Games of No Chance
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Championship-Level Play of Dots-and-Boxes

JULIAN WEST

Abstract. A single-elimination Dots-and-Boxes tournament was held during the MSRI meeting, with a $500 purse. This is an analysis of the finals, in which Daniel Allcock defeated Martin Weber, playing both first and second player. A systematic notation is developed for the analysis.

Dots-and-Boxes, described in Chapter 16 of *Winning Ways* [Berlekamp et al. 1982], is a game played on a finite rectangular unit lattice of dots (or, in dualized form, on an arbitrary graph). A move consists of joining two adjacent dots, that is, dots at distance one; if this completes one or more squares, a point ("box") is awarded for each and the player retains her turn. A move after which turn does not pass to the opponent is known as a *complimenting move* and, under so-called normal win conditions—last player to move wins—leads to so-called *loony* values (explained below). However, Dots-and-Boxes does not have a normal win condition; indeed, analysis is complicated considerably by the unusual who-dies-with-the-most-wins condition.

Under various names, this game is popular with children in many countries. As played by most practitioners, it is a fairly uninteresting game, consisting of a phase of randomly segmenting the board followed by a phase of greedily dividing up the spoils. So it would seem an odd choice for a tournament between serious game theory researchers. But Dots-and-Boxes is a classic example of a game that is Harder Than You Think. *Winning Ways* (p. 535) gives an account of all the stages you will go through in becoming a Dots-and-Boxes expert:

- abandon the greedy approach in favour of double-dealing moves;
- learn the parity rule for long chains;
- become an expert at Strings-and-Coins;
- (and so) become an expert at Nimstrings;
- apply Twopins theory; become an expert at Kayles;
- recognize the exceptional cases where Nimstrings does not suffice to analyse Dots-and-Boxes.

Finally, it goes on to demonstrate that Dots-and-Boxes, played on general graphs, is NP-hard (p. 543).

This is rather daunting, but for our present purposes we shall only really need to know the parity rule and one observation drawn from Nimstrings. We'll be able to illustrate each by using the games to hand, so we exhibit these games now: see Figure 1. The games were played on a grid of six dots by six, and are shown here near completion, well beyond the point where the outcome has become obvious. (Since future play by a sensible player can now be predicted, these diagrams give a complete account of each game. Nevertheless, we'll also present a diagram of one game at an earlier stage, in order to illustrate crucial stages in the play.)

In each of these games, the moves of the first player (Dodie) are shown with Dotted lines; therefore the first edge drawn is the dotted one numbered 1, and the second edge drawn is the solid one numbered 1. Sometimes multiple edges are played in one turn; when this happens we have labelled them as, say, 15a, 15b, 15c. When either player claims a square, a D (for Dodie) or an E (for Evie) is entered; we have placed the letter slightly off centre in the direction of the edge that completed the square. Every edge approached by a D or an E bears a letter, since it was a complimenting move, though not every edge bearing a letter completed a square, since it may have been the last edge of a turn. Note that (by coincidence) Dodie's move 16a completed two boxes in each game. Moves that claim two boxes are called *double-crossing moves*; they are fundamental in the analysis of Dots-and-Boxes.

Consider the situation in Figure 1, top. Evie has just claimed the box at upper right (18a) and with her move at 18b opens up two boxes for Dodie. Such a move is called a *hard-hearted handout*; Dodie has no reason not to take the two free boxes, and therefore no real choice. But after this she is stuck: she must either move in the four-box *chain* or the six-box *loop*, and any move here allows Evie to capture boxes. We say that Evie *has control*. In either region, Evie will be able to *retain control* by "politely declining" the last squares, making a *double-dealing* move—that is, one that invites a double-crossing one in reply. If you don't know how to decline the last two boxes in a chain, you should work it out as an exercise: you will then be able to defeat most casual players of Dots-and-Boxes!

So if Dodie moves in the chain, Evie has a choice between taking all four boxes and surrendering the remaining six, or taking only two boxes and getting the remaining six on her following turn. Alternatively, if Dodie moves in the loop, Evie can either claim six boxes and surrender four, or claim only two boxes (since it requires four boxes—in other words, two double-dealing moves—to decline a loop) and get four on her next turn. Evie thus has a chance to get eight boxes if Dodie moves in the chain, and at best six if Dodie moves in the loop, so Dodie should move in the loop. Nevertheless, Evie will win the game, with 13 boxes to Dodie's 12.

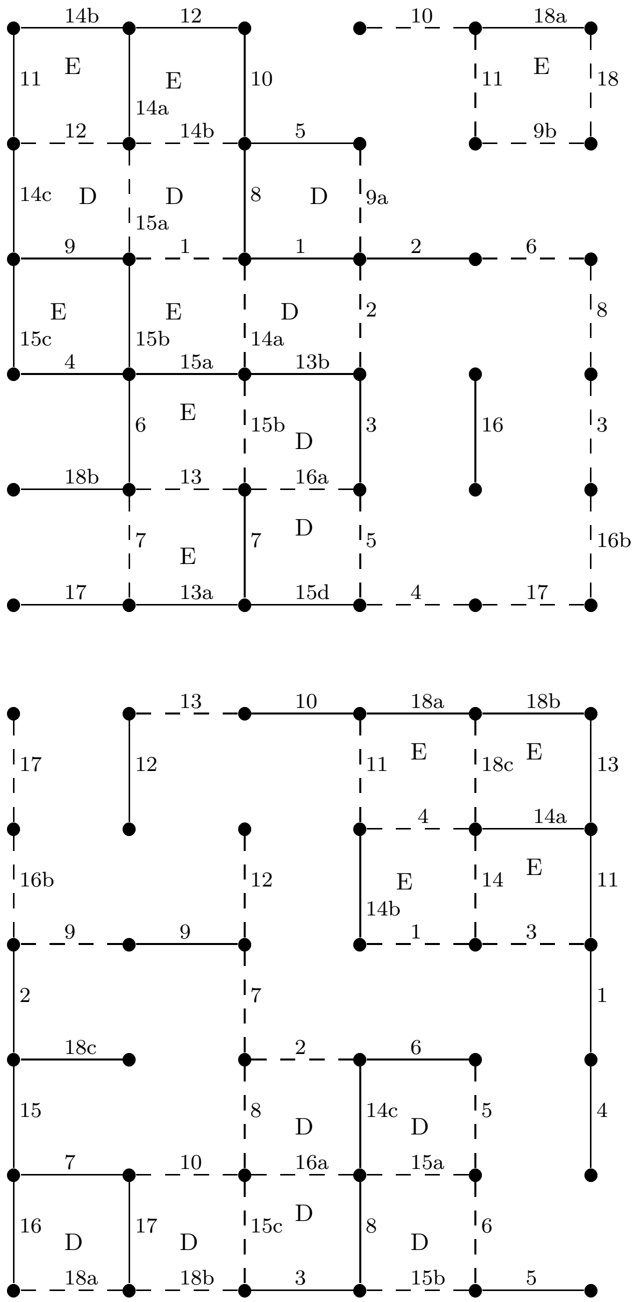

Figure 1. Near-ending position in the two games of the MSRI Dots-and-Boxes tournament finals. Dodie (D) to move in each case; dashed lines indicate her moves.

If there were two long chains on this 6×6 dot board, they would both be claimed by Dodie, not Evie. (A *long chain* is one with at least three boxes in it, so that it is always possible to decline the last two with a double-dealing move.) This is because of the *parity rule* mentioned earlier: Dodie wishes to make the number of initial dots plus double-crossed moves *odd*, whereas Evie wishes to make it even. In the most straightforward type of game, the number of double-crossing moves is one fewer than the number of long chains, and we have 36 dots, so a single long chain is (usually) a win to Evie. If the players feel that the number of eventual long chains will be just one or two, Evie will try to keep the chain intact, and Dodie will try to split it up. If they feel that the board is headed for two or three long chains, it is Evie who will be trying to split things up.

A game of Dots-and-Boxes is usually a fight to gain control by forcing the number of evenutal long chains to have the right parity. Generally, there are enough long chains, and the chains sufficiently long, that this is where the game is settled. However, in game 1 very few of the boxes were claimed in long chains: there is the chain of length four at upper right, and the chain of length three that went at Evie's move 15. (There is also the loop of length six). Game 1 was a game dominated by sacrifices; both players had to surrender boxes in their efforts to gain control. After the game, Evie announced that she had won by counting individual boxes, rather than paying exclusive attention to the parity rule.

The outcome of game 2 is confusing at first sight: there appears to be a single long chain, but it will be claimed by Dodie, not Evie as we might expect. The reason is that one long chain has already been claimed, as can be seen from Dodie's double-crossed move at 16a. (Evie will make two more double-crossed moves, because Dodie's next move will be to decline the loop. But the total number will be odd, as Dodie wants it to be.) By turn 11, both players could see that there was one chain of at least six boxes, and another, at the bottom, of at least four boxes. Since there will be at least two long chains, Evie hoped to create a third one, but Dodie prevented this with a series of strong moves. First she extended the six-box chain to nine boxes with her moves 11, 12 and 13. This left only two boxes at the upper left *not* in this chain—not enough to form a long chain on their own. Then, by sacrificing two boxes with a hard-hearted handout at turn 14, she prevented the development of a long chain at the upper right. At this point, Evie surrendered and moved at turn 14c in a long chain—an explicitly suicidal move, as we will explain. Even though she didn't have to move in a long chain yet, she knew she would have to eventually, because of the parity rule. So her turn 14c is much like a player resigning in chess as soon as the checkmate becomes evident. Both players then played out the game with intentionally cavalier moves—the two boxes that Dodie takes on turn 16 could have been taken by either player on turn 15, but both declined as a gesture of futility. For clarity, we redraw in Figure 2 the situation just after this move.

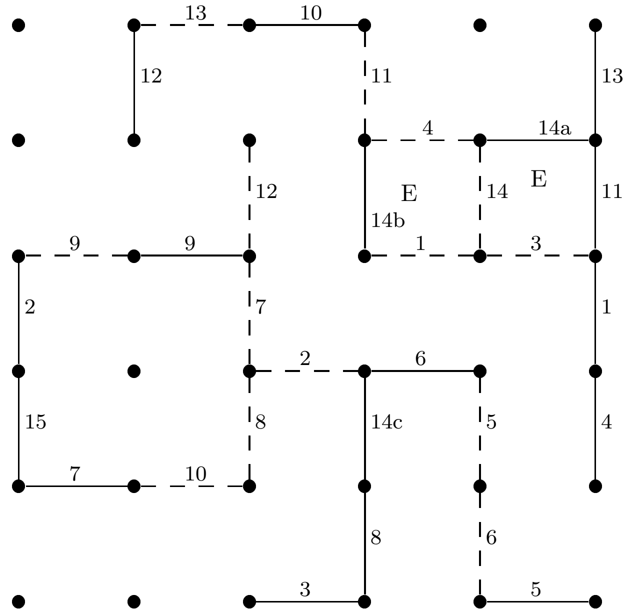

Figure 2. Position in game 2 just after Evie (full lines) plays her move 14c, at which point both players knew Dodie would win.

The reason Evie's move 14c is suicidal comes from a simple strategy-stealing argument in the theory of Nimstrings. Nimstrings is just Dots-and-Boxes with the normal win condition of last-to-move-wins. Move 14c allows the choice of two replies (those edges labelled 15a and 15b in our example). Consider the entire game position with the exception of these two boxes: it must be a win for either the first player or the second player (though it may in general be very hard to know which). Arrange to be the appropriate player by either: moving at 15a, then 15b, then going on to be player 1 on the remaining position; *or* moving at 15b, then stopping so as to become player 2 on the remaining position. (One player or another will eventually claim the two free boxes left by this second option, but in Nimstrings we don't care who.)

In Nimstrings, 14c is therefore an example of a *loony* move. The value loony is an extension to the field of nimbers from Sprague–Grundy theory. It has the peculiar property that its sum with any nimber is again loony. This can be seen in action here, because leaving a loony position is a losing move, whatever is happening elsewhere on the board. (Note that also the sum of two loonies is loony.)

In the language of Dots-and-Boxes, a move such as Evie's 14c is called a *half-hearted handout.* Since it is fatal (i.e., loony) in Nimstrings, it is nearly always fatal in Dots-and-Boxes, amounting as it does to offering your opponent the choice of parity of the number of double-crossed moves, and thus of eventual long chains.

Understanding about hard-hearted handouts enables us to see why a chain of length three or more counts as long, while one of length two does not. If you are forced to move in a chain of length two, you have the choice between making a half-hearted handout (generally suicidal) or taking the middle edge to make a *hard-hearted handout*. Moving in a chain of length three or more, drawing even an interior edge creates a suicidal situation on at least one side of the edge. Therefore, the first player to offer up boxes in a long chain has lost control, and will almost always lose.

When these two games were actually played at MSRI, Daniel Allcock, a Berkeley graduate student in geometric group theory, was Evie in game 1 and Dodie in game 2. His opponent was Martin Weber, a differential geometer.

The pictures in this note were set in LaTeX with some assistance from Jennifer Overington.

Reference

[Berlekamp et al. 1982] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways For Your Mathematical Plays*, vol. II, Academic Press, London, 1982.

Julian West
Julian West
Malaspina University-College
and
Department of Mathematics and Statistics
University of Victoria
Victoria, B.C.
Canada V8W 3P4
julian@math.uvic.ca