Comparison Geometry
MSRI Publications
Volume **30**, 1997

Applications of Quasigeodesics and Gradient Curves

ANTON PETRUNIN

Abstract. This paper gathers together some applications of quasigeodesic and gradient curves. After a discussion of extremal subsets, we give a proof of the Gluing Theorem for multidimensional Alexandrov spaces, and a proof of the Radius Sphere Theorem.

This paper can be considered as a continuation of [Perelman and Petrunin 1994]. It gathers together some applications of quasigeodesic and gradient curves. The first section considers extremal subsets; in the second section we prove the Gluing Theorem for multidimensional Alexandrov spaces; in the third we give another proof of the Radius Sphere Theorem. Our terminology and notation are those of [Perelman and Petrunin 1994] and [Burago et al. 1992]. We usually formulate the results for general Alexandrov space, but for simplicity give proofs only for nonnegative curvature.

Notation. We denote by M a complete n-dimensional Alexandrov space of curvature $\geq k$. As in [Burago et al. 1992], we denote by p'_q the direction at q of a shortest path to p. If H is a subset of M and $p, q \in H$, we denote by $|pq|_H$ the distance between p and q in the intrinsic metric of H. Finally, if X is a metric space with metric ρ, we denote by X/c denote the space X with metric ρ/c; where no confusion will arise, we may use the same notation for points in X and their images in X/c.

1. Intrinsic Metric of Extremal Subsets

The notion of an extremal subset was introduced in [Perelman and Petrunin 1993, 1.1], and has turned out to be very important for the geometry of Alexandrov spaces. It gives a natural stratification of an Alexandrov space into open topological manifolds. Also, as is shown in recent results of G Perelman, extremal subsets in some sense account for the singular behavior of collapse. Therefore

This material is part of the author's Ph.D. thesis [Petrunin 1995].

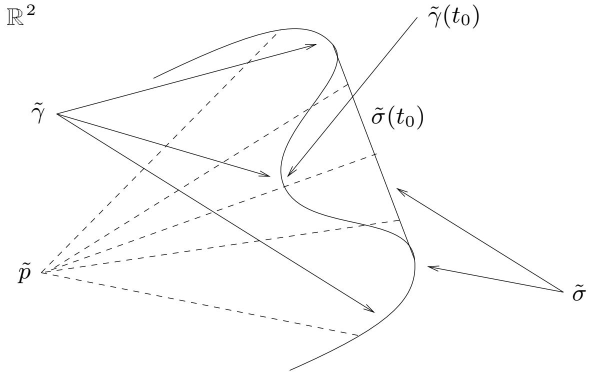
the intrinsic metric of such subsets turns out to be important. Moreover, there is hope that extremal subsets with intrinsic metric will give a way to approach the idea of multidimensional generalized spaces with bounded integral curvature.

In this section we give a new proof of the generalized Lieberman lemma, prove a kind of "stability" property for extremal subsets, and prove the first variation formula for the intrinsic metric of extremal subsets. The Lieberman lemma can be understood as a totally quasigeodesic property of extremal subsets and therefore offers some hope that extremal subsets with the intrinsic metric might be Alexandrov spaces with the same curvature bound; at the end of this section we give a counterexample to this conjecture for extremal subsets with codimension at least 3. This question is still open for codimension one (i.e., for a boundary) and for codimension two.

Theorem 1.1 (Generalized Lieberman lemma). *Any shortest path in the intrinsic metric of an extremal subset $F \subset M$ is a quasigeodesic in M.*

The first proof of this theorem was given in [Perelman and Petrunin 1993, 5.3].

Proof. Assume γ is a shortest path in the length metric of some extremal subset F. Suppose γ is not a quasigeodesic. Then there is a point p such that the development $\tilde\gamma(t)$ from p is not convex in any neighborhood of some t_0. Now for any $\varepsilon > 0$ it is easy to find a "rounded" curve $\tilde\delta(t)$ such that $\tilde\delta(t) = \tilde\gamma(t)$ if $|t - t_0| > \varepsilon$, $\operatorname{length}(\tilde\delta) < \operatorname{length}(\tilde\gamma) = \operatorname{length}(\gamma)$, and for any t the points $\tilde p$, $\tilde\gamma(t)$, and $\tilde\delta(t)$ are collinear in this order.



Now consider the curve in M given by

$$\delta(t) = \alpha_{\gamma(t)} \circ \rho_t^{-1}(|\tilde p\, \tilde\delta(t)|),$$

where $\alpha_{\gamma(t)} : [0, \infty) \to M$ is the dist_p-gradient curve that goes through $\gamma(t)$ such that $\alpha_{\gamma(t)}|_{[0,|p\gamma(t)|]}$ is a shortest path, and where ρ_t is its reparametrization, as in [Perelman and Petrunin 1994, 3.3(1)].

By [Perelman and Petrunin 1994, Theorem 6.3(a)], which states that if such a gradient curve starts at a point of an extremal subset F then it is contained in F, we obtain $\delta \subset F$. From [Perelman and Petrunin 1994, 3.3.3] (expansion along gradient curves is not more than in the model space)

$$\text{length}(\delta) \leq \text{length}(\tilde{\delta}) < \text{length}(\tilde{\gamma}) = \text{length}(\gamma).$$

Therefore γ is not a shortest path in F. $\square$

THEOREM 1.2. *Let M_n converge to M in the Gromov–Hausdorff topology without collapse* (*that is*, $\dim M_n = \dim M$), *and let $F_n \subset M_n$ be extremal subsets. Assume $F_n \to F \subset M$ as subsets. Then $F_n \xrightarrow{GH} F$ as length metric spaces with intrinsic metrics induced from M_n and M.*

PROOF. Let x and y lie in an extremal subset G. By the equivalence of the intrinsic metric of an extremal subset and the metric of the ambient space [Perelman and Petrunin 1993, 3.2(2)], we have for any open subset U in M an $\varepsilon = \varepsilon(\mathrm{Vol}_n(U), \mathrm{Diam}(U)) > 0$ such that $|xy|_G \leq \varepsilon^{-1}|xy|$ if $x, y \in U$. (The dependence on $\mathrm{Vol}_n(U)$ and $\mathrm{Diam}(U)$ can be easily obtained from the proof).

Consider $p, q \in F$ and $p_n, q_n \in F_n$ such that $p_n \to p$ and $q_n \to q$. It is easy to see that $|pq|_F \leq \liminf_{n\to\infty} |p_n q_n|_{F_n}$. Therefore we need to show only that $|pq|_F \geq \limsup_{n\to\infty} |p_n q_n|_{F_n}$. Set $\|pq\| = \limsup_{n\to\infty} |p_n q_n|_{F_n}$; this is easily seen to be a metric. From the previous paragraph, $\|pq\|$ does not depend on the choice of sequences $\{p_n\}$ and $\{q_n\}$, and we have $\|pq\| < \varepsilon^{-1}|pq|$, because from above ε can be found uniformly for all M_n in the absence of collapse.

Let $\gamma : [a, b] \to F$ be a shortest path in F between p and q parametrized by arclength. Assume $|pq|_F < \|pq\|$. Then, from [Busemann 1958, 5.14], for some $t_0 \in [a, b]$ and $\varepsilon > 0$ there is a sequence $t_i \to t_0\pm$ such that

$$\|\gamma(t_0)\gamma(t_i)\| \geq (1 + \varepsilon)|t_i - t_0|.$$

Setting $r = \gamma(t_0)$ and $s = \gamma(t_i)$, take sequences $r_n, s_n \in F_n$ such that $r_n \to r$ and $s_n \to s$. Let γ_i in F be the limit curve to the shortest paths between r_n and s_n in F_n. By [Perelman and Petrunin 1994, 2.3(3)] and the generalized Lieberman lemma, γ_i is a quasigeodesic between $\gamma(t_0)$ and $\gamma(t_i)$. From above, $\text{length}(\gamma_i) \geq (1 + \varepsilon)|t_i - t_0|$. Now consider the limit $(M/|t_0 - t_i|, r) \to (C_r, 0)$. Consider the curve in C_r given by

$$\gamma_*(t) = \lim_{i\to\infty} \left(\frac{\gamma_i}{|t_0 - t_i|}\right)(t\,|t_0 - t_i|) \in \frac{M}{|t_0 - t_i|},$$

where $(\gamma_i/|t_0 - t_i|)$ denotes the image of γ_i in $M/|t_0 - t_i|$. Then γ_* is a quasigeodesic between 0 and the tangent vector $\gamma^{\pm}(t_0)$ which has length not less then $1+\varepsilon$. This is a contradiction since $|\gamma^{\pm}(t_0)| = 1$ by [Perelman and Petrunin 1994, 2.3(2)]. $\square$

REMARK 1.3. The author does not know a counterexample for the following conjecture: *Let $M_n \xrightarrow{GH} M$, with $\dim M_n \leq C < \infty$, and let $F_n \subset M_n$ be extremal subsets. Assume that $F_n \to F \subset M$ as subsets and that $F_n \xrightarrow{GH} \bar{F}$. Then there is a discrete group of isometries G on $\bar{F}$ such that $F = \bar{F}/G$.*

As an example, consider the collapse of spaces with boundary $M_i \xrightarrow{GH} M$ *such that* $\dim M = \dim M_i - 1$. *Then* $\partial M_i \to M$ *as subsets and* $\partial M_i \xrightarrow{GH} \tilde{M}$, *where* $\tilde{M}$ *is the double of* M.

Now let M be an Alexandrov space and $F \subset M$ be an extremal subset. By the generalized Lieberman lemma, every shortest path in the length metric of F is a quasigeodesic as a curve in M, and every quasigeodesic at every point has directions of exit and entrance [Perelman and Petrunin 1994, 2.1(b) and 2.3(2)]. Thus if p and q lie in F we can define $q^\circ (= q_p^\circ)$ as the set of all directions of entrance in $\Sigma_p(F)$ of shortest paths between p and q in the length metric of F. It is easy to see that q° is compact.

Theorem 1.4 (The first variation formula). *Let F be an extremal subset of the Alexandrov space M. Let $p, q \in F$, and let $\xi(t)$ be a curve in F starting from p in direction $\xi_0' \in \Sigma_p(F)$. Assume that $|p\,\xi(t)| = t + o(t)$. Then*

$$|\xi(t)\,q|_F = |pq|_F - \cos|\xi_0' q^\circ|_{\Sigma_p(F)}\,t + o(t).$$

Proof. To prove this we have to prove two inequalities:

$$|\xi(t)q|_F \le |pq|_F - \cos|\xi_0' q^\circ|_{\Sigma_p(F)}\,t + o(t), \tag{1.1}$$
$$|\xi(t)q|_F \ge |pq|_F - \cos|\xi_0' q^\circ|_{\Sigma_p(F)}\,t + o(t). \tag{1.2}$$

Proof of (1.1). Take some $R \gg 1$. Set $\alpha = |\xi_0' q^\circ|_{\Sigma_p(F)}$ and $|pq|_F = l$. Take $\eta \in q^\circ$ such that $\alpha = |\xi_0' q^\circ|_{\Sigma_p(F)} = |\xi_0' \eta|$ and let $\gamma : [0, l] \to F$ be a shortest path between p and q in F such that $\gamma(0) = p$ and $\gamma^+(0) = \eta$. Then, by the triangle inequality,

$$|\xi(t)\,q|_F \le l - Rt + |\xi(t)\,\gamma(Rt)|_F.$$

The cosine rule gives us

$$|\xi_0'\,R\eta|_{C_p(F)} = \sqrt{R^2 + 1 - 2R\cos\alpha}.$$

Now, using Theorem 1.2, for the limit $(M/t, p) \to C_p$, we obtain

$$\lim_{t\to 0} |\xi(t)\gamma(Rt)|_F / t = |\xi_0'\,R\eta|_{C_p(F)}.$$

Therefore

$$\begin{aligned} |\xi(t)\,q|_F &\le l - Rt + t\sqrt{R^2 + 1 - 2R\cos\alpha} + o(t) \\ &\le l - \cos\alpha\, t + \frac{t}{R-1} + o(t). \end{aligned}$$

When $R \to \infty$ we obtain

$$|\xi(t)q|_F \le |pq|_F - \cos|\xi_0' q^\circ|_{\Sigma_p(F)}\,t + o(t). \qquad \square$$

LEMMA 1.5. *Let $C = C(\Sigma)$ be a cone with curvature ≥ 0 (so the curvature of Σ is ≥ 1). Let γ be a quasigeodesic in C not passing through the vertex o. Then the projection of γ on Σ parametrized by the arclength is a quasigeodesic in Σ and the development of γ in the plane with respect to the vertex of C is a straight line.*

PROOF. To prove the second part of this lemma we have to prove that

$$(|\gamma(t)|^2)'' = 2.$$

In order to prove that $(|\gamma(t)|^2)'' \leq 2$, it is enough to consider the development of γ with respect to the vertex o of the cone. We prove that $(|\gamma(t)|^2)'' \geq 2$. Consider the Busemann function for $\theta \in \Sigma$, namely,

$$f_\theta = \lim_{\lambda \to \infty} (\operatorname{dist}_{\lambda\,\theta} - \lambda).$$

The condition of convexity of the development with respect to $\lambda\,\theta$ gives the concavity of the function $f_\theta \circ \gamma(t)$ for every quasigeodesic γ in C. Using this for $\theta = \gamma(t)/|\gamma(t)|$ we get the needed inequality.

Therefore if γ^* is the projection of γ on Σ, we can choose a unique arclength parameter x on γ^* such that

$$\operatorname{pr}\big(\gamma(c \tan x + d)\big) = \gamma^*(x)$$

for some constants $c > 0$ and d; without loss of generality we can set $d = 0$.

Now we have to prove that the development of γ^* in a standard sphere with respect to any $\theta \in \Sigma$ is convex, i.e., that $\cos(|\theta\gamma^*(x)|)'' + \cos(|\theta\gamma^*(x)|) \geq 0$. By [Perelman and Petrunin 1994, 1.7] it is enough to prove this only for $|\theta\gamma^*(x)| < \pi/2$. It is easy to see that

$$\cos(|\theta\gamma^*(x)|) = -\frac{f_\theta(\gamma(c \tan x))}{|\gamma(c \tan x)|}.$$

Then direct calculation gives what we need, because $f_\theta \circ \gamma$ is convex and because $|\gamma(c \tan x)| = c/\cos x$. $\square$

PROOF OF (1.2). Assume that (1.2) is false. Then one can find a sequence $\{t_i\}$, $t_i \to 0^+$, such that

$$|\xi(t_i)q|_F < |pq|_F - \cos|\xi_0' q^\circ|_{\Sigma_p(F)}\, t_i - \varepsilon\, t_i$$

for some fixed $\varepsilon > 0$.

Assume $|pq|_F = l$ and $|\xi(t_i)q|_F = l_i$. Let $\gamma_i : [0, l_i] \to F$ be the shortest paths between $\xi(t_i)$ and q in F such that $\gamma(0) = \xi(t_i)$. We can pass to a subsequence of $\{\gamma_i\}$ such that the shortest paths γ_i approach some shortest path $\gamma : [0, l] \to F$ between q and p. Let $\eta \in q^\circ$ be the direction of this shortest path γ. By Theorem 1.1, γ_i and γ are quasigeodesics.

Now consider the Gromov–Hausdorff limit $(M/t_i, p) \xrightarrow{GH} C_p$, and pass to a subsequence again, so that there exists $\hat\gamma : [0,\infty) \to C_p$ satisfying

$$\hat\gamma(t) = \lim_{i\to\infty} (\gamma_i/t_i)(tt_i) \in M/t_i,$$

where (γ_i/t_i) denotes the image of γ_i in M/t_i.

By [Perelman and Petrunin 1994, 2.3(3)], $\hat\gamma$ is a quasigeodesic in C_p, and it is easy to see that $\hat\gamma(0) \in \Sigma_p \subset C_p$.

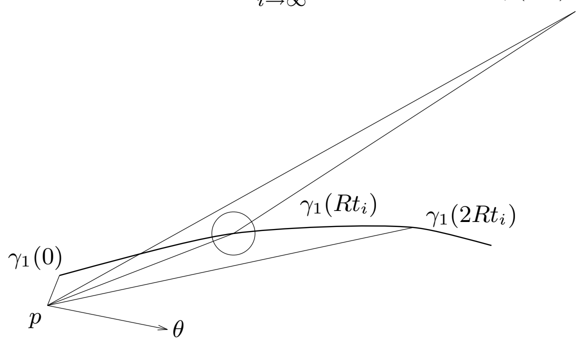
We define the direction at infinity of the curve $\hat\gamma$ in C_p by

$$\lim_{t\to\infty} \frac{\hat\gamma(t)}{|o\,\hat\gamma(t)|}.$$

By Lemma 1.5 this is well defined for quasigeodesics.

We claim that the direction at infinity of $\hat\gamma$ is η. Indeed, let θ be the direction of $\hat\gamma$ at infinity. By the cosine rule we obtain, for $R \gg 1$,

$$\begin{aligned}
|\hat\gamma(2R)|^2 &= \lim_{i\to\infty} (|p\,\gamma_i(2Rt_i)|/t_i)^2 \\
&\le \lim_{i\to\infty} \big(|p\,\gamma_i(Rt_i)|^2 + (Rt_i)^2 - 2Rt_i|p\,\gamma_i(Rt_i)| \cos\angle(\gamma_i^+(Rt_i), p'_{\gamma_i(Rt_i)})\big)/t_i^2 \\
&= |\hat\gamma(R)|^2 + R^2 - 2R\,|\hat\gamma(R)| \lim_{i\to\infty} \cos\angle(\gamma_i^+(Rt_i), p'_{\gamma_i(Rt_i)}).
\end{aligned}$$



Now, by Lemma 1.5, we have for some β

$$\begin{aligned}
\lim_{i\to\infty} \angle(\gamma_i^+(Rt_i), p'_{\gamma_i(Rt_i)}) &\ge \arccos \frac{|\hat\gamma(R)|^2 + R^2 - |\hat\gamma(2R)|^2}{2R\,|\hat\gamma(R)|} \\
&= \arccos \frac{\big((R^2+1-2R\cos\beta) + R^2 - (4R^2+1-4R\cos\beta)\big)}{2R\sqrt{R^2+1-2R\cos\beta}} \\
&= \arccos\left(-\sqrt{\frac{R^2-2R\cos\beta+\cos^2\beta}{R^2-2R\cos\beta+1}}\right) \ge \arccos\left(-\sqrt{1-\frac{1}{(R-1)^2}}\right) \\
&\ge \arccos\big(-1+1/(R-1)^2\big) > \pi(1-1/R).
\end{aligned}$$

Taking $r_k \to p$ such that $(r_k)'_p \to \theta$, we have

$$\lim_{i\to\infty} \tilde\angle p\,\gamma_i(Rt_i)\,r_k \ge \pi - \angle(\hat\gamma(R), (r_k)'_p) > \pi - \pi/R - \angle(\theta, (r_k)'_p).$$

The latter inequality is a corollary of Lemma 1.5, since $\sin\angle(\hat\gamma(R),\theta) \le 1/R$. Therefore, since the perimeter of any triangle in the space of directions is at most 2π, we get

$$\begin{aligned}\lim_{i\to\infty} \angle(\gamma_i^+(Rt_i),\,(r_k)'_{\gamma_i(Rt_i)}) &\le 2\pi - \lim_{i\to\infty} \tilde\angle p\,\gamma_i(Rt_i)\,r_k - \lim_{i\to\infty} \angle\big(\gamma_i^+(Rt_i), p'_{\gamma_i(Rt_i)}\big)\\ &\le \pi/R + \angle(\theta,(r_k)'_p) + \pi/R.\end{aligned}$$

Using [Perelman and Petrunin 1994, 1.4(G2)] for γ_i with respect to the points r_k and starting at $\gamma_i(Rt_i)$, we obtain the estimates

$$\begin{aligned}|r_k\gamma(|pr_k|)| &= \lim_{i\to\infty}\big|r_k\,\gamma_i\big(Rt_i+|\gamma_i(Rt_i)r_k|\big)\big|\\ &\le \lim_{i\to\infty}|\gamma_i(Rt_i)r_k|\,\lim_{i\to\infty}\angle(\gamma_i^+(Rt_i),\,(r_k)'_{\gamma_i(Rt_i)})\\ &\le |pr_k|\,(2\pi/R+\angle(\theta,(r_k)'_p)).\end{aligned}$$

This means that η is $2\pi/R$-close to θ. Sending R to infinity we obtain $\theta=\eta$.

Now fix $R \gg 1$ and divide γ_i into two pieces using a parameter value $x_i \in [0,l_i)$ such that $|p\gamma_i(x_i)| = Rt_i$. We estimate the length of each part separately.

By Theorem 1.2 the length of the first part $|q\gamma_i(x_i)|_F$ is possible to estimate from the triangle inequality:

$$|q\gamma_i(x_i)|_F \ge |pq|_F - |p\gamma_i(x_i)|_F = |pq|_F - Rt_i + o(t_i).$$

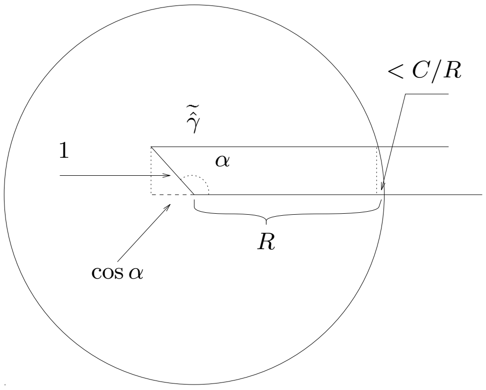
The length of the second part is estimated using the fact that the limit of lengths of quasigeodesics is the length of the limit quasigeodesic [Perelman and Petrunin 1994, 2.2, 2.3(3)]. Therefore

$$|\xi(t_i)\gamma_i(x_i)|_F/t_i \to \mathrm{length}(\hat\gamma\cap B_R(o)\subset C_p).$$

By Lemma 1.5 the last expression can be estimated from below as

$$R-\cos\angle(q^\circ,\xi_0')-C/R.$$

This estimate is easily deduced from the following diagram in the plane of the development $\tilde{\hat\gamma}$ of $\hat\gamma$ from o. Here α is the angle at $\tilde o$ subtended by $\tilde{\hat\gamma}$. Clearly α is not less than $\angle(q^\circ,\xi_0')$.



From these two estimates we obtain

$$|\xi(t_i)q|_F \geq |pq|_F - \cos|\xi_0' q^\circ|_{\Sigma_p(F)}\, t_i - C/R\, t_i + o(t_i),$$

which for $C/R < \varepsilon$ contradicts the assumption. This completes the proof of (1.2) and of the first variation formula. □

A counterexample. In [Perelman and Petrunin 1993, 6.1] we conjectured that the intrinsic metric of a primitive extremal subset has curvature bounded from below. Here we show a counterexample to this conjecture for $\operatorname{codim} F \geq 3$. Therefore this question is still open for $\operatorname{codim} F = 1$ (i.e., for a boundary) and for $\operatorname{codim} F = 2$. Sergei Buyalo [1976] has settled the first of these questions affirmatively for a “smooth” Alexandrov space, i.e., for a convex subset in a Riemannian manifold with curvature bounded from below.

Consider a right simplex $\operatorname{conv}\{a_1 a_2 a_3 a_4 a_5\}$ in a standard S^4 such that $|a_i a_j| = \pi/2$ for $i \neq j$. Assume $a_5 = a_0$, take some $\varepsilon > 0$, and consider the closed broken geodesic

$$F = a_0^+ a_1^- a_1^+ a_2^- a_2^+ a_3^- a_3^+ a_4^- a_4^+ a_5^- a_0^+,$$

where $a_i^\pm$ is the point on the geodesic $a_i a_{i\pm 1}$ such that $|a_i a_i^\pm| = \varepsilon$. Let $\Sigma = \operatorname{conv}\{F\}$. Then direct calculation shows that F is a primitive extremal subset of Σ and that, for ε sufficiently small, $\operatorname{length}(F) > 2\pi$. In particular, $C(F)$ is an extremal subset of $C(\Sigma)$, which has a singular point of negative curvature.

2. The Gluing Theorem

The Gluing Theorem for the two dimensional case is due to A. D. Alexandrov (see [Pogorelov 1973, § 11], for example). Later Perelman [1991, 5.2] proved the Doubling Theorem for multidimensional Alexandrov spaces; this is a special case of the theorem formulated below. The original Alexandrov's Theorem had a lot of applications to the bending of convex surfaces with boundary, which are currently impossible to generalize to the multidimensional case, because they are supported by the Theorem about convex embeddings [Pogorelov 1973, Sect6–7]. Formally, the following theorem gives new examples of Alexandrov spaces, but unfortunately we have not too many examples of Alexandrov spaces with isometric boundaries.

THEOREM 2.1. *Let M_1 and M_2 be Alexandrov spaces with nonempty boundary and curvature $\geq k$. Let there be an isometry* $\mathrm{is} : \partial M_1 \to \partial M_2$, *where ∂M_1 and ∂M_2 are considered as length-metric spaces with the induced metric from M_1 and M_2. Then the glued space $X = M_1 \cup_{\mathrm{is}(x)=x} M_2$ is an Alexandrov space with curvature $\geq k$.*

LEMMA 2.2. *Let $p \in \partial M$ and $\eta \in \partial\Sigma_p$. Then there exists a shortest path in ∂M starting at p in a direction arbitrarily close to η.*

Proof. Let $N = \partial M$. The boundary is an extremal subset and therefore we can use notation $q^\circ(= q_p^\circ)$ for the set of all directions of entrance in $\Sigma_p(N)$ of shortest paths between p and q in the length metric of N.

Choose a sequence of points $q_n \in N$ such that $q_n \to p$ and $\angle(q_n', \eta) \to 0$ (where $q_n' = (q_n)_p'$ is the direction at p of the shortest path pq). Assume that $\angle(\eta q_n^\circ) \geq \varepsilon$ for all n. Pass to a subsequence such that $\lim_{n\to\infty} \angle(\theta\, q_n^\circ) \to 0$ for some direction θ.

Find a point $r \in M$ such that $\angle(r', \theta) < \varepsilon/6$. Let $\{r_n\}$ be points on the shortest path pr such that $|pr_n| = |pq_n|$. Since the shortest path from p to q_n in N is a quasigeodesic (see Theorem 1.1), we conclude by using [Perelman and Petrunin 1994, 1.4(G2), 1.5] that $|r_n q_n| < (\varepsilon/5)|pq_n|$ for n sufficiently large, hence that $\lim_{n\to\infty} \angle(q_n', r') < \varepsilon/3$ for $\varepsilon \leq \pi/4$. Therefore

$$\lim_{n\to\infty} \angle(q_n', \theta) < \varepsilon/2.$$

We obtain a contradiction because $\lim_{n\to\infty} q_n' = \eta$ and $\angle(\eta, \theta) \geq \varepsilon$. □

The rest of this section will be devoted to the proof of Theorem 2.1. Let $N = M_1 \cap M_2 = \partial M_i \subset X$.

Definition 2.3. The *m-predistance* $|pq|_m$ between points p and q in X is the minimal length of broken geodesics with vertices $p = p_0, p_1, \ldots, p_{k+1} = q$, where $k \leq m$, $p_l p_{l\pm1}$ is a shortest path that lies completely in one of M_i for every $l \in \{1, 2, \ldots, k\}$, and p_l lies in N. A broken geodesic that realizes this minimum is called an *m-shortest path.*

Remark 2.4. It is easy to see that $|pq|_m \geq |pq|_{m+1} \geq |pq|$, $\lim_{m\to\infty} |pq|_m = |pq|$,

$$\begin{aligned} |pq|_m + |qr|_l &\geq |pr|_{m+l} && \text{if } q \in X \setminus N, \\ |pq|_m + |qr|_l &\geq |pr|_{m+l+1} && \text{if } q \in N\ . \end{aligned} \tag{2.1}$$

For every interior vertex $p = p_l$, $l \in \{1, 2, \ldots, k\}$, of an m-shortest path, we can define directions of exit and entrance ξ_i as directions in $\Sigma_p(M_i)$ of shortest paths in M_i.

By Theorem 1.2 the isometry $\mathrm{is} : \partial M_1 \to \partial M_2 = N$ gives an isometry $\mathrm{is}_p' : \partial\Sigma_p(M_1) \to \partial\Sigma_p(M_2) = \Sigma_p(N)$ and $\mathrm{is}_p : \partial C_p(M_1) \to \partial C_p(M_2) = C_p(N)$. Set

$$\begin{aligned} \Sigma_p^\#(X) &:= \Sigma_p(M_1) \cup_{\mathrm{is}_p'(x)=x} \Sigma_p(M_2), \\ C_p^\#(X) &:= C(\Sigma_p^\#(X)) = C_p(M_1) \cup_{\mathrm{is}_p(x)=x} C_p(M_2). \end{aligned}$$

From the induction hypothesis, $\Sigma_p^\#(X)$ will be an Alexandrov space with curvature ≥ 1, and therefore $C_p^\#(X)$ will be a cone with curvature ≥ 0.

Notation. If K_1 and K_2 are two compact metric spaces, we say that $K_1 \leq K_2$ if there is a noncontracting map $m : K_1 \to K_2$. If (L_1, p_1) and (L_2, p_2) are two locally compact metric spaces with base points, we say that $(L_1, p_1) \leq (L_2, p_2)$ if for any $R > 0$ there is a noncontracting map $m : B_R(p_1) \to B_R(p_2)$.

We will write $\limsup_{i\to\infty} K_i \le K$ if for any Hausdorff subsequence $K_{i_k} \xrightarrow{GH} K'$ we have $K' \le K$. Similarly one can write $\liminf_{i\to\infty} K_i \ge K$. We write $\limsup_{i\to\infty}(L_i, p_i) \le (L, p)$ if for any $R > 0$ we have $\limsup_{i\to\infty} B_R(p_i) \le B_R(p)$ (compare with [Burago et al. 1992, 7.13]).

PROOF. Proof of Theorem 2.1 As a base we can take the classical Gluing Theorem of A. D. Alexandrov in dimension 2 [Pogorelov 1973, § 11] Assume we have already proved Theorem 2.1 for dimensions less than n.

LEMMA 2.5. *For any point $p \in N$ we have* $\limsup_{\delta\to 0}(X/\delta, p) \le (C_p^{\#}(X), o)$.

It is easy to see that as a corollary of Theorem 2.1 we will actually have equality in this theorem, instead of inequality.

PROOF OF LEMMA 2.5. Consider the gradient-exponential maps [Perelman and Petrunin 1994, 3.5] $\mathrm{gexp}_1 : C_p(M_1) \to M_1$ and $\mathrm{gexp}_2 : C_p(M_2) \to M_2$. By [Perelman and Petrunin 1994, 6.4(a)], we have $\exp_i(C_p(N)) \subset N$. We construct an exponential map $\exp : C_p^{\#}(X) \to X$ by setting

$$\exp(v) = \begin{cases} \mathrm{gexp}_1(v) & \text{for } v \in C_p(M_1) \subset C_p^{\#}(X), \\ \mathrm{gexp}_2(v) & \text{for } v \notin C_p(M_1). \end{cases}$$

Define $\exp_\delta : C_p^{\#}(X) \to X/\delta$ by $\exp_\delta(v) = i_\delta \circ \exp \circ (v\delta)$, where $i_\delta : X \to X/\delta$ is the canonical mapping.

Let $x = x_0, x_1, \ldots, x_k, x_{k+1} = y$ be vertices of an m-shortest path in $C_p^{\#}(X)$. It is easy to see that $|x_l x_{l+1}| \ge |\exp_\delta(x_l)\exp_\delta(x_{l+1})| + o(\delta)/\delta$. Therefore for the m-predistance in $C_p^{\#}(X)$ we have $|xy|_m \ge |\exp_\delta(x)\exp_\delta(y)| + o(\delta)/\delta$. Now $|xy| = \lim_{m\to\infty} |xy|_m$ for any $x, y \in C_p^{\#}(X)$. Hence

$$\lim_{\delta\to 0} |\exp_\delta(x)\exp_\delta(y)| \le \lim_{m\to\infty} |xy|_m = |xy|.$$

Now in order to complete the proof we need to verify that

$$\lim_{\delta\to 0} \exp_\delta^{-1}(B_R(p) \subset X/\delta) \subset B_R(o) \subset C_p^{\#}(X).$$

for any $R > 0$, or, equivalently, that $\lim_{\delta\to 0} |p\exp_\delta(x)| \ge |x|$ for any $x \in C_p^{\#}(X)$.

Assume otherwise. Therefore we can find $x \in C_p^{\#}(X)$ and a sequence $\delta_n \to 0$ such that for some $\varepsilon > 0$ we have

$$|p\exp_{\delta_n}(x)| \le (1-\varepsilon)\,|x|.$$

Consider shortest paths $p\exp_{\delta_n}(x) \subset X/\delta_n$ for all n. No subsequence lies completely in M_i/δ_n for fixed i. Let $y_n \in N/\delta_n \subset X/\delta_n$ be the closest point of N/δ_n to $\exp_{\delta_n}(x)$ on $p\exp_{\delta_n}(x)$. Pass to a subsequence of $\{\delta_n\}$ such that $\exp_{\delta_n}^{-1}(y_n) \to x^*$. By [Perelman 1991, 4.7], $x^* \in C(\Sigma_p(N)) = C(\partial M_i)$ and

$$\lim_{\delta_n\to 0} |\exp_{\delta_n}(x)\exp_{\delta_n}(x^*)| = |xx^*|$$

(because a shortest path $\exp_{\delta_n}(x)y_n$ completely lies in one of the M_i and because $|y_n \exp_{\delta_n}(x^*)| = o(\delta_n)/\delta_n$). Therefore $|p\exp_{\delta_n}(x^*)| \le (1-\varepsilon)\,|x^*|$ for n sufficiently large, By Lemma 1.5, a limit of shortest paths in N/δ_n between p and $\exp_{\delta_n}(x^*)$ (which is a quasigeodesic by the generalized Lieberman lemma, Theorem 1.1) is a shortest path ox^* in $C_p(M_i)$. Because limits preserve lengths of quasigeodesics [Perelman and Petrunin 1994, 2.3(3)], we have

$$\lim_{n\to\infty} |p\exp_{\delta_n}(x^*)|_{N/\delta} = |x^*|.$$

Hence for n sufficiently large we get

$$|p\exp_{\delta_n}(x^*)| \le (1-\varepsilon)\,|p\exp_{\delta_n}(x^*)|_{N/\delta}.$$

Therefore we can find a segment $s_n r_n$ on a shortest path $p\exp_{\delta_n}(x^*)$ that completely lies in one of the M_i/δ_n, such that $s_n, r_n \in N/\delta_n$ and

$$|s_n r_n|_{M_i} \le (1-\varepsilon)(|pr_n|_N - |ps_n|_N) \tag{2.2}$$

(where we use the same notation for points in N and N/δ).

We can easily pass to a subsequence such that $\lim_{n\to\infty} |ps_n|_N/|pr_n|_N = c$. for some $0 \le c \le 1$.

Now we consider two cases, $c \ne 1$ and $c = 1$.

Suppose $c \ne 1$, and consider limit $(M_i/|pr_n|_N, p) \xrightarrow{GH} C_p(M_i)$. Pass to a subsequence such that $s_n \to s$ and $r_n \to r$. The boundary N is an extremal subset; therefore, by Theorem 1.2, $(N/|pr_n|_N, p) \xrightarrow{GH} C_p(N)$ as length-metric spaces. Hence

$$\lim_{n\to\infty} \frac{|s_n r_n|_{M_i}}{|pr_n|_N} = |sr| \ge |r| - |s| = |r|_{C(N)} - |s|_{C(N)} = 1 - \lim_{n\to\infty} \frac{|ps_n|_N}{|pr_n|_N},$$

contradicting (2.2).

Suppose instead that $c = 1$. Pass to a subsequence such that there exists a limit $(M_i/|s_n r_n|_{M_i}, s_n) \xrightarrow{GH} (M_s, s)$. (We remark that M_s need not be the tangent cone.) Set $N_s = \partial M_s$. By Theorem 1.2 we have

$$(N/|s_n r_n|_{M_i}, s_n) \xrightarrow{GH} (N_s, s).$$

Let $f_n : N/|s_n r_n|_{M_i} \to R$ be functions defined by

$$f_n(x) = |px|_{N/|s_n r_n|_{M_i}} - |ps_n|_{N/|s_n r_n|_{M_i}}.$$

Pass to a subsequence such that there exists a limit $f : N_s \to R$, $f = \lim_{n\to\infty} f_n$.

It is easy to see that M_s can be represented as a product $R \times M_s'$ such that $f(x) \le pr_R(x)$, where pr_R is the projection $M_s \to R$. Indeed a sequence of quasigeodesics that prolong shortest paths ps_n in N easily goes to a straight line in M_s, so by the Toponogov splitting theorem we have such a representation. Therefore N_s is split as well, $N_s = R \times N_s'$.

Let σ_n be a shortest path in N between p and s_n, parametrized by distance from s_n, and let σ be a limit of $\{\sigma_n/|r_n s_n|_{M_i}\}$. By the triangle inequality, for any

$T > 0$ we have $|xp|_N - |s_n p| \le |x\sigma_n(|s_n r_n|T)| - |s_n r_n|T$. As a limit we obtain that $f(x) \le |x\sigma(T)| - T$. For $T \to \infty$ the right side goes to the Busemann function of σ which coincides with pr_R.

Pass to a subsequence such that there is a limit as $r_n \to r$. We obtain

$$1 = |rs| \ge pr_R(r) \ge f(r) = \lim_{n\to\infty} (|pr_n|_N - |ps_n|_N)/|r_n s_n|_{M_i},$$

again contradicting (2.2). This concludes the proof of the lemma. $\square$

LEMMA 2.6. *The directions of exit and entrance* (ξ_i) *of any* m*-shortest path at every interior vertex* $p = p_l$, *for* $l \in \{1, 2, \dots, k \le m\}$ (see Definition 2.3), *are opposite in* $C^\#_p(X)$ (that is, $|\xi_1\xi_2| = 2|\xi_1| = 2|\xi_2|$; see [Perelman and Petrunin 1994, 2.1]).

PROOF. Let $\xi_i \in \Sigma_p(M_i)$ be directions of exit/entrance of the m-shortest path at the interior vertex p. We first prove that $|\xi_1\nu|_0 + |\xi_2\nu|_0 = \pi$ for any $\nu \in \Sigma_p(N) \subset \Sigma^\#_p(X)$. Here the left side is the sum of two 0-distances in the glued space $\Sigma^\#_p(X)$, each of which, by Definition 2.3, is measured in one of the $\Sigma_p(M_i)$. Assume we have proved the lemma for $\dim < n$, and let $\dim \Sigma^\#_p(X) = n$. From the first variation formula we obtain

$$f(\nu) := |\xi_1\nu|_0 + |\nu\xi_2|_0 \ge \pi$$

for any $\nu \in \Sigma_p(N)$. Assume $\bar\nu$ is the minimum point in $\Sigma_p(N)$ of the last function. Thus, $\xi_1\bar\nu\xi_2$ is a 1-shortest path. Let γ be a shortest path in $\Sigma_p(N)$ such that $\gamma(0) = \bar\nu$ with arbitrary initial data $\gamma^+(0) = \eta$. Assume $f(\bar\nu) > \pi$. By the induction assumption, $|(\xi_1)'_{\bar\nu}\eta|_0 + |\eta(\xi_2)'_{\bar\nu}|_0 = \pi$. By the generalized Lieberman lemma, Theorem 1.1, γ is a quasigeodesic as a curve in $\Sigma_p(M_1)$ and $\Sigma_p(M_2)$. By [Perelman and Petrunin 1994, 1.4(G1)], the condition $f(\bar\nu) > \pi$ implies $(f\circ\gamma)(x) < (f\circ\gamma)(0) = f(\bar\nu)$ for sufficiently small x. This contradicts the assumption that f has a minimum at $\bar\nu$.

Therefore $f(\bar\nu) = \pi$. Take any shortest path γ in $\Sigma_p(N)$ such that $\gamma(0) = \bar\nu$. Then γ is a quasigeodesic for $\Sigma_p(M_1)$ and $\Sigma_p(M_2)$. Set

$$g(\nu) := \cos|\xi_1\nu|_0 + \cos|\nu\xi_2|_0$$

for $\nu \in \Sigma_p(N)$. By the preceding arguments, $g(\bar\nu) = g\circ\gamma(0) = 0$, $(g\circ\gamma)'(0) = 0$ and $g\circ\gamma \le 0$. By [Perelman and Petrunin 1994, 1.3(L2)], $(g\circ\gamma)'' + g\circ\gamma \ge 0$. Therefore $(g\circ\gamma)'' \ge 0$ and so $g\circ\gamma \equiv 0$; in particular for any ν, $g(\nu) = 0$. Therefore $f \equiv \pi$, that is, $|\xi_1\nu|_0 + |\xi_2\nu|_0 = \pi$ as claimed.

In order to prove that ξ_1 and ξ_2 are opposite, it is enough to show that $2|\xi_1| = 2|\xi_2| = |\xi_1\xi_2|$ holds in $C^\#_p(X)$, or equivalently that $|\xi_1\xi_2| = \pi$ holds in $\Sigma^\#_p(X)$. If this is false, there is m such that $|\xi_1\xi_2|_m < \pi$ in $\Sigma^\#_p(X)$. Let θ be the closest vertex to ξ_1 of the m-shortest path $\xi_1\xi_2$. By the preceding discussion, there is a 1-shortest path through θ of length π. Therefore we have two distinct directions at θ which are opposite to $(\xi_1)'_\theta$, a contradiction to the fact that $\Sigma^\#_p$ is an Alexandrov space. This completes the proof of the lemma. $\square$

COROLLARY 2.7. *Let $\xi_i \in \Sigma_p(M_i)$ be directions of exit/entrance of an m-shortest path at an interior vertex. For any $\eta \in \Sigma_p(M_i)$ there is a unique $\eta^* \in \Sigma_p(N)$ such that*

$$|\xi_1\eta|_0 + |\eta\eta^*|_0 + |\eta^*\xi_2|_0 = \pi$$

or

$$|\xi_1\eta^*|_0 + |\eta^*\eta|_0 + |\eta\xi_2|_0 = \pi.$$

PROOF. Suppose $\eta \in \Sigma_p(M_1)$. Consider the 1-shortest path $\eta\xi_2$. Applying Lemma 2.6 to $\Sigma_p^{\#}(X)$ we see that the directions at the vertex are opposite; therefore this 1-shortest path is a part of a 1-shortest path $\xi_1\xi_2$. □

LEMMA 2.8. *Let $\gamma : [a,b] \to X$ be a quasigeodesic in one of the $\operatorname{int} M_i$ or a shortest path in the length metric of N. Then*

$$\rho_k(|p\gamma(t)|_m)'' + k\rho_k(|p\gamma(t)|_m) \le 1$$

for any $p \in X$

For the definition of ρ_k see [Perelman and Petrunin 1994, 1.4(L2)].

PROOF. We consider the case $k = 0$; we must show that $(|p\gamma(t)|_m^2)'' \le 2$.

This is true for $m = 0$ because

$$|pq|_0 = \begin{cases} |pq|_{M_i} & \text{if } p \in M_i, q \in \operatorname{int} M_i \text{ or } q \in M_i, p \in \operatorname{int} M_i, \\ \min_i |pq|_{M_i} & \text{if } p, q \in N, \\ \infty & \text{otherwise.} \end{cases}$$

(Recall that a shortest path in N is a quasigeodesic in both M_i by the generalized Lieberman Lemma).

Suppose the claim is true for all $l < m$ and false for m. Then the standard idea shows that in this case there exists $t_0 \in (a,b)$ and $\varepsilon > 0$ such that for $|t - t_0| < \varepsilon$

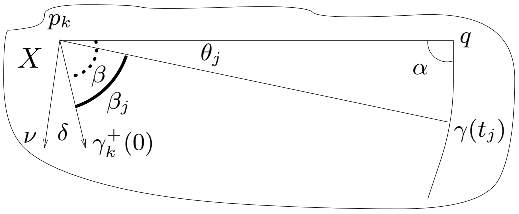
$$|p\gamma(t)|_m^2 \ge |p\gamma(t_0)|_m^2 - A(t - t_0) + (t - t_0)^2 + \varepsilon(t - t_0)^2,$$

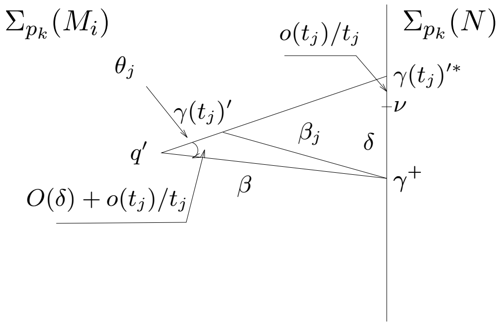
for some constant A.

Assume $t_0 = 0$. Set $q = \gamma(0)$ and let $p = p_0p_1 \dots p_kp_{k+1} = q$ be an m-shortest path. Take a sequence $t_j \to 0$ such that the sequence $((\gamma(t_j)'_{p_k})^*$ (as in Corollary 2.5) goes to some direction $\nu \in \Sigma_{p_k}(N)$. Using Lemma 2.2 we can find a shortest path γ_k in N which goes from p_k in a direction arbitrarily close to ν.

In the following proof one might get lost in calculations and lose the main idea. If we assume that all $((\gamma(t_j)'_{p_k})^*$ coincide with ν and there is a shortest path (in the intrinsic metric of N) that goes in this direction, one can ignore the residue terms below.

Set $\alpha = \angle((p_k)'_q, \gamma^+(0))$, $\beta = \angle(q'_{p_k}, \gamma_k^+(0))$, $\beta_j = \angle(\gamma_k^+(0)\,(\gamma(t_j))'_{p_k})$, $\theta_j = \angle((\gamma(t_j))'_{p_k}\, q'_{p_k})$, and $\delta = \angle(\gamma_k^+, \nu)$, as in the figure below.





It is easy to see that

$$\theta_j \geq \frac{t_j \sin\alpha}{|p_k q|_0} + o(t_j).$$

We can assume that $q'_{p_k} \notin \Sigma_{p_k}(N)$; otherwise our m-shortest path lies completely in N. By the cosine rule applied to the triangle $\triangle q'_{p_k}(\gamma(t_j))'_{p_k}\gamma_k^+(0)$, we have

$$\beta - \beta_j \geq \big(1 + o(\delta) + o(t_j)/t_j\big)\theta_j \geq t_j\left(\frac{\sin\alpha}{|p_k q|_0} + o(\delta)\right) + o(t_j).$$

Hence

$$\cos(\beta - \beta_j) \leq 1 - \frac{t_j^2 \sin^2\alpha}{2\,|pq|_0^2} + o(\delta)t_j^2 + o(t_j^2).$$

From the induction assumption and Lemma 2.6 we have

$$|p\,\gamma_k(\tau)|^2_{m-1} \leq |p\,p_k|^2_{m-1} + 2\tau|p\,p_k|_{m-1}\cos\beta + \tau^2.$$

Because γ_k is a quasigeodesic for both of the M_i, we obtain

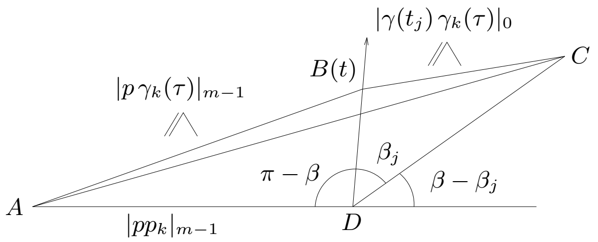
$$|\gamma(t_j)\gamma_k(\tau)|_0^2 \leq |\gamma(t_j)\,p_k|_0^2 - 2\cos\beta_j\,\tau\,|\gamma(t_j)\,p_k|_0 + \tau^2,$$

where these distances are measured in a fixed M_i.

Therefore, using (2.1) and the previous two inequalities, we have

$$\begin{aligned}
|p\,\gamma(t_j)|_m^2 &\leq \min_\tau(|p\,\gamma_k(\tau)|_{m-1} + |\gamma(t_j)\gamma_k(\tau)|_0)^2\\
&\leq \min_\tau(|AB(\tau)| + |B(\tau)C|)^2\\
&= |AC|^2 = |p\,p_k|^2_{m-1} + |\gamma(t_j)\,p_k|_0^2 + 2\,|p\,p_k|_{m-1}|\gamma(t_j)\,p_k|_0\cos(\beta - \beta_j),
\end{aligned}$$

where $A, B(\tau)$ and C are as shown in the following diagram in the plane:



Because γ is either a quasigeodesic in one of the M_i, or a shortest path in N and therefore a quasigeodesic in both of the M_i (see Theorem 1.1), we conclude that

$$|p_k\,\gamma(t_j)|_0^2 \le |p_k q|_0^2 + t_j^2 - 2t_j|p_k q|_0 \cos\alpha$$

and so

$$|p_k\,\gamma(t_j)|_0 \le |p_k q|_0 - t_j\cos\alpha + \frac{t_j^2 \sin^2\alpha}{2|p_k q|_0} + o(t_j^2).$$

Hence

$$\begin{aligned}
|p\,\gamma(t_j)|_m^2 &\le |p\,p_k|_{m-1}^2 + |q\,p_k|_0^2 + t_j^2 - 2t_j|q\,p_k|_0\cos\alpha \\
&\qquad + 2|p\,p_k|_{m-1}\left(|p_k q|_0 - t_j\cos\alpha + \frac{t_j^2\sin^2\alpha}{2|p_k q|_0} + o(t_j^2)\right) \\
&\qquad\qquad \times\left(1 - \frac{t_j^2\sin^2\alpha}{2|p_k q|_0^2} + t_j^2 o(\delta) + o(t_j^2)\right) \\
&\le (|pp_k|_{m-1}+|p_k q|_0)^2 - 2t_j(|pp_k|_{m-1}+|p_k q|_0)\cos\alpha + t_j^2 + t_j^2 o(\delta) + o(t_j^2) \\
&= |p\,q|_m^2 - 2t_j|pq|_m\cos\alpha + t_j^2 + t_j^2 o(\delta) + o(t_j^2).
\end{aligned}$$

This inequality for two sequences $t_j \to 0^+$ and $t_j \to 0^-$ contradicts our assumption for sufficiently small δ. □

We continue the proof of Theorem 2.1, showing that every m-shortest path is a k-quasigeodesic. Indeed, using [Perelman and Petrunin 1994, 1.4(L2), 1.5], we only need to verify that $\rho_k(|\gamma(t)p|)'' \le 1 - k\rho_k(|\gamma(t)p|)$. Now $|\gamma(t)p| = \lim_{n\to\infty}|\gamma(t)p|_n$, and using Lemma 2.8 and [Perelman and Petrunin 1994, 1.3(4)] we obtain the needed inequality for all $t \ne t_l$ (where $\gamma(t_l) = p_l$).

Let σ be a shortest path between an arbitrary point x and $\gamma(t_l)$, parametrized by distance from $\gamma(t_l)$. By Lemmas 2.5 and 2.6 we conclude that, for fixed ε,

$$|\sigma(T)\gamma(t_l + T\varepsilon)| + |\sigma(T)\gamma(t_l - T\varepsilon)| \le 2T + CT\varepsilon^2 + o(T).$$

Therefore

$$\mathrm{dist}_x \circ \gamma(t_l + T\varepsilon) + \mathrm{dist}_x \circ \gamma(t_l - T\varepsilon) \le 2\mathrm{dist}_x \circ \gamma(t_l) + CT\varepsilon^2 + o(T).$$

Therefore, for $T \to 0$,

$$(\mathrm{dist}_x \circ \gamma)^+(t_l) \le (\mathrm{dist}_p \circ \gamma)^-(t_l) + C\varepsilon.$$

Hence, for $\varepsilon \to 0$, we obtain $(\mathrm{dist}_x \circ \gamma)^+(t_l) \le (\mathrm{dist}_x \circ \gamma)^-(t_l)$. From this, using [Perelman and Petrunin 1994, 1.3(2)], we obtain the needed inequality for any t.

Let γ_m be an m-shortest path between $p, q \in X$. Then $\gamma = \lim_{m\to\infty} \gamma_m$ is a shortest path between p and q. It is easy to see that γ is convex (as a limit of convex curves) and parametrized by the arclength (because $\mathrm{length}(\gamma_m) \to \mathrm{length}(\gamma)$); hence γ is a quasigeodesic. Therefore by [Perelman and Petrunin 1994, 1.6] we obtain that X is an Alexandrov space of curvature $\ge k$. This completes the proof of the Gluing Theorem. ☐

3. The Radius Sphere Theorem

Theorem 3.2 below was proved independently by Karsten Grove and Peter Petersen [Grove and Petersen 1993]. Another proof follows immediately from [Perelman and Petrunin 1993, 1.2, 1.4.1]. The following proof is only a good demonstration of how beautiful quasigeodesics are.

PROPOSITION 3.1. *Let Σ be an Alexandrov space of curvature ≥ 1, with radius greater than $\pi/2$. Then for any $p \in \Sigma$ the space of directions Σ_p has a radius greater than $\pi/2$.*

PROOF. Assume that Σ_p has radius $\le \pi/2$, and let $\xi \in \Sigma_p$ be a direction such that $\mathrm{clos}\, B_\xi(\pi/2) = \Sigma_p$. Take a quasigeodesic of length $\pi/2$ starting at p in the direction ξ. Then the other endpoint q of this quasigeodesic satisfies $\mathrm{clos}\, B_q(\pi/2) = \Sigma$. (Indeed, for any point $r \in \Sigma$ we have $\angle rpq \le \pi/2$; therefore $|rq| \le \pi/2$ by the comparison inequality [Perelman and Petrunin 1994, 1.4(G2)]. This contradicts our assumption that Σ has radius $> \pi/2$. ☐

THEOREM 3.2. *Let Σ be an Alexandrov space of curvature ≥ 1, with radius $> \pi/2$. Then Σ is homeomorphic to the sphere S^n.*

PROOF. Assume we have proved the theorem for $\dim \Sigma < n$. We now prove it for $\dim \Sigma = n$.

Let xy be a diameter of Σ. Let z be a critical point of dist_x. Then $\tilde{\angle} xzy \le \angle xzy \le \pi/2$. By assumption $|xz|, |zy|, \pi/2 \le |xy|$. Therefore the last inequality can hold only for $z = y$. Therefore dist_x has no critical points but x and y. By [Perelman 1994], Σ is homeomorphic to $S(\Sigma_x)$. By Proposition 3.1 we have $\mathrm{Rad}\,(\Sigma_x) > \pi/2$. Hence by the induction assumption Σ_x is homeomorphic to S^{n-1}. Therefore Σ is homeomorphic to S^n. ☐

References

[Buyalo 1976] S. V. Buyalo, "Кратчайшие на выпуклых гиперповерхностях риманова пространства" [Shortest lines on convex hypersurfaces of a Riemannian space], *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov* **66** (1976), 114–131.

[Busemann 1958] H. Busemann, *Convex Surfaces*, Tracts in Pure and Appl. Math. **6**, Interscience, New York, 1958.

[Burago et al. 1992] Y. Burago, M. Gromov, and G. Perelman, "A. D. Aleksandrov spaces with curvatures bounded below", *Uspehi Mat. Nauk* **47**:2 (1992), 3–51, 222; translation in *Russian Math. Surveys* **47**:2 (1992), 1–58.

[Grove and Petersen 1993] K. Grove and P. Petersen, "A radius sphere theorem", *Invent. Math.* **112** (1993), 577–583.

[Liberman 1941] I. M. Liberman, "Geodesics on convex surfaces" (in Russian), Doklady Akademii Nauk **32**:5 (1941), 310–313.

[Perelman 1994] G. Perelman, "Elements of Morse theory on Alexandrov spaces" (Russian), *Algebra i analiz* **5**:1 (1993), 232–241; translation in *St. Petersburg Math. J.* **5**:1 (1994), 207–214.

[Perelman 1991] G. Ya. Perelman, "Alexandrov's spaces with curvatures bounded below, II", preprint, 1991.

[Perelman and Petrunin 1993] G. Ya. Perelman and A. M. Petrunin, "Extremal subsets in Aleksandrov spaces and the generalized Liberman theorem" (Russian), *Algebra i Analiz* **5**:1 (1993), 242–256; translation in *St. Petersburg Math. J.* **5**:1 (1994), 215–227.

[Perelman and Petrunin 1994] G. Ya. Perelman and A. M. Petrunin, "Quasigeodesics and gradient curves in Alexandrov spaces", preprint, University of California at Berkeley, 1994.

[Petrunin 1995] A. Petrunin, "Quasigeodesics in multidimensional Alexandrov spaces", Ph.D. thesis, Steklov Institute at St. Petersburg and University of Illinois at Urbana–Champaign, 1995.

[Pogorelov 1973] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, Transl. Math. Monographs **35**, American Mathematical Society, Providence, RI, 1973.

Anton Petrunin
St. Petersburg Branch
V. A. Steklov Institute of Mathematics (POMI)
Russian Academy of Sciences
Fontanka 27
St. Petersburg, 191011
Russia