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Infinitesimals and Coin-Sliding

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Abstract. We define and solve a two-player perfect information game, the *coin-sliding game*. One reason why this game is of interest is that its positions generate a large family of infinitesimals in the group of two-player partizan combinatorial games under disjunctive composition.

1. The Simplest Form of the Game

Consider the following game, played by two players, Left and Right: Coins of various (positive numeric) monetary values, colored red or blue, are placed on a semi-infinite strip. We call the red coins Right's and the blue coins Left's. The playing field and possible moves are indicated in Figure 1. Each player can, on his turn, either slide one of his coins down one square, or remove one of his opponent's coins from the strip. Each player gets to keep all the money he moves off the bottom of the strip. The winner is the player who ends up with the most money, or, in the event of a tie, the player who moved last.

Since each player moves just one coin at each turn, the overall game is a disjunctive composition of subgames corresponding to the individual positions. In fact, we can assign a partizan combinatorial game, in the sense of *Winning Ways* [Berlekamp et al. 1982], Chapters 1–8, or *On Numbers and Games* [Conway 1976], to each coin on a square, and the overall game will then be the sum of these individual games, which we call *terms*. Figure 1 has names for these terms [Berkekamp and Wolfe 1994, §§ 4.1, 4.11; Wolfe 1994, §§ 3.1, 3.10]. We have also numbered our squares, with square 0 at the bottom. When we refer to these squares by number, we will also call them *rows* (rows with more than one square will appear later.) In the figure and in the rest of the paper,

$$+_x \text{ means the game } 0\,\|\,0\,|\,{-x},$$
$$0^2\,|\,{+_x} \text{ means the game } 0\,\|\,0\,|\,{+_x},$$
$$0^3\,|\,{+_x} \text{ means the game } 0\,\|\!|\,0\,\|\,0\,|\,{+_x},$$

and so on.

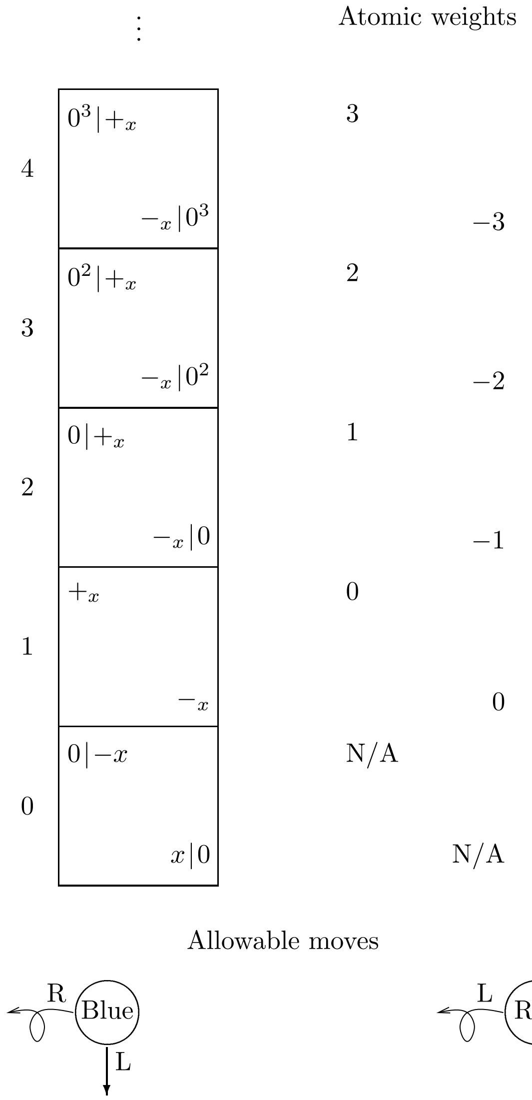

Figure 1. The simplest form of coin-sliding. The name for the term corresponding to a red coin of value x in a square is in the upper left corner of the square, and for a blue coin, the corresponding term name is in the lower right.

Replacing a coin on a square with a coin of the same monetary value and other color on the same square replaces a term by its negative. Hence

$$-_x \text{ is the negative of } +_x,$$
$$-_x|0^2 \text{ is the negative of } 0^2|+_x,$$
$$-_x|0^3 \text{ is the negative of } 0^3|+_x,$$

and so on.

2. Infinitesimals and Atomic Weight

A partizan combinatorial game (henceforth just a "game") G is *infinitesimal* if $-\varepsilon < G < \varepsilon$ for every positive rational number ε. The simplest nonzero infinitesimal is $* = 0|0$, but we will see many others in the sequel. All our coin-sliding terms are infinitesimal except for those coming from row 0—in fact, except for the row 0 terms, Right's terms (terms coming from Right's coins) are positive infinitesimals and Left's terms are negative infinitesimals. For the games we are considering, which always end in a bounded amount of time, infinitesimals are also called *small* (*On Numbers and Games*, Chapter 9), and a game is called *all small* if it and all its subpositions are small. All small games G that end in a bounded amount of time have an *atomic weight* G'', defined as follows (*Winning Ways*, Chapter 8; *On Numbers and Games*, Chapter 16):

$$G'' = \{G^{L''}{-2}\,|\,G^{R''}{+2}\},$$

except when more than one integer is permitted by this definition. (By a number n being *permitted* we mean that it is $\mathrel{|\!\!\rhd}$ every $G^{L''}-2$ and is $\mathrel{\lhd\!\!|}$ every $G^{R''}+2$, where $G \mathrel{|\!\!\rhd} H$ when $G-H$ is positive or fuzzy, that is, is a first-player win for Left, and $G \mathrel{\lhd\!\!|} H$ when $G-H$ is negative or fuzzy, that is, is a first-player win for Right.) If more than one integer is permitted we pick the

smallest permitted, zero, or largest permitted

integer according to whether G is

less than, incomparable with, or greater than

any $*m$ not equal to a subposition of G—such a $*m$ is called a *remote star* (*Winning Ways*, Chapter 8; *On Numbers and Games*, Chapter 16).

Atomic weight is then an additive homomorphism—that is, $(G+H)'' = G''+H''$, $(-G)'' = -G''$, and $0'' = 0$. The positive all small game $\uparrow\, = 0|*$ is the natural unit of atomic weight, with $\uparrow'' = 1$. We write $\downarrow\, = -\uparrow$, so $\downarrow'' = -1$. Positive atomic weights are favorable for Left. In fact, if $G'' \geq 2$, then $G > 0$, and if $G'' \mathrel{|\!\!\rhd} 0$, then $G \mathrel{|\!\!\rhd} 0$. Similarly, if $G'' \leq -2$, then $G < 0$, and if $G'' \mathrel{\lhd\!\!|} 0$, then $G \mathrel{\lhd\!\!|} 0$.

Given these facts, we can compute the atomic weight of $+_{n\cdot\uparrow}$, for n large:

$$\{0|{-n}\cdot\uparrow\}'' = \{0{-2}|{-n}{+2}\} = \{-2|2{-n}\}$$

and

$$+''_{n\cdot\uparrow} = \{0{-}2\,|\,\{-2\,|\,2{-}n\}{+}2\} = \{-2\,\|\,0\,|\,4{-}n\},$$

and here 0 and -1 are permitted; since $+_{n\cdot\uparrow}\,\|\,{*2}$, we find that $+''_{n\cdot\uparrow} = 0$. Now $+_x$ is a positive infinitesimal that decreases very rapidly as x increases. Hence it is reasonable to take $+''_x = 0$ for the positive numbers x that we encounter in coin-sliding, and we do so. We can then compute the atomic weights for all our coin-sliding terms except those coming from row 0. The results are given in Figure 1, the upper left atomic weight being for Right's terms and the lower right being for Left's terms.

3. The Atomic Weight Strategy

Each player can observe that sliding down one of his own coins usually changes the atomic weight by 1 in his favor, and that removing one of his opponent's coins usually changes the atomic weight in his disfavor. In conjunction with the idea that moves that immediately give one money or prevent one's opponent from getting money are more important than other moves, this gives the *atomic weight strategy*:

(i) If the opponent has a coin in row 0, remove it.
(ii) Otherwise, if we have a coin in row 0, slide it down.
(iii) Otherwise, if we have a coin anywhere, slide it down.
(iv) Otherwise, remove any one of the opponent's coins.

Suppose that Left starts from an infinitesimal position (which we assume to have all coins in row 1 or above) and plays the atomic weight strategy, and that he has coins somewhere on the board. He will then usually increase the atomic weight by 1 on his move, and Right's return can decrease the atomic weight by no more than 1. The only exceptions to this are when coins are moved into row 0, in which case the position becomes non-infinitesimal. Right can repeatedly move his coins into row 0, which Left will repeatedly remove; after both players have moved, there is no change in the atomic weight, because row 1 coins had atomic weight 0 to begin with. Also, Left himself may slide a coin into row 0. Right can respond at first by sliding his own coins into row 0, which Left will immediately remove, but eventually he must remove Left's coin or lose the game. Again, there is no net change in atomic weight after the exchanges have ended.

If Left starts from an infinitesimal position with atomic weight at least 1, then, and plays the atomic weight strategy, he will eventually come to an infinitesimal position that also has atomic weight at least 1, but that has none of his own coins. There must then be an opponent's coin present, which he will remove, and Right's subsequent moves will slide his own coins down, which will always leave Left a coin to remove on his turn. Thus, Left will win.

If we start from an infinitesimal position, then, the atomic weight strategy will win for Left if the total atomic weight is at least 1 and Left moves first.

Since Right can decrease the atomic weight by no more than 1 on his turn, the atomic weight strategy will also win for Left if the total atomic weight is at least 2, whoever moves first. Similarly, it will win for Right if the total atomic weight is -2 or less, or if it is -1 or less and Right moves first.

The atomic weight strategy thus furnishes a proof of our assertions about atomic weight—namely, that $G'' \geq 2$ implies $G > 0$, $G'' \mathrel{|\!\!\rhd} 0$ implies $G \mathrel{|\!\!\rhd} 0$, $G'' \leq -2$ implies $G < 0$, and $G'' \mathrel{\lhd\!\!|} 0$ implies $G \mathrel{\lhd\!\!|} 0$—in the special case of simple coin-sliding. We need a more complicated strategy, the *desirability strategy*, to show who wins in the remaining cases: positions with atomic weight -1, Left starting, atomic weight 1, Right starting, and atomic weight 0, either player starting.

4. The Desirability Strategy

The atomic weight strategy takes no notice of the monetary value of the coins, but this will obviously influence our strategy. The desirability order in Figure 2 shows what terms we prefer to move on; it can be summarized as follows:

(i) A term coming from rows 0 or 1 is always more desirable than a term coming from row 2 or above.

(ii) Among terms coming from rows 0 or 1, terms with bigger monetary values are more desirable.

(iii) Among terms coming from row 2 or above, terms with smaller monetary values are more desirable.

(iv) If none of the above three rules apply, terms from a row closer to row 2 are more desirable.

(v) If none of the above four rules apply, the terms are equal or negatives of each other and are equally desirable.

$$
\begin{array}{c}
\dots\ ; \\
+_3,\ -_3;\ \ 0|{-3},\ 3|0;\ \ \dots\ ; \\
+_2,\ -_2;\ \ 0|{-2},\ 2|0;\ \ \dots\ ; \\
+_1,\ -_1;\ \ 0|{-1},\ 1|0;\ \ \dots\ ; \\
\dots\ ; \\
0|{+_1},\ {-_1}|0;\ \ 0^2|{+_1},\ {-_1}|0^2;\ \ 0^3|{+_1},\ {-_1}|0^3;\ \ \dots\ ; \\
\dots\ ; \\
0|{+_2},\ {-_2}|0;\ \ 0^2|{+_2},\ {-_2}|0^2;\ \ 0^3|{+_2},\ {-_2}|0^3;\ \ \dots\ ; \\
\dots\ ; \\
0|{+_3},\ {-_3}|0;\ \ 0^2|{+_3},\ {-_3}|0^2;\ \ 0^3|{+_3},\ {-_3}|0^3;\ \ \dots\ ; \\
\dots
\end{array}
$$

Figure 2. Simple coin terms ordered from most desirable to least desirable (reading in the normal way, from left to right and from top down). A comma separates terms of equal desirability, and a semicolon those of unequal desirability.

Given this ordering, we can formulate the *desirability strategy*:

(i) If, on the board, there are coins that belong to us or are in row 0, move on the one with the most desirable term.
(ii) Otherwise, remove any one of the opponent's coins.

When applying this strategy, pairs of cancelling terms—that is, pairs of terms that come from a red and a blue coin of the same monetary value on the same square—must be ignored. (If the opponent makes a move in a pair of cancelling terms, the player should respond with the corresponding move in the other.) One can make a complete classification of who wins for all infinitesimal positions, with either player starting. Given this classification, it can be shown that the desirability strategy is optimal: if a player starts from an infinitesimal position won for him and plays it thereafter, it will win for him. (The classification is somewhat complicated, but we can say, for example, that an atomic weight 1 position is positive if the least desirable term in it (after cancelling pairs of terms) is Left's, or that in an atomic weight 0 position where Left has the least desirable term, Left can win provided that he moves first, provided that this least desirable term is not of the form $-_x$.)

5. A Generalization

We generalize the game: instead of playing on a semi-infinite strip, we now play on an infinite quarter-plane, as shown in Figure 3. Also, we allow coins to have zero monetary values as well as positive numeric monetary values. As before, the red coins are Right's and the blue are Left's. Each player can move each of his own coins from one row to the leftmost square on the next row down (or off the bottom of the board if the coin started in the bottom row.) Each player can move each of his opponent's coins one square to the left, or off the board if the coin starts in the leftmost column. Each player gets to keep the money he moves off the bottom (but not the left) of the board.

As before, each square on the board is marked with names [Berlekamp and Wolfe 1994, §§ 4.1, 4.11; Wolfe 1991, §§ 3.1, 3.10] of terms coming from coins on the square, a red coin of value x producing the term named in the upper left, and a blue coin producing the term named in the lower right. For a term $G = 0\,|\,H$ coming from a red coin in the leftmost column, $G^{\to n}$ is the term for a red coin of the same monetary value $n-1$ columns to the right; it is defined by taking $G^{\to 1} = 0\,|\,H$ and $G^{\to n+1} = G^{\to n}\,|\,H$ for all $n \in \mathbb{Z}_{>\nvdash}$. For a leftmost-column blue coin term $G = H\,|\,0$, $G^{\to n}$, the term for a blue coin of the same monetary value $n-1$ columns to the right, is defined by $G^{\to n} = -\{0\,|\,-H\}^{\to n}$. As before, except for the row 0 terms, Right's terms are positive infinitesimals and Left's terms are negative infinitesimals.

To make this version of the game easier to analyze, we alter the playing field and rules somewhat. Figure 4 shows our revision of the playing field. In the

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	$0^3\|+_x$ $-_x\|0^3$	$\{0^3\|+_x\}^{\to 2}$ $\{-_x\|0^3\}^{\to 2}$	$\{0^3\|+_x\}^{\to 3}$ $\{-_x\|0^3\}^{\to 3}$	$\{0^3\|+_x\}^{\to 4}$ $\{-_x\|0^3\}^{\to 4}$	$\cdots$
3	$0^2\|+_x$ $-_x\|0^2$	$\{0^2\|+_x\}^{\to 2}$ $\{-_x\|0^2\}^{\to 2}$	$\{0^2\|+_x\}^{\to 3}$ $\{-_x\|0^2\}^{\to 3}$	$\{0^2\|+_x\}^{\to 4}$ $\{-_x\|0^2\}^{\to 4}$	$\cdots$
2	$0\|+_x$ $-_x\|0$	$\{0\|+_x\}^{\to 2}$ $\{-_x\|0\}^{\to 2}$	$\{0\|+_x\}^{\to 3}$ $\{-_x\|0\}^{\to 3}$	$\{0\|+_x\}^{\to 4}$ $\{-_x\|0\}^{\to 4}$	$\cdots$
1	$+_x$ $-_x$	$+_x^{\to 2}$ $-_x^{\to 2}$	$+_x^{\to 3}$ $-_x^{\to 3}$	$+_x^{\to 4}$ $-_x^{\to 4}$	$\cdots$
0	$0\|-x$ $x\|0$	$\{0\|-x\}^{\to 2}$ $\{x\|0\}^{\to 2}$	$\{0\|-x\}^{\to 3}$ $\{x\|0\}^{\to 3}$	$\{0\|-x\}^{\to 4}$ $\{x\|0\}^{\to 4}$	$\cdots$

Players' possessions

Allowable moves

Figure 3. Another playing field for coin-sliding.

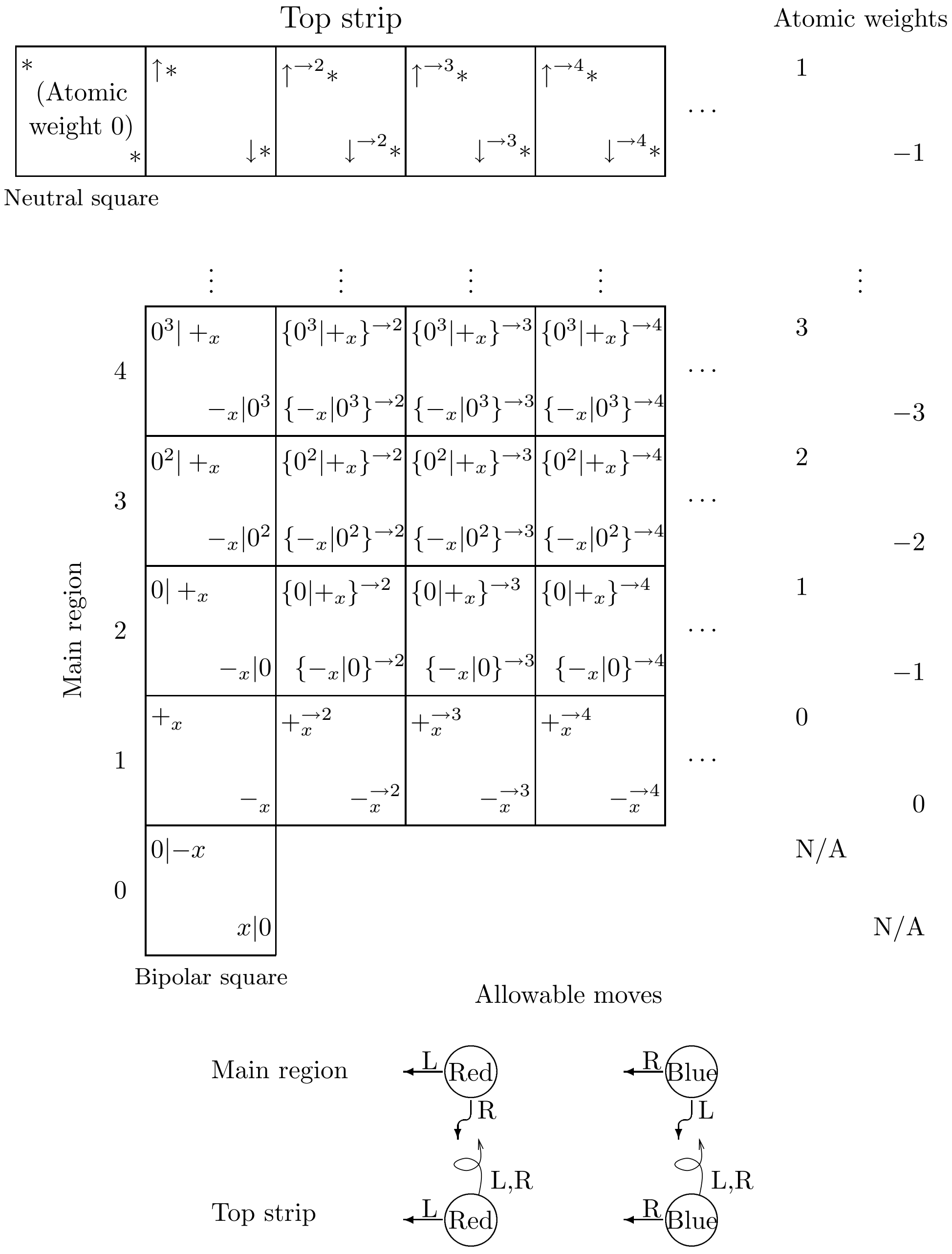

Figure 4. The revised playing field for coin-sliding.

main region, we have omitted the bottom row squares other than the leftmost. This is because a red (or blue) coin n squares to the right of the bipolar square behaves the same in play as a blue (or red) coin of the same monetary value n squares above the bipolar square together with a red (or blue) coin of the same monetary value that Right (or Left) already has; in other words, $\{0\,|\,-x\}^{\rightarrow n+1} = -x+\{x\,|\,0^{n+1}\}$, and $\{x\,|\,0\}^{\rightarrow n+1} = x+\{0^{n+1}\,|\,-x\}$.

We have also created a new row of squares at the top, the *top strip*. All zero monetary value coins are now placed in the top strip. The top strip behaves peculiarly in that both Left and Right are allowed to remove both red and blue coins from the strip at any time, as well as Left (or Right) being able to move red (or blue) coins left one square. The top strip terms $\uparrow^{\rightarrow n}*$ and $\downarrow^{\rightarrow n}*$ are defined by $\uparrow^{\rightarrow n}* = \uparrow^{\rightarrow n} + *$ and $\downarrow^{\rightarrow n}* = \downarrow^{\rightarrow n} + *$, where $\uparrow^{\rightarrow n}$ and $\downarrow^{\rightarrow n}$ are as above: $\uparrow^{\rightarrow 1} = \uparrow$, $\uparrow^{\rightarrow n+1} = \uparrow^{\rightarrow n}|*$ (for $n \geq 1$), and $\downarrow^{\rightarrow n} = -\uparrow^{\rightarrow n}$. It can be shown that a coin of zero value on the main board, m rows above and n columns to the right of the bipolar square, behaves the same as one coin on the neutral square, one coin $n+1$ squares to the right of the neutral square (if $m > 0$), and $m-1$ coins one square to the right of the neutral square (if $m > 1$); this fact is the reason for these definitions. (We assume here that the transformation in the last paragraph has already been made, so that $n = 0$ if $m = 0$.)

6. Ordinal Sums

The new terms we have introduced can be viewed as *ordinal sums* of our old terms with integers. The ordinal sum $G\!:\!H$ of games G and H, is defined (*On Numbers and Games*, Chapter 15) by

$$G\!:\!H = \{G^L, G\!:\!H^L \,|\, G^R, G\!:\!H^R\}.$$

We find that $G\!:\!0 = G$. In fact, $G\!:\!H$ is usually very close to G; Norton's Lemma (*On Numbers and Games*, Chapter 16) says that $G\!:\!H$ and G have the same order-relations with all games $K \neq G$ that do not contain a game equal to G as a subposition.

The significance here is that

$$G^{\rightarrow k} = \begin{cases} G\!:\!(k-1) & \text{if } G > 0,\ k \geq 1, \\ G\!:\!(-(k-1)) & \text{if } G < 0,\ k \geq 1, \end{cases}$$

so coin terms don't change much with a change in the column of the coin. We define more infinitesimals to tell us just how much they differ (see also [Berlekamp and Wolfe 1994, § 4.11; Wolfe 1991, § 3.10]):

$$\begin{aligned} &G^1 = G, \\ &G^k = G^{\rightarrow k} - G^{\rightarrow k-1} \quad \text{for } k \geq 2. \end{aligned}$$

The order relation between $G\!:\!H$ and $G\!:\!K$ is always the same as that between H and K. Since $G^k = (G\!:\!(k-1)) - (G\!:\!(k-2))$ for $k \geq 2$ and $G > 0$, it follows that G^k is always positive for $G > 0$. After generalizing our desirability strategy to our new version of coin-sliding, we will be able to compare the G^k's with each other, where G can be $\uparrow$, $\{0^m|+_x\}$, or $+_x$.

7. The Desirability Strategy Generalized

The desirability ordering for our new terms is the same as for our original terms, except that

(i) all top strip terms are less desirable than all main region terms, and
(ii) among terms coming from coins in the same row and of the same monetary value, terms coming from coins on the right are more desirable than terms coming from coins on the left.

The resultant desirability ordering is shown in Figure 5. The desirability strategy that goes with this ordering is somewhat more complicated than in the simple case, and now runs as follows:

(i) If, on the board, there are coins not in the neutral square that belong to us or are in row 0, move on the one giving the most desirable term, unless it is "crucial", giving a term of $G^{\rightarrow k}$, say, and there is a $-G^{\rightarrow l}$ present, for some $l > k \geq 1$.

$$
\begin{array}{c}
\ldots; \\
\ldots;\ \ +_3^{\rightarrow 3},\ -_3^{\rightarrow 3};\ \ +_3^{\rightarrow 2},\ -_3^{\rightarrow 2};\ \ +_3,\ -_3;\ \ 0|{-3},\ 3|0; \\
\ldots; \\
\ldots;\ \ +_2^{\rightarrow 3},\ -_2^{\rightarrow 3};\ \ +_2^{\rightarrow 2},\ -_2^{\rightarrow 2};\ \ +_2,\ -_2;\ \ 0|{-2},\ 2|0; \\
\ldots; \\
\ldots;\ \ +_1^{\rightarrow 3},\ -_1^{\rightarrow 3};\ \ +_1^{\rightarrow 2},\ -_1^{\rightarrow 2};\ \ +_1,\ -_1;\ \ 0|{-1},\ 1|0; \\
\ldots; \\
\ldots;\ \ \{0|+_1\}^{\rightarrow 3},\ \{-_1|0\}^{\rightarrow 3};\ \ \{0|+_1\}^{\rightarrow 2},\ \{-_1|0\}^{\rightarrow 2};\ \ 0|+_1,\ -_1|0; \\
\ldots;\ \ \{0^2|+_1\}^{\rightarrow 3},\ \{-_1|0^2\}^{\rightarrow 3};\ \ \{0^2|+_1\}^{\rightarrow 2},\ \{-_1|0^2\}^{\rightarrow 2};\ \ 0^2|+_1,\ -_1|0^2; \\
\ldots;\ \ \{0^3|+_1\}^{\rightarrow 3},\ \{-_1|0^3\}^{\rightarrow 3};\ \ \{0^3|+_1\}^{\rightarrow 2},\ \{-_1|0^3\}^{\rightarrow 2};\ \ 0^3|+_1,\ -_1|0^3; \\
\ldots; \\
\ldots;\ \ \{0|+_2\}^{\rightarrow 3},\ \{-_2|0\}^{\rightarrow 3};\ \ \{0|+_2\}^{\rightarrow 2},\ \{-_2|0\}^{\rightarrow 2};\ \ 0|+_2,\ -_2|0; \\
\ldots;\ \ \{0^2|+_2\}^{\rightarrow 3},\ \{-_2|0^2\}^{\rightarrow 3};\ \ \{0^2|+_2\}^{\rightarrow 2},\ \{-_2|0^2\}^{\rightarrow 2};\ \ 0^2|+_2,\ -_2|0^2; \\
\ldots;\ \ \{0^3|+_2\}^{\rightarrow 3},\ \{-_2|0^3\}^{\rightarrow 3};\ \ \{0^3|+_2\}^{\rightarrow 2},\ \{-_2|0^3\}^{\rightarrow 2};\ \ 0^3|+_2,\ -_2|0^3; \\
\ldots; \\
\ldots; \\
\ldots;\ \ \uparrow^{\rightarrow 3}*,\ \downarrow^{\rightarrow 3}*;\ \ \uparrow^{\rightarrow 2}*,\ \downarrow^{\rightarrow 2}*;\ \ \uparrow*,\ \downarrow*; \\
*
\end{array}
$$

Figure 5. Terms ordered from most desirable to least desirable (reading in the normal way, from left to right and from top down). A comma separates terms of equal desirability, and a semicolon those of unequal desirability.

(a) In this case, move on the coin as above giving the next most desirable term, if present.

(b) If such a coin is not present, move the $-G^{\to l}$ coin one square leftwards.

(ii) Otherwise, if there are exactly two coins in the top strip, move a coin in the top strip and not in the neutral square one square leftwards.

(iii) Otherwise, if there are coins in the top strip, remove any one.

(iv) Otherwise, move on any coin.

As before, pairs of cancelling coins must be ignored when applying this strategy. Since, on the neutral square, the color of a coin is irrelevant, we must also cancel a red coin with a red coin or a blue coin with a blue coin when more than one coin appears on the neutral square.

This desirability strategy can be proved to be optimal as before by first deriving a complete classification of who wins and loses for all infinitesimal positions, whoever starts. This classification, which provides the definition of "crucial", can be found in [Moews 1993, Chapter 5] or [Moews a]. Typically, no term is crucial. The classification is rather more complicated than in the simple case, but it is still the case that Left can win if he has an atomic weight 1 position and the least desirable term, or an atomic weight 0 position, the least desirable term, and the first move, provided that this least desirable term is not of the form $-\vec{_x}^{k}$. (Here, if there is a $*$ term present after cancellation, it should be counted as Left's, regardless of the color of coin it comes from.)

8. A Resultant Ordering of Some Infinitesimals

Our classification of positions lets us compare the infinitesimals $\uparrow^{k+1}$, $+_x^k$, and $\{0^m|+_x\}^{k+1}$ with each other. For positive x and positive integers k and m, these infinitesimals are all positive and have atomic weight 0. Also, it follows from our classification that $G^k \gg G^{k+1}$ for $k \geq 1$ and $G = \uparrow$, $+_x$ or $0^m|+_x$; here by $H \gg K$ we mean that $H > r \cdot K$ for all positive integers r. Define $H \,||||\, K$ to mean that $r \cdot H \,\|\, s \cdot K$ for all positive integers r and s. Then we can say as well that

$$\begin{array}{rll}
\uparrow^k \gg \{0^m|+_x\}^l & \text{for } l \geq 2, \\
\{0^m|+_x\}^k \gg +_y^l & \text{for } m \geq 2, \\
\{0^m|+_x\}^k \gg \{0^n|+_y\}^l & \text{for } x > y,\ m \geq 2,\ l \geq 2, \\
\{0^m|+_x\}^k \gg \{0^n|+_x\}^l & \text{for } m > n,\ m \geq 2,\ l \geq 2, \\
+_x^k \gg +_y^l & \text{for } x < y, \\
\{0^n|+_x\}^k \,||||\, \{0|+_y\}^l & \text{for } x < y,\ k \geq 2,\ l \geq 2, \\
\{0|+_x\}^k \,||||\, +_y^l & \text{for } x > y,\ k \geq 2, \\
\{0|+_x\}^k \,||||\, +_x & \text{for } k \geq 2, \\
\{0|+_x\}^k \gg +_x^l & \text{for } l \geq 2.
\end{array}$$

Here are examples of some of these relations between infinitesimals:

$$\begin{aligned}
&\uparrow \gg \uparrow^2 \gg \uparrow^3 \gg \cdots \\
&\gg \cdots \\
&\gg \{0^4|+_3\}^2 \gg \{0^4|+_3\}^3 \gg \{0^4|+_3\}^4 \gg \cdots \\
&\gg \{0^3|+_3\}^2 \gg \{0^3|+_3\}^3 \gg \{0^3|+_3\}^4 \gg \cdots \\
&\gg \{0^2|+_3\}^2 \gg \{0^2|+_3\}^3 \gg \{0^2|+_3\}^4 \gg \cdots \\
&\gg \cdots \\
&\gg \{0^4|+_2\}^2 \gg \{0^4|+_2\}^3 \gg \{0^4|+_2\}^4 \gg \cdots \\
&\gg \{0^3|+_2\}^2 \gg \{0^3|+_2\}^3 \gg \{0^3|+_2\}^4 \gg \cdots \\
&\gg \{0^2|+_2\}^2 \gg \{0^2|+_2\}^3 \gg \{0^2|+_2\}^4 \gg \cdots \\
&\gg \cdots \\
&\gg \{0^4|+_1\}^2 \gg \{0^4|+_1\}^3 \gg \{0^4|+_1\}^4 \gg \cdots \\
&\gg \{0^3|+_1\}^2 \gg \{0^3|+_1\}^3 \gg \{0^3|+_1\}^4 \gg \cdots \\
&\gg \{0^2|+_1\}^2 \gg \{0^2|+_1\}^3 \gg \{0^2|+_1\}^4 \gg \cdots \\
&\gg \cdots \\
&\gg +_1 \gg +_1^2 \gg +_1^3 \gg \cdots \\
&\gg +_2 \gg +_2^2 \gg +_2^3 \gg \cdots \\
&\gg +_3 \gg +_3^2 \gg +_3^3 \gg \cdots,
\end{aligned}$$

$$\{0|+_3\}^k \,||||\, \{0^n|+_2\}^l, \{0^n|+_1\}^l, +_1^j, +_2^j, +_3 \quad \text{for } k \ge 2,\ l \ge 2,$$

$$\{0|+_2\}^k \,||||\, \{0^n|+_1\}^l, +_1^j, +_2 \quad \text{for } k \ge 2,\ l \ge 2,$$

$$\{0|+_1\}^k \,||||\, +_1 \quad \text{for } k \ge 2.$$

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