Comparison Geometry
MSRI Publications
Volume **30**, 1997

Collapsing with No Proper Extremal Subsets

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Abstract. This is a technical paper devoted to the investigation of collapsing of Alexandrov spaces with lower curvature bound. In a previous paper, the author defined a canonical stratification of an Alexandrov space by the so-called extremal subsets. It is likely that if the limit of a collapsing sequence has no proper extremal subsets, then the collapsing spaces are fiber bundles over the limit space. In this paper a weaker statement is proved, namely, that the homotopy groups of those spaces are related by the Serre exact sequence. A restriction on the ideal boundary of open Riemannian manifolds of nonnegative sectional curvature is obtained as a corollary.

We assume familiarity with the basic notions and results about Alexandrov spaces with curvature bounded below [Burago et al. 1992; Perelman 1994; Perelman and Petrunin 1994], and with earlier results on collapsing with lower curvature bound [Yamaguchi 1991]. For motivation for the collapsing problem, see [Cheeger et al. 1992] and references therein.

1. Background

Notation. Throughout the paper we denote by M a fixed m-dimensional compact Alexandrov space with curvature $\geq k$; by N a variable n-dimensional compact Alexandrov space with curvature $\geq k$, where $n > m$; by $\Phi : M \to N$ a ν-approximation. For $p \in M$ we denote by $\bar{p}$ the image $\Phi(p)$. We set

$$I^l_a(v) = \{x \in \mathbb{R}^l : |x_i - v_i| < a \text{ for all } i\}.$$

For a map $f : M \to \mathbb{R}^{l_1}$ and $1 \leq l \leq l_1$ we denote by $f_{[l]}$ a map from M to $\mathbb{R}^l$ whose coordinate functions are the first l coordinate functions of f.

Admissible functions, maps, and regular points. A function $f : M \to \mathbb{R}$ is called *admissible* if $f(x) = \sum_\alpha \phi_\alpha(\mathrm{dist}_{q_\alpha}(x))$, where $q_\alpha \in M$, the ϕ_α are smooth, increasing, concave functions, and the set of indices α is finite.

This paper was prepared while the author was a Miller fellow at the University of California at Berkeley.

A map $\hat{f} : M \to \mathbb{R}^l$ is said to be *admissible* in a domain $U \subset M$ if it can be represented there as $\hat{f} = H \circ f$, where H is a homeomorphism of $\mathbb{R}^l$ and the coordinate functions f_i of f are admissible.

A map $\hat{f} = H \circ f$ admissible in U is *regular* at $p \in U$ if the coordinate functions $f_i = \sum_{\alpha_i} \phi_{\alpha_i} \circ \mathrm{dist}_{q_{\alpha_i}}$ satisfy

(1) $\sum_{\alpha_i,\alpha_j} \phi'_{\alpha_i}\big(\mathrm{dist}_{q_{\alpha_i}}(p)\big)\, \phi'_{\alpha_j}\big(\mathrm{dist}_{q_{\alpha_j}}(p)\big) \cos \widetilde{\angle} q_{\alpha_i} p q_{\alpha_j} < 0$ for any $i \neq j$, and
(2) there exists $\xi \in \Sigma_p$ such that $f'_i(\xi) > 0$ for all i.

(Note that $\hat{f}$ has no regular points if $l > m$.)

An admissible map $\hat{f} = H \circ f$ is said to be regular in a domain $V \subset U$ if it is regular at all points $p \in V$.

Conditions (1) and (2) are slightly more restrictive than those given in the definitions in [Perelman 1994]; however, it is easy to check that the arguments of [Perelman 1994] go through with this modified definition. The reason for this modification is that our new definition is stable: if f_i satisfies (1) and (2) then $\bar{f}_i = \sum_{\alpha_i} \phi_{\alpha_i}(\mathrm{dist}_{\bar{q}_{\alpha_i}}(x))$ satisfies the same conditions at $\bar{p} \in N$, provided that $\nu > 0$ is small enough. Therefore if $\hat{f} = H \circ f$ is regular at p then $\bar{\hat{f}} = H \circ \bar{f}$ is regular at $\bar{p}$. Now if $\hat{f} = H \circ f$ is regular in U and, for some subset $\tilde{K} \subset \mathbb{R}^l$, the set $K = \hat{f}^{-1}(\tilde{K}) \cap U$ has compact closure in U, then, assuming $\nu > 0$ to be small enough, we can unambiguously define $\bar{K} \subset N$ as a union of those components of $\bar{\hat{f}}^{-1}(\tilde{K})$ that are Hausdorff-close to $\Phi(K)$.

Canonical neighborhoods. Let $\hat{f} = H \circ f : M \to \mathbb{R}^{l+1}$ be admissible in U; let $p \in U$, and $a > 0$. Suppose that

(i) $\hat{f}_{l+1}(U) \subset (-\infty, 0]$ and $\hat{f}_{l+1}(p) = 0$;
(ii) the coordinate functions of $f_{[l]}$ satisfy conditions (1) and (2) in U, and the coordinate functions of f satisfy the same conditions in $U \backslash \hat{f}_{l+1}^{-1}(0)$;
(iii) the set $K_p(a)$ of points $x \in U$ such that $|\hat{f}_i(x) - \hat{f}_i(p)| < a$ for $i = 1, \dots l+1$ has compact closure in U; and
(iv) $\hat{f}$ is one-to-one on $K_p(a) \cap \hat{f}_{l+1}^{-1}(0)$.

Then $K_p(a)$ is called a canonical neighborhood of p with respect to $\hat{f}$. (In particular we allow $l = m$ and $\hat{f}_{l+1} \equiv 0$.)

It has been proved in [Perelman 1994] that if $f : M \to \mathbb{R}^l$ is regular at p, then f is an open map near p and there exists a canonical neighborhood of p with respect to some map $g : M \to \mathbb{R}^{l_1}$ with $l_1 > l$, such that $g_{[l]} = f$.

Regular fibers and fiber data. Let $f : M \to \mathbb{R}^m$ be regular at p. Then f is a local homeomorphism near p. If $\nu > 0$ is small enough, $\bar{f}$ is regular in a neighborhood of $\bar{p}$ of size independent of N. Therefore, according to [Perelman 1994, Main Theorem (B)], we have a trivial bundle $\bar{f} : \bar{f}^{-1}\big(I_a^m(f(p))\big) \to I_a^m(f(p))$ for small $a \gg \nu$. In this case the fiber $\bar{f}^{-1}(f(p))$ is called regular and the pair (f, p) is called *fiber data*. It is easy to see that a regular fiber is connected.

It is easy to see that if (f_1, p) and (f_2, p) are fiber data then the corresponding regular fibers F_1, F_2 are homotopy equivalent, provided that $\nu > 0$ is small enough. Since a point $p \in M$ is a part of fiber data if and only if Σ_p contains $m + 1$ directions making obtuse angles with each other, and since the set of such points is open, dense and convex (convexity follows from Petrunin's result on parallel translation), we can conclude that the statement about homotopy equivalence is valid for arbitrary fiber data (f_1, p_1) and (f_2, p_2).

If N is a smooth manifold, a regular fiber is a topological manifold. In general one can check that a regular fiber has $\mathbb{Z}/2\mathbb{Z}$-fundamental class in $(n - m)$-dimensional singular homology (relative to the boundary if $\partial N \neq \varnothing$).

Homotopy groups. Fix an integer $l \geq 0$. Let (f, p) be fiber data and let $F = \bar{f}^{-1}(f(p))$ be the corresponding regular fiber. We can assume that $\bar{p} \in F$. We can try to define a homomorphism $\pi_l(N, F, \bar{p}) \to \pi_l(M, p)$ in the following way. Given a spheroid in $\pi_l(N, F, \bar{p})$, consider its fine triangulation and for each vertex x_α find a point $y_\alpha \in M$ so that $|x_\alpha \Phi(y_\alpha)| \leq \nu$; now span the corresponding spheroid in $\pi_l(M, p)$ using vertices y_α. The local geometric contractibility of M implies that this procedure gives a correctly defined homomorphism provided that $\nu > 0$ is small enough.

The main result of this paper is that the constructed homomorphism has an inverse, namely the lifting map constructed after Proposition 2.4 below, provided that M has no proper extremal subsets.

2. The Lifting Map

A point $p \in M$ is called *good* if it satisfies the following condition.

Condition PN. For any $R > 0$ there exists a number $\rho = \rho(p, R) > 0$ such that, for any fiber data (f, q) with $q \in B_p(\rho)$, one can find $\bar{\nu} = \bar{\nu}(p, R, f, q) > 0$ so that, if $\nu \leq \bar{\nu}$, then N contains a *product neighborhood*, that is, a domain U with $B_{\bar{p}}(\rho) \subset U \subset B_{\bar{p}}(R)$ such that the inclusion $\bar{f}^{-1}(f(q)) \hookrightarrow U$ induces isomorphisms of homotopy and homology groups.

Let *NPN* denote the set of all bad points of M.

Proposition 2.1. *The closure of NPN, if nonempty, is a proper extremal subset of M.*

Observe that the closure of *NPN* is not all of M, since p is definitely good if Σ_p contains $m+1$ directions making obtuse angles with each other. The extremality of clos(*NPN*) follows from two lemmas.

Lemma 2.2. *Let K be a canonical neighborhood with respect to $f : M \to \mathbb{R}^{l+1}$. Assume that $p, q \in K \cap f_{l+1}^{-1}(0)$, and that p is good. Then q is also good.*

PROOF. Fix $R > 0$. Find $a > 0$ so that $K_p(3a) \subset K$ and $K_q(3a) \subset B_q(R) \cap K$. Take $r > 0$ such that $B_p(r) \subset K_p(a)$. Choose $d > 0$ so that $K_p(3d) \subset B_p(\rho(p,r))$. Finally, define $\rho(q,R) > 0$ so that $B_q(\rho(q,R)) \subset K_q(d)$.

To check Condition PN, let $L = K \cap f_{l+1}^{-1}(-d,0]$ and observe that, according to [Perelman 1994, Main Theorem (A)], the map $\bar{f}_{[l]}$ is a (topological) submersion in $\bar{K}$, whereas $\bar{f}$ is a submersion in $\bar{K} \setminus \bar{L}$, if $\nu > 0$ is small enough. Therefore, applying the results of [Siebenmann 1972], we can construct a homeomorphism $\psi : F \times I^l \to \bar{K}$ that respects $\bar{f}_{[l]}$ and, in addition, respects $\bar{f}_{l+1}$ over $\bar{K} \backslash \bar{L}$. Thus we can use the homeomorphism $\theta = \psi \circ \operatorname{transl}(\overline{f_{[l]}(p) f_{[l]}(q)}) \circ \psi^{-1}$ to transfer $\bar{K}_p(\rho)$ to $\bar{K}_q(\rho)$ for $d \le \rho \le 3a$, and it is easy to see that if U is a product neighborhood, $\bar{K}_p(2d) \subset U \subset \bar{K}_p(2a)$, then $\theta(U)$ is also a product neighborhood that satisfies Condition PN with respect to given q, R and chosen $\rho(q,R)$. $\square$

LEMMA 2.3. *Let $f : M \to \mathbb{R}^l$ be regular in a neighborhood U of p, and let $L = \bigcap_{i>1} f_i^{-1}(f_i(p))$. Then f_1 restricted to clos$(NPN \cap L) \cap U$ cannot attain its minimum.*

PROOF. We use reverse induction on l. If $l = m$, our assertion is clear since all points in U are good in this case. Assume that $l < m$. If the minimum is attained, we can assume that it is attained at p. Consider a canonical neighborhood $K_p(a) \subset V \subset U$ with respect to a map $g : M \to \mathbb{R}^{l_1}$, where $l_1 > l$, such that $g_{[l]} = f$ in V. Since p is the point of minimum, we have $K_p(a) \cap L \cap g_{l_1}^{-1}(0) \not\subset NPN$. Therefore Lemma 2.2 implies that $K_p(a) \cap NPN \cap g_{l_1}^{-1}(0) = \varnothing$. On the other hand, since $p \in \operatorname{clos}(L \cap NPN)$, we can find $v \in \mathbb{R}^{l_1}$ such that $v_i = f_i(p)$ for $1 < i \le l$, $|v_i - f_i(p)| < a$ for $l+1 \le i \le l_1$, $v_{l_1} \ne 0$, and $K_p(a) \cap NPN \cap L_1 \ne \varnothing$, where $L_1 = \bigcap_{1<i\le l_1} g_i^{-1}(v_i) \subset L$. Let $U_1 = K_p(a) \backslash g_{l_1}^{-1}(0)$, and $p_1 \in U_1 \cap L_1$. Then g is regular in $U_1 \ni p_1$, and $g_1 (= f_1)$ restricted to $\operatorname{clos}(NPN \cap L_1) \cap U_1$ attains its minimum value because $f_1(p) \le f_1(x) < f_1(p) + a$ for all $x \in \operatorname{clos}(NPN \cap L_1) \cap U_1$, whereas $f_1(L_1 \cap \partial U_1) \subset \{f_1(p)+a, f_1(p)-a\}$. This proves the induction step. $\square$

The extremality of clos(NPN) follows immediately from the case $l = 1$ of the lemma above and the definition.

Now assume that M has no proper extremal subsets. According to Proposition 2.1, all points of M are good. Moreover, the compactness of M implies that we can define a function $\rho(R)$ satisfying Condition PN and independent of p. (Choose a finite covering of M by balls $B(p_\alpha, \rho(p_\alpha, R_\alpha/10)/10)$ and let $\rho(R) = \min_\alpha \rho(p_\alpha, R/10)/10$).

Fix a positive integer l and fiber data (f_0, p_0). Choose $R_{2l+3} > 0$ so small that f_0 is regular in a R_{2l+3}-neighborhood of p_0 and for any finite simplicial complex K of dimension at most l and its subcomplex L, any two maps of (K, L) into (M, p_0) that are uniformly R_{2l+3}-close are homotopic relative to L. Define $R_i > 0$ for $0 \le i \le 2l+2$ inductively, in such a way that $\rho(R_{i+1}/10) \ge 10R_i$ for all i. Take a finite family of fiber data (f_α, q_α) such that q_α form an R_0-net in M. Repeating the compactness argument, we can choose a universal $\bar{\nu}$, with

$0 < \bar{\nu} < R_0$, so that Condition PN is satisfied for all $p \in M$, $R = R_i$, $f = f_\alpha$, $q = q_\alpha$. From now on we assume $\nu \le \bar{\nu}$.

PROPOSITION 2.4. *Let K be a finite simplicial complex of dimension $\le l$. Suppose $\phi : K \times I \to M$ and $\bar{\phi} : K \to N$ satisfy $|\bar{\phi}(x)\Phi \circ \phi(x)| < R_{i+1}$ for $x \in \mathrm{skel}_i K$, for $0 \le i \le l$, and that $\mathrm{diam}\phi(\Delta) < R_0$ for all simplices $\Delta \subset K$. Then $\bar{\phi}$ can be extended to a map from $K \times I$ to N such that $\left|\bar{\phi}(x)\Phi \circ \phi(x)\right| < R_{i+1}$ for $x \in \mathrm{skel}_i(K \times I)$, for $0 \le i \le l+1$.*

PROOF. A standard argument reduces our extension problem to the case when $K = \Delta^i$, $0 \le i \le l$, $\mathrm{diam}(\phi(\Delta^i \times I)) < R_0$, and $\bar{\phi}$ has already been defined on $\Delta^i \times \{0\} \cup \partial\Delta^i \times I$. Take any $p \in \phi(\Delta^i \times I)$ and let U, V be the product neighborhoods such that $B_{\bar{p}}(10R_{i+1}) \subset U \subset B_{\bar{p}}(R_{i+2}/10)$ and $B_{\bar{p}}(10R_i) \subset V \subset B_{\bar{p}}(R_{i+1}/10)$. Clearly $\bar{\phi}(\Delta^i \times \{0\} \cup \partial\Delta^i \times I) \subset U$ and $\bar{\phi}(\partial\Delta^i \times \{1\}) \subset V$. Since there exists a regular fiber F such that $F \hookrightarrow V$ and $F \hookrightarrow U$ induce isomorphisms of homotopy groups, we conclude that $V \hookrightarrow U$ has the same property; in particular, $\pi_{i+1}(U, V) = 0$. Therefore we can easily extend $\bar{\phi}$ so that $\bar{\phi}(\Delta^i \times I) \subset U$ and $\bar{\phi}(\Delta^i \times \{1\}) \subset V$. ☐

Now we can define a lifting map $\pi_l(M, p_0) \to \pi_l(N, F_0, \bar{p}_0)$: Given a spheroid ϕ in $\pi_l(M, p_0)$, let $\bar{\phi}$ be its image if $\left|\bar{\phi}(x)\Phi \circ \phi(x)\right| < R_{l+1}$ for all x. Existence of such $\bar{\phi}$ follows from Proposition 2.4 and the fact that the inclusion of F_0 into its appropriate neighborhood of size R_{2l+3} is a homotopy equivalence; correctness follows from Proposition 2.4 with R_{l+i+1} substituted for R_{i+1} in the assumption and the conclusion. It is clear that this lifting homomorphism is an inverse of the one described at the end of Section 1.

3. Corollaries

COROLLARY 3.1. *Let N be a complete noncompact Riemannian manifold of nonnegative sectional curvature that does not admit isometric splitting and is not diffeomorphic to $\mathbb{R}^n$. Then its asymptotic cone M has proper extremal subsets. In particular, the radius of its ideal boundary is at most $\pi/2$.*

REMARK. The last assertion was conjectured by Shioya [1993]. Recently Sérgio Mendonça independently obtained a direct proof of the same result.

PROOF OF COROLLARY 3.1. The manifold N with rescaled metrics collapses to M. If M has no proper extremal subsets, then, in particular, its apex is a good point. Therefore we can easily construct neighborhoods $U_1 \supset V \supset U_2 \supset S$ of the soul S, such that U_1, U_2 are product neighborhoods, whereas V is a convex neighborhood. Thus S must have the homology groups of the regular fiber, whence $\dim S = n - m$. Now an easy packing argument shows that the normal bundle of S has finite holonomy, which is therefore a quotient of the fundamental group of S and N. Thus a finite cover $\tilde{N}$ of N has normal bundle with trivial holonomy, and, according to [Strake 1988], $\tilde{N}$ splits isometrically. It follows that

M is the quotient of the asymptotic cone of $\tilde{N}$ by an isometric action of a finite group, which fixes the apex. Thus, according to [Perelman and Petrunin 1994, §4.2], M has proper extremal subsets, unless the group action is trivial and $\tilde{N} = N$.

The second statement of the corollary follows from [Perelman and Petrunin 1994, Proposition 1.4.1]. $\square$

Corollary 3.2. *If Σ^{m-1} is a limit of a collapsing sequence of compact $(n-1)$-dimensional Alexandrov spaces with curvature ≥ 1, then either the diameter of Σ^{m-1} is at most $\pi/2$ or Σ^{m-1} has proper extremal subsets.*

In particular, according to [Perelman and Petrunin 1994, Proposition 1.4.1], the radius of Σ^{m-1} cannot exceed $\pi/2$. This conclusion was obtained earlier by Petrunin (in his unpublished Master's Thesis), and it also follows immediately from [Grove and Petersen 1993, Theorem 3(3)].

Proof. Consider the collapsing of the corresponding cones. If the conclusion is false, then $M = \operatorname{cone}(\Sigma^{m-1})$ has no proper extremal subsets; in particular, its apex is a good point. Thus we can construct neighborhoods $U_1 \supset V \supset U_2$ in the collapsing cone, such that U_1, U_2 are product neighborhoods, whereas V is a spherical neighborhood of the apex. Therefore the inclusion $U_2 \hookrightarrow U_1$ factors through a contractible space—a contradiction. $\square$

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