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Construction of Manifolds of Positive Ricci Curvature with Big Volume and Large Betti Numbers

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Abstract. It is shown that a connected sum of an arbitrary number of complex projective planes carries a metric of positive Ricci curvature with diameter one and, in contrast with the earlier examples of Sha–Yang and Anderson, with volume bounded away from zero. The key step is to construct complete metrics of positive Ricci curvature on the punctured complex projective plane, which have uniform euclidean volume growth and almost contain a line, thus showing topological instability of the splitting theorem of Cheeger–Gromoll, even in the presence of the lower volume bound. In the absence of such a bound, the topological instability was earlier shown by Anderson; metric stability holds, even without the volume bound, by the recent work of Colding–Cheeger.

1. Outline

We start from a singular space of positive curvature, namely the double spherical suspension of a small round two-sphere. The size of that sphere can be estimated explicitly and is fixed in our construction. The singular points of our space fill a circle of length 2π. We smooth our space near the singular circle in a symmetric way. Then we remove a collection of disjoint small metric balls centered at the former singular points, and glue in our "building blocks" instead. A building block is a metric on $\mathbb{C}P^2 \setminus$ ball, having positive Ricci curvature and strictly convex boundary. We arrange that the boundary of the building block is isometric to the boundary of the removed ball and is "more convex". This allows us to smooth the resulting space to get a manifold of positive Ricci curvature. Its volume is close to the volume of the double suspension we started from, whereas its second Betti number equals the number of building blocks it contains. This number can be made arbitrarily large by localizing the initial

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smoothing of the double suspension to smaller neighborhoods of the singular circle, and by removing a larger number of smaller balls.

The main difficulty is to construct building blocks. This is carried out in Section 2. In Section 3 we construct the ambient space, i.e., we make precise the relation between the number and the size of balls to be removed and the smoothing of our double suspension. In Section 4 we explain how to smooth the result of gluing two manifolds with boundaries, so as to retain positive Ricci curvature.

Notation. $K(X \wedge Y)$ denotes the sectional curvature in the plane spanned by X, Y. We abbreviate $\mathrm{Ric}(X,Y)/|X||Y|$ as $\mathrm{Ri}(X,Y)$.

2. Construction of the Building Block

Our building block is glued from two pieces: the core and the neck. The core is a metric on $\mathbb{C}P^2 \setminus$ ball with positive Ricci curvature and strictly convex boundary; moreover, the boundary is isometric to a round sphere. The neck is a metric on $S^3 \times [0,1]$ with positive Ricci curvature, such that the boundary component $S^3 \times \{0\}$ is concave and is isometric to a round sphere, whereas the boundary component $S^3 \times \{1\}$ is convex and looks like a lemon, that is, like a smoothed spherical suspension of a small round two-sphere. There are some additional conditions on the normal curvatures of the boundary of the neck; they will be made clear below.

Construction of the core. It is well known that the canonical metric of $\mathbb{C}P^2$ in a neighborhood of $\mathbb{C}P^1$ can be expressed as

$$ds^2 = dt^2 + A^2(t)\,dx^2 + B^2(t)\,dy^2 + C^2(t)\,dz^2,$$

where t is the distance from $\mathbb{C}P^1$ and X, Y, Z is the standard invariant framing of S^3, satisfying $[X,Y] = 2Z$, $[Y,Z] = 2X$, $[Z,X] = 2Y$. The canonical metric has $A = \sin t \cos t$ and $B = C = \cos t$, but we are going to consider general A, B, C.

The curvature tensor in this presentation is listed on page 165 of this volume. Assuming now that $B \equiv C$, we deduce from those formulas that

$$\mathrm{Ri}(X,X) = (-A''/A - 2A'B'/AB + 2A^2/B^4),$$

$$\mathrm{Ri}(Y,Y) = \mathrm{Ric}(Z,Z) = (-B''/B - A'B'/AB - B'^2/B^2 + (4B^2 - 2A^2)/B^4),$$

$$\mathrm{Ri}(T,T) = (-A''/A - 2B''/B),$$

and all off-diagonal entries vanish.

Now let $A = \sin t \cos t$, $B = \frac{1}{100}\cosh(t/100)$. Clearly this choice gives a smooth metric. We have, for $0 < t < \frac{1}{10}$,

$$-A''/A = 4, \quad -B''/B = -10^{-4}, \quad 0 < A'B'/AB < \tfrac{1}{10}, \quad 0 < B'/B < \tfrac{1}{100},$$

and $(4B^2 - 2A^2)/B^4 > 100$ provided that $A \le B$.

Thus if $0 < t_0 < \frac{1}{10}$ is such that $A(t_0) = B(t_0)$, $A(t) < B(t)$ for $t < t_0$, then Ricci curvatures are positive for $0 \le t \le t_0$. Such t_0 exists because $A(0) = 0 < B(0)$, whereas $A(\frac{1}{10}) > \frac{1}{20} > B(\frac{1}{10})$. Therefore the core is constructed.

Construction of the neck. We first prove the following result.

ASSERTION. *Let (S^n, g) be a rotationally symmetric metric of sectional curvature > 1, distance between the poles πR and waist $2\pi r$; that is, g can be expressed as $ds^2 = dt^2 + B^2(t)\, d\sigma^2$, where $d\sigma^2$ is the standard metric of S^{n-1}, $t \in [0, \pi R]$, and $\max_t B(t) = r$. Let $\rho > 0$ be such that $\rho < R$ and $r^{n-1} < \rho^n$. Then there exists a metric of positive Ricci curvature on $S^n \times [0,1]$ such that (a) the boundary component $S^n \times \{1\}$ has intrinsic metric g and is strictly convex; moreover, all its normal curvatures are > 1; (b) the boundary component $S^n \times \{0\}$ is concave, with normal curvatures equal to $-\lambda$, and is isometric to a round sphere of radius $\rho\lambda^{-1}$, for some $\lambda > 0$.*

(Note that, if ρ is small enough, the core constructed earlier in this section can be glued, after rescaling, to the neck along $S^n \times \{0\}$, so that the resulting space can be smoothed with positive Ricci curvature; see Section 4.)

To prove the assertion, we start by rewriting our metric g in the form

$$ds^2 = r^2 \cos^2 x \, d\sigma^2 + A^2(x)\, dx^2,$$

where $-\pi/2 \le x \le \pi/2$ and A is a smooth positive function satisfying $A(\pm\pi/2) = r$, $A'(\pm\pi/2) = 0$. Clearly $A(x) \ge R > r$ for some x, so we can write

$$A(x) = r(1 - \eta(x) + \eta(x) \cdot a_\infty),$$

where $\max_x \eta(x) = 1$ and $a_\infty \ge R/r$, $\eta(\pm\pi/2) = 0$, and $\eta'_x(\pm\pi/2) = 0$.

Consider a metric on $S^n \times [t_0, t_\infty]$ of the form

$$ds^2 = dt^2 + A^2(t,x)\, dx^2 + B^2(t,x)\, d\sigma^2,$$

where $B = tb(t)\cos x$, $A = tb(t)\big(1 - \eta(x) + \eta(x)a(t)\big)$, $a(t_0) = 1$, $a'(t_0) = 0$, $b(t_0) = \rho$, $b'(t_0) = 0$, $a(t_\infty) = a_\infty > 1$, $b(t_\infty) > r$. That metric will satisfy the conditions of our assertion after rescaling by a multiple $r/(t_\infty \cdot b(t_\infty))$, provided that the functions $a(t)$ and $b(t)$ are chosen appropriately.

The curvatures of our metric can be computed as follows:

$$\begin{aligned}
K(T \wedge X) &= -A_{tt}/A \\
K(T \wedge \Sigma) &= -B_{tt}/B \\
K(X \wedge \Sigma) &= -A_t B_t/AB + A_x B_x/A^3 B - B_{xx}/BA^2 \\
K(\Sigma_1 \wedge \Sigma_2) &= 1/B^2 - B_x^2/A^2B^2 - B_t^2/B^2,
\end{aligned}$$

and the Ricci tensor has only one nonzero off-diagonal term, namely

$$\mathrm{Ri}(T,X) = (n-1)(A_t B_x/A^2 B - B_{xt}/AB).$$

Further computations give

$$\begin{aligned}
\mathrm{Ri}(T,T) &= -n\left(\frac{b''}{b}+\frac{2b'}{tb}\right)-\left(\frac{\eta a''}{1-\eta+\eta a}+\frac{2\eta a'}{t(1-\eta+\eta a)}+2\frac{b'}{b}\frac{\eta a'}{1-\eta+\eta a}\right),\\
\mathrm{Ri}(T,X) &= -(n-1)\ \mathrm{tg}\, x\,\frac{\eta a'}{tb(1-\eta+\eta a)^2}
\end{aligned}$$

for the Ricci tensor, and

$$\begin{aligned}
K(X\wedge\Sigma) &= \frac{1}{t^2b^2}\left(\frac{1}{(1-\eta+\eta a)^2}-\frac{\eta_x\ \mathrm{tg}\, x\,(a-1)}{(1-\eta+\eta a)^3}\right)\\
&\qquad -\left(\frac{1}{t}+\frac{b'}{b}\right)\left(\frac{1}{t}+\frac{b'}{b}+\frac{\eta a'}{1-\eta+\eta a}\right),\\
K(\Sigma_1\wedge\Sigma_2) &= \frac{1}{t^2b^2}\left(\frac{1}{\cos^2 x}-\frac{\mathrm{tg}^2\, x}{(1-\eta+\eta a)^2}\right)-\left(\frac{1}{t}+\frac{b'}{b}\right)^2
\end{aligned}$$

for the sectional curvatures. Note that the first terms in the last two formulas are the intrinsic curvatures of the hypersurfaces $t=\mathrm{const}$, $K_i(X\wedge\Sigma)$, $K_i(\Sigma_1\wedge\Sigma_2)$.

We construct the functions $b(t)$ and $a(t)$ as follows:

$$\begin{aligned}
&b'/b = -\beta(t-t_0)/2t_0^2\log 2t_0 \quad \text{for } t_0\le t\le 2t_0,\\
&b'/b = -\beta\log 2t_0/t\log^2 t \quad \text{for } t\ge 2t_0 \quad (b \text{ must be smoothed near } t=2t_0),\\
&a'/a = -\alpha b'/b \quad \text{for } t\ge t_0.
\end{aligned}$$

Here $\beta=(1-\varepsilon)(\log\rho-\log r)/(1+1/(4\log 2t_0))$ and $\alpha=(1+\delta)\beta^{-1}\log a_\infty/(1+1/(4\log 2t_0))$, so that $\int_{t_0}^\infty b'/b=(1-\varepsilon)(\log r-\log\rho)$ and $\int_{t_0}^\infty a'/a=(1+\delta)\log a_\infty$; the small positive numbers ε,δ are to be determined later.

At this point we still have freedom in the choice of t_0,ε,δ, and t_∞. Note, however, that t_∞ is determined by δ from the relation $a(t_\infty)=a_\infty$. We choose $\varepsilon>0$ first in such a way that the metric $(\rho/r)^\varepsilon\cdot g$ still has sectional curvatures >1. In this situation we prove below that the curvatures $K_i(X\wedge\Sigma)$ and $K_i(\Sigma_1\wedge\Sigma_2)$ can be estimated from below by c/t^2 for some $c>1$ independent of t_0, t_∞, and δ. Then we show that the Ricci curvatures of our metric are strictly positive if t_0 is chosen large enough, independently of δ, if δ is small. Finally, we choose δ in such a way that the normal curvatures of $S^n\times\{t_\infty\}$ are sufficiently close to t_∞^{-1}.

First of all we need to check that $1<\alpha<n$. Indeed, at the maximum point of $\eta(x)$ we have $\mathrm{tg}\, x\,\eta_x=0$, so $K_g(X\wedge\Sigma)=1/r^2a_\infty^2$. Thus $a_\infty<1/r$ and $\log a_\infty<n(\log\rho-\log r)$ since $r^{n-1}<\rho^n$. On the other hand $a_\infty\ge R/r>\rho/r$, so $\log a_\infty>\log\rho-\log r$.

Now we are in a position to check that $K_i(X\wedge\Sigma)\ge c/t^2$ and $K_i(\Sigma_1\wedge\Sigma_2)\ge c/t^2$ for some $c>1$. Let $\psi=\log\big(t^2K_i(\Sigma_1\wedge\Sigma_2)\big)$. We know that $\psi|_{t=t_0}>0$ and

$\psi|_{t=t_\infty} > 0$. A computation gives

$$\psi_t = -\frac{2b'}{b}\left(1 + \frac{\alpha\eta a \sin^2 x}{(1-\eta+\eta a)((1-\eta+\eta a)^2 - \sin^2 x)}\right).$$

This expression is positive if $\eta \geq 0$, and it is decreasing in a if $\eta < 0$. Therefore ψ cannot have a local minimum on (t_0, t_∞) for any fixed x, so $\psi > 0$ as required.

Now let $\phi = \log\bigl(t^2 K_i(X \wedge \Sigma)\bigr)$. Again consider the behavior of this function for $t \in [t_0, t_\infty]$ and fixed x. Suppose that $\operatorname{tg} x\, \eta_x \geq 0$. A computation gives

$$\begin{aligned}\phi_t &= -\frac{b'}{b}\frac{1}{1-\eta+\eta a}\left[2(1-\eta) + 2\eta a(1-\alpha) - \frac{\alpha a \operatorname{tg} x\, \eta_x}{1-\eta+\operatorname{tg} x\,\eta_x + a(\eta - \operatorname{tg} x\, \eta_x)}\right] \\ &= -\frac{b'}{b}\left(2 - \frac{\alpha a}{1-\eta+\eta a}\left[2\eta + \frac{\operatorname{tg} x\,\eta_x}{1-\eta+\operatorname{tg} x\,\eta_x + a(\eta-\operatorname{tg} x\,\eta_x)}\right]\right).\end{aligned}$$

If $\eta \geq 0$ the first expression in large brackets is clearly decreasing in a, whereas if $\eta < 0$ the second expression in brackets, and also $\alpha a/(1-\eta+\eta a)$, are clearly increasing in a, so ϕ cannot have a local minimum in (t_0, t_∞), and therefore $\phi > 0$ since $\phi|_{t=t_0} > 0$ and $\phi|_{t=t_\infty} > 0$. Now suppose that $\operatorname{tg} x\,\eta_x < 0$. Then $\phi > -2\log ab > 0$, because $a(t_\infty)b(t_\infty) < 1$ and $\log(ab)$ is increasing in t (note that $\log(ab)' = -(b'/b)(\alpha - 1) > 0$). The estimate for $K_i(X \wedge \Sigma)$ and $K_i(\Sigma_1 \wedge \Sigma_2)$ is proved.

Now we can estimate the Ricci curvatures of our metric. Note that the normal curvatures $1/t + b'/b$ and $1/t + b'/b + \eta a'/(1-\eta+\eta a)$ can be estimated by $1/t + O(1/t\log t)$. It follows that $K(X \wedge \Sigma) \geq c/t^2$ and $K(\Sigma_1 \wedge \Sigma_2) \geq c/t^2$, for $c > 0$. It is also clear that $|\mathrm{Ri}(T,X)|$, $|K(X \wedge T)|$, and $|K(\Sigma \wedge T)|$ are all bounded above by $c\log t_0/(t^2\log^2 t)$; hence, in particular, $\mathrm{Ri}(X,X) \geq c/t^2$ and $\mathrm{Ri}(\Sigma,\Sigma) \geq c/t^2$, with $c > 0$.

To estimate $\mathrm{Ri}(T,T)$, write it as

$$\begin{aligned}\mathrm{Ri}(T,T) = &\left(\frac{\alpha\eta a}{1-\eta+\eta a} - n\right)\left((b'/b)' + \frac{2}{t}\frac{b'}{b}\right) \\ &- n(b'/b)^2 - \frac{\eta a}{1-\eta+\eta a}\left(2\frac{b'}{b}\frac{a'}{a} + \left(\frac{a'}{a}\right)^2\right).\end{aligned}$$

Note that, since $\eta \leq 1$ and $\alpha < n$, the first multiple is negative and bounded away from 0. Therefore, the first term is $\geq c\log t_0/(t^2\log^2 t)$, for $c > 0$. The remaining terms are of order $c\log^2 t_0/(t^2\log^4 t)$, so we get

$$\mathrm{Ri}(T,T) \geq \frac{c\log t_0}{t^2\log^2 t}.$$

Since $|\mathrm{Ri}(T,X)| \leq c\mathrm{Ri}(T,T) \ll \mathrm{Ri}(X,X)$, we conclude that Ricci curvature is positive, if t_0 was chosen large enough.

To complete the construction it remains to choose $\delta > 0$ so small, and correspondingly t_∞ so large, that normal curvatures of $S^n \times \{t_\infty\}$ are $> (r/\rho)^\varepsilon t_\infty^{-1}$; this is possible since they are estimated by $1/t + O(1/t\log t)$.

3. Construction of the Ambient Space

The metric of the double suspension of a sphere of radius R_0 near singular circle can be written as

$$ds^2 = dt^2 + \cos^2 t\, dx^2 + R^2(t)\, d\sigma^2,$$

where $t \geq 0$ is the distance from the singular circle, $R(t) = R_0 \sin t$. We smooth this metric by modifying $R(t)$ in such a way that $R(0) = 0$, $R'(0) = 1$, and $R''(0) = 0$. The curvatures of this metric are easily computed to be $K(T \wedge X) = 1$, $K(T \wedge \Sigma) = -R''/R$, $K(X \wedge \Sigma) = (R'/R)\operatorname{tg} t$, $K(\Sigma_1 \wedge \Sigma_2) = (1 - R'^2)/R^2$, and all mixed curvatures in this basis vanish. We will choose $R(t)$ in such a way that $1 > R' > 0$ and $R'' < 0$ when $t > 0$, thus making all sectional curvatures positive.

Consider a metric ball of small radius r_0 centered on the axis $t = 0$. We assume that $-R'' \geq R$ for $0 \leq t \leq r_0$, so that $K(T \wedge \Sigma) \geq 1$. In this case the normal curvatures of the boundary of this ball do not exceed $\operatorname{ctg} r_0$. We'll show that the function $R(t)$ can be chosen so that the intrinsic curvatures of the boundary are strictly bigger than $\operatorname{ctg}^2 r_0$. It would follow that for sufficiently small $R_0 > 0$, we can construct a building block that, after rescaling by $\operatorname{tg} r_0$, can be glued (using the results of Section 4) in our ambient space instead of the metric ball.

At a point of the boundary of our metric ball, which is at a distance t from the axis, we can compute the intrinsic curvatures as follows:

$$\begin{aligned} K_i(\Sigma_1 \wedge \Sigma_2) &= (1/R^2)(1 - R'^2 \sin^2 \phi), \\ K_i(Y \wedge \Sigma) &= -\frac{R''}{R} \sin^2 \phi + \frac{R'}{R} \operatorname{tg} t \cos^2 \phi + \operatorname{ctg} r_0 \frac{R'}{R} \cos \phi, \end{aligned}$$

where $Y \in X \wedge T$ is a tangent vector to the boundary, ϕ is the angle between T and the normal vector, $\cos \phi = \operatorname{ctg} r_0 \operatorname{tg} t$.

Since $K(T \wedge \Sigma) \geq 1$, it follows from the comparison theorem that

$$K_i(\Sigma_1 \wedge \Sigma_2) \geq 1/\sin^2 t - \operatorname{ctg}^2 t(1 - \cos^2 \phi) = 1 + \operatorname{ctg}^2 r_0 > \operatorname{ctg}^2 r_0.$$

To make sure that $K_i(Y \wedge \Sigma) > \operatorname{ctg}^2 r_0$, we have to choose $R(t)$ more carefully. Namely, let $R(t) = R_0 \sin\bigl(t + \delta\gamma(t/r_0 - 1)\bigr)$ for $t \geq \varepsilon$, where γ is a standard smooth function interpolating between $\gamma(x) = 1$ for $x \leq 0$ and $\gamma(x) = 0$ for $x \geq 1$. Extend $R(t)$ to the segment $[0, \varepsilon]$ in such a way that $R(0) = 0$, $R'(0) = 1$, and $-R''/R \geq 2r_0^{-2}$ on $[0, \varepsilon]$. (This is possible with an appropriate choice of ε and δ—for example, if $\varepsilon = \frac{1}{2}r_0^2$ and $\delta = \frac{1}{4}r_0^4$.) Now it is clear that already the first term in the expression for $K_i(Y \wedge \Sigma)$ is bigger than $\operatorname{ctg}^2 r_0$ if $0 \leq t \leq \varepsilon$. For $\varepsilon \leq t \leq r_0$ we have

$$\begin{aligned} K_i(Y \wedge \Sigma) &= \sin^2 \phi + \operatorname{ctg}(t + \delta) \operatorname{tg} t \cos^2 \phi + \operatorname{ctg}^2 r_0 \operatorname{ctg}(t + \delta) \operatorname{tg} t \\ &\geq \operatorname{ctg}(t + \delta) \operatorname{tg} t\, (\operatorname{ctg}^2 r_0 + 1). \end{aligned}$$

So we need to check that $\operatorname{tg}(t+\delta)/\operatorname{tg} t < 1+\operatorname{tg}^2 r_0$. This is true for $t \geq \varepsilon = \frac{1}{2}r_0^2$ and $\delta = \frac{1}{4}r_0^4$ and for small r_0. Thus, both $K_i(\Sigma_1 \wedge \Sigma_2)$ and $K_i(X \wedge \Sigma)$ are bigger than $\operatorname{ctg}^2 r_0$, and the construction of the ambient space is complete.

4. Gluing and Smoothing

To justify gluing and smoothing in our construction we need the following fact.

Let M_1, M_2 be compact smooth manifolds of positive Ricci curvature, with isometric boundaries $\partial M_1 \simeq \partial M_2 = X$. Suppose that the normal curvatures of ∂M_1 are bigger than the negatives of the corresponding normal curvatures of ∂M_2. Then the result $M_1 \cup_X M_2$ of gluing M_1 and M_2 can be smoothed near X to produce a manifold of positive Ricci curvature.

To prove this, express the metric of $M_1 \cup_X M_2$ in normal coordinates with respect to X; let t be the normal coordinate. Introduce a small parameter $\tau > 0$, and for arbitrary coordinate vectors X_1, X_2 tangent to X replace our given function $\langle X_1, X_2\rangle(x,t)$ by its interpolation on the segment $[-\tau, \tau]$. At first we can take a C^1 interpolation given by a cubic polynomial in t, whose coefficients are linear functions of $\langle X_1, X_2\rangle(x, \pm\tau)$ and $\langle X_1, X_2\rangle'_t(x, \pm\tau)$. Clearly this procedure is independent of the choice of coordinates in X, and it gives a C^1 metric, which is C^2 outside two hypersurfaces $X_\tau, X_{-\tau}$, corresponding to $t = \pm\tau$. It is easy to see that in the segment $[-\tau, \tau]$ we get all $K(T \wedge X)$ positive of order $c\tau^{-1}$ and all $K(X_1 \wedge X_2)$ and $\mathrm{Ri}(T, X)$ bounded, so the Ricci curvature is positive. Now we can use a similar procedure to smooth our manifold near hypersurfaces $t = \pm\tau$. This time we choose another $\tau' \ll \tau$ and construct a C^2-interpolation. It is clear that only the components $R(T, \cdot, \cdot, T)$ of the curvature tensor were discontinuous on, say, X_τ, and, up to an error of order τ', these components now interpolate linearly between their original values on the different sides of X_τ. Since positivity of Ricci curvature is open and convex condition, the smoothed manifold has positive Ricci curvatures.

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