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Aspects of Ricci Curvature

TOBIAS H. COLDING

This article is dedicated to my parents.

Abstract. We describe some new ideas and techniques introduced to study spaces with a given lower Ricci curvature bound, and discuss a number of recent results about such spaces.

Introduction

In studying spaces with a given lower sectional curvature bound we have a very powerful tool in the Toponogov triangle comparison theorem. This allows us to study metric properties of such spaces (see for instance [Toponogov 1964; Burago et al. 1992; Perelman 1995]), and topological properties (see for instance [Cheeger 1991; Cheeger and Gromoll 1972; Grove and Shiohama 1977; Gromov 1981a; Grove and Petersen 1988; Perelman 1991]). Perhaps the most important tool for studying topological properties of such manifolds is the notion of critical points of distance functions in connection with the Toponogov triangle comparison theorem; see [Grove and Shiohama 1977] and compare with the remarks at the end of Section 1.

When we only assume a lower Ricci curvature bound, no such estimate is available. Classically, the only known general estimates of this type for Ricci curvature are the volume comparison theorem [Bishop and Crittenden 1964; Gromov 1981b] and the Abresch–Gromoll inequality [Abresch and Gromoll 1990].

In order to study manifolds with a given lower Ricci curvature bound there are at least two obstacles to overcome. First, many results from the sectional curvature case do not remain true for Ricci curvature. Second, due to the lack of a good estimate on the distance function we do not have good control on the

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local geometry in this case. We will see in this survey that in some sense the second obstacle is the most serious.

In Section 1 we will discuss a new estimate of the distance function. In later sections we will see that this type of estimate has a large number of consequences, some of which are given in Sections 2 and 3.

In Section 2 our main focus is the geometry and topology of manifolds with a lower Ricci curvature bound.

In Section 3 our focus is on regularity properties of general metric spaces that are (Gromov–Hausdorff) limits of n-dimensional manifolds with a given lower Ricci curvature bound. This is in part motivated by Gromov's compactness theorem [1981b], saying that the space of n-dimensional manifolds with a given lower Ricci curvature bound is precompact in the Gromov–Hausdorff topology.

In Section 4 we discuss some analytic properties of manifolds with a given lower Ricci curvature bound. The Harnack inequality and the gradient estimate of Yau and Cheng [Yau 1975; Cheng and Yau 1975] discussed in that section play a crucial role in the results of Section 1, 2 and 3.

The results described in this survey are to be found in the references listed under "Direct Sources" (page 94).

1. Integral Estimate of Angles and Distances Using the Hessian

Our main technical tool is a new estimate for distance functions. The first such estimate appeared in [Colding 1996a], and the reader should consult that reference for a more precise statement than we are about to give here (and in particular for the notion of "almost equal").

Suppose M is an n-dimensional closed manifold with $\mathrm{Ric}_M \geq n-1$. Consider the space of geodesics of some fixed length $l < \pi$, and identify each such geodesic with its initial velocity. Equip this space with the probability measure coming from the normalized Liouville measure.

Let $p, q \in M$ be points with $d(p,q)$ almost equal to π, where $d(p,q)$ denotes the distance between p and q. For any geodesic γ (not necessarily minimizing) of fixed length l, let Δ be the geodesic triangle of which γ is the side opposite the vertex p, and let $\underline{\Delta}$ be the comparison triangle on the unit sphere, in the sense of Toponogov (see Figure 1). For $0 \leq t \leq l$, let d_t be the distance between $\gamma(t)$ and p, and let $\underline{d}_t$ be the corresponding distance on the sphere.

(If the triangle in M does not satisfy the triangle inequality, so that no comparison triangle exists, we use an alternative analytical definition of $\underline{d}_t$ and $\underline{\angle}_t$ that still makes sense in this case.)

Then, in an L^2-sense (or equivalently for a set of geodesics of nearly full measure), d_t is almost equal to $\underline{d}_t$, and $\angle_t$ is almost equal to $\underline{\angle}_t$. More formally:

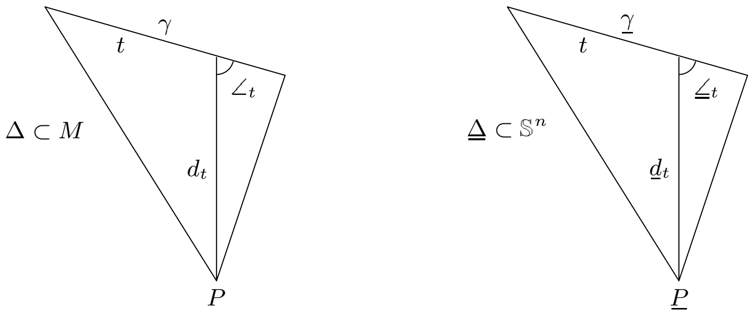

Figure 1. Estimate of distances and angles for positive Ricci curvature.

THEOREM 1.1. *Given $\varepsilon > 0$ and $0 < l < \pi$, there exists a $\delta = \delta(\varepsilon, n) > 0$ such that, for any M of dimension n with* $\mathrm{Ric}_M \geq n-1$, *any $p, q \in M$ with $d(p,q) > \pi - \delta$, and any $0 \leq t \leq l$, we have $\|d_t - \underline{d}_t\|_2 < \varepsilon$ and $\|\angle_t - \underline{\angle}_t\|_2 < \varepsilon$.*

See [Colding 1996a] for details, and compare [Colding 1996b; Cheeger and Colding 1996] for later versions.

In the sequel we will let $\psi(\varepsilon \mid \dots)$ and $\psi_i(\varepsilon \mid \dots)$ denote nonnegative functions depending on ε and possibly on some additional parameters (written after the bar), satisfying the property that when the other parameters are fixed the function tends to 0 as ε approaches 0.

SKETCH OF THE PROOF OF THEOREM 1.1. First we use the fact that $d(p,q) > \pi - \delta$ and the bound $\mathrm{Ric}_M \geq n-1$ to approximate $\cos d(p,\cdot)$ in the $(2,1)$-Sobolev norm by a smooth function that satisfies $\|\Delta f + nf\|_2 < \psi_1(\delta \mid n)$; here the L^2-norm is normalized so that $\|1\|_2 = 1$. Next, from the Bochner formula for f and the fact that $\mathrm{Ric}_M \geq n-1$, we get

$$\tfrac{1}{2}\Delta|\nabla f|^2 \geq |\mathrm{Hess}(f)|^2 + \langle \nabla \Delta f, \nabla f\rangle + (n-1)|\nabla f|^2.$$

Integrating by parts over M gives

$$0 \geq \int_M |\mathrm{Hess}(f)|^2 - \int_M |\Delta f|^2 - (n-1)\int_M f\Delta f.$$

Now since $\|\Delta f + nf\|_2 < \psi_1(\delta \mid n)$ we get from the Cauchy–Schwarz inequality

$$\psi_2(\delta \mid n) > \frac{1}{\mathrm{Vol}(M)}\int_M |\mathrm{Hess}(f)|^2 - \frac{n}{\mathrm{Vol}(M)}\int_M f^2.$$

From the Cauchy–Schwarz inequality we therefore get, again using the fact that $\|\Delta f + nf\|_2 < \psi_1(\delta \mid n)$,

$$\begin{aligned}\psi_3(\delta \mid n) &> \frac{1}{\mathrm{Vol}(M)}\int_M |\mathrm{Hess}(f)|^2 + \frac{n}{\mathrm{Vol}(M)}\int_M f^2 + \frac{2}{\mathrm{Vol}(M)}\int_M f\Delta f \\ &= \frac{1}{\mathrm{Vol}(M)}\int_M |\mathrm{Hess}(f) + fg|^2,\end{aligned}$$

where g is the metric tensor. By integrating this along geodesics we can show the theorem for f. Finally using that f approximates $\cos d(p,\cdot)$ we can show the theorem for the distance function. □

This proves, in particular, that if $d(p,q)$ is almost equal to π then the Hessian of a function that approximates $\cos d(p,\cdot)$ is almost a diagonal form (in fact it is almost fg).

The constant δ in Theorem 1.1 can be explicitly estimated in terms of n and ε. This explicit dependence will be the case throughout.

It is interesting to compare Theorem 1.1 with the following example:

EXAMPLE 1.2 [Anderson 1990b]. There exist metrics on $\mathbb{CP}^n\#\mathbb{CP}^n$, with $\mathrm{Ric} \geq 2n-1$ and $\mathrm{Vol} \geq v > 0$, that are arbitrary close to a metric on $\mathbb{S}^{2n}$ with two conical singularities. Moreover, the diameter of $\mathbb{S}^{2n}$ with this metric is π.

The metric on $\mathbb{S}^4$ is constructed as follows. Let $\Pi : \mathbb{S}^3 \to \mathbb{S}^2$ be the Hopf fibration, and let $\sigma_X, \sigma_Y, \sigma_Z$ the standard left invariant coframing of $\mathbb{S}^3 = \mathrm{SU}(2)$, where σ_Z is tangent to the Hopf fibers. Define a metric on $\mathbb{S}^3$ by $C_1^2\sigma_Z^2 + C_2^2(\sigma_X^2+\sigma_Y^2)$, where $C_1 \approx 0.08$ and $C_2 \approx 0.25$ are constants. Then $(\mathbb{S}^4, g)$ is the spherical suspension of $\mathbb{S}^3$ with this Berger metric.

In the metrics of [Anderson 1990b] the two embedded $\mathbb{CP}^1$'s in $\mathbb{CP}^2\#\mathbb{CP}^2$ are totally geodesic and round. Furthermore their curvature converges to infinity as $\mathbb{CP}^2\#\mathbb{CP}^2$ converges to $\mathbb{S}^4$. See [Anderson 1990b] for further details and examples of similar metrics in dimension > 4.

We will now consider a different measure on the set of all minimal geodesics with unit speed, one that behaves better when the volume of the manifold is small. For a manifold M, a point $p \in M$, and $r_0 > 0$, we identify the minimal geodesics contained in $B_{r_0}(p)$ with their endpoints: $\gamma \to (\gamma(0), \gamma(l))$, where $l = \mathrm{length}(\gamma)$. We equip this space of minimal geodesics with the natural measure coming from the product measure on $M \times M$, and we normalize this measure so that the space of minimal geodesics with endpoints in $B_{r_0}(p)$ has measure one.

The next theorem appeared in [Cheeger and Colding 1996]. It follows an earlier version given in [Colding a]. For $p, q, x \in M$, define the *excess* by $e_{p,q}(x) = d(p,x) + d(x,q) - d(p,q)$.

THEOREM 1.3. *Fix* $r_0 > 0$. *Given* $\varepsilon > 0$, *there exist* $R = R(\varepsilon, r_0, n) > 0$, $\Lambda = \Lambda(\varepsilon, r_0, n) > 0$ *and* $\delta = \delta(\varepsilon, r_0, n) > 0$ *such that, for any* M *of dimension* n *with* $\mathrm{Ric}_M \geq -(n-1)R^{-2}\Lambda$ *and any* $p, q, x \in M$ *with* $e_{p,q}(x) < \delta$ *and* $d(p,x), d(q,x) > R$, *we have* $\|d_t - \underline{d}_t\|_2 < \varepsilon$ *and* $\|\angle_t - \underline{\angle}_t\|_2 < \varepsilon$.

To prove Theorem 1.3, we show first that we can approximate the distance function in the $(2,1)$-Sobolev norm by a harmonic function. We then show that the L^2-norm of the Hessian of this harmonic function is small. Finally we integrate this Hessian estimate along geodesics to get the theorem for the harmonic function, and we use the fact that the harmonic function approximates the distance function to get the theorem for the distance function.

One should compare these three steps with the corresponding three steps in the proof of Theorem 1.1.

It is important to note that in both Theorem 1.1 and Theorem 1.3 the difference in the angles is not necessarily uniformly small, but only small with respect to the L^2-norm. For examples where this difference is not uniformly small, see [Anderson 1992; Perelman 1997]. Such examples also show that we cannot control the critical points of the distance function for Ricci curvature (compare with the discussion in the Introduction).

For other estimates along these lines, see [Colding 1996a; Colding 1996b; Colding a; Cheeger and Colding 1996; Cheeger, Colding, and Tian b].

2. Almost Maximal Manifolds

Recall that the set of all metric spaces can be made into a metric space by means of the *Gromov–Hausdorff distance* d_{GH}. Denoting by $T_\varepsilon(X)$ the ε-neighborhood of a subset X of a metric space Y, this distance is defined as follows:

DEFINITION 2.1 [Gromov 1981b]. The Gromov–Hausdorff distance between two metric spaces (X_1, d_1) and (X_2, d_2) is the infimum of all $\varepsilon > 0$ such that there exist a metric space Y and isometric embeddings $j_1 : X_1 \to Y$ and $j_2 : X_2 \to Y$ with $j_1(X_1) \subset T_\varepsilon(j_2(X_2))$ and $j_2(X_2) \subset T_\varepsilon(j_1(X_1))$.

For noncompact metric spaces we say that a pointed sequence (X_i, x_i) converges to (X, x) in the pointed Gromov–Hausdorff topology if, for all $r > 0$, the sequence $X_i \cap B_r(x_i)$ converges to $X \cap B_r(x)$ in the Gromov–Hausdorff topology. This convergence should be thought of as convergence on compact subsets.

For our purposes the importance of this definition is that $d_{GH}(X_1, X_2) < \varepsilon$ if and only if there exist maps $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_1$ such that, for $i = 1, 2$ and all $a_i, b_i \in X_i$,

$$|d(f_i(a_i), f_i(b_i)) - d(a_i, b_i)| < \psi(\varepsilon),$$

and for $i, j = 1, 2$ with $i \neq j$,

$$d(f_j \circ f_i(a_i), a_i) < \psi(\varepsilon).$$

One of the most useful properties of the Gromov–Hausdorff distance is that it is a good tool to measure the rough geometry of a metric space. In particular, we will see in Theorems 2.3 and 2.5 that if we have a space with a lower Ricci curvature bound then in many instances the rough (large-scale) geometry controls the small-scale geometry.

For the next few results the L^2-estimate on distance functions (and of the Hessian of a function that approximates the distance function) mentioned in Section 1 is crucial.

By $V^n_\Lambda(r)$ we will mean the volume of a ball of radius r in the n-dimensional simply connected space form of constant sectional curvature Λ.

THEOREM 2.2 [Colding 1996a]. *Given* $\varepsilon > 0$, *there exists* $\delta = \delta(n, \varepsilon) > 0$ *such that*, *if an* n-*dimensional manifold* M *has* $\mathrm{Ric}_M \geq n-1$ *and* $\mathrm{Vol}(M) > V^n_1(\pi) - \delta$, *then* $d_{GH}(M, \mathbb{S}^n) < \varepsilon$.

Theorem 2.2 was conjectured by Anderson–Cheeger and Perelman.

A key point in the proof of Theorem 2.2 is that the Bishop volume comparison theorem and the assumption on the volume imply that, for any $p \in M$, there exists $q \in M$ with $d(p, q) > \pi - \psi(\delta \,|\, n)$. We can therefore apply Theorem 1.1 for all $p \in M$.

Theorem 2.2 is the first result about Ricci curvature proved using "synthetic" techniques (see Section 3 for more on this).

The case of Alexandrov spaces, which are possibly singular spaces with a lower sectional curvature bound in the triangle comparison sense, was treated systematically from such a point of view in [Burago et al. 1992]; compare [Perelman 1995].

THEOREM 2.3 (VOLUME CONVERGENCE [Colding 1996b; a]). *For* $r > 0$, *consider all metric balls of radius* r *in all complete* n-*dimensional Riemannian manifolds with Ricci curvature greater or equal to* $-(n-1)$. *Equip this space with the Gromov–Hausdorff topology*. *Then the volume function is a continuous function*.

This result was conjectured by Anderson and Cheeger.

Using a covering argument and the volume comparison theorem we can reduce the proof of Theorem 2.3 to showing that, if a ball in an n-dimensional manifold for which the infimum of the Ricci curvature is almost zero is close to the corresponding ball in $\mathbb{R}^n$, the volumes are also close.

It is easy to see that a lower Ricci curvature bound is needed in Theorem 2.3; see for instance the examples in [Colding 1996b]. In the same reference we showed that Vol is continuous at the unit n-sphere, $\mathbb{S}^n$: more precisely, if M_i is a sequence of n-dimensional manifolds with $\mathrm{Ric}_{M_i} \geq n-1$ and $M_i \xrightarrow{d_{GH}} \mathbb{S}^n$, then $\mathrm{Vol}(M_i) \to \mathrm{Vol}(\mathbb{S}^n)$.

THEOREM 2.4 [Colding a]. *There exists an* $\varepsilon = \varepsilon(n) > 0$ *such that if* M *is a closed* n-*dimensional manifold with* $\mathrm{Ric}_M \,\mathrm{diam}^2_M > -\varepsilon$ *and* $b_1(M) = n$ *then* M *is homeomorphic to a torus if* $n \neq 3$ *and homotopically equivalent to a torus if* $n = 3$.

This result was conjectured by Gromov [1981b, p. 75].

To prove Theorem 2.4, we show first that a finite cover of M is close to a flat n-dimensional torus. This allows us to apply Theorem 2.3 to a finite cover of M to conclude that a finite cover is a homotopy torus. See also Theorem 2.6.

Theorem 2.4 should be compared with Bochner's theorem [Bochner 1946; Bochner and Yano 1953] and with Gromov's theorem [1981b]. Recall that

Bochner's theorem says that a closed n-manifold with nonnegative Ricci curvature has $b_1 \leq n$, and equality holds if and only if the manifold is isometric to a flat torus. Later Gromov showed (see also [Gallot 1983]) that there exists an $\varepsilon = \varepsilon(n) > 0$ such that any closed n-manifold M with $\mathrm{Ric}_M \operatorname{diam}_M^2 > -\varepsilon$ has $b_1 \leq n$.

Yamaguchi [1988] proved that, given $D > 0$ and $k \leq n$, there exists an $\varepsilon = \varepsilon(k, D, n) > 0$ such that if M is a closed n-manifold with $\operatorname{diam}_M \leq D$, $b_1(M) = k$, $\mathrm{Ric}_M \geq -\varepsilon$, and $|\mathrm{Sec}_M| \leq 1$, then a finite cover of M fibers over a k-torus. In [Yamaguchi 1991] he showed that if $\mathrm{Sec}_M \operatorname{diam}_M^2 > -\varepsilon$, then a finite cover of M fibers over a k-torus. Actually, in [Yamaguchi 1988] he made a stronger conjecture than the original one by Gromov, namely, that this should remain true for almost nonnegative Ricci curvature. This was disproved by Anderson [1992], who gave counterexamples for $k \leq n-1$; Gromov's original conjecture, however, was left open.

Let M be an n-dimensional open Riemannian manifold with nonnegative Ricci curvature. By Gromov's compactness theorem [1981b], any sequence $r_i \to \infty$ has a subsequence $r_j \to \infty$ such that the rescaled manifolds $(M, p, r_j^{-2} g)$ converge in the pointed Gromov–Hausdorff topology to a length space M_∞.

Every such limit (an example of Perelman shows that M_∞ is not unique in general) is said to be a *tangent cone at infinity* of M. Even though uniqueness fails in general, one expects it to hold in the maximal case. In fact, we have the following result:

THEOREM 2.5 [Colding a]. *If an n-dimensional manifold M has nonnegative Ricci curvature and some M_∞ is isometric to $\mathbb{R}^n$ then M is isometric to $\mathbb{R}^n$.*

Theorem 2.5 was conjectured by Anderson and Cheeger [Anderson 1992].

SKETCH OF PROOF. Fix $p \in M$. By the assumption, if we rescale a large ball centered at p to unit size the rescaled ball is close to the unit ball in $\mathbb{R}^n$. Therefore, by Theorem 2.3, the volumes are close. Using the volume comparison theorem we can now conclude that a ball of a fixed size in M with center at p has the same volume as the corresponding ball in $\mathbb{R}^n$. From equality in the volume comparison theorem we conclude that M is isometric to $\mathbb{R}^n$. $\square$

As an immediate consequence of Theorem 2.3, small balls have almost maximal volume. The importance of this was pointed out to us by Anderson and Cheeger. In particular they observed that this, together with [Anderson 1990c] (see also [Anderson 1992]), implies Theorem 2.7 below. Moreover, this together with the result of [Perelman 1994] and methods from controlled topology [Ferry 1979; Ferry and Quinn 1991], as in [Grove et al. 1989; 1990; Petersen 1990], gives the next theorem:

THEOREM 2.6 (TOPOLOGICAL STABILITY [Colding a]). *If M is a closed n-manifold, there exists $\varepsilon(M) > 0$ such that, if N is a n-manifold with* $\mathrm{Ric}_N \geq -(n-1)$ *and $d_{GH}(M, N) < \varepsilon$, then M and N are homotopically equivalent* (*and even homeomorphic if $n \neq 3$*).

In the case of sectional curvature, Perelman [1991] proved that Theorem 2.6 remains true in the class of Alexandrov spaces. However, for Ricci curvature, if one allows M in Theorem 2.6 to have singularities, the conclusion does not hold. Indeed, Anderson [1990a] gave examples of metrics on $\mathbb{S}^2 \times \mathbb{S}^2$ with $|\mathrm{Ric}| \leq C$ that converge to a metric with two conical singularities on the suspension of $\mathbb{RP}^3$ (the diagonal and antidiagonal in $\mathbb{S}^2 \times \mathbb{S}^2$ are each collapsed to a point). Numerous further examples have since been considered. For instance, Example 1.2 shows that there exist metrics on $\mathbb{CP}^2 \# \mathbb{CP}^2$ arbitrary close to a metric suspension of a Berger metric on $\mathbb{S}^3$ and having $\mathrm{Ric} \geq 3$. Similarly, Otsu [1991] constructed metrics on $\mathbb{S}^3 \times \mathbb{S}^2$ with $\mathrm{Ric} \geq 4$ and arbitrarily close to a metric suspension of a metric on $\mathbb{S}^2 \times \mathbb{S}^2$. Even for Einstein metrics the topological stability of Theorem 2.6 fails if M has singularities; see for instance [Tian and Yau 1987].

Note that the ε in Theorem 2.6 must depend on M for the same reason that the conclusion fails if one allows M to have singularities. For instance, a trivial modification of Anderson's examples [1990b] gives a sequence where the manifolds are alternately homeomorphic to $\mathbb{S}^4$ and to $\mathbb{CP}^2 \# \mathbb{CP}^2$, and all metrics have $\mathrm{Ric} \geq 3$, but the limit is still the same spherical suspension of a Berger sphere.

Theorem 2.7 (Metric stability [Colding a]). *Let M_i be a sequence of n-dimensional Einstein manifolds with* $\mathrm{Ric}_{M_i} = c_i g_i$, *for* $|c_i| \leq n-1$, *converging to a closed n-manifold in the Gromov–Hausdorff topology. Then the M_i converge in the C^∞-topology.*

See also [Colding 1996a; 1996b; a] for further applications of these estimates.

The following theorem gives a generalization of Cheng's maximal diameter theorem [1975] to singular spaces that are limits.

Theorem 2.8 (Almost maximal diameter [Cheeger and Colding 1996]). *Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, n) > 0$ such that if* $\mathrm{Ric}_M \geq n-1$ *and* $\mathrm{diam}_M > \pi - \delta$ *then for some metric space X we have $d_{GH}(M, S(X)) < \varepsilon$.*

Here $S(X) = (0,\pi) \times_{\sin r} X$ is the metric suspension of X.

Theorem 2.9 (Almost splitting [Cheeger and Colding 1996]). *Suppose that M_i is a sequence of n-dimensional manifolds with* $\mathrm{Ric}_{M_i} \geq -(n-1)\varepsilon_i$, *where $\varepsilon_i \to 0$. If M_i converges in the pointed Gromov–Hausdorff topology to a metric space X that contains a line, then X splits isometrically, that is, $X = Y \times \mathbb{R}$ for some metric space Y.*

This was conjectured in [Fukaya and Yamaguchi 1992]. It generalizes the splitting theorem of [Cheeger and Gromoll 1971] to singular spaces that are limits.

Theorem 2.10 (Almost volume cone implies almost metric co ne [Cheeger and Colding 1996]). *If M has nonnegative Ricci curvature and Euclidean volume growth, every tangent cone at infinity is a Euclidean cone.*

Here a Euclidean cone is the metric completion of a space of the form $C(X) = (0, \infty) \times_r X$, for some metric space X.

The next theorem, which was conjectured by Gromov, was proved for manifolds of almost nonnegative sectional curvature by Fukaya and Yamaguchi [1992], who also observed that their proof would go through for almost nonnegative Ricci curvature provided that two conjectures could be established. One of these conjectures follows from Theorem 2.6; see [Colding a] for the exact statement. The other conjecture is Theorem 2.10. Therefore:

THEOREM 2.11 [Cheeger and Colding 1996]. *There exists an* $\varepsilon = \varepsilon(n) > 0$ *such that if* M *is a closed* n*-dimensional manifold with* $\mathrm{Ric}_M \operatorname{diam}_M^2 > -\varepsilon$ *then* $\pi_1(M)$ *is almost nilpotent* (that is, has a nilpotent subgroup of finite index).

3. The Structure of Spaces with Ricci Curvature Bounded Below

As mentioned in Section 2, the estimates in Section 1 and the way they occur give the possibility of treating Ricci curvature from a "synthetic" point of view (compare [Gromov 1980]). The first applications of such ideas were given in Section 2. In this section we will explore this further. Due to Gromov's compactness theorem we can think of the results of this section in two equivalent ways: as the study of smooth manifolds with a given lower Ricci curvature bound on a small but definite scale; or as the study of spaces that are Gromov–Hausdorff limits of such manifolds.

Throughout this section M_i will always be a sequence of n-manifolds with $\mathrm{Ric}_{M_i} \geq -(n-1)$, having M_∞ as a Gromov–Hausdorff limit. Unless otherwise stated, the examples, theorems, and definitions are to be found in [Cheeger and Colding a].

A tangent cone at $p_\infty \in M_\infty$ is a pointed metric space (X, x) that is a Gromov–Hausdorff limit of the rescaled metrics $(M_\infty, p_\infty, r_j d_\infty)$, where $r_j \to \infty$. Such limits exist by the Gromov compactness theorem.

We will sometimes also require that for all i and all $p_i \in M_i$ we have

$$\mathrm{Vol}(B_1(p_i)) \geq v > 0. \tag{3.1}$$

By the volume comparison theorem, this condition gives a uniform lower bound on the volume of all balls of a fixed radius. Condition (3.1) will often be referred to by saying that the sequence M_i does not collapse. Using in part Theorem 2.3, we can show that (3.1) is equivalent to requiring that M_∞ has Hausdorff dimension n.

The next example should be compared with the example of Perelman mentioned in Section 2.

EXAMPLE 3.2. There exists M_∞ where all M_i satisfy (3.1) and $p_\infty \in M_\infty$ such that the tangent cone at p_∞ is not unique.

An important feature in the noncollapsing case is that even though tangent cones are not unique, as seen in the preceding example, they are Euclidean cones. Indeed:

THEOREM 3.3. *Suppose that all M_i satisfy* (3.1). *Then, for all* $p_\infty \in M_\infty$, *every tangent cone at* p_∞ *is a Euclidean cone.*

The next example shows that, if the volume of balls of a fixed size is not uniformly bounded from below, tangent cones may not be Euclidean cones.

EXAMPLE 3.4. There exists M_∞ and $p_\infty \in M_\infty$ such that no tangent cone at p_∞ is a Euclidean cone.

We next define two notions of regular points.

DEFINITION 3.5. We say that $p \in M_\infty$ is a *weakly k-Euclidean point* if some tangent cone at p splits off a factor $\mathbb{R}^k$ isometrically.

DEFINITION 3.6. A point $p \in M_\infty$ is called *regular* if, for some k, every tangent cone at p is isometric to $\mathbb{R}^k$. In this case we write $p \in \mathcal{R}$.

DEFINITION 3.7. A point $p \in M_\infty$ is called *singular* if it is not regular. In this case we write $p \in \mathcal{S}$.

Note that, if the volume of balls of a fixed size is uniformly bounded from below (that is, if (3.1) is satisfied), weakly n-Euclidean implies regular by Theorem 2.5.

We next stratify the points of M_∞ according to how regular their tangent cones are.

DEFINITION 3.8. A point $p \in M_\infty$ is called *k-degenerate* if it is not $(k+1)$-weakly Euclidean. We let $\mathcal{D}_k$ denote the set of k-degenerate points.

Let dim denote the Hausdorff dimension.

THEOREM 3.9. *If all M_i satisfy* (3.1), $\dim(\mathcal{D}_k) \le k$.

If all M_i satisfy (3.1), then $\mathcal{S} = \mathcal{D}_{n-1}$. This follows from the fact that weakly n-Euclidean imply regular if the sequence M_i does not collapse.

THEOREM 3.10. *If all M_i satisfy* (3.1), *then* $\mathcal{S} \subset \mathcal{D}_{n-2}$, *so Theorem* 3.9 *implies* $\dim(\mathcal{S}) \le n-2$.

In the next theorem the volume of subsets of M_∞ are measured with respect to the n-dimensional Hausdorff measure.

THEOREM 3.11 (VOLUME COMPARISON FOR LIMIT SPACES). *If all M_i satisfy* (3.1) *then M_∞ satisfies the relative volume comparison theorem.*

If we do not assume a lower bound on the volume, this is not always the case:

EXAMPLE 3.12. There exists M_∞ that does not satisfy the volume comparison theorem.

The next theorem was proved earlier by Fukaya and Yamaguchi [1994], under the additional assumption that all M_i have a uniform lower sectional curvature bound.

THEOREM 3.13 [Cheeger and Colding b]. *If all M_i satisfy* (3.1), *the isometry group of M_∞ is a Lie group.*

By assuming that all M_i are Einstein manifolds we get further regularity of the limit M_∞. This is the topic of the next two theorems.

THEOREM 3.14. *Suppose that all M_i satisfy* (3.1) *and are all Einstein with uniformly bounded Einstein constants. Then $\mathcal{S}$ is a closed subset and* $\dim(\mathcal{S}) \leq n-2$. *Further*, *$\mathcal{R}$ is a smooth Einstein manifold and the convergence is in the C^∞-topology on compact subsets of $\mathcal{R}$.*

THEOREM 3.15 [Cheeger, Colding, and Tian b]. *Suppose that all (M, g_i) satisfy* (3.1) *and are Kähler–Einstein on M* (*where* $\dim_{\mathbb{C}} M = n$), *with uniformly bounded Einstein constants. Then* $\dim(\mathcal{S}) \leq 2n-4$. *Further*, *there exists a subset $S \subset M_\infty$ with $H_{2n-4}(S) = 0$* (*where H_{2n-4} is the Hausdorff measure*) *such that for all $p \in M_\infty \setminus S$ the tangent cone at p is unique and is equal to $\mathbb{C}^{n-2} \times \mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SU}(2)$.*

Finally, in [Cheeger and Colding b] we give a generalization to the collapsed case of the volume convergence theorem of [Colding a].
See [Cheeger and Colding a; b; Cheeger, Colding, and Tian b] for more results in the spirit of those given in this section.

4. Function Theory on Spaces with a Lower Ricci Curvature Bound

In this section we will touch on some of the results on function theory on spaces with Ricci curvature bounded below (see also [Li 1993] for a discussion of this subject). For further details on the results of this section, see [Colding and Minicozzi 1996; a; Cheeger, Colding, and Minicozzi 1995].

DEFINITION 4.1. Let M be a open (complete, noncompact) manifold, and fix $p \in M$. Let r be the distance function from p. A harmonic function u on M has *polynomial growth of order at most d* if there exists some $C > 0$ so that $|u| \leq C(1 + r^d)$. We denote by $\mathcal{H}_d(M)$ the linear space of such functions.

Yau [1975] generalized the classical Liouville theorem of complex analysis to open manifolds with nonnegative Ricci curvature. Specifically, he proved that a positive harmonic function on such a manifold must be constant. This theorem was generalized in [Cheng and Yau 1975] by means of a gradient estimate that implies the Harnack inequality; in fact this gradient estimate played an important role in the proof of the results of Section 1. As a consequence, one sees that harmonic functions which grow less than linearly must be constant. In his study of these functions, Yau was motivated to conjecture that the space of harmonic functions of polynomial growth of a fixed rate is finite-dimensional on an open manifold with nonnegative Ricci curvature. On this conjecture of Yau we have the following result:

THEOREM 4.2 [Colding and Minicozzi a]. *For an open manifold with nonnegative Ricci curvature and Euclidean volume growth*, *the space of harmonic functions with polynomial growth of a fixed rate is finite-dimensional*.

Other important results on this conjecture of Yau can be found in [Christiansen and Zworski 1996; Donnelly and Fefferman 1992; Kasue a; Li 1995; Li and Tam 1989; 1991; Lin 1996; Wang 1995; Wu 1991].

For $d = 1$ we have the following theorem, which was proved earlier by Peter Li [1995] in the case where M is Kähler.

THEOREM 4.3 [Cheeger, Colding, and Minicozzi 1995]. *If* $\dim \mathcal{H}_1(M) = n+1$ *for an open* n*-dimensional manifold* M *with nonnegative Ricci curvature*, M *is isometric to* $\mathbb{R}^n$.

To prove this, we show first that if M is an open manifold with nonnegative Ricci curvature and u is a nonconstant harmonic function with linear growth then any tangent cone at infinity M_∞ splits off a line. This implies in particular that if $\dim \mathcal{H}_1(M^n) = n+1$ then $M_\infty = \mathbb{R}^n$; then Theorem 2.5 gives $M = \mathbb{R}^n$.

There exist manifolds with nonnegative (or even positive) Ricci curvature that do not split off a line, but admit a nonconstant linear growth harmonic function. In contrast, if M has nonnegative sectional curvature and M_∞ splits off a line, then M must split off a line.

As a final remark, I note that after the original version of this survey was written several related results were shown. In particular, jointly with Bill Minicozzi we settled the general case of Yau's conjecture mentioned above [Colding and Minicozzi c; d; e].

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Tobias H. Colding
Courant Institute
251 Mercer Street
New York, NY 10012
colding@cims.nyu.edu