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Loopy Games and Go

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ABSTRACT. Berlekamp, Conway and Guy have developed a theory of partizan loopy combinatorial games—that is, partizan combinatorial games that allow infinite play—under disjunctive composition. We review this theory of loopy games and show how it can be adapted to the two-person strategy game of Go, which also has the feature that situations involving infinitely long play often arise.

1. Introduction

In the two-player strategy game of Go, it can happen that an endgame position splits up into several non-interacting subpositions. Since each player must then move in just one of the subpositions on his turn, the whole position will then be the so-called *disjunctive compound*, or *sum*, of the subpositions. As it turns out, we can then apply to these Go endgames the theory of partizan combinatorial games with finite play under disjunctive composition, as found in *Winning Ways* [Berlekamp et al. 1982], Chapters 1–8, or *On Numbers and Games* [Conway 1976].

This paper assumes that the reader is already somewhat familiar with Go and with the application of this theory to Go, as given in [Wolfe 1991; Berlekamp and Wolfe 1994]. In Chapter 11 of *Winning Ways* there is a theory of partizan combinatorial games with possibly infinite play under disjunctive composition. These games are there called *loopy*, since what was a game tree in the finite play case is now a game graph, perhaps with cycles. We review this theory of loopy games and show how it can be applied to Go, which also has cycles.

2. Loopy Games

We review the definitions of loopy games and basic theorems about them, as given in *Winning Ways*, pp. 314–357. A *loopy game* is played on a directed pseudograph (that is, a directed graph that is allowed to have multiple edges from one vertex to another, or edges from a vertex to itself). Each vertex of this graph corresponds to a position of the game. We partition the edges into two sets, Left's and Right's, corresponding to our two players, Left and Right. Each player plays along his own edges only, changing the position from the vertex at the start of the edge to that at the end. Left and Right play alternately; if either is ever left without a move, he loses.

To specify a loopy game completely, we need to specify a start-vertex as well as the graph and its edge-partition. We also need to specify for each legal infinite sequence of moves—whether it has alternating play by Left and Right or not—whether it is won by Left, won by Right, or is drawn. We assume that two infinite plays that are the same except for a finite initial segment are similarly designated. In general, such games can be very complicated. For example, assuming the Axiom of Choice, there are games with no plays drawn for which there is a winning strategy for neither Left nor Right. However, our infinite play criteria will always be so simple that this sort of thing will never arise.

The distinctive features of this theory of loopy games are the partizan nature of the game—i.e., that the two players may have different moves—and the disjunctive composition that we will define now. Given two loopy games, G and H, we define their disjunctive compound, or sum, $G + H$, as in the case where play must end in a finite length of time: it is just G and H placed side by side, with each player moving on just one of the games—whichever he chooses—on his turn. If there is infinite play in a sum, to determine who wins, we need to look at the subplays in each component:

> A player *wins* the sum just if he wins *all* the components in which there is *infinite* play. (*Winning Ways*, p. 315)

If we interchange the roles of Left and Right in a game G, we call the result $-G$. Moreover, G^+ will be G with draws redefined as wins for Left, and G^- will be G with draws redefined as wins for Right. We say that $G \geq H$ if Left wins or draws both $G^+ - H^+$ and $G^- - H^-$ moving second. This relation is reflexive and transitive, and $G \geq H$ implies that $G + K \geq H + K$ for all K (*Winning Ways*, pp. 328–330). As in the finite play case, we define 0 as the game where neither player can ever move, and $G = H$ just if $G \geq H$ and $G \leq H$. Under $+$, the loopy games form a monoid with additive identity 0, but unlike the finite play case, they do not form a group. For example, let **on** be the game where there is only one position and a move by Left from the position to itself, infinite play being drawn. For any G, Left can always move from **on** to **on** in $\mathbf{on} + \mathbf{G}$, and at least draw, whoever starts. Hence $\mathbf{on} + \mathbf{G}$ can never be 0.

3. Sidling

Sidling, defined in *Winning Ways*, pp. 318 ff., is a means of approximating loopy games by simpler games—hopefully, even *enders*, which are what partizan combinatorial games with finite play are called in this new context. The basic idea is that we cut play off after a finite number of moves.

Let's say we have a loopy game G and a set M of moves in G, and that infinite play is always won for Left if there are infinitely many moves in M. Let S be the set of positions of G. Now if we have a loopy game H whose set of positions includes S, let $G^*(H)$ be what we get when we put G next to H and redirect the moves in M from their destinations in G to the corresponding positions in H. Infinite play for $G^*(H)$ is handled by the criterion for H or for G, according to whether the play ever enters H or not. After doing this, we consider the set of positions of the altered copy of G within $G^*(H)$ to be S. Let K_v be the game K with the start vertex redefined to be the position v of K. Then, if $H_v \geq K_v$ for all $v \in S$, we have $G^*(H)_v \geq G^*(K)_v$ for all $v \in S$. (You can see this by means of a simple reflection strategy in the difference game.) In this sense, G^* is an order-preserving operator. Let N have set of positions S and a move by Left from each position to itself. We consider all infinite play in N to be won for Left. Then $N_v = \mathbf{on}^+$ for each $v \in S$, and $\mathbf{on}^+$ is a maximal game, that is, $\mathbf{on}^+ \geq \mathbf{H}$ for all H. Hence $N_v \geq G^*(N)_v$ for all $v \in S$, and then since G^* is order-preserving it follows that

$$N_v \geq G^*(N)_v \geq G^*(G^*(N))_v \geq \cdots$$

for all $v \in S$. If we are in luck, the sequence N, $G^*(N)$, ... will reach a fixed point U—i.e., a U such that $U_v = G^*(U)_v$ for all $v \in S$. This is then a maximal fixed point, since we got it by iterating G^* on the maximal N. But $G^*(G)_v = G_v$ for all $v \in S$; another obvious reflection strategy proves this, once you recall that we required our infinite play criterion in G to be independent of any finite initial play in G. Hence $U_v \geq G_v$ for all $v \in S$. But we have the following result:

Sidling Theorem. *For all R such that $R_v \leq G^*(R)_v$ for all $v \in S$, we have $R_v \leq G_v$ for all $v \in S$.*

Proof. See *Winning Ways*, pp. 351–353, for the case when M contains every move in G. The extension to the general case is easy [Moews 1993, Theorem 1]. □

Hence our U above must in fact have $U_v = G_v$ for all $v \in S$, and we have found a simplified form for G. Often, U will even be an ender. For example, let G be the game in Figure 1, where we take all infinite play to be drawn. G^+ will then

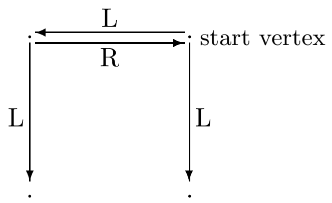


Figure 1. A loopy game.

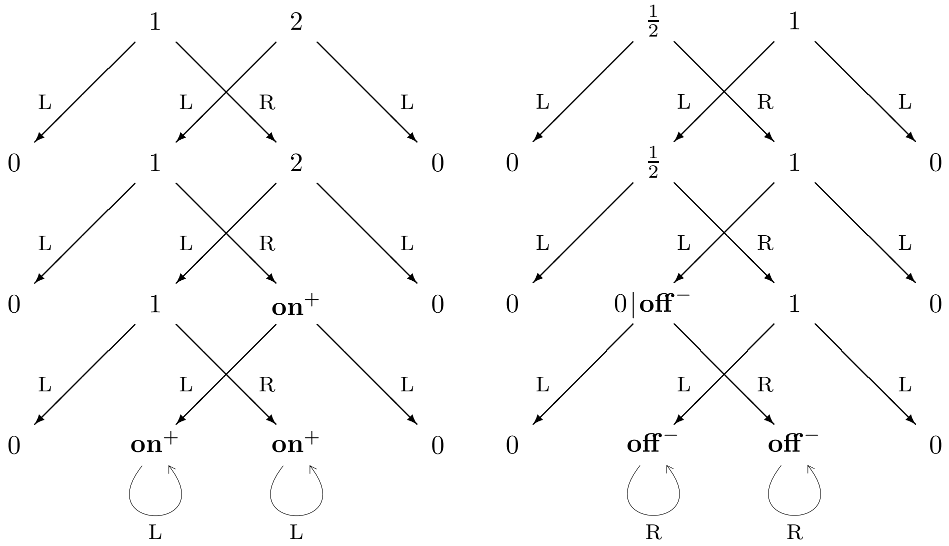


Figure 2. Left: Computing G^+. Right: Computing G^-.

have all infinite play won for Left. Taking M to be the set of the two horizontal moves in G, we can compute $U = G^*(G^*(G^*(N)))$ as shown in Figure 2, left. We have labelled the vertices of U with their values—that is, we have labelled each v with the name of a game equal to U_v. You can see from the figure that

$$G^*(G^*(N))_v = G^*(G^*(G^*(N)))_v$$

for all $v \in S$. It follows that $G^+ = 2$.

Similarly, suppose we start with the game N', which consists of moves for Right from positions to themselves, all infinite play being won for Right. We will then have $N'_v = \mathbf{off}^-$ for all v ($\mathbf{off} = -\mathbf{on}$ being a game with only one position, a move for Right from that position to itself, and all infinite plays drawn), and if we reach a fixed point by iterating G^* starting at N', it will be a minimal fixed point of G^*. If we negate everything and apply the Sidling Theorem, we can see that this fixed point contains a value for G^-. The result in our case can be seen in Figure 2, right (where $0\,|\,\mathbf{off}^-$ is a game from which Left can move to 0 and Right can move to $\mathbf{off}^-$). We find that $G^- = 1$. From these values, we can find the outcome of $G + E$ for all enders E. For example, $G - \frac{1}{2}$ is won by Left moving first, since $G^+ - \frac{1}{2}$ and $G^- - \frac{1}{2}$ are both positive; $G - 1$ is a draw for Left moving first, since $G^+ - 1$ is positive but $G^- - 1$ is zero; and $G - 2$ is a loss for Left moving first, since both $G^+ - 2$ and $G^- - 2$ are less than or equal to zero.

4. Go and Kos

As we mentioned in the introduction, the board can split up into independently analyzable subpositions in Go endgames, and given such a subposition, we can treat it as a partizan combinatorial game by identifying the integer scores of the terminal positions arising from this subposition with the integer combinatorial games that occur in this theory [Wolfe 1991, Chapter 3; Berlekamp and Wolfe 1994, Chapter 2; Moews 1993, Chapter 1]. (We identify Black and White in Go with Left and Right in this theory, so that a negative final score is favorable to White.) We will assume that these subpositions are separated by stones, called *immortal* in [Berlekamp and Wolfe 1994, § 4.1] and [Moews 1993, Chapter 1], which are uncapturable and will hence remain on the board until the end of the game. In diagrams of these subpositions we show the separating immortal stones on grid lines that extend off the edge of the subposition.

Consider the subposition shown at B in Figure 3. Right (White) can move on the empty vertex to a zero position, but if Left (Black) moves, he will capture a White stone and move to A. From A, Right can move back to a position equivalent to B, and Left can move to a position with no moves and one White stone captured, which is 1. This position is called *one-point ko* in Go, since the total value at stake is $1 - 0 = 1$. The moves to 1 or 0 are said to *fill* the ko.

Naïvely, it looks as though play could continue forever in the ko, if both players move back and forth without filling it. In Go, there are various *ko-ban* rules used to prevent this. We will use the Japanese ko-ban rule, which states that a player can't move back to the position that occurred immediately before the last player's move (considering the whole board position, and not just our subpositions.)

If G is a loopy game, we let $\phi(G)$ be the loopy game G with the constraint of the Japanese ko-ban rule added. We take all infinite play to be drawn in $\phi(G)$.

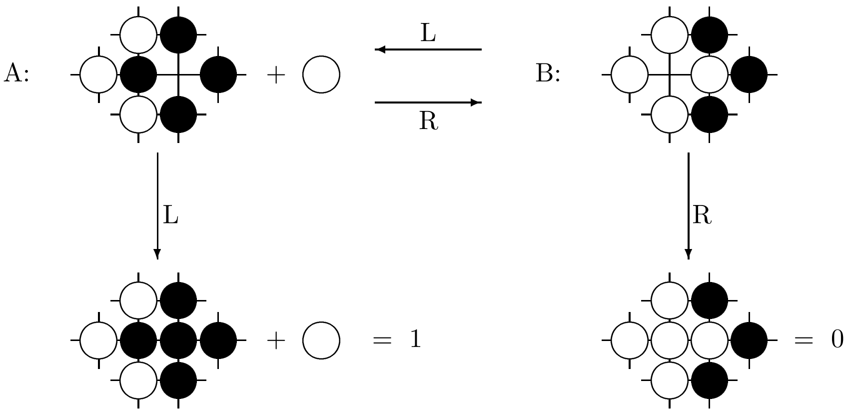


Figure 3. Ko.

A position of $\phi(G)$ will be either an ordered pair $\varepsilon[\{\lambda\}]$ of positions ε and λ of G, by which we mean the position ε of G with the position λ occurring before the last move, or $\varepsilon[\varnothing]$, by which we mean the position ε of G when there has been no previous move. It should be clear which moves are allowed in $\phi(G)$.

The difficulty with ϕ is that if H is an ender, we do not have the property $\phi(G+H) = \phi(G) + H$, even if $H = 0$. For example, let H be $\{100|0\|\}$, a game equal to 0. Right moving first wins on $\phi(A)$, since he moves from A to B and then Left has no move. Moving first on $\phi(A+H)$, Right's first move must still be from A to B. However Left can then move from H to $100|0$, and Right is then forced to move from $100|0$ to 0. Left can then move back from B to A without violating the ko-ban in the sum, and Right then has no move and loses. In Go terminology, H was a *ko-threat* for Left, since by using it Left was able to make an otherwise ko-banned move. We let $\phi_L(G)$ be $\phi(G)$ with ko-banned moves adjoined for Left, and similarly for $\phi_R(G)$ and Right. Since to make a ko-banned move in G Left must use up a ko-threat, and in a subposition of a real position there could only be finitely many ko-threats like H available, we make all infinite play in $\phi_L(G)$ with an infinite number of ko-banned moves a loss for Left, although all other infinite play remains a draw, as in $\phi(G)$. The rule for $\phi_R(G)$ is analogous.

Evidently, for all G,

$$\phi_L(G) \geq \phi(G) \geq \phi_R(G),$$

so ϕ_L and ϕ_R are bounds for ϕ, and for all G that can occur in Go, we have the desirable properties

$$\begin{aligned} \phi_L(G+H) &= \phi_L(G) + H \qquad \text{for } H \text{ an ender,} \\ \phi_R(G+H) &= \phi_R(G) + H \qquad \text{for } H \text{ an ender.} \end{aligned}$$

In Figure 4, we see the graph of ϕ_L of the one-point ko of Figure 3. The graph of ϕ_L of a single ko is acyclic, so all positions in Figure 4 will equal enders. In fact, if we start at $A[\varnothing]$, it can be seen that the resultant value will equal 2, and if we start at $B[\varnothing]$, it will equal $2|0$.

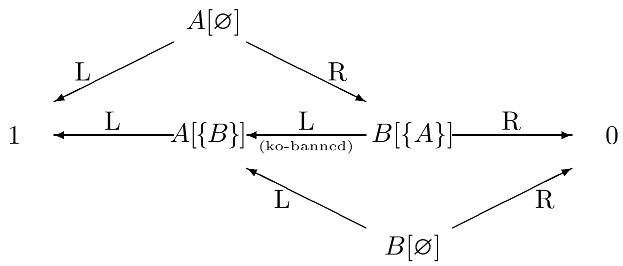
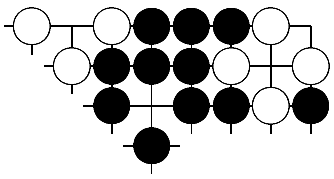


Figure 4. The graph of ϕ_L of a one-point ko.

5. An Example of Sidling in Go

When we have more than one ko in G, the game-graph of $\phi_L(G)$ or $\phi_R(G)$ may contain loops, so we will need to sidle to compute enders equal to them. For example, let T be the position on the right, consisting of the sum of a one-point and a seven-point ko. The decomposition of T as a sum of kos is shown in Figure 5. We wish to compute $\phi_L(T)$.



The game graph of $P = \phi_L(T)$ is the large connected component in the center-right of Figure 6. We have made certain simplifications, such as precomputing the values of subgraphs where only one ko is still alive. We have also omitted nodes of the form $\varepsilon[\varnothing]$, since their values can be shown to be equal to the value of $\varepsilon[\{\lambda\}]$, for any λ, and there are no moves back from the $\varepsilon[\{\lambda\}]$'s to the $\varepsilon[\varnothing]$'s. For reference, ϕ_L of the one-point ko is shown at the top and lower left. (Here, again, values of $\varepsilon[\varnothing]$ vertices are the same as those of $\varepsilon[\{\lambda\}]$ vertices, and the $\varepsilon[\varnothing]$ vertices are again omitted.) Also, ϕ_L of the seven-point ko is shown at the upper left and bottom. In the top half of P, a move on the one-point ko is shown by a horizontal line, and a move on the seven-point ko is shown by a vertical line. In the bottom half of P, this convention is reversed. The moves for Left violating the ko-ban are indicated as such and are always shown as vertical lines going from one half to the other. We have labelled the positions in the center A through H.

We try to approximate P's value by sidling. We will start by approximating P^+. We will let M be the set of Left's ko-banned moves. Remembering that all plays using infinitely many moves in M are lost for Left, and referring to Section 3, we see that we have to start sidling from $\mathbf{off}^-$. Let K_0 have moves for Right from every position to itself, with infinite play won for Right. The next step is to construct $(P^+)^*(K_0) = K_1$, which is itself loopy. In fact, if we let $\bar{P}$ (shown in Figure 7) be P with the moves in M removed, then any $(K_1)_v$ will be the same as $(\bar{P}^+)_v$, since Left will never want to take the moves in K_1 from the embedded P^+ to $\mathbf{off}^-$. So we need to sidle $\bar{P}$, and now we can let M be the set of all the moves in $\bar{P}$ that do not destroy a ko, that is, all the moves from one to another of the positions $A, \ldots, H$.

$$\begin{tikzcd} \alpha \arrow[r, shift right, "\mathrm{R}"'] \arrow[d, "\mathrm{L}"'] & \beta\ \text{(start vertex)} \arrow[l, shift right, "\mathrm{L}"'] \arrow[d, "\mathrm{R}"] \\ 7 & 0 \end{tikzcd} \quad + \quad \begin{tikzcd} \delta \arrow[r, shift right, "\mathrm{R}"'] \arrow[d, "\mathrm{L}"'] & \gamma\ \text{(start vertex)} \arrow[l, shift right, "\mathrm{L}"'] \arrow[d, "\mathrm{R}"] \\ 1 & 0 \end{tikzcd}$$

Figure 5. How the position shown above is the sum of two kos.

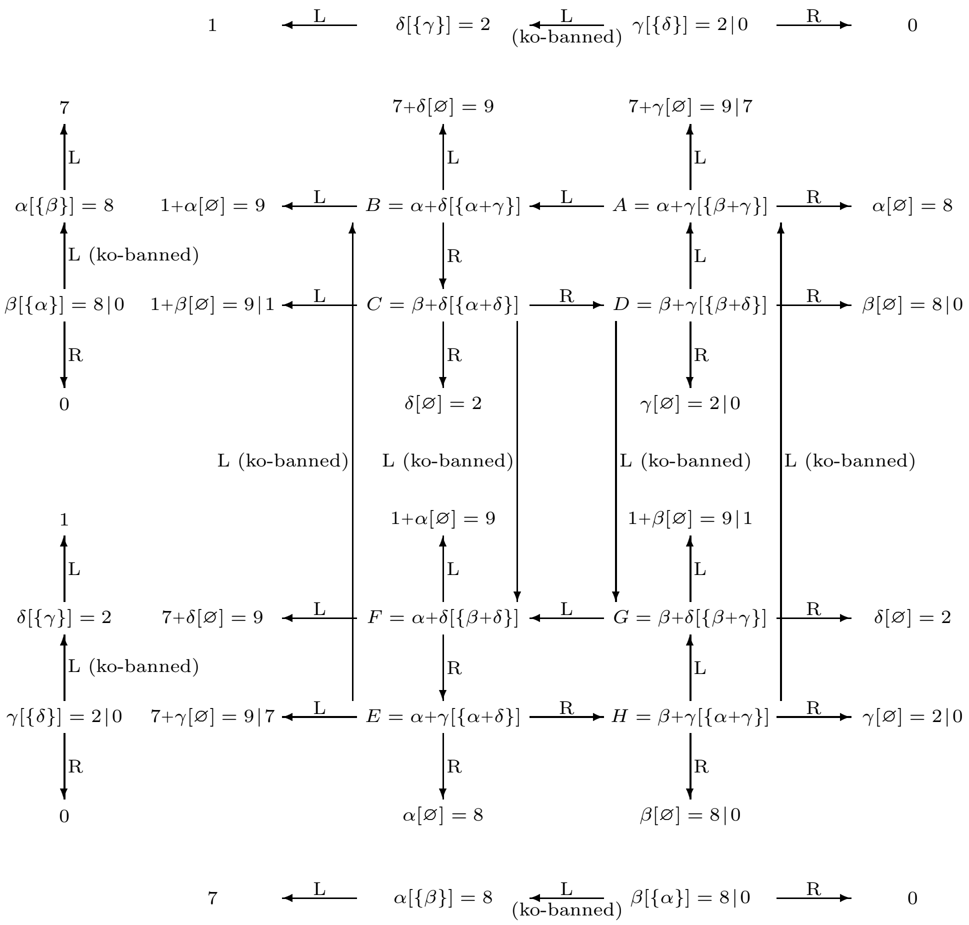


Figure 6. An example of sidling. The large connected component in the center-right is P.

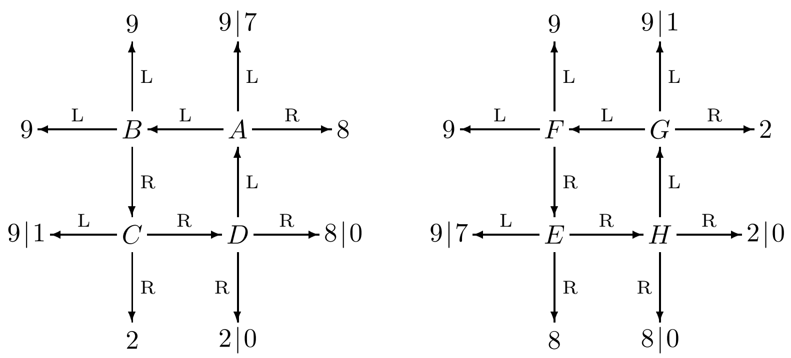


Figure 7. The graph $\bar{P}$.

Our first approximation to K_1, K_{10}, will have moves for Left from A, ... , H to themselves, with infinite play won for Left. (We can omit the other vertices since they are not moved to by moves in M.) We must then set $K_{11} = (\bar{P}^+)^*(K_{10})$. Given a sequence of approximate values for the vertices of $\bar{P}$, applying $(\bar{P}^+)^*$ refines it by taking just one more move in $\bar{P}$. For example, we have $(\bar{P}^+)^*(Q)_A = \{\{9|7\}, Q_B|8\}$, so $(K_{11})_A = \{\mathbf{on}^+|\mathbf{8}\}$. The other $(K_{11})_v$'s are computed similarly. Their values are given in Table 1, and their graphs are shown in Figure 8, for $v \in \{A, B, C, D\}$. (The graphs for $v \in \{E, F, G, H\}$ are similar.)

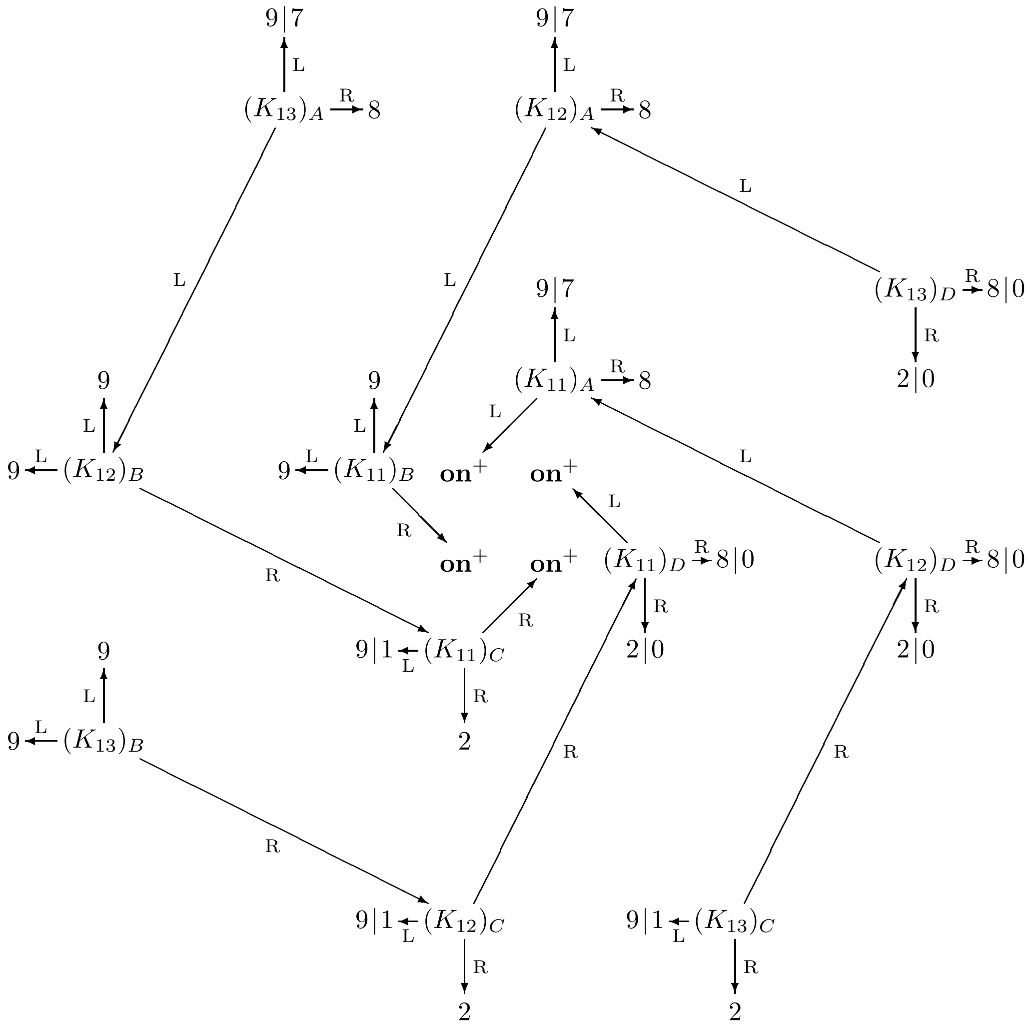


Figure 8. The graphs of $(K_{1i})_v$, for $i = 1$, 2, 3 and $v = A$, B, C, D.

v	A	B	C	D	E	F	G	H
$(K_0)_v$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$
$(K_{10})_v$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$
$(K_{11})_v$	$\mathbf{on}^+ \mid \mathbf{8}$	10	1	$\mathbf{on}^+ \Vert \mathbf{2} \mid \mathbf{0}$	7	10	$\mathbf{on}^+ \mid \mathbf{2}$	$\mathbf{on}^+ \Vert \mathbf{2} \mid \mathbf{0}$
$(K_{12})_v$	$10 \mid 8$	$9 \mid 1$	1	$\mathbf{on}^+ \mid \mathbf{8} \Vert \mathbf{2} \mid \mathbf{0}$	7	$9 \mid 7$	$10 \mid 2$	2
$(K_{13})_v$	7	$9 \mid 1$	1	$10 \mid 8 \Vert 2 \mid 0$	$9 \mid 7 \Vert 2$	$9 \mid 7$	$9 \mid 7 \Vert 2$	2
$(K_{14})_v$	7	$9 \mid 1$	1	$7 \Vert 2 \mid 0$	$9 \mid 7 \Vert 2$	$9 \mid 7$	$9 \mid 7 \Vert 2$	2
$(K_{15})_v$	7	$9 \mid 1$	1	$7 \Vert 2 \mid 0$	$9 \mid 7 \Vert 2$	$9 \mid 7$	$9 \mid 7 \Vert 2$	2
$(K_{20})_v$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$
$(K_{21})_v$	$\mathbf{on}^+ \mid \mathbf{8}$	10	$9 \mid 7 \Vert 2$	$\mathbf{on}^+ \Vert \mathbf{2} \mid \mathbf{0}$	7	10	$\mathbf{on}^+ \mid \mathbf{2}$	$\mathbf{on}^+ \Vert \mathbf{2} \mid \mathbf{0}$
$(K_{22})_v$	$10 \mid 8$	$9 \mid 7$	$9 \mid 7 \Vert 2$	$\mathbf{on}^+ \mid \mathbf{8} \Vert \mathbf{2} \mid \mathbf{0}$	7	$9 \mid 7$	$10 \mid 2$	$7 \Vert 2 \mid 0$
$(K_{23})_v$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$10 \mid 8 \Vert 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$
$(K_{24})_v$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$
$(K_{25})_v$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$
$(K_{30})_v$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$
$(K_{31})_v$	$\mathbf{on}^+ \mid \mathbf{8}$	10	$9 \mid 7 \Vert 2$	$\mathbf{on}^+ \Vert \mathbf{2} \mid \mathbf{0}$	7	10	$\mathbf{on}^+ \mid \mathbf{2}$	$\mathbf{on}^+ \Vert \mathbf{2} \mid \mathbf{0}$
$(K_{32})_v$	$10 \mid 8$	$9 \mid 7$	$9 \mid 7 \Vert 2$	$\mathbf{on}^+ \mid \mathbf{8} \Vert \mathbf{2} \mid \mathbf{0}$	7	$9 \mid 7$	$10 \mid 2$	$7 \Vert 2 \mid 0$
$(K_{33})_v$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$10 \mid 8 \Vert 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$
$(K_{34})_v$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$
$(K_{35})_v$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \Vert 2$	$7 \Vert 2 \mid 0$

Table 1. Sidling values from Figure 6.

We must then compute $K_{12} = (\bar{P}^+)^*(K_{11})$, $K_{13} = (\bar{P}^+)^*(K_{12})$, and so on. These values are also in Table 1; the graphs of the $(K_{12})_v$'s and $(K_{13})_v$'s are shown in Figure 8 for $v \in \{A, B, C, D\}$. Eventually, we find that $(K_{14})_v = (K_{15})_v$ for all v. We may then conclude, from the Sidling Theorem, that $(\bar{P}^+)_v = (K_1)_v = (K_{15})_v$ for all v.

(As we can see from Figure 8, the graphs of the $(K_{1j})_v$'s for $v \in \{A, B, C, D\}$ form a set of four spirals. To compute the $(K_1)_v$'s for $v \in \{A, B, C, D\}$, it would in fact suffice to look at any one of the four spirals. This would correspond to letting M contain just, e.g., the move from C to D, instead of all the moves between A, B, C, and D.)

We return to sidling with M equal to Left's ko-banned moves, and set $K_2 = (P^+)^*(K_1) = (P^+)^*(\bar{P}^+)$. K_2 is also loopy. $(K_2)_v$ will be the same as $(\bar{P}^+)_v$, except that extra moves are present: if P has a ko-banned move for Left going from E to B (say), then $(K_2)_v$ has a move for Left from E to $(K_1)_B = (\bar{P}^+)_B$. In fact, we can write $(K_2)_v = Q_v$ for all $v \in \{A, \ldots, H\}$, where Q is the graph in Figure 9. By replacing the bottom copy of $\bar{P}$ in Q with the enders in K_{15}, we get the simplified graph Q_2 in Figure 10. To approximate K_2, we can then sidle with all those moves in $\bar{P}$ that do not destroy a ko, as we did with K_1. We

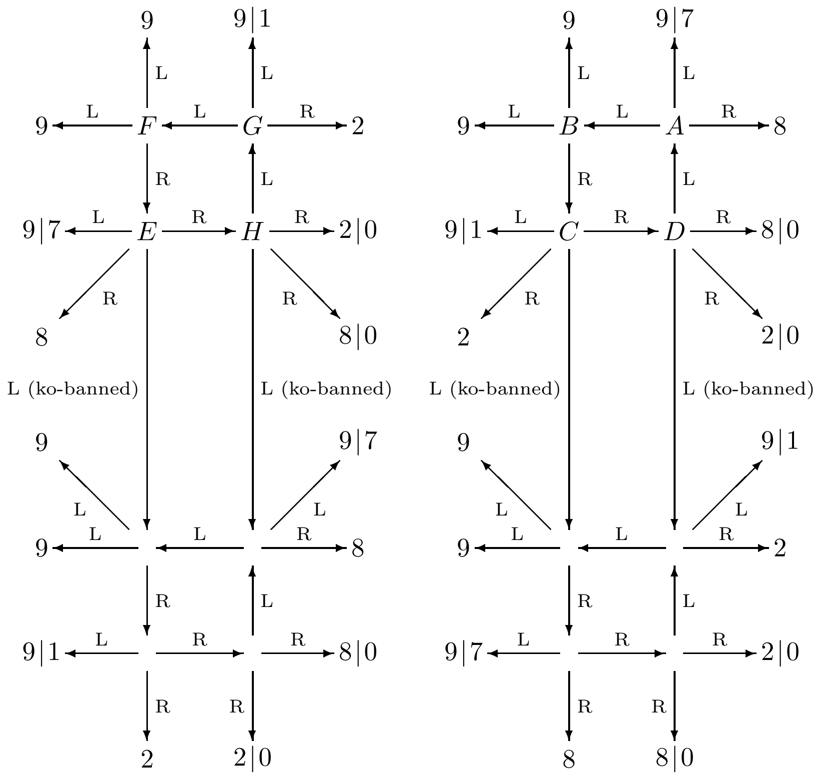


Figure 9. The graph Q.

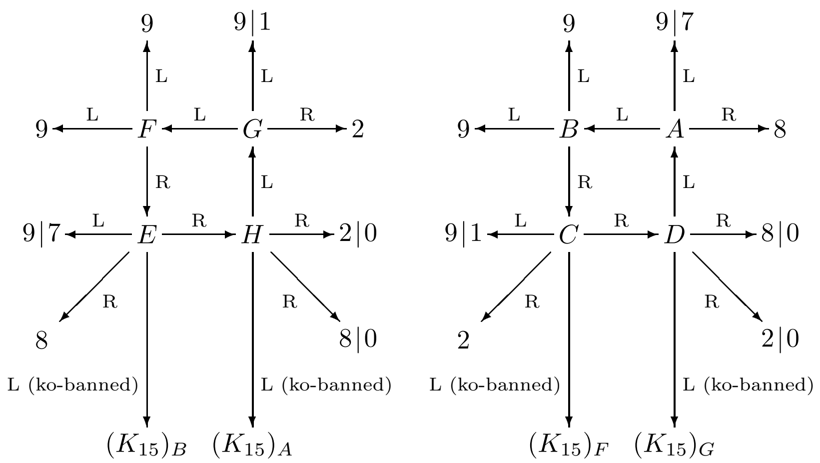


Figure 10. The graph Q_2.

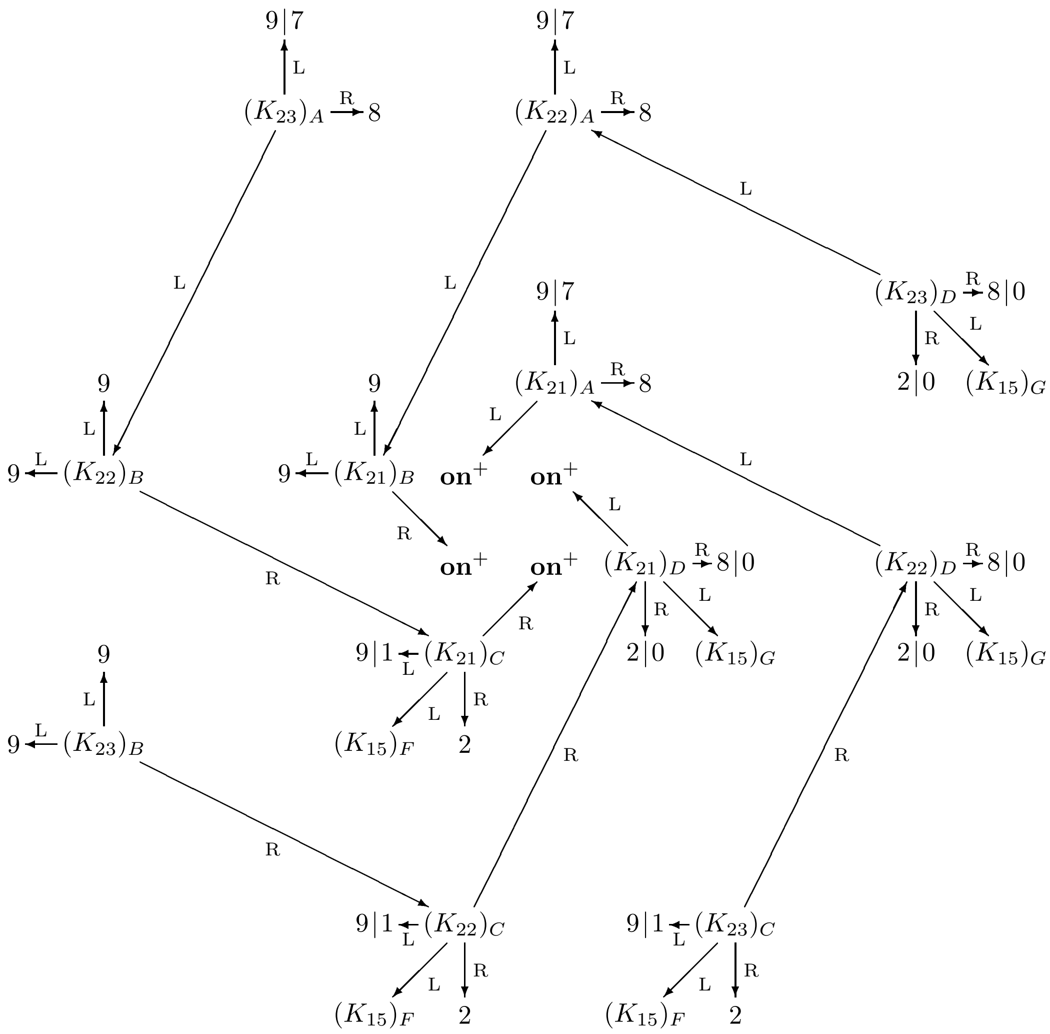


Figure 11. The graphs of $(K_{2i})_v$, for $i = 1, 2, 3$ and $v = A, B, C, D$.

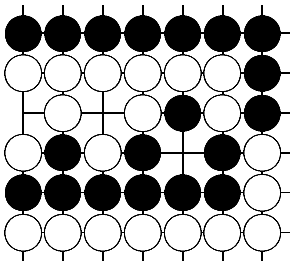
can then approximate K_2 by a sequence $K_{20}, K_{21}, K_{22}, \ldots$, as we did with K_1; we display graphs of some $(K_{2i})_v$'s in Figure 11. We find that $(K_{24})_v = (K_{25})_v$ for all v, so $(K_2)_v = (P^+)^*(K_1)_v = (K_{25})_v$ for all v.

We can compute K_3, and its approximations, $K_{30}, K_{31}, K_{32}, \ldots$, similarly. We find that $(K_{34})_v = (K_{35})_v$ for all v and that $(K_3)_v = (K_{35})_v = (K_2)_v = (K_{25})_v$ for all v, so that we can write $(P^+)_v = (K_2)_v = (K_{25})_v$ for all v.

In an analogous manner, we may compute $(P^-)_v$. In this case, we find that $(P^-)_v = (K_{25})_v$ for all v as well, so $P_v = (K_{25})_v$ for all v, since we have $X^+ \geq X \geq X^-$ for all games X. As we remarked earlier, $\phi_L(T)$ will equal both P_H and P_D. Hence $\phi_L(T) = 7\|2|0$.

6. Final Remarks

In the example T above, we found that $\phi_L(T)$ was an ender. The full power of the loopy theory was thus unnecessary, since our games effectively had finite play all along. However this will not always be so. Consider the *triple ko* position on the right, which we call A [Berlekamp and Wolfe 1994, Figure A.6]. As you can see, there are three kos, all involving the same pair of groups. Black's filling is suicidal, and if White fills, Black can immediately capture White's group. We will thus assume that no one ever fills a ko.



If White plays in the ko on the right, he will capture Black's group, plus one black stone. Assign the resultant position value W; then Right (White) has a move from A to W. If Left (Black) plays in a ko on the left, he reaches a position that we call E or F, depending on which he plays in; if Left plays again, he can capture White's group (plus two white stones); assign this position value V, so that Left has moves from E or F to V. We can apply similar reasoning to that concluding that no one fills in A to conclude that no one fills in E or F either. If we continue along these lines, we get the game graph in Figure 12, left.

In play, positions A, B, and C will all be equivalent, as will position D, E, and F, and players will thus be effectively moving on the game graph G of Figure 12, right, except that thereto E, Right can always move back to C, regardless of ko-ban. If we take G to have starting vertex ABC and infinite play drawn, sidling gives $G^+ = \{V|\}|W \neq G^- = \{|W\}$, so G is not an ender.

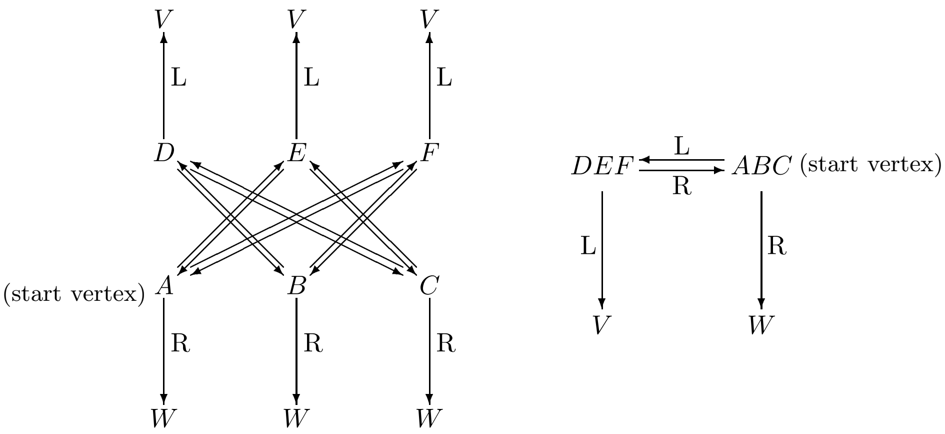


Figure 12. Left: A simplified game graph for the triple ko position. Arrows sloping upward are Left's moves; those sloping downward are Right's moves. Right: An equivalent graph G.

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