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A Complete Riemannian Manifold of Positive Ricci Curvature with Euclidean Volume Growth and Nonunique Asymptotic Cone

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Consider the metric $ds^2 = dt^2 + A^2(t)\,dx^2 + B^2(t)\,dy^2 + C^2(t)\,dz^2$, where t is the radial coordinate and x, y, z are "spherical coordinates" with $[X,T]=[Y,T]=[Z,T]=0$, $[X,Y]=2Z$, $[Y,Z]=2X$, and $[Z,X]=2Y$. (Taking $A(t)=B(t)=C(t)=t$ we get the standard Euclidean metric.) A straightforward computation gives

$$\langle R(X,T)T,X\rangle = -\frac{A''}{A}\|X\|^2\|T\|^2,$$

$$\langle R(X,Y)Y,X\rangle = \|X\|^2\|Y\|^2 \times \Big(-\frac{A'B'}{AB}+\frac{1}{A^2B^2C^2}(A^4+B^4-3C^4+2A^2C^2+2B^2C^2-2A^2B^2)\Big)$$

and similar equalities obtained by permutation of the pairs (X,A), (Y,B), (Z,C); similarly

$$\langle R(X,Y)Z,T\rangle = \|X\|\,\|Y\|\,\|Z\|\,\|T\| \times \frac{1}{ABC}\Big(-\frac{A'}{A}(C^2+A^2-B^2)+\frac{B'}{B}(A^2-B^2-C^2)+2C'C\Big),$$

while $\langle R(X,T)T,Y\rangle = \langle R(T,Y)T,Z\rangle = \langle R(Z,T)T,X\rangle = \langle R(X,Y)Y,Z\rangle = \langle R(Y,Z)Z,X\rangle = \langle R(Z,X)X,Y\rangle = \langle R(T,X)X,Y\rangle = \langle R(T,X)X,Z\rangle = \langle R(T,Y)Y,X\rangle = \langle R(T,Y)Y,Z\rangle = \langle R(T,Z)Z,X\rangle = \langle R(T,Z)Z,Y\rangle = 0$. In particular, the matrix of Ricci curvature in these coordinates is diagonal. Now take

$$A(t) = \tfrac{1}{10}\,t\big(1+\phi(t)\sin(\ln\ln t)\big),$$
$$B(t) = \tfrac{1}{10}\,t\big(1+\phi(t)\sin(\ln\ln t)\big)^{-1},$$
$$C(t) = \tfrac{1}{10}\,t\big(1-\gamma(t)\big),$$

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where $\phi(t)$ is a smooth function such that $\phi(t)=0$ for $t \in [0,T]$ (where $T>0$ is a sufficiently big number), $\phi(t)>0$ for $t>T$, $0 \le \phi'(t) \le t^{-2}$, and $|\phi''(t)| \le t^{-3}$; and $\gamma(t)$ is a smooth function such that $\gamma(t)=0$ for $t \in [0,T/2]$, $\gamma'(t), \gamma''(t) > 0$ for $t \in (T/2,T)$, and $\gamma'(t) = (t \ln^{3/2} t)^{-1}$ for $t > T$.

Computation shows that $\|\mathrm{Rm}\| = O(t^{-2})$ and $\mathrm{Ric}(T,T) \ge C/(t^2 \ln^{3/2} t)$, while $\mathrm{Ric}(X,X)$, $\mathrm{Ric}(Y,Y)$, $\mathrm{Ric}(Z,Z)$ are all $\ge C/t^2$. It is also clear that the asymptotic cone is not unique. It remains only to smooth off the vertex ($t=0$), where our space is isometric to a cone over a sphere of constant curvature 100.

Remark. Mike Anderson has pointed out to me that a similar construction was used earlier by Brian White, in the context of surfaces in euclidean spaces.

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