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Geography
Played on Products of Directed Cycles

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ABSTRACT. We consider the game of Geography played on $G = C_n \times C_m$, the product of two directed cycles. The analysis is easy for $n = 2$ and in the case where both n and m are even. Most of the paper is devoted to solving the game on the graphs $C_n \times C_3$.

1. Introduction

The game called *Kotzig's Nim* in *Winning Ways* [Berlekamp et al. 1982] and *Modular Nim* in [Fraenkel et al. 1995] consists of a directed cycle of length n with the vertices labelled 0 through $n-1$, a coin placed initially on vertex 0, and a set of integers called the move set. There are two players, who alternate moves; a move consists of moving the coin from the vertex i on which it currently resides to vertex $i+m$ mod n, where m is a member of the move set. However, the coin can only land on a vertex once. Thus, the game is finite. The last player to move wins. Most of the known results concern themselves with move sets of small cardinality and consisting of small numbers (see p. 481 of *Winning Ways*, and [Fraenkel et al. 1995]).

Obviously, this game can be extended to more general directed graphs, the move set being indicated by directed edges, for clarity. This has become known as *Geography* [Fraenkel et al. 1993; Fraenkel and Simonson 1993].

The *Grundy value* of a game G, denoted $\mathcal{G}(G)$, is either $\mathcal{P}$ for a previous-player win or $\mathcal{N}$ for a next-player win. Throughout, we call the first player Algois and the second Berol.

One general strategy for Geography can be identified quickly. A set A of independent edges in an undirected graph is called a *perfect matching* if every vertex is incident with an edge in A. A set of edges $\{(a_i, b_i) : i = 1 \dots, n\}$ is a *directed perfect matching* of a directed graph G if the edges form a matching of the underlying undirected graph and if, for each $i = 1, \dots, n$, the vertex b_i has

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no edges directed to b_j for any $j \neq i$. The following is a variant of results found in [Fraenkel et al. 1993] and [Bondy and Murty 1976, p. 71, problem 5.1.4].

THEOREM 1.1. *Let* $\{(a_i, b_i) : i = 1, \dots, n\}$ *be a directed perfect matching of a graph* G. *If the coin is placed initially on* a_i *for some* i, *then* $\mathcal{G}(G) = \mathcal{N}$.

PROOF. Algois's strategy is always to move from the present position a along this matching edge. He has this option available on the first move since e is the start of a matching edge. This forces Berol to be the first to move to the start of a new matching edge. $\square$

In the sequel we concentrate on those graphs that are the products of two cycles.

2. Cartesian Products of Directed Cycles

We consider the cases where $G = C_n \times C_m$. Throughout, we will put $C_k = \{0, 1, 2, \dots, k-1\}$. The graph $G = C_n \times C_m$ has vertex set

$$\{(i, j) : 0 \le i \le n-1,\ 0 \le j \le m-1\},$$

and (i, j) is adjacent to (k, l) (that is, there is an edge directed from (i, j) to (k, l)) if i is adjacent to k and $j = l$, or if $i = k$ and j is adjacent to l. Throughout this section (i, j) will be taken modulo n in the first coordinate and modulo m in the second.

When a vertex (a, i) is occupied, its Grundy value $\mathcal{G}(a, i)$ depends on the preceding moves. Since the context will indicate the history of the position, we do not add anything to our notation to take this into account. However, a useful notation is $[a, i]$, which will indicate that this is the first time that a vertex with a as the first coordinate has been visited. Let $\mathcal{G}[b, i]$ refer to the Grundy value of the position obtained by moving to $[b, i]$ from $(b-1, i)$. Again, the context will fill in the history.

THEOREM 2.1. *Let* $G = C_n \times C_m$. *If* $n = 2$, *or if both* n *and* m *are even*, *then* $\mathcal{G}(C_n \times C_m) = \mathcal{N}$.

PROOF. If $n = 2$, Algois follows a variant of the strategy given in Theorem 1.1. He plays moves that only change the first coordinate. He can do this on the first move. Berol has to move so as to change the second coordinate whereupon Algois can again change the first coordinate.

When both n and m are even, Algois always moves from a vertex where both coordinates are of the same parity to a vertex where the parities are different. Let $P = \{(ab,\ a(b+1)) : a \equiv b \bmod 2\}$. The set P is a directed perfect matching for G and the result follows from Theorem 1.1. $\square$

The interesting cases are therefore the ones where one or both of n and m is odd. The rest of the section is devoted to showing the following result:

THEOREM 2.2. *Let* $G = C_n \times C_3$, *where* $n \ge 3$. *Then the Grundy values for* G *are as follows*, *where* n *is taken modulo* 42:

n	$\mathcal{G}$	n	$\mathcal{G}$	n	$\mathcal{G}$	n	$\mathcal{G}$	n	$\mathcal{G}$	n	$\mathcal{G}$	n	$\mathcal{G}$
1	$\mathcal{P}$	7	$\mathcal{P}$	13	$\mathcal{N}$	19	$\mathcal{N}$	25	$\mathcal{N}$	31	$\mathcal{P}$	37	$\mathcal{P}$
2	$\mathcal{N}$	8	$\mathcal{P}$	14	$\mathcal{P}$	20	$\mathcal{P}$	26	$\mathcal{P}$	32	$\mathcal{N}$	38	$\mathcal{N}$
3	$\mathcal{P}$	9	$\mathcal{P}$	15	$\mathcal{N}$	21	$\mathcal{N}$	27	$\mathcal{N}$	33	$\mathcal{P}$	39	$\mathcal{P}$
4	$\mathcal{N}$	10	$\mathcal{N}$	16	$\mathcal{N}$	22	$\mathcal{N}$	28	$\mathcal{N}$	34	$\mathcal{N}$	40	$\mathcal{N}$
5	$\mathcal{P}$	11	$\mathcal{N}$	17	$\mathcal{N}$	23	$\mathcal{N}$	29	$\mathcal{P}$	35	$\mathcal{P}$	41	$\mathcal{P}$
6	$\mathcal{N}$	12	$\mathcal{P}$	18	$\mathcal{P}$	24	$\mathcal{P}$	30	$\mathcal{P}$	36	$\mathcal{N}$	42	$\mathcal{N}$

PROOF. The vertices of the directed graph will be denoted by (a, i), where a is taken modulo n and i modulo 3. During the game, the subgraph consisting of the unused vertices we call the *field*; this excludes the vertex occupied by the coin.

We define two move sequences:

$$S = (a,i),\ (a+1,\ i),\ (a+1,\ i+1),\ (a+1,\ i+2),$$
$$H = (a,i),\ (a,\ i+1),\ (a+1,\ i+1),\ (a+1,\ i+2).$$

Both are called *closing-off* sequences. Such a sequence changes the field graph from being strongly connected to just connected. This also means that the game can only go around (in the first coordinate) once more. Most of the proof revolves around whether and when one of the players should simplify the game by completing one of these closing-off sequences. The next lemma is technical but develops a useful tool for the later analysis.

LEMMA 2.3. *Suppose that player Y completes a closing-off sequence by moving to vertex (a, i). Then $\mathcal{G}(a-1,\ i) = \mathcal{P}$. Moreover, if (b, j) is in the field graph and player Y has previously played to $(b,\ j+1)$, then $\mathcal{G}(b, j) = \mathcal{P}$.*

PROOF. Suppose that a closing-off sequence has been completed by player Y moving to (a, i). Note that in either case, player X had moved to $(a-1,\ i+1)$. Then only $(a-1,\ i)$ has no outgoing edges and thus it is a $\mathcal{P}$ position.

The induced subgraph H of the field graph whose vertices have first coordinate b, for $0 \le b \le a$ (not taken modulo n), is bipartite. By working backwards and inducting on the distance from (b, j) to $(a-1,\ i)$, we see that all vertices in the same color class as $(a-1,\ i)$ are $\mathcal{P}$ positions, and all the other vertices are $\mathcal{N}$ positions. Adjoining to H the vertices used by the players but adding only the edges into these vertices results in another bipartite graph. All the vertices that X moved to are in one color class, and those moved to by Y are in the other. Since $(a-1,\ i)$ has an edge to a vertex once played to by X all the other such vertices are in the same color class and, thus, are $\mathcal{P}$ positions. □

The next result considers the other case in which the Grundy values can be easily found.

LEMMA 2.4. *If $\mathcal{G}[a, i] = \mathcal{P}$ and $\mathcal{G}[a,\ i+1] = \mathcal{N} = \mathcal{G}[a,\ i-1]$, it follows that $\mathcal{G}[a-1,\ i-1] = \mathcal{P}$ and $\mathcal{G}[a-1,\ i] = \mathcal{N} = \mathcal{G}[a-1,\ i+1]$.*

PROOF. Suppose that in a game it happens that $\mathcal{G}[a, i] = \mathcal{P}$ and $\mathcal{G}[a, i+1] = \mathcal{N} = \mathcal{G}[a, i-1]$. Then $\mathcal{G}(a-1, i) = \mathcal{N}$, since it has $[a, i]$ as a follower. The followers of $[a-1, i-1]$ are $[a, i-1]$ and $(a-1, i)$, both of which are $\mathcal{N}$ positions; thus $\mathcal{G}[a, i-1] = \mathcal{P}$.

The followers of $[a-1, i+1]$ are $[a, i+1]$, which is an $\mathcal{N}$ position, and $(a-1, i-1)$, if it exists. From this latter position the followers are both $\mathcal{N}$ positions; thus $\mathcal{G}[a, i+1] = \mathcal{N}$. □

Algois has two possible first moves. The status of moving from $(0,0)$ to $(0,1)$ is the subject of the next lemma.

LEMMA 2.5. *Algois wins by moving from* $(0,0)$ *to* $(0,1)$ *if and only if* $n \equiv 4 \bmod 6$.

PROOF. Suppose that the first two moves are $(0,0) \to (0,1) \to (0,2)$. Then $\mathcal{G}[n-1, i] = \mathcal{P}$, since when the players reach that position there are exactly two moves left in the game.

Now, $\mathcal{G}[n-j, i]$ is a $\mathcal{P}$ position if j is odd and an $\mathcal{N}$ position otherwise. The proof is as follows. We assume it is X to move. If $\mathcal{G}[n-j, i] = \mathcal{P}$ with j odd, then $\mathcal{G}[n-j+1, i] = \mathcal{N}$, and X immediately moves to $(n-j, i)$. If $\mathcal{G}[n-j, i] = \mathcal{P}$ with j even, X has two options: (i) to move to $(n-j+1, i)$, which is not possible if $j = 1$ and otherwise, by induction, is an $\mathcal{N}$ position; or (ii) to move to $(n-j, i+1)$; but the good reply to the latter is for Y to move to $(n-j, i+2)$, leaving X with only a move to $(n-j+1, i+2)$, which again is an $\mathcal{N}$ position.

Thus Algois will move to $(0,1)$ only if n is even; otherwise he loses when Berol moves to $(0,2)$.

Assume n is even. Berol can also move to $(1,1)$ and Algois can close off by moving to $(1,2)$. At this point in the game, $\mathcal{G}(0,2) = \mathcal{P}$, and so $\mathcal{G}(n-1, 2) = \mathcal{N}$. If $(n-1, 1)$ is the first position in the $(n-1)$-st column to be occupied, its only follower is $(n-1, 2)$, and thus $\mathcal{G}[n-1, 1] = \mathcal{P}$. Similarly, it follows that $\mathcal{G}[n-1, 0] = \mathcal{N}$.

Thus Algois will win if $\mathcal{G}[2,0] = \mathcal{P}$ and $\mathcal{G}[2,1] = \mathcal{N} = \mathcal{G}[2,2]$. From Lemma 2.4 the $\mathcal{P}$ positions are periodic with period 3 in the first coordinate. Since $\mathcal{G}[n-2, 0] = \mathcal{P}$, Algois wins when $n-2-2 \equiv 0 \bmod 3$, that is, when $n \equiv 1 \bmod 3$. We have already determined that n is even; therefore Algois wins by moving from $(0,0)$ to $(0,1)$ just if $n \equiv 4 \bmod 6$. □

The rest of the proof is devoted to the other case: that is, Algois moves from $(0,0)$ to $(1,0)$. The first part of the analysis deals with the question of when a player can move safely in the second coordinate.

LEMMA 2.6. *Suppose player* X *makes the non-closing-off move* (a, i) *to* $(a, i+1)$. *Player* X *will not lose to the closing-off move* $(a, i+1)$ *to* $(a, i+2)$ *if and only if either* $n-a+i \equiv 0 \bmod 3$ *and* X *is Algois*, *or* $n-a+i \equiv 1 \bmod 3$ *and* X *is Berol*. *If* Y *moves from* $(a, i+1)$ *to* $(a+1, i+1)$ *then in neither case can*

X win by completing the closing-off sequence by moving from $(a+1,\, i+1)$ to $(a+1,\, i+2)$.

PROOF. Suppose Berol has moved from $(a-1,\, i)$ to (a, i) and then Algois moves from (a, i) to $(a,\, i+1)$ and this does not complete a closing-off sequence. He will only do this if $(a,\, i+1)$ to $(a,\, i+2)$ is a losing move for Berol. If Berol does close off with this move, we have $\mathcal{G}(a-1,\, i+2) = \mathcal{P}$, since it has no followers.

Since Algois moved to $(a-1,\, i)$, Lemma 2.3 gives $\mathcal{G}(0, 1) = \mathcal{P}$. In turn, it follows that $\mathcal{G}(n-1,\, 0) = \mathcal{P}$ and that $\mathcal{G}[n-1,\, 1] = \mathcal{N} = \mathcal{G}[n-1,\, 2]$. By Lemma 2.4, $\mathcal{G}[a+1,\, i+1] = \mathcal{P}$ just if $(n-1) - a + i \equiv 2 \bmod 3$. Thus Algois will move from (a, i) to $(a,\, i+1)$ just if $(n-1) - a + i \equiv 2 \bmod 3$. Note that if indeed $(n-1)-a+i \equiv 2 \bmod 3$ then Berol will not close off but will move to $(a+1,\, i+1)$. The closing-off move to $(a+1,\, i+2)$ is now a bad move for Algois since from Lemma 2.3 and Lemma 2.4 we have that $\mathcal{G}[a+2,\, i+2] = \mathcal{P}$.

Suppose that Berol is player X. She will only move from (a, i) to $(a,\, i+1)$ (when this does not complete a closing-off sequence) if the move $(a,\, i+1)$ to $(a,\, i+2)$ is a losing move for Algois. As in the previous paragraph, Lemmas 2.3 and 2.4 show that this is the case when $(n-1) - a + i \equiv 0 \bmod 3$. Moreover, if Algois does not close off but moves from $(a,\, i+1)$ to $(a+1,\, i+1)$, then again from Lemma 2.3 and Lemma 2.4 Berol will lose by closing-off on her next move. $\square$

This leads to a sequence of forcing moves that are actually three repetitions of a sequence going from (a, i) to $(a+7,\, i+1)$, one for each $i = 0, 1, 2$. See Figure 1.

B_0*									
A_1	$[B_0]$	A_2	B_1	A_0	B_2	A_1*	B_0*		
						B_2	$[A_1]$	B_0	$\cdots$

A_1*									
B_2	$[A_1]$	B_0*	A_2	B_1	A_0	B_2	A_1*		
		A_1	B_0	A_2	B_1	A_0	$[B_2]$	A_1	$\cdots$

Figure 1. Forcing sequences for Algois (top) and Berol (bottom). The entry labels A and B indicate who moved to that vertex; the subscripts are $n - a + i \bmod 3$. The brackets indicate the beginning and end of a sequence. In each case, the other player can determine part of the sequence, but an $*$ indicates the only time a choice is available.

Since neither can be forced into a disadvantageous closing-off move, it remains to analyze the situations when the players have to move to the last column before a closing-off move. In addition, we have to consider which player *drops* first, that is, makes a move in the second coordinate.

In what follows, the $\mathcal{N}$ and $\mathcal{P}$ positions are calculated using Lemma 2.3 as soon as the status of one position can be identified. All the information will be summarized in Table 1.

Case 1. There is no closing-off, and there is a move to $[n-1,\,0]$.

The following cases are independent of who dropped first:

$$\begin{array}{rrr|ll} B & A & B & 0 & A \\ & & A & & \\ & \mathcal{P} & B & \mathcal{N} & \mathcal{P} \end{array} \qquad \begin{array}{rr|ll} B & A & 0 & A \\ \mathcal{P} & B & \mathcal{P} & \\ A & & & \end{array} \qquad \begin{array}{rr|ll} A & B & 0 & A \\ \mathcal{P} & A & \mathcal{N} & \mathcal{P} \\ B & \mathcal{N} & & \end{array}$$

subcase 1: B wins subcase 2: A wins subcase 3: A wins

The 0 indicates the initial vertex. In each case, the $\mathcal{P}$ position to the left of the vertical line is formed by the closing-off move. The other $\mathcal{P}$ and $\mathcal{N}$ positions are obtained from Lemma 2.3. No more need be said about these cases.

In the next two cases it matters who drops first.

$$\begin{array}{rrr|lll} A & B & A & 0 & A\ldots & \underline{B} \\ & & B & & & A \\ & & & & & \end{array} \qquad \begin{array}{rrr|lll} A & B & A & 0 & A\ldots & \underline{A} \\ & & B & & & B \\ & & & & & \end{array}$$

subcase 4: B wins subcase 5: A wins

The situation presented in subcase 4 is a losing one for Algois. He has played to $[n-1,\,0]$, so Berol is forced to move to $(n-1,\,1)$. However, Berol does not have to move off the line $i=1$: she can force Algois to have that doubtful privilege. Whenever he does, he either moves to $(n-1,\,2)$ and Lemma 2.3 shows that $\mathcal{G}(0,2)=\mathcal{P}$; or he moves to $(2k+1,\,2)$ and thus $\mathcal{G}(2j,2)=\mathcal{P}$ for $j=0,1,\ldots k$. But in this case $\mathcal{G}(n-1,\,2)=\mathcal{P}$ and, again by Lemma 2.3, $\mathcal{G}(2a,2)=\mathcal{P}=\mathcal{G}(2b,2)$ for $b=j+1,\,j+2,\,\ldots,\,a-1$, and thus Berol moves to a $\mathcal{P}$ position on her next move after she has forced Algois to drop.

In subcase 5, Algois moves to $(0,1)$ and forces Berol to drop first. The analysis follows as in subcase 4, except that now Algois wins.

Case 2. No closing-off and a move to $[n-1,\,1]$: Berol wins in all cases.

Since $(n-1,\,0)$ has not been visited, it follows that $\mathcal{G}(n-1,\,0)=\mathcal{P}$ and also that $\mathcal{G}(n-1,\,2)=\mathcal{N}$. If Algois is forced to move to $[n-1,\,1]$, Lemma 2.3 implies that $\mathcal{G}(0,1)=\mathcal{P}$, and therefore Berol wins by moving to $(0,1)$. If Berol moves to $[n-1,\,1]$, Lemma 2.3 implies that $\mathcal{G}(0,2)=\mathcal{P}$ and $\mathcal{G}(0,1)=\mathcal{N}$. Thus Algois has no good move, and Berol wins again.

Case 3. No closing-off, and a move to $[n-1,\,2]$.

These cases are independent of who dropped first:

$$\begin{array}{rrr|ll} & & A & 0 & A \\ & \mathcal{P} & B & \mathcal{P} & \\ B & A & B & & \mathcal{P} \end{array} \qquad \begin{array}{rrr|ll} & \mathcal{P} & B & 0 & A \\ B & A & \mathcal{N} & \mathcal{P} & \\ & B & A & & \mathcal{P} \end{array}$$

subcase 6: A wins subcase 7: B wins

Again, in each case, the $\mathcal{P}$ position to the left of the line is formed by the closing-off move. The other $\mathcal{P}$ and $\mathcal{N}$ positions are obtained from Lemma 2.3. No more need be said about these cases.

Here are the cases where it matters who drops first:

$$\begin{array}{r|l} & 0\ A\ldots\underline{A}\ B \\ A\ B\ \mathcal{P} & \qquad\mathcal{P}\ A \\ A\ B & A\ B\ldots\underline{B}\ \mathcal{P} \end{array} \qquad\qquad \begin{array}{r|l} & 0\ A\ldots\underline{B}\ A \\ A\ B\ \mathcal{N} & \mathcal{P}\qquad\ \ \mathcal{P}\ B \\ B\ A & B\qquad\ \ B\ \mathcal{P} \end{array}$$

subcase 8: A wins   subcase 9: B wins

$$\begin{array}{r|l} B & 0\ A \\ P\ A & \mathcal{N} \\ A\ B\ A & \end{array} \qquad \begin{array}{r|l} & 0\ A\ldots\underline{A}\ B \\ \mathcal{P} & \qquad\quad\mathcal{P}\ A \\ A\ B\ A & B\ A\qquad A\ \mathcal{N} \end{array} \qquad \begin{array}{r|l} & 0\ A\ldots\underline{B}\ A \\ \mathcal{N} & \mathcal{P}\qquad\quad\mathcal{P}\ B \\ A\ B\ A & B\ A\qquad B\ \mathcal{N} \end{array}$$

subcase 10a: A wins   subcase 10b: A wins   subcase 10c: B wins

In subcase 8, Algois moves to $(0, 2)$; then the players must remain on the bottom row, with Berol moving to $(2a+1,\ 2)$. Thus $\mathcal{G}(2a+1,\ 1) = \mathcal{P}$, and so $\mathcal{G}(n-1,\ 1) = \mathcal{P}$. From Lemma 2.3 we see that $\mathcal{G}(2a + 2,\ 2) = \mathcal{P}$, and thus Berol loses.

The analysis in subcase 9 is similar, but leads to the conclusion that Berol wins.

In subcase 10a, if Berol moves to $(n-1,\ 0)$ Algois is forced to move to $(n-1,\ 1)$. Now, $(n-2,\ 1)$ has no followers and so is a $\mathcal{P}$ position. By Lemma 2.3, $\mathcal{G}(0, 2) = \mathcal{P}$ and so now Berol has no winning move. This is regardless of which player dropped first.

Suppose that Algois drops first, from $(2a, 0)$ to $(2a, 1)$. If Berol moves to $(0, 2)$ instead of $(n-1,\ 0)$, the moves $(0, 2) \mapsto (1, 2) \mapsto (2, 2) \ldots$ are forced until Algois moves to $(2a-1,\ 2)$. Thus $(2a-1,\ 1)$ is a $\mathcal{P}$ position, since it has no followers. Working back we see that $(n-1,\ 2)$ is a $\mathcal{P}$ position. Thus by Lemma 2.3, we conclude that $(2a, 2)$ is a $\mathcal{N}$ position and Berol loses.

In a similar way, we see that, if Berol drops first, from $(2a+1,\ 0)$ to $(2a+1,\ 1)$, and Berol moves to $(0, 2)$, Berol wins by moving to $(0, 2)$.

The preceding information is summarized in Table 1. We combine the information from these cases and the forcing sequence argument in the next tables. There are only three possible starting points for the sequences. Also, the dropping moves can only occur at those values of $n-a+i \bmod 3$ given in Lemma 2.6.

In Figure 2, we assume that Berol drops first. The arrows at the top (bottom) of the figure indicate in what positions of the top (bottom) forcing sequence Berol would win if that column were the column $n - 1$. From this we see that the winning positions for Berol are 2, 3, 4, 5, 7, 9, 13, 15, 17, 18, 19 or 20 along the forcing sequences, thus repeating with period 21. The initial starting vertex could be any one of those indicated by $\Downarrow$. In any of those cases, Algois does not have a chance to drop before Berol.

		$[n-1,\,0]$	$[n-1,\,1]$	$[n-1,\,2]$
A drops first	ABA	B	B	A
	BAB	B	B	A
	AB/AB	A	B	A
	BA/BA	A	B	B
B drops first	ABA	A	B	B
	BAB	B	B	A
	AB/AB	A	B	B
	BA/BA	A	B	B

Table 1. The results if there are no closing-off moves. The second column gives the sequence of moves, with / just before the move where the drop occurs.

The start adds 1, 3 or 5 to these values, except that there are extra congruence conditions to be satisfied: that is, since $a = i = 0$ then $n \equiv 2$, 1 or 0 modulo 3 as indicated by the number in the initial position. Thus Berol can win if $n = 21k+j$ for $j \in \{1, 3, 5, 7, 8, 9, 12, 14, 18, 20\}$ for $k \geq 0$.

In the other cases, Berol will not want to drop first. But Berol has little influence on Algois's forcing sequence. In Figure 3, we assume that Algois drops first. The arrows at the top (bottom) of the figure indicate in what positions of the top (bottom) forcing sequence Algois would win if that column were the column $n-1$. Algois will win when $n = 21k + j$ where $j \in \{11, 13, 14, 15, 16, 17, 18, 19, 21\}$.

Since the case of $n \equiv 4 \bmod 6$ was settled first, the only cases left are $n \equiv 2, 6, 23, 25, 27 \bmod 42$. In all of these cases neither player wishes to be the one who drops first, because they will then lose via the forcing sequences. Neither can drop until the congruence conditions are right, which happens six moves

$\Downarrow\ \Downarrow\ \Downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow$
$0\ 2\ 1\ 0\ 2\ A \downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\ \ A\,B\,A\,B\,A\,B\,A\,B\,A\,B\,A\,B\,A$
$B\,A\,B\,A\,B\,A\,B\,A \downarrow \qquad\qquad A\,B\,A\,B\,A\,B$
$\uparrow A\,B\,A\,B\,A\,B\,A\,B\,A\,B\,A\,B\,A \quad \uparrow\uparrow\uparrow\uparrow\uparrow$
$\uparrow\uparrow\ \uparrow\ \uparrow\ \uparrow\ \uparrow\ \uparrow$

Figure 2. Berol's Winning Positions.

$\Downarrow\ \Downarrow\ \Downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow\downarrow$
$0\ 2\ 1\ 0\ 2\ 1\ B \qquad\qquad\qquad\qquad B\,A\,B\,A\,B\,A\,B\,A\,B$
$A\,B\,A\,B\,A\,B\,A\,B \downarrow\downarrow\downarrow\downarrow\downarrow \quad \uparrow$
$B\,A\,B\,A\,B\,A\,B\,A\,B$
$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$

Figure 3. Algois's Winning Positions.

later. This allows an initial segement of up to length 12. This changes the cases and they become equivalent to that of the case 6 less modulo 21. So modulo 42 they become respectively equivalent to the cases $n \equiv 36, 0, 17, 19, 21 \bmod 42$. But all these are $\mathcal{N}$ positions.

To complete the proof, it is now enough to note that the cases $n = 2$ and $n = 6$ were done by hand and were found to be $\mathcal{N}$ positions. Thus there are no anomalous Grundy values for small values of n. $\square$

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