8	n_0	n_2	n_2	p_1	n_4	n_4	p_3	n_5	f_5	n_6	n_6
9	n_0	n_2	n_2	p_1	n_4	n_4	p_3	f_5	n_6	p_5	p_5
10	n_0	n_2	n_2	p_1	n_4	n_4	p_3	n_6	n_6	p_5	f_6

Table 5. Fine classification of three-player two-pile nim position $i + j$. Regions associated with the four win types are outlined.

Without loss of generality we assume $i \le j$.

$i = 0$, $j = 0$: Fine type p_0. The previous player has already won.
$j > 0$: Fine type n_0. The next player wins immediately.

$i = 1$, $j = 1$: Fine type f_0. Forced win for the following player.
$j > 1$: Fine type n_0. Stable type FP. Queer game.

$i = 2$, $j = 2$: Fine type f_1. Stable type NP. Queer game.
$j > 2$: Fine type n_2. Stable type *NF NP*. Queer game.

$i = 3$: Fine type p_1. Stable type *NF*. Queer game.

$i \ge 4$: Queer game. Fine type x_{n+2m}, where x_n is the fine type of the nim game $(i - 3m) + (j - 3m)$ and $m = \lfloor \frac{1}{3}(i - 1) \rfloor$. Stable type *NF FP* if $i = j = 4$, and *NF FP NP* otherwise.

There is no two-pile nim position with stable type *NP FP*, or fine type p_{2k} for $k \ge 1$.

Table 6. Classification of the three-player two-pile nim position $i + j$.

The fine classification is so named since it is a refinement of the stable classification and Straffin classification [Straffin 1985], both of which refine the three-player win classification (Table 3). There are three countable families of fine types (Table 2). For $i > 0$, n_i corresponds to Straffin's "N decides". Note that the game n_2 favors player N according to our theory (N wins half of the time if F and P are not biased), but should be a loss for N according to Straffin's theory.

Calculations with fine types are rather complicated. For example, whereas two-pile two-player nim is completely trivial (the next player wins unless the two piles are of equal size), the fine type of the nim game $i + j$ can only be calculated by a complicated rule (see Tables 5 and 6).

No general pattern is known for three-player nim with an arbitrary number of piles. Moreover, any attempt to generalize this theory to four or more players, (even with a simplifying Revenge Rule) seems hopelessly complicated.

Rules PR1–PR6 and RR1–RR4 are sufficient to play a three-player game. However, to obtain numerical data concerning the probability of victory for a given player, say player N, we must make an additional "fairness" assumption (in addition to PR2). Namely, we must assume that in a position of fine type f_{i+1}, the option n_i will be chosen half of the time, and likewise in a position of fine type p_{i+1}. The probabilities that follow from these assumptions are given in Table 4. Note that this probability approaches $\frac{1}{3}$ but never attains this limit. Thus, we have the counter-intuitive result that *no* game is symmetric under the hypotheses above.

6. A Probabilistic Model

6.1. Queer Games. A simple heuristic argument shows that virtually all games are queer.

Any game can be represented by a tree, and without loss of generality a complete binary tree of some finite depth k. To generate a random game tree of depth k, we independently assign random labels to all nodes of a complete binary tree of depth k.

Let a_k, b_k, and c_k represent the probability that a random game of depth k is a forced win for player A, B, or C, respectively. Let q_k represent the probability that a random game of depth k is queer. For example, $a_0 = b_0 = c_0 = \frac{1}{3}$, and $q_0 = 0$.

Obviously, there are forced win games with arbitrarily large game trees. However, most large random games are queer (Table 3).

PROPOSITION 6.1. *As the depth of a random binary three-player game tree increases, the probability that the game is queer approaches a certainty*:

$$\lim_{k\to\infty} q_k = 1.$$

PROOF. Let $k > 0$. If player A moves first, he can force a win in the entire game tree if and only if he can force a win in one of the two principal subtrees; this occurs with probability $2a_{k-1} - a_{k-1}^2$. However, if player B or C moves first, in order to force a win A needs a forced win in both principal subtrees; this occurs with probability a_{k-1}^2. Thus, the *a priori* probability of A forcing a win is

$$a_k = \tfrac{1}{3}(2a_{k-1} - a_{k-1}^2) + \tfrac{2}{3}a_{k-1}^2 = a_{k-1}(\tfrac{2}{3} + \tfrac{1}{3}a_{k-1}).$$

However, by symmetry, $a_k = b_k = c_k$. Thus $a_k \le \frac{1}{3}$. Hence $\frac{2}{3} + \frac{1}{3}a_{k-1} \le \frac{7}{9}$, so

$$a_k \le \tfrac{7}{9}a_{k-1} \le \tfrac{1}{3}(\tfrac{7}{9})^k,$$

which tends exponentially to zero. Thus q_k rapidly converges to one. □

6.2. Win Types. In general, consider a large random n-player binary game tree G, and choose a subset $C \subseteq V$ of k players, with $k < \frac{1}{2}n$. What is the probability p that C is a winning coalition?

There are two cases: either a member of C must move (probability $x = k/n$), in which case C loses with probability $(1 - p_{k-1})^2$, or else a member of $V \setminus C$ must move, in which case C wins with probability p_{k-1}^2. Thus

$$p_k = x(2p_{k-1} - p_{k-1}^2)^2 + (1-x)p_{k-1}^2 = p_{k-1}(2x + p_{k-1}(1-2x)).$$

Now, $p_{k-1} \leq \frac{1}{2}$ by symmetry, for $V \setminus C$ must have at least as great a chance of winning as C does. Thus $(2x + p_{k-1}(1-2x)) \leq 2x < 1$. Therefore,

$$p_k \leq 2xp_{k-1} \leq \tfrac{1}{2}(2x)^k.$$

Thus the probability of a win rapidly approaches zero for all minority coalitions. The probability approaches one for all majority coalitions. By symmetry, the probability is one-half for all $(\frac{1}{2}n)$-player coalitions.

We thus deduce the following theorem.

THEOREM 6.2. *Let $\mathcal{M}$ be a maximal intersecting family of subsets of $V = \{1, 2, \ldots, 2n+1\}$. Let T_k be a random $(2n+1)$-player game tree of depth k. The limit win type probability*, $\lim_{k=\infty} P(\mathcal{M} = \mathcal{F}(T_k))$, *is one if*

$$\mathcal{M} = \{A \subseteq V : |A| > n\}$$

and zero otherwise. □

Thus, the maximal intersecting family corresponding to a typical large combinatorial game with an odd number of players is a "democracy." Its winning coalitions are exactly the majority coalitions. Thus, of the 81 different five-player win types, and of the 1,422,564 seven-player win types [Loeb 1992] only the "democracy" is likely to occur as a large random game.

In the case of an even number of players $2n$, all winning coalitions are likely to contain at least n players. For example, of the twelve four-player win types listed on Table 1, the forced wins (win types 1–4) have probability zero, and of the 229,809,982,112 eight-player win-types, at most $2^{35} = 34,359,738,368$ may have non-zero probability. In order to compute the probabilities of the remaining

$$2^{\frac{1}{2}\binom{2n}{n}}$$

possible win types additional calculations are necessary.

For example, consider $n = 4$. Let p be the probability of win types 5–8, and q that for types 9–12. We consider the 64 possible pairs of win types that might arise for a player to choose from as left subtree and right subtree in a game, and determine the win type of the complete game tree. The resulting table leads to the system of equations

$$p = 10p^2 + 12pq + 6q^2,$$
$$q = 6p^2 + 20pq + 10q^2,$$

from which we deduce the win type probabilities

$$p = \tfrac{1}{2} - \tfrac{1}{8}\sqrt{10} \sim 10.47\%,$$
$$q = \tfrac{1}{2} + \tfrac{1}{8}\sqrt{10} \sim 14.53\%.$$

For six players, 1024 of the 2646 maximal intersecting families may have nonzero probabilities of occurring as the win type of a random game. Analysis of the 1,048,576 possible combinations of these games is still tractable. The result would be a system of thirteen quadratic equations in thirteen variables (once symmetries are taken into account, there are thirty classes of maximal intersecting families of subsets of a six-element set, seventeen of which involve at least one winning minority coalition, and thus have probability zero). This system of equations could be solved numerically, although probably not formally.

For $n = 2k \geq 8$, the calculation of win type probabilities is intractable, with over three million simultaneous quadratic equations to be solved.

The exact values of these probabilities should not be taken too seriously, since they are influenced by small changes in the model (such as using complete trinary tree instead of binary trees). In particular, this probabilistic model should not be used to study any particular game (such as nim).

6.3. Stable Types. The probability of a large random three-player game having a certain stable type was calculated as follows. The forced wins have probability zero and can be ignored. We consider the remaining $7 \times 7 = 49$ possible pairs of stable types (G_1, G_2) that might arise for a player to choose from a left subtree and right subtree in a game, and determine the stable type of the complete game. Identifying the probabilities of certain of the remaining seven stable types, the resulting table leads to the system

$$x = \tfrac{7}{3}x^2 + \tfrac{8}{3}xy + \tfrac{4}{3}xz,$$
$$y = \tfrac{2}{3}x^2 + \tfrac{10}{3}xy + \tfrac{7}{3}y^2 + \tfrac{2}{3}yz,$$
$$z = 2x^2 + 2xz + 4yz + z^2,$$

from which we deduce the probabilities $x = \frac{3}{23}$, $y = \frac{2}{23}$, and $z = \frac{8}{23}$. Thus, with probability $z = \frac{8}{23}$, the game will be symmetrical, and all three pairs of players will be able to form stable coalitions.

Four-player games can be divided into 79 stable types. Up to permutation of players, there are thirteen stable classes. Only five stable classes (25 stable types) have non-zero probability, which we can approximate numerically (Table 7).

Note that for twelve of the 79 stable types of four-player games, all winning coalitions are stable: $\mathcal{F}(P) = \mathrm{St}(P)$. However, unlike the case of three-player

representative class $\mathrm{St}(P)$	types in class	type prob.	class prob.	$\mathcal{F}(P) = \mathrm{St}(P)$?	nim example
$AB\ AC\ BC$	4	0.0485%	0.194%	yes	2+2+2+2+2+1
$AB\ AC\ BCD$	12	1.1818%	14.182%	no	6+6+1+1
$AB\ ACD\ BCD$	6	7.1996%	43.197%	no	6+6+1
$ABC\ AD\ BD\ CD$	4	0.4326%	1.731%	yes	none known
$ABC\ ABD\ ACD\ BCD$	1	40.595%	40.595%	no	7+6

Table 7. The five "common" four-player stable classes, with examples from nim.

games, these stable types have relatively low probabilities (less than 2% in total) of occurring as the stable type of a large random game.

Numerical approximation of the 2644 five-player stable type probabilities is probably feasible. However, even that seems impractical for the 1,422,562 six-player stable type probabilities.

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