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Tight Submanifolds, Smooth and Polyhedral

THOMAS F. BANCHOFF AND WOLFGANG KÜHNEL

ABSTRACT. We begin by defining and studying tightness and the two-piece property for smooth and polyhedral surfaces in three-dimensional space. These results are then generalized to surfaces with boundary and with singularities, and to surfaces in higher dimensions. Later sections deal with generalizations to the case of smooth and polyhedral submanifolds of higher dimension and codimension, in particular highly connected submanifolds. Twenty-six open questions and a number of conjectures are included.

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Introduction

The theory of tight submanifolds starts with attempts to generalize theorems about convex surfaces to topologically more complex surfaces such as the torus. For surfaces, it is possible to develop this generalization in terms of an elementary notion, the two-piece property, which then leads to the study of critical points of height functions and the theory of total absolute curvature. These notions can then be applied for higher-dimensional objects in higher-dimensional Euclidean spaces, producing a rich collection of examples and theorems in the global geometry of submanifolds.

An object in ordinary three-dimensional space is said to have the *two-piece property*, or TPP, if any plane cuts it into at most two pieces. Examples of surfaces with the TPP are spheres and ellipsoids and, more generally, the boundary of any bounded convex body. There are also nonconvex objects with boundaries that have the TPP: for example, a torus of revolution (Figure 1), or, more generally, a surface of revolution obtained by revolving a convex curve around an axis in the plane of the curve and not meeting the curve. If we deform a sphere into a nonconvex surface, for example a U-shaped object, or a sphere with a dent in it, the resulting surfaces (Figure 2) will not have the TPP.

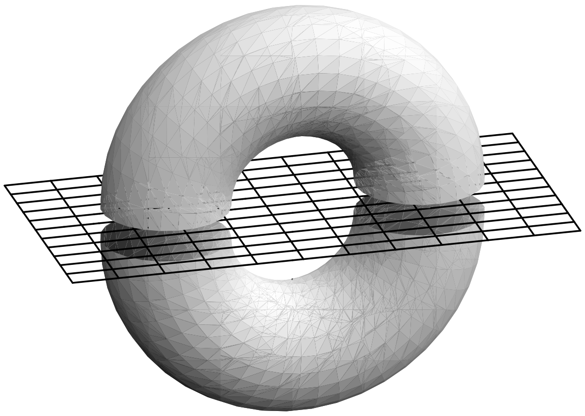

Figure 1. A torus of revolution has the two-piece property.

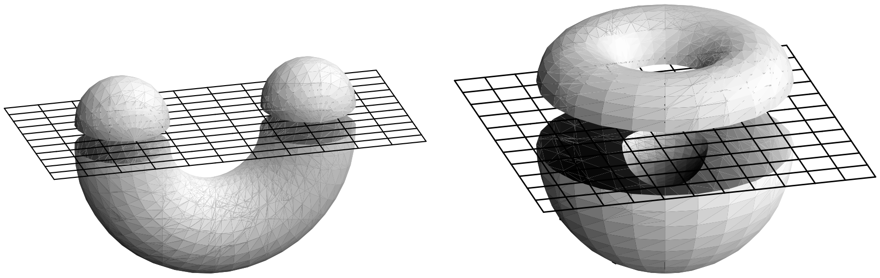

Figure 2. Smooth spheres without the two-piece property.

For closed subsets, the TPP is equivalent to the condition that the intersection of the object with every closed half-space is connected.

For compact surfaces (without boundary), the TPP is closely related to the study of *critical points of height functions*. Any plane in space can be considered a level set of a height function in a direction perpendicular to the plane. If a plane cuts a surface into more than two pieces, a height function perpendicular to this plane must have at least one maximum or minimum on each piece. It follows that if a surface does not have the TPP, there must be a height function with at least two (strict) local maxima on the surface. Conversely, if a height function has two strict local maxima on a surface, the half-space above the level set containing the lower of the two will intersect the surface in at least two pieces. It follows that a surface has the TPP if and only if no height function restricted to the surface has more than one strict local maximum. A surface with this property is called *tight*. An equivalent definition for a surface to be tight is that every local support plane be a global support plane.

The TPP is a topological condition, so it applies to any surface in space, whether it be smooth, polyhedral, or just a one-to-one continuous image of such a surface. If the surface happens to be sufficiently smooth, it is possible to characterize the tightness condition in terms of the surface's *total* or *Gaussian curvature*. Any point of *positive* curvature is a local extremum of the height function perpendicular to the tangent plane at the point. So the tightness condition implies that in any direction there is at most one point on the surface with positive curvature that is critical for that direction. For almost any direction, the maximum of the height function in that direction on the surface will occur at a point of positive curvature. It follows that, if a smooth surface is tight, the only strict local maxima of any height function must occur on the "outside", where the surface intersects its *convex hull*, the smallest convex set containing the surface.

For a smooth surface embedded in three-dimensional space, tightness can be expressed in terms of the *Gauss spherical image mapping*, which sends each point of the surface to the point of the unit sphere centered at the origin having the same outer unit normal vector. (This definition assumes that a consistent field of unit normals has been chosen over the whole surface.) For any smooth surface without boundary, almost every point of the sphere is the image of at least one point with positive curvature, so the total area of the spherical image of the part with positive curvature is at least 4π. For a tight smooth surface, almost every point of the sphere is the image of *exactly* one point of the surface with positive curvature, so the total area of the spherical image of the positive curvature part of the surface achieves the minimum value, namely 4π. Originally this property was used as the definition of tightness for smooth surfaces in ordinary space.

Although the first definition of tightness was given in terms of curvature, the critical point or TPP reformulation is much broader in scope. It applies not only to smooth surfaces in space but also to polyhedral surfaces. The critical point condition and the TPP extend naturally to surfaces embedded in higher-dimensional Euclidean spaces, and to immersions and to mappings with singularities.

It was Nicolaas Kuiper who made the first wide-ranging and systematic study of tight embeddings and immersions of surfaces, in three dimensions and higher. He produced tight embeddings of all orientable surfaces and tight immersions of all but three nonorientable surfaces in three-dimensional space. He proved that two of the remaining surfaces, the real projective plane and the Klein bottle, could not be immersed tightly in three-space even as topological surfaces, and he conjectured that the final case, a real projective plane with one handle, could not be immersed tightly into three-space. Only recently was François Haab able to prove that conjecture for smooth immersions, and even more recently Davide Cervone produced a surprising example to show that this surface can be immersed tightly as a polyhedral surface.

There are still a number of unsolved problems concerning immersions of surfaces into four-space, but thanks to the work of Kuiper the situation for five-space is better understood. First of all, Kuiper showed that any smooth immersion of a surface into Euclidean n-space for $n \geq 5$ must lie in a five-dimensional affine space; moreover, if the image does not lie in a four-dimensional subspace, the surface is the real projective plane and the immersion is affinely equivalent to the Veronese embedding, an algebraic surface. An even stronger result by Kuiper and William Pohl states that any topological tight embedding of the real projective plane into five-space whose image does not lie in a four-space must be either the smooth Veronese embedding or a simplexwise linear embedding of a triangulation with exactly six vertices.

In order to appreciate the nature of the theorems of Kuiper, it is useful to consider the TPP and tightness for closed curves in Euclidean spaces. A convex curve in the plane has the TPP, whether it is smooth or polygonal or a more general topological embedding of the circle. If an embedded curve in the plane is not convex, it does not coincide with the boundary of its convex hull, and there is a segment in the convex hull boundary containing points not in the curve. A line containing this segment bounds a half-space meeting the curve in at least two pieces, so a nonconvex plane curve does not have the TPP. Furthermore, if a curve is not planar, then there are four points on the curve, in cyclic order, not lying in a plane, and it is possible to find a plane with the first and third of these points on one side and the second and fourth on the other; this plane separates the curve into at least four pieces. Thus a TPP curve in n-space is necessarily contained in an affine two-space.

When the curve is smooth, we may recast the preceding result in terms of curvature to obtain a famous theorem in global differential geometry due to Werner Fenchel: the total curvature of a smooth closed curve in any dimension is at least 2π, and if it is exactly 2π, the curve is a convex plane curve.

For two-dimensional surfaces, the TPP restricts not only the number of maxima and minima of height functions but also the total number of critical points. This follows from the *critical point theorem* of elementary Morse theory: for almost every height function on a smooth surface, the only critical points are maxima, minima, and ordinary saddle points, and the number of maxima plus the number of minima minus the number of saddles is constant and equal to the *Euler characteristic* of the surface, also described as the number of vertices minus the number of edges plus the number of triangles in any triangulation of the surface. By "integrating" this theorem over all height functions, we obtain one of the most famous of all theorems in global differential geometry, the *Gauss–Bonnet Theorem*, relating the integral of the total curvature of a smooth surface to its Euler characteristic. We may use this fact to obtain other characterizations of tightness.

For higher-dimensional manifolds, the situation is quite different. The two-piece property no longer places such a strong restriction on the nature of the critical points of height functions. We say that an n-manifold is *tight* if the intersection of the object with any half-space is no more complicated than it has to be, that is, if the homology of the intersection is not greater than the homology of the whole object. For example, if a manifold is simply connected, we require that the intersection with every half-space also be simply connected. Thus a closed hemisphere has the TPP, but it is not tight since it is simply connected but there is a half-space that intersects it in a circle.

Morse theory gives lower bounds for the numbers of critical points of various types for almost all smooth functions defined on a higher-dimensional manifold. Tightness for higher-dimensional submanifolds of a Euclidean space requires that almost all height functions have the minimal number of critical points. Fenchel's theorem was generalized by Chern and Lashof by considering the *Lipschitz–Killing curvature* of a submanifold of a higher-dimensional Euclidean space. The total measure of the absolute value of this curvature is equal to the integral over the sphere of the number of critical points of height functions on the submanifold. Fenchel's theorem is generalized by the result that an m-dimensional sphere is immersed with minimum total absolute curvature if and only if it is a convex hypersurface in an affine subspace of dimension $m + 1$.

In this article, we will develop the theory of tight submanifolds primarily in the smooth and polyhedral situations. A related article based on the work of Nicolaas Kuiper will develop this theory for topological immersions.

1. Tight Surfaces

1.1. Definitions and notations. By a *surface*, we mean a connected two-dimensional manifold without boundary, unless stated otherwise (Section 1.5). We use the term *closed surface* to denote a compact surface without boundary.

For any closed surface M embedded in Euclidean three-space $\mathbb{E}^3$, and for any unit vector z, the height function in the direction of z achieves its absolute maximum on some subset of M. A point of M is a *local maximum* for z when a neighborhood of the point is contained in the half-space below the plane perpendicular to z through the point.

We say that the surface M is *tight* if, for almost every unit vector z, the height function in the direction of z has a unique local maximum on M. This condition is equivalent to the *Two-Piece Property*, or TPP, which states that any plane in space cuts the surface into at most two pieces. Equivalently, a surface M has the TPP if the intersection of M and any open or closed half-space is connected. Note that the property of tightness is invariant under projective transformations of three-space, since such transformations are homeomorphisms that send planes to planes.

If the closed surface satisfies some additional properties, we may find equivalent formulations of the tightness condition in terms of familiar quantities like curvature and critical points. We will be especially concerned with two important classes of surfaces: *smooth* and *polyhedral*.

For a smooth surface M in $\mathbb{E}^3$, at every point there is a well-defined tangent plane and a pair of unit normal vectors perpendicular to the tangent plane. When M is smooth and embedded, the surface is orientable and it is possible to make a choice of unit normal vector at each point over the entire surface. The *Gauss mapping* assigns each point of M to the point on the unit sphere having the same unit normal vector. For a region U, the *algebraic area* of the image of U under the Gauss mapping is given by the integral

$$\int_U K\, dA$$

of the *Gaussian curvature* function K, where the Gauss mapping preserves the orientation in a neighborhood of a point when the sign of K is positive at that point and it reverses orientation in a neighborhood of a point when the sign of K is negative at that point.

For almost all directions z on the unit sphere, only a finite number of points of M are sent to z under the Gauss mapping. At the one among those points that is highest in the direction for z, the orientation of the surface will be preserved. It follows from this observation that

$$\int_{M\cap\{K\geq 0\}} |K|\, dA \geq 4\pi.$$

Equality occurs when there is exactly one local maximum for almost every height function, so this condition gives an equivalent definition of tightness in the case of a smooth surface. This definition first appears in the work of A. D. Aleksandrov [1938].

One of the most fundamental results in elementary differential geometry is the *Gauss–Bonnet Theorem*, which states that the total algebraic area over an entire smooth closed surface M is independent of the embedding, and is given by $\int_M K\,dA = 2\pi\chi(M)$, where $\chi(M)$ denotes the *Euler characteristic* of the surface M, the number of vertices minus the number of edges plus the number of triangles in a triangulation of the surface.

It follows that

$$\begin{aligned}\int_M |K|\,dA &= \int_{M\cap\{K\geq 0\}} K\,dA - \int_{M\cap\{K\leq 0\}} K\,dA \\ &= 2\left(\int_{M\cap\{K\geq 0\}} K\,dA\right) - 2\pi\chi(M) \geq 2\pi\big(4-\chi(M)\big).\end{aligned}$$

The integral $\int_M |K|$ is called the *total absolute curvature* of M, and M is *tight* when this functional achieves its minimum value.

By a fundamental result in elementary Morse theory, for almost all unit vectors z, the height function in the direction of z, when restricted to the smooth closed surface M, is *nondegenerate*, with isolated singularities that are either local maxima, local minima, or ordinary saddle points. For any such function, the number of maxima plus the number of minima minus the number of (ordinary) saddles equals the Euler characteristic $\chi(M)$.

If M is tight, almost every height function will have exactly one maximum and one minimum, so the number of saddles will be $2-\chi(M)$ and the total number of critical points will be $4-\chi(M)$, the minimum number of nondegenerate critical points a function can have on a closed surface. The total absolute curvature of a surface M equals the integral over the unit sphere, in the sense of Lebesgue, of the number of critical points of height functions on M corresponding to unit vectors on the sphere.

We may summarize these comments about smooth surfaces as follows:

Definition 1.1.1 (smooth tight closed surfaces). A smooth (at least C^2) immersion of a closed surface $f : M \to \mathbb{E}^3$ is said to be *tight* if one of the following equivalent conditions is satisfied:

(i) $\frac{1}{2\pi}\int_M |K|\,dA = 4-\chi(M)$.
(ii) Every nondegenerate height function $zf : x \mapsto \langle fx, z\rangle$ in the direction of a unit vector $z \in \mathbb{R}^3$ has exactly one local minimum and one local maximum (and, consequently, $2-\chi(M)$ saddle points).
(iii) $\frac{1}{2\pi}\int_M |K|\,dA = \sum_i b_i(M;\mathbb{Z}_2)$. Here $b_i(M;\mathbb{Z}_2)$ denotes the *i-th Betti number* $b_i = \dim_{\mathbb{Z}_2} H_i(M;\mathbb{Z}_2)$.
(iv) f has the Two-Piece Property (TPP): For every plane $H \subset \mathbb{E}^3$, the complement $f^{-1}(M \setminus H)$ has at most two connected components.
(v) For every open half-space h bounded by a plane H, the induced morphism $H_*(f^{-1}(h)) \to H_*(M)$ is injective, where H_* denotes the singular homology with coefficients in $\mathbb{Z}_2$.

Note. The TPP condition is also called 0-*tightness* because in this case the homomorphism $H_0(f^{-1}(h)) \to H_0(M)$ is injective for any open half-space h.

The equivalence of (i) and (ii) and (iv) has been established above. The equivalence of (ii) and (iii) follows from elementary Morse theory, and that of (iv) and (v) follows from Poincaré duality. For orientable surfaces one can replace $\mathbb{Z}_2$ by any field or by $\mathbb{Z}$ in condition (v).

In the case of a smooth immersion of a nonorientable surface M into $\mathbb{E}^3$, we will not have a field of unit normal vectors, and even in the case of an immersion of an orientable surface, it may be that the Gauss mapping is not surjective. We can take care of each of these situations by considering not just one but both unit normal vectors at each point of the surface. We define a double spherical image mapping that assigns to each point the pair of unit vectors perpendicular to the tangent plane at the point, and we get the total curvature by taking half the total integral of this double spherical image mapping.

For an immersion $f : M \to \mathbb{E}^N$ of the closed surface M for $N > 3$, the equivalences in Definition 1.1.1 still hold if we replace the word "plane" by "hyperplane" and if we no longer use the Gaussian curvature of the surface itself but rather the Lipschitz–Killing curvature of the unit normal bundle of the surface; that is, if we replace the total absolute curvature $\frac{1}{2\pi}\int_M |K|\,dA$ by

$$\mathrm{TA}(f) := \frac{1}{c_{N-1}} \int_{\perp_f} |K|\,dV,$$

where K denotes the Lipschitz–Killing curvature (the determinant of the shape operator) in a normal direction, $\perp_f$ denotes the unit normal bundle with its canonical volume element, c_{N-1} denotes the volume of the unit sphere S^{N-1}. Compare [Willmore 1971; Cecil and Ryan 1985] for more details.

For the total absolute curvature of immersions into the sphere see [Teufel 1982]. For a generalization of tightness to the case of hyperbolic space or manifolds without conjugate points other than $\mathbb{E}^N$ see [Cecil and Ryan 1979; Bolton 1982].

The Two-Piece Property is a purely topological condition, and as such it can be applied even to topological embeddings or immersions where it does not make sense to talk about a curvature measure K or the collection of nondegenerate height functions. In particular, for polyhedral surfaces embedded in $\mathbb{E}^3$ we may use the TPP as the definition of tightness. Condition (ii) still applies, since we may consider surfaces for which almost all height functions have exactly one local maximum. We can no longer assume that almost all height functions restrict to functions on the surface with only maxima, minima, and ordinary saddles, since there might be more complicated isolated critical points. Nonetheless, for each unit vector not perpendicular to any edge of a closed polyhedral surface M, it is possible to assign an index of singularity to each vertex such that maxima and minima have index 1 and such that the sum of the indices equals $\chi(M)$. The number of critical points, counted with multiplicities, once again is greater than or equal to the sum of the Betti numbers of the surface, with equality when M

has the TPP. For further discussion, see [Banchoff 1970a; Banchoff and Takens 1975].

Definition 1.1.2 (polyhedral curvature). In the case of a polyhedral surface, we can also find analogues for the concept of curvature. Although there is not a well-defined Gauss mapping at the vertices of a polyhedron, we may assign to each vertex v the integral of the indices of singularity at v for all height functions determined by points of the unit sphere. This average can be shown to equal the *polyhedral curvature* $K(v)$ given by

$$K(v) := 2\pi - \textstyle\sum_i \alpha_i,$$

where the α_i, for $i = 1, 2, \dots$ denote the interior angles of the faces at v.

The analogue of the Gauss–Bonnet formula for a closed polyhedral surface is then $\sum_v K(v) = 2\pi\chi(M)$.

In the case of a smooth immersion, a fundamental theorem of Gauss states that the curvature K is intrinsic, dependent only on measurements made along the surface and not depending on the way the surface sits in space. The situation is quite different for polyhedra. Although the quantity $K(v)$ depends only on intrinsic measurements, these measurements do not give as much information about the way the surface is embedded in space. For example, if two smooth surfaces are intrinsically the same and if one is convex, then the other is as well. This is not true for polyhedra: for example, a regular icosahedron is intrinsically the same as a nonconvex polyhedron where one of the vertices is pushed in.

Nevertheless, we can define a concept of the positive curvature at a vertex v of a polyhedral surface M in $\mathbb{E}^3$: Namely, K_+ is the polyhedral curvature of the local convex hull around v. If v happens to be an interior vertex of the convex hull of its neighbors, we set $K_+ := 0$. Thus K_+ represents the area of the collection of unit vectors such that the associated height function has index 1 at v. Define K_- by the condition $K = K_+ - K_-$. Then $K_* := K_+ + K_-$ is the analogue of the absolute Gaussian curvature. As in the smooth case, $\sum_v K_+(v) \geq 4\pi$, and we obtain the inequality [Brehm and Kühnel 1982]

$$\textstyle\sum_v K_*(v) \geq 2\pi(4 - \chi(M)).$$

As before, the polyhedral surface M will be tight if equality is achieved.

One of the consequences of tightness for polyhedral surfaces is that a vertex v with $K(v) > 0$ (or $K_+(v) > 0$) has to lie on the boundary of its convex hull. More precisely, it has to be an *extreme vertex* of the convex hull, not contained in the interior of any segment in the convex hull.

We say that a height function in the direction of the unit vector z is *general with respect to M* if it takes distinct values at distinct vertices (so it is never constant on an edge). For an embedded polyhedral surface M, the nongeneral height functions correspond to unit vectors contained in a finite union of great circles. So almost all height functions are general for M.

We may summarize our discussion of tight polyhedral surfaces as follows:

Definition 1.1.3 (polyhedral tight closed surfaces). A polyhedral immersion of a closed surface $f : M \to \mathbb{E}^3$ is said to be *tight* if one of the following equivalent conditions is satisfied:

(i) $\frac{1}{2\pi}\sum_v K_*(v) = 4 - \chi(M) = \sum_{i\geq 0} b_i(M;\mathbb{Z}_2)$.
(ii) Every height function $zf : x \mapsto < fx, z >$ in the direction of a unit vector $z \in \mathbb{E}^3$ that is general for M has exactly one local minimum and one local maximum (and, consequently, $b_1 = 2 - \chi(M)$ saddle points, counted with multiplicity).
(iii) $\frac{1}{2\pi}\int_M |K|\, dA = \sum_i b_i(M;\mathbb{Z}_2)$.
(iv) f has the TPP.
(v) For every open half-space h bounded by a plane H, the induced morphism $H_*(f^{-1}(h)) \to H_*(M)$ is injective.

(The last three conditions are the same as their counterparts in Definition 1.1.1.)

In contrast to the smooth situation, where we had to modify the definitions of curvature for immersions of surfaces, or surfaces in higher codimension, these five conditions remain equivalent for any polyhedral mapping of a triangulated surface into Euclidean space of any dimension. For polyhedral surfaces in $\mathbb{E}^N$, with $N \geq 4$, condition (i) is not well defined, but the conditions (ii), (iii), (iv) still remain unchanged, and they are still equivalent.

Definition 1.1.4 (topsets, substantial codimension). A *supporting hyperplane* for a compact subset M of $\mathbb{E}^N$ is a hyperplane that contains at least one point of M such that M is entirely contained in one of the two half-spaces determined by the hyperplane. The intersection of M with a supporting hyperplane is called an *i-topset* if it is contained in an i-dimensional linear subspace but not in any $(i-1)$-dimensional linear subspace. A *topset* is defined as an i-topset for some i. A topset of an immersion is defined as the preimage of a supporting hyperplane. A topset is thus a subset of M where a certain height function attains its maximum (or minimum).

A closed surface M is called *substantial in* $\mathbb{E}^N$ if it is not contained in any hyperplane. An immersion f is called *substantial* if its image is substantial. If f is polyhedral and substantial in $\mathbb{E}^N$, the convex hull of $f(M)$ is an N-dimensional convex polytope, or N-polytope for short. The topsets are the preimages under $f(M)$ of the faces of this polytope. For general facts about convex polytopes see [Grünbaum 1967; Brøndsted 1983].

If f is smooth, its convex hull admits a stratification into smooth i-topsets of various dimensions. Not necessarily every i between 0 and $N-1$ has to occur in this case: The standard sphere has only 0-topsets, while the convex hull of a torus of revolution in $\mathbb{E}^3$ and the Veronese surface in $\mathbb{E}^5$ (see Example 1.2.4 below) possess only 0-topsets and 2-topsets. A piece of cylinder capped off by two hemispheres has 0- and 1-topsets. An analytic tight closed surface can never have 1-topsets or 2-topsets.

1.2. Basic examples: smooth and polyhedral.

Example 1.2.1 (Genus 0). The boundary of any bounded three-dimensional convex body is a tightly embedded two-sphere. The easiest condition to verify is the TPP, which works no matter what degree of smoothness is present. It is possible to find smooth examples of any order C^k, for $k \geq 0$. The unit two-sphere is an example of a real analytic tight embedding. The boundary polyhedral surface of a Platonic solid, or of any convex three-polytope, is a polyhedral tightly embedded two-sphere.

Example 1.2.2 (Genus 1). A torus of revolution, obtained by revolving a circle around a disjoint axis in its plane, is a tight analytic surface in $\mathbb{E}^3$. Such a torus can be obtained by stereographic projection of the Clifford torus $S^1 \times S^1 \subset \mathbb{E}^2 \times \mathbb{E}^2 = \mathbb{E}^4$, a tight torus contained on the three-sphere S^3 of radius $\sqrt{2}$ in four-space. (Any tight surface situated on a sphere satisfies a stronger condition, called *tautness* or the *spherical TPP*; see [Cecil 1997] in this volume.) A polyhedral analogue of the Clifford torus is the Cartesian product of two square polygons in two-dimensional planes in four-space, regarded as a subcomplex of the four-cube, which it itself a Cartesian product of two square regions. The vertices of this polyhedron lie on a three-sphere, and stereographic projection from the north pole of this sphere sends the vertices of the tight "polyhedral torus of revolution" to the vertices of a Schlegel diagram of the four-cube.

An essentially different example of a tight polyhedral torus in 3-space is Császár's torus, a polyhedron without diagonals [Császár 1949; Bokowski and Eggert 1991]; see Figure 3. Any two of the seven vertices are joined by an edge. As a consequence, any simplexwise linear embedding of this particular triangulation into $\mathbb{E}^4$, $\mathbb{E}^5$ or $\mathbb{E}^6$ is tight. The abstract 7-vertex triangulation of the torus was already known to Möbius [1886], and, as the union of two disjoint triple systems on 7 points, it was mentioned by Cayley [1850].

Example 1.2.3 (Higher genus). In order to construct smooth tight orientable surfaces of higher genus in $\mathbb{E}^3$, we start with the boundary of the convex hull of a thin torus of revolution (an ε neighborhood of a horizontal circle). This convex smooth surface contains horizontal circular discs. We may then remove smaller circular discs from these discs and attach rotationally symmetric handles given by the parts of negative curvature of other tori of revolution. This is an instance of what is known as "tight surgery" $S^0 \times B^2 \to B^1 \times S^1$. One can see that condition (i) of Definition 1.1.1 remains satisfied. Polyhedral analogues are obtained by removing pairs of small square discs from parallel faces of a cube, then connecting the square boundaries by tubes with square cross-section. We can also carry out a similar polyhedral construction to join pairs of convex polygons in disjoint planar faces of a tight polyhedral surface.

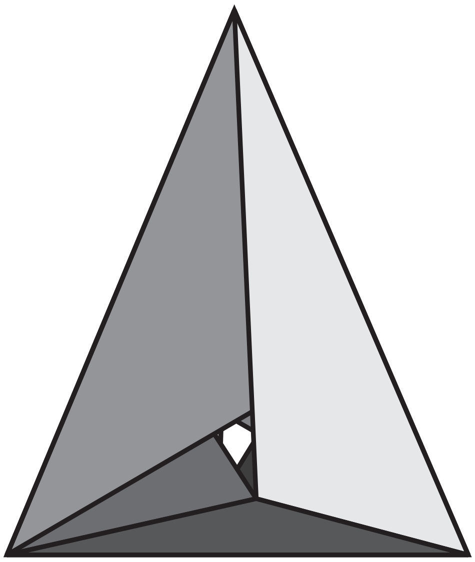

Figure 3. Császár's seven-vertex embedding of the torus.

A similar type of construction is possible in the case of a tight smooth surface in $\mathbb{E}^4$ that contains a pair of parallel planar pieces. We can obtain such a tight torus as the product of a pair of curves consisting of a pair of parallel segments capped by two semicircles. This surface contains a pair of parallel square regions lying in a three-dimensional affine subspace of $\mathbb{E}^4$, so we may add negative curvature tubes as above. Note that these examples will not be real analytic, and in fact there are no real analytic surfaces of genus greater than 1 in $\mathbb{E}^4$ not lying in any affine hyperplane [Thorbergsson 1991]. This leads to an open problem:

QUESTION 1. *For a polyhedral tight surface in* $\mathbb{E}^4$, *it is possible to join two polygons in different faces by a polyhedral tube*, *preserving tightness*, *even if the two faces do not lie in any three-dimensional subspace. Is an analogous construction possible for smooth surfaces*, *to produce examples of tight surfaces of higher genus by adding handles to nonparallel flat pieces*?

For any N it is possible to construct tight polyhedral surfaces in $\mathbb{E}^N$ not lying in any hyperplane. We can obtain such examples recursively as follows: Start with any tight polyhedral closed surface substantially embedded in $\mathbb{E}^N$, and translate a copy of it into a parallel hyperplane of $\mathbb{E}^{N+1}$. Then remove a number of open two-faces with pairwise disjoint boundaries such that every vertex of the given surface is contained in one of the boundaries. Remove the corresponding faces in the parallel copy, and join corresponding boundaries by polyhedral tubes. The resulting surface has the TPP, so this procedure gives a tight polyhedral surface substantially embedded in $\mathbb{E}^{N+1}$, of at least twice the genus of the original surface. Applying this procedure to the boundary of the three-cube gives the polyhedral torus in the four-cube constructed earlier. We may repeat the procedure to obtain a series of highly symmetric tight surfaces embedded as subcomplexes of higher-dimensional cubes. This series of examples has been developed independently by various authors [Banchoff 1965; Coxeter 1937; Kühnel and Schulz 1991; Kühnel 1995]. Surprisingly, these examples can also be realized as embedded polyhedra in $\mathbb{E}^3$ where the number of vertices can be smaller than the genus [McMullen et al. 1983].

Example 1.2.4 (The Klein bottle). According to a theorem of Kuiper [1960; 1983b], it is not possible to find a tight immersion of the Klein bottle into $\mathbb{E}^3$, even as a topological surface. However, following a procedure analogous to the one used in the previous paragraph, we can construct a tight polyhedral embedding of the Klein bottle into $\mathbb{E}^4$. We start with a Möbius band in $\mathbb{E}^3$ with five vertices, all ten edges connecting pairs of vertices, and five triangular faces. Take a copy in a parallel hypersurface in $\mathbb{E}^4$ and connect corresponding vertices by segments and corresponding boundary segments by rectangles. The resulting tight polyhedral surface contains all edges of the convex hull of the ten points in $\mathbb{E}^4$. Compare [Ba7].

Question 2. *Is there a tight smooth embedding or immersion of the Klein bottle substantially into* $\mathbb{E}^4$? (Haab [1994/95] has announced that the answer is negative.)

Example 1.2.5 (The projective plane). There is no tight immersion of the real projective plane into $\mathbb{E}^3$. This was proved by Kuiper [1960], who showed that it was not possible even for topological immersions. One of the most famous and most beautiful examples in the theory of tight surfaces is the *Veronese surface* contained substantially in a five-dimensional hyperplane in $\mathbb{E}^6$. It is defined as the immersion of the unit sphere

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

into $\mathbb{E}^6$ by the formula

$$(x, y, z) \mapsto \big(x^2, y^2, z^2, \sqrt{2}\,xy, \sqrt{2}\,yz, \sqrt{2}\,zx\big).$$

Since antipodal points of the unit sphere are sent to the same image, and no other disjoint pairs are sent to the same point, this defines an embedding into $\mathbb{E}^6$ of the real projective plane, thought of as the unit sphere with antipodal points identified. The sum of the first three coordinates is 1, so the image is contained in a five-dimensional hyperplane, but it is not contained in any four-dimensional affine subspace. It is, however, contained in the five-sphere of radius 1, so also in the small four-sphere obtained by intersecting the unit sphere with the hyperplane. It follows that this surface is not only tight but also taut; see [Cecil 1997] in this volume.

The Veronese surface has the TPP. We can see this using projective geometry, since any hyperplane cuts the surface in a projective quadric that separates the projective plane into at most two pieces. Alternatively, one can verify condition (ii) in Definition 1.1.1 because any nondegenerate height function on the Veronese surface can be regarded as a quadratic function on S^2. Using Lagrange multipliers, we see that there are exactly six critical points on S^2, three pairs of antipodal points, giving the minimum number of critical points on the real projective plane $\mathbb{RP}^2$.

By stereographic projection, we can obtain a taut (therefore tight) embedding of the real projective plane into $\mathbb{E}^4$. The Veronese surface has a very special property called *secant-tangency.* Any secant joining two points of the Veronese surface is parallel to a line in a tangent plane to the surface, and from this it follows that any orthographic projection into four-space that leads to an immersion does in fact lead to an embedding. Such an embedding is automatically tight since almost all height functions on the projected image will still have the minimal number of critical points.

QUESTION 3. *Is it possible to attach a handle tightly to the Veronese surface projected into* $\mathbb{E}^4$?

By a result of Kuiper, it is not possible to carry out this construction smoothly in $\mathbb{E}^5$, and the construction in $\mathbb{E}^4$ would require very subtle details; compare Question 1.

A tight polyhedral embedding of the real projective plane into $\mathbb{E}^5$ is given by any simplexwise linear embedding of the unique six-vertex triangulation obtained by identifying antipodal points on a regular icosahedron (this is called the *hemi-icosahedron* in [Coxeter 1970]). As an abstract triangulation, the six-vertex $\mathbb{RP}^2$ was already known to Möbius [1886]. In this embedding the TPP follows from the fact that any two vertices are joined by an edge. As in the case of the Veronese surface, any orthogonal projection of this surface into $\mathbb{E}^4$ that results in an immersion automatically produces a tight embedding.

EXAMPLE 1.2.6. Nonorientable surfaces with even Euler characteristic $\chi \leq -2$

To construct a smooth tight immersion of a Klein bottle with one handle [Kuiper 1961], start with a tight torus smoothly embedded into $\mathbb{E}^3$ such that some line intersects the surface in four parallel flat pieces. We can then construct a tight tube as above, intersecting the original torus in a curve, thus producing an immersion of a nonorientable surface of Euler characteristic -2 (see Figure 4). Attaching additional handles produces tight immersions of all nonorientable surfaces with negative even Euler characteristic. Similar constructions can be carried out on polyhedral tori to obtain tight polyhedral immersions of such surfaces.

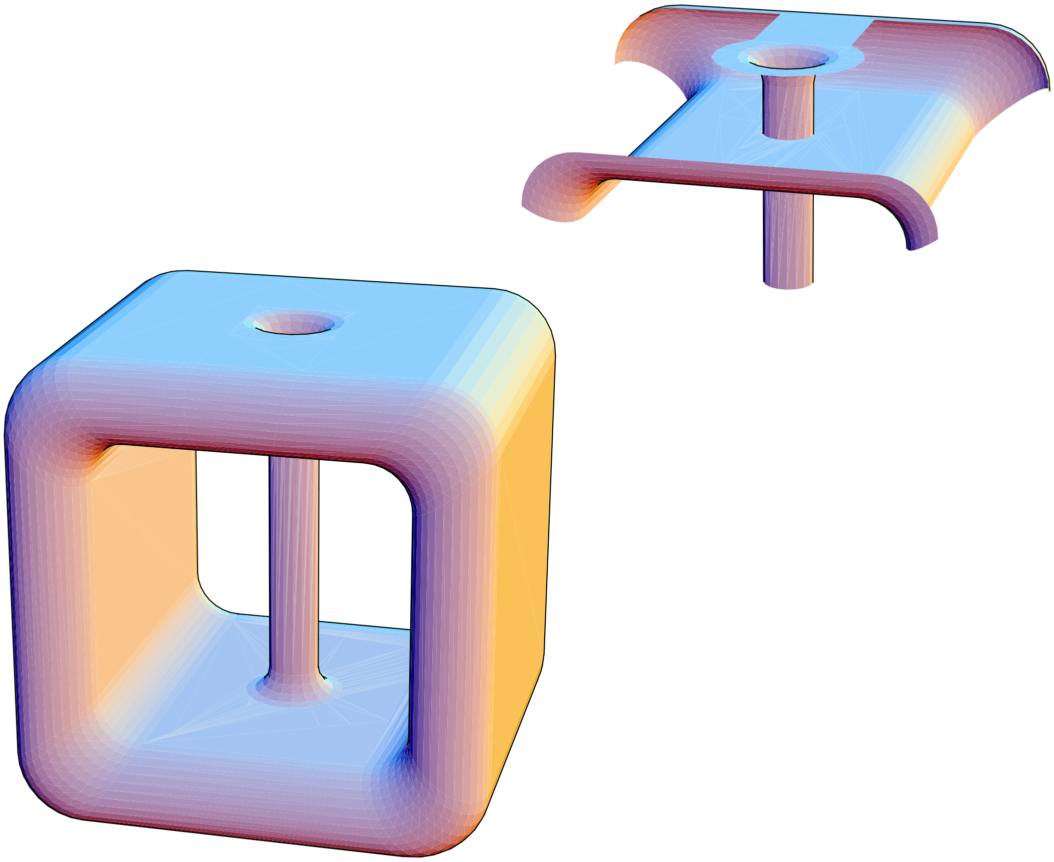

Figure 4. Kuiper's tight immersed Klein bottle with one handle, with a detail showing the self-intersection.

Example 1.2.7 (Nonorientable surfaces with odd Euler characteristic $\chi \leq -1$). An example with $\chi = -3$ can be obtained by attaching a certain interior part (two handles together with one cross-cap) to a convex outer part. The idea was described in [Kuiper 1961]; a more concrete description of a polyhedral example admitting a tight smoothing was given in [Kühnel and Pinkall 1985]. Additional handles can be attached tightly.

This covers the case of odd Euler characteristic $\chi \leq -3$. The case $\chi = -1$, a projective plane with one handle, was mentioned as an open problem in [Kuiper 1961] and remained open until recently. For a long time it has been conjectured that existence or nonexistence would be the same for the smooth and the polyhedral case. The solution was quite surprising: F. Haab [1992] proved that no smooth tight surface with $\chi = -1$ can exist, and recently D. Cervone [1994] found a polyhedral tight example (see also [Cervone 1997] in this volume).

Example 1.2.8 (Combinatorial). Assume there is given an abstract simplicial triangulation of a closed surface M with n vertices and $\binom{n}{2}$ edges (that is, any two vertices are joined by an edge). Regard the union of the triangles as a subcomplex of the $(n-1)$-dimensional simplex spanned by n vertices in $\mathbb{E}^{n-1}$. Then M is tightly and substantially embedded into $\mathbb{E}^{n-1}$ [Banchoff 1974; Kühnel 1980]. Special cases are the 7-vertex torus and the 6-vertex real projective plane.

1.3. Tight smooth surfaces. In a series of papers starting in 1958, Kuiper studied tight smooth embeddings and immersions of surfaces using a variety of approaches. He obtained existence and uniqueness results, obstructions to tight immersions, and characterizations of special examples. One of his basic observations is the following:

Proposition 1.3.1 [Kuiper 1962]. *If $f : M \to \mathbb{E}^N$ is a smooth and substantial 0-tight immersion of a closed surface, then $N \le 5$. In other words, a smooth closed surface immersed tightly in a Euclidean space must be contained in a three-, four-, or five-dimensional affine subspace.*

It is instructive to consider the analogue of this statement for a lower-dimensional situation, that of a smooth immersion of a closed one-dimensional manifold (a closed curve) into $\mathbb{E}^N$ possessing the TPP. We know that even for a topological immersion the TPP for a closed curve implies that the image is contained in a two-dimensional affine subspace. In the case of a smooth immersion, we observe that if the curve is not contained in the plane of the velocity and the acceleration vectors at a particular point, we can find a different plane containing the tangent line that cuts the curve away from a neighborhood of the point. This violates the TPP.

In the case of a two-dimensional surface in $\mathbb{E}^N$, at any given point all the tangent vectors to curves in the surface are contained in a two-dimensional affine subspace, and all the acceleration vectors are contained in a five-dimensional affine subspace containing that plane. If the surface is not wholly contained in that five-dimensional space, there is an affine hyperplane through the tangent plane meeting the surface away from a neighborhood of the point. This violates the TPP.

We can reformulate this argument in terms of the space of second fundamental forms as follows (compare [Cecil and Ryan 1985, p. 34]):

Let zf denote a nondegenerate height function attaining its maximum at a point $p \in M$. For any normal ξ at p, let A_ξ denote the second fundamental form in direction ξ. If the mapping $\xi \mapsto A_\xi$ is injective, the codimension $N-2$ cannot exceed the dimension of symmetric bilinear forms on the tangent plane, which for a two-dimensional surface is 3.

If this mapping were not injective, then for some ξ, the associated bilinear form would be zero. For any t, the height function in the direction of $zf + t\xi$ would have a local maximum at p; but since f is assumed to be substantial, the height function in the direction of ξ is not constant, so for some point q of M, it has a higher value at q than at p, contradicting the TPP.

PROPOSITION 1.3.2 [Chern and Lashof 1957; Kuiper 1960]. *Let $f : S^2 \to \mathbb{E}^N$ be a smooth and substantial tight immersion. Then $N = 3$, and $f(S^2)$ is the boundary of a convex body in $\mathbb{E}^3$.*

PROOF. Let $\mathcal{H}$ denote the convex hull of $f(S^2)$. A nonempty intersection of $\mathcal{H}$ with a hyperplane bounding a half-space containing the image of f is called a *topset*. If f is tight, every topset A of $\mathcal{H}$ is a convex set. A 0-topset is a point that must be contained in $f(S^2)$. A 1-topset is an interval that, by the TPP, is entirely contained in $f(S^2)$. A 2-topset of $\mathcal{H}$ has the property that its boundary is contained in $f(S^2)$, and the preimage of this boundary curve will separate the two-sphere into two pieces, each topologically equivalent to a disc. If the 2-topset is not contained in the image, the TPP is violated. Inductively, it follows that $f(S^2)$ would have to contain all of the topsets of $\mathcal{H}$ and finally the boundary of $\mathcal{H}$ itself. But this is impossible if $N \geq 4$. Hence $N = 3$, and $f(S^2)$ is a closed surface that contains the boundary of the convex body $\mathcal{H}$. This can only happen if $f(S^2)$ coincides with the boundary of its convex hull. ☐

THEOREM 1.3.3 (EXISTENCE OF SMOOTH TIGHT SURFACES). *A tight and smooth immersion $f : M \to \mathbb{E}^3$ exists if*

(i) *M is orientable (and in this case, f can be chosen to be algebraic* [Banchoff and Kuiper 1981]), *or*
(ii) *M is nonorientable with $\chi(M) \leq -2$ (and if χ is even, f can be chosen to be algebraic* [Kuiper 1983a]).

Moreover, if M is not topologically equivalent to a two-sphere, then there exists a tight and substantial smooth immersion $f : M \to \mathbb{E}^4$ if

(iii) *M is orientable (but f can only be analytic in the case where M is the torus* [Thorbergsson 1991]), *or*
(iv) *M is nonorientable with $\chi = 1$, $\chi = -2$ or $\chi \leq -4$.*

PROOF. The existence in (i) and (ii) follows from the basic examples in Section 1.2. Case (iii) is also contained in Example 1.2.3, and the case $\chi = -1$ in (iv) is mentioned in Example 1.2.4.

For the nonorientable case $\chi = -2$ in $\mathbb{E}^4$ we start with the 4×4 torus as a subcomplex of the 4-cube $[0,1]^4$ of Example 1.2.2, and consider the diagonally opposite squares with vertices $(0,0,x,y)$ and $(1,1,x,y)$. Since these squares lie in a common three-plane, it is possible to attach a polyhedral or smooth handle tightly. The outer torus can be chosen to be smooth as well; see Example 1.2.2. Additional handles can be attached tightly. This covers the case of even Euler characteristic $\chi \leq -2$. Note that these examples are embedded.

For the case of odd Euler characteristic one would like to have a starting example with $\chi = -3$ (or, even better, $\chi = -1$), which does not seem to be known. For $\chi = -5$ one can start with the torus in $\mathbb{E}^4$ as above and then attach in a certain three-plane the inner part of the surface with $\chi = -3$ in $\mathbb{E}^3$, as in Example 1.2.7. Then additional handles can be attached. This construction leads to self-intersections, which are not really necessary from the differential topological point of view; compare Question 7. □

Note that in each of cases (i)–(iv) (except possibly for $\chi = 1$) the smooth tight surface can be approximated by polyhedral tight surfaces, and it can also be obtained as the smoothing of a certain polyhedral example.

PROPOSITION 1.3.4 (TOP-CYCLES [Cecil and Ryan 1984]). *Let $f : M \to \mathbb{E}^3$ be a tight immersion of a closed surface (smooth or polyhedral), and let $\mathcal{H}$ denote the convex hull of $f(M)$. Then $\partial\mathcal{H} \setminus f(M)$ consists of a finite number of convex planar discs. Their boundaries are called top-cycles. The number $\alpha(f)$ of these top-cycles satisfies $2 \le \alpha(f) \le 2 - \chi(M)$. Moreover, if $\alpha(f) = 2 - \chi(M)$, the top-cycles are joined pairwise by cylinders of nonpositive Gaussian curvature, or by cylinders of $K_+ = 0$ in the polyhedral case. For nonorientable surfaces the sharper inequality $2 \le \alpha(f) \le 1 - \chi(M)$ holds.*

SKETCH OF PROOF. If the surface is not an immersed sphere, obviously there must exist at least one top-cycle. In a first step one has to show that the number of top-cycles is finite. Furthermore, each of the top-cycles lies in the part of the surface with vanishing Gaussian curvature. By the tightness, the part with positive Gaussian curvature is contained in the boundary of the convex hull, and the part with negative Gaussian curvature lies in the interior, connecting the various top-cycles with one another. Finally, their number can be related to the topology of the surface by a decomposition argument and the additivity of the Euler characteristic. □

In fact, any value for $\alpha(f)$ within the range of the inequalities above can be realized by a tight surface; for examples, see [Cecil and Ryan 1985, § 7.27].

EXAMPLE 1.3.5. *There exists a tight smooth immersion of the torus into $\mathbb{E}^3$ that is not an embedding (i.e., that has double points).*

The key to creating a tight smooth immersion of the torus that is not an embedding is to find a nonsingular tube with everywhere nonpositive curvature joining two convex curves in parallel planes. If $X(t)$ and $Y(t)$ are two convex closed curves both defined over the same interval $\{a \le t \le b\}$, with $X(t)$ in the horizontal plane through the origin and $Y(t)$ in the horizontal plane at height 1, then we may define a surface

$$Z(t,u) = uX(t) + (1-u)Y(t).$$

The partial derivative with respect to t will be horizontal, while the partial with respect to u is a nonzero vector from the plane at height 0 to the plane at height 1. These vectors can only be linearly dependent when the first is zero,

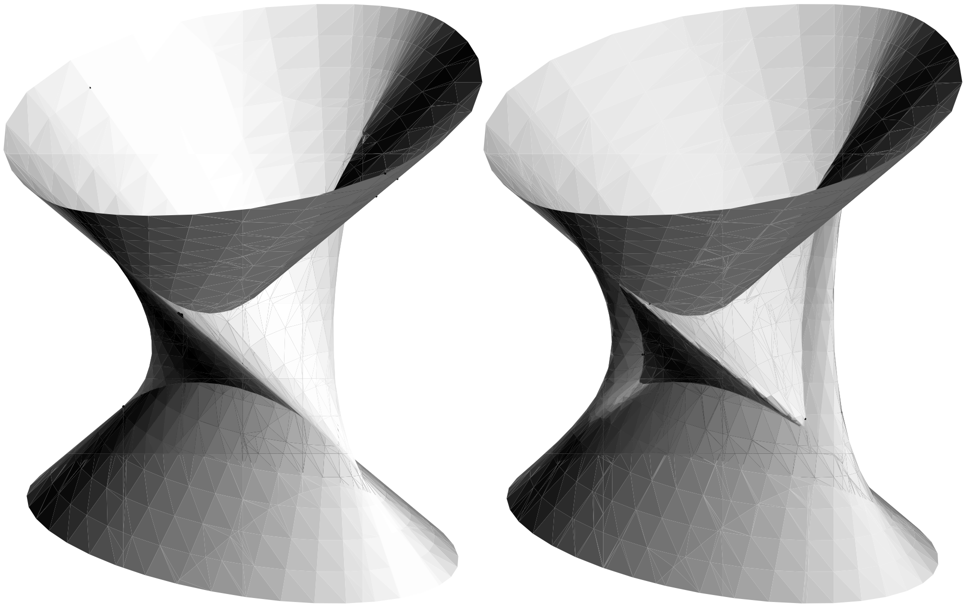

Figure 5. Construction of the inner part of a tight smooth immersion of a torus with self-intersections.

that is, when $uX'(t) + (1-u)Y'(t) = 0$ for some t and u, so the tube will be nonsingular if $X'(t)$ is never parallel to $Y'(t)$. One way to arrange this is to start with an ellipse $X(t)$ with unequal axes, then let $Y(t)$ be the parallel curve at distance r, where the distance is greater than the radius of curvature at any point of the ellipse. The tube constructed for these two curves will have four cuspidal edges meeting pairwise in a set of four swallowtail points, and there are two curves of double points. If we then rotate $Y(t)$ slightly, no $X'(t)$ will be parallel to the corresponding $Y'(t)$, so there will be no singularities. However for small enough rotations, there will still be intersection points near the original double point arcs. See Figure 5.

A similar example of a tight polyhedral immersion of a torus that is not an embedding is described in [Kuiper 1983b].

THEOREM 1.3.6 [Kuiper 1962; 1997]. *A tight and substantial smooth immersion* $f : M \to \mathbb{E}^5$ *exists only if* M *is the real projective plane*, *and its image is the Veronese surface* (*up to projective transformations of* $\mathbb{E}^5$).

This result is quite surprising, and the proof is difficult; compare the higher dimensional generalization given below as Theorem 2.4.2. It is also surprising that the tight Veronese surface $f : \mathbb{RP}^2 \to \mathbb{E}^5$ cannot be approximated by polyhedral tight surfaces [Kuiper and Pohl 1977]. Conversely, the tight polyhedral $\mathbb{RP}^2$ in 5-space cannot be approximated by smooth tight surfaces.

Theorem 1.3.7 (nonexistence results). *There is no tight and smooth immersion into* $\mathbb{E}^3$ *of the projective plane* [Kuiper 1960], *the Klein bottle* [Kuiper 1960], *or the surface with* $\chi = -1$ [Haab 1992].

The assertions for the projective plane and the Klein bottle follow from Proposition 1.3.4, because there is no possibility for $\alpha(f)$. The proof in the case $\chi = -1$ is much more involved.

The remaining open cases lead to the following questions.

Question 4. *Is there a smooth tight immersion of the surface with* $\chi = -1$ *or* $\chi = -3$ *into* $\mathbb{E}^4$?

Question 5. *Are there tight algebraic surfaces in* $\mathbb{E}^3$ *with odd Euler characteristic*?

Question 6. *Are there tight analytic immersions of nonorientable surfaces into* $\mathbb{E}^4$ *if* $\chi \neq 1$? See [Haab 1994/95].

Question 7. *Does there exist a smooth tight embedding into four-space of a surface with odd Euler characteristic* $\chi \leq -1$?

Question 8. *Is it possible to approximate the Veronese surface in* $\mathbb{E}^4$ *by tight polyhedral surfaces*? A positive answer would also shed some light on Question 2.

Question 9. *Is it possible to approximate Császár's seven-vertex torus in* $\mathbb{E}^3$ *by tight smooth surfaces*? One may start with any version of this polyhedron [Bokowski and Eggert 1991].

Question 10. *Is there any difference between the case of* C^2*-immersions and the case of* C^1*-immersions, as far as existence or nonexistence of tight immersions is concerned*?

1.4. Tight polyhedral surfaces. Tight polyhedral surfaces were introduced in [Banchoff 1965]. There are a number of analogies with the smooth case, but also a number of significant differences. In particular, the structure of the topsets can be different from those in the smooth case, and the substantial codimension can be arbitrarily large (see Example 1.2.3). In [Banchoff 1974] the relationship with Heawood's map color problem was mentioned, and this was developed in a systematic way in [Kühnel 1980]. In this section we summarize the main results. For the details and proofs see [Kühnel 1995].

The 1-*skeleton* of a convex polytope, denoted by Sk_1, is defined as the set of all extreme vertices and extreme edges of the polytope. For example, the 1-skeleton of an N-dimensional simplex is the *complete graph* K_{N+1} on $N+1$ vertices.

Lemma 1.4.1. (i) [Banchoff 1965] *Let* $M \subset \mathbb{E}^d$ *be a* 0*-tight and connected polyhedron. Then* M *contains the* 1*-skeleton of its convex hull*: $\mathrm{Sk}_1(\mathcal{H}) \subset M$.
(ii) *A polyhedral surface* M *with convex faces is* 0*-tight if and only if its* 1*-skeleton is* 0*-tight*.

PROOF. (i) Let e be an extreme edge of $\mathcal{H}$ with the extreme vertices v, w as its endpoints. By construction M contains v and w. There is a half-space h of $\mathbb{E}^d$ such that $h \cap \mathcal{H} = e$. Consequently we have $\{v, w\} \subset h \cap M \subset h \cap \mathcal{H} = e$. By the 0-tightness, $h \cap M$ must be connected. It follows that $h \cap M = e$.

(ii) If the one-skeleton is 0-tight, M is 0-tight because adding higher dimensional faces preserves the connectedness of $M \cap h$. Conversely, if $M \cap h$ is connected then $\mathrm{Sk}_1(M) \cap h$ must be connected because the faces are convex.

Note that this is not true if there are nonconvex faces. For example, if we remove two square regions from opposite faces of a cube and connect by a polyhedral tube, the resulting one-skeleton is not even connected, and if we make it connected by adding some diagonals, it is possible that the resulting one-skeleton will not be tight. □

The tightness of the one-skeleton of a polyhedron essentially means that (i) the one-skeleton of the convex hull is contained in the surface, and (ii) every vertex that is not a vertex of the convex hull lies in the relative interior of some of its neighbors. For example, a vertex might lie in the interior of a segment determined by two neighboring vertices, or in the interior of a triangle determined by three neighbors. These situations are not stable, in that tightness can be lost by small perturbations of the vertex. If, however, a vertex is in the interior of a tetrahedron spanned by four neighboring vertices, this situation is preserved under small perturbations of the vertex.

THEOREM 1.4.2 (EXISTENCE RESULTS IN SMALL CODIMENSION). *Let M be an abstract surface with Euler characteristic $\chi(M)$.*

(i) *There is a tight polyhedral embedding $M \to \mathbb{E}^3$ if M is orientable.*
(ii) *There is a tight polyhedral immersion $M \to \mathbb{E}^3$ if M is nonorientable and $\chi(M) \leq -1$* (see [Cervone 1994; 1997] for the case $\chi = -1$).
(iii) *There are tight and substantial polyhedral embeddings $M \to \mathbb{E}^4$ and $M \to \mathbb{E}^5$ if M is not topologically equivalent to the two-sphere.*
(iv) *There is a tight and substantial polyhedral embedding $M \to \mathbb{E}^6$ if M is orientable and distinct from the two-sphere.*

In particular, any closed surface admits a tight polyhedral embedding into some Euclidean space $\mathbb{E}^N$.

PROOF. The proof consists in a series of examples, most of them already sketched in Section 1.2. The most difficult case is the nonorientable surface in $\mathbb{E}^3$ with $\chi = -1$, settled only recently by D. Cervone [1994; 1997]. The case $\chi = -3$ is also special, although it can now be obtained from the case $\chi = -1$ by attaching a handle. In [Kühnel and Pinkall 1985], a symmetric immersion of the surface with $\chi = -3$ is constructed in such a way that it can be smoothed to give a tight smooth immersion of the surface. By the nonexistence result of Haab, the Cervone immersions cannot be smoothed while maintaining tightness. □

For a tight torus in three-space, the intersection with the convex hull is a convex sphere with two open convex discs removed. In the case of a smooth torus, the closures of these two convex discs are disjoint, although in the polyhedral case they may intersect at a point or along a line segment. In any case, the complement of this outer part is an open cylinder strictly contained in the interior of the convex hull.

Although the outer part is necessarily a one-to-one image of a subset of the torus, the inner part, surprisingly, can have self-intersections. The first example of this phenomenon in the smooth case was observed by L. Rodríguez, and in the polyhedral case by Banchoff. Since this example has not been published elsewhere, we include it here.

Start with a tetrahedron inscribed in a cube, with vertices

$$v_1 = (1,1,1),\ v_2 = (1,-1,-1),\ v_3 = (-1,1,-1),\ v_4 = (-1,-1,1).$$

Shift the triangles $v_1v_3v_4$ and $v_2v_3v_4$ in the direction $(t,0,0)$ and the triangles $v_1v_2v_3$ and $v_1v_2v_4$ in the direction $(-t,0,0)$. For t between 0 and 1, the intersection of these triangles will be a skew quadrilateral with vertices $(1-t)v_1, (1-t)v_3, (1-t)v_2, (1-t)v_4$. To these four triangles, we may attach two parallelograms $v_1 + (t,0,0)$, $v_1 + (-t,0,0)$, $v_3 + (-t,0,0)$, $v_3 + (t,0,0)$ and $v_2 + (t,0,0)$, $v_2 + (-t,0,0)$, $v_4 + (-t,0,0)$, $v_4 + (t,0,0)$. This produces a self-intersecting polyhedral cylinder with boundary given by two parallelograms $v_1 + (t,0,0)$, $v_1 + (-t,0,0)$, $v_4 + (-t,0,0)$, $v_4 + (t,0,0)$ and $v_2 + (t,0,0)$, $v_2 + (-t,0,0)$, $v_3 + (-t,0,0)$, $v_3 + (t,0,0)$. We may obtain a tightly immersed polyhedral torus by attaching this cylinder to the surface of a square prism with vertices $(\pm 2, \pm 2, \pm 1)$, with the regions removed that are bounded by the parallelograms $v_1 + (t,0,0)$, $v_1 + (-t,0,0)$, $v_4 + (-t,0,0)$, $v_4 + (t,0,0)$ and $v_2 + (t,0,0)$, $v_2 + (-t,0,0)$, $v_3 + (-t,0,0)$, $v_3 + (t,0,0)$. See Figure 6.

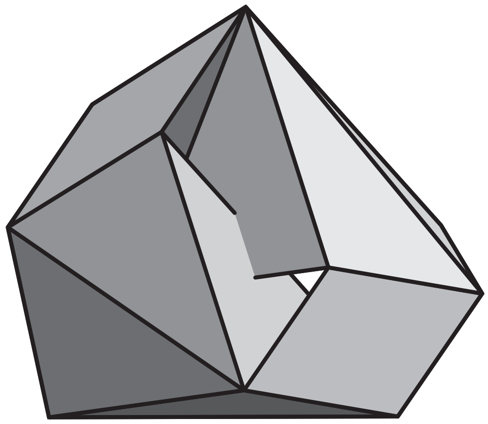

Figure 6. The inner part of a tight polyhedral torus with self-intersections.

ATTACHING LEMMA 1.4.3 [Kühnel 1980]. *Let M be a tight polyhedral surface, substantial in $\mathbb{E}^N$, for $N \geq 4$. Then there is a tight polyhedral $M \# S^1 \times S^1$ and a tight $M \# \mathbb{RP}^2$ in the same space $\mathbb{E}^N$, obtained just by local modifications of M (attaching a handle or a cross-cap).*

A handle can be attached tightly also in $\mathbb{E}^3$; see Example 1.2.3. For the construction of attaching a cross-cap one can replace the cone over a skew pentagon by a five-vertex Möbius band.

LEMMA 1.4.4 [Grünbaum 1967]. *The one-skeleton of any convex N-polytope contains the complete graph K_{N+1} as a subset (not necessarily as a subgraph).*

COROLLARY 1.4.5 [Banchoff 1971a]. *Let $f : S^2 \to \mathbb{E}^d$ be a tight and substantial polyhedral embedding or immersion. Then $d = 3$, and $f(S^2)$ is the boundary of a convex three-polytope.*

PROOF. By Lemma 1.4.1, $f(S^2)$ contains the one-skeleton of its convex hull. This in turn contains a K_{d+1} by Lemma 1.4.4. On the other hand, S^2 does not contain a K_5 because K_5 is not planar, hence it does not contain a K_{d+1} for any $d \geq 4$. Therefore $d + 1 = 4$. Assume that $f(S^2)$ is not identical with its convex hull $\mathcal{H}$. Then one of the two-dimensional faces of $\mathcal{H}$ is not contained in $f(S^2)$. On the other hand, the boundary of this two-face is certainly contained in $f(S^2)$ because this boundary contains all edges of $\mathcal{H}$. It separates S^2 into two pieces. Therefore it follows that a certain plane in $\mathbb{E}^3$, parallel to this two-face, would separate $f(S^2)$ into more than two pieces, a contradiction to 0-tightness. □

THEOREM 1.4.6 (NONEXISTENCE RESULTS IN SMALL CODIMENSION). (i) *There is no tight polyhedral immersion of the real projective plane or of the Klein bottle into $\mathbb{E}^3$ (not even topologically* [Cecil and Ryan 1985; Kuiper 1983b]).
(ii) *There is no tight and substantial polyhedral immersion of S^2 into $\mathbb{E}^N$, for $N \geq 4$.*

Assertion (i) follows from Proposition 1.3.4. For assertion (ii), see Corollary 1.4.5.

For the case of higher codimension, see Theorem 1.4.8 below.

By Lemma 1.4.4, a necessary condition for tightness of a polyhedral surface into $\mathbb{E}^d$ is the embeddability of the complete graph on $d+1$ vertices in the surface. This in turn is closely related with the Heawood map color problem [Heawood 1890; Ringel 1974; White 1984]. The following result of G. Ringel and J. W. T. Youngs expresses the embeddability of the complete graph in n vertices in terms of an inequality, known as *Heawood's inequality*, between n and the genus of the surface.

THEOREM 1.4.7 [Ringel 1974]. *For every abstract surface M of genus g, apart from the Klein bottle, the following conditions are equivalent:*

(i) *There exists an embedding $K_n \to M$.*
(ii) $\chi(M) \leq n(7 - n)/6$.

(iii) $n \leq \frac{1}{2}\bigl(7 + \sqrt{49 - 24\chi(M)}\,\bigr)$.
(iv) $\binom{n-3}{2} \leq 3\bigl(2 - \chi(M)\bigr) = 6g$.

For the Klein bottle, condition (i) *is equivalent to* $n \leq 6$. *Moreover if equality holds in* (ii)–(iv), *the embedding of* K_n *induces an abstract triangulation of* M.

THEOREM 1.4.8 [Kühnel 1980]. *For an abstract surface* M *and a number* $n \geq 6$, *the following conditions are equivalent*:

(i) *There exists a tight and substantial polyhedral embedding* $M \to \mathbb{E}^{n-1}$.
(ii) *There exists an embedding* $K_n \to M$.

Combining this with Theorem 1.4.7, we obtain:

COROLLARY 1.4.9 [Kühnel 1978, Theorem A]. *Let* M *be an abstract surface that is not topologically equivalent to the Klein bottle . Then there exists a tight substantial polyhedral embedding* $M \to \mathbb{E}^d$ *if and only if*

$$d_0 \leq d \leq \tfrac{1}{2}\bigl(5 + \sqrt{49 - 24\chi(M)}\,\bigr),$$

where $d_0 = 3$ *if* M *is orientable and* $d_0 = 4$ *if* M *is nonorientable. The Klein bottle can be embedded tightly and substantially into* $\mathbb{E}^4$ *and* $\mathbb{E}^5$ *but not into* $\mathbb{E}^k$, $k \geq 6$ [Franklin 1934; Banchoff 1974].

SKETCH OF PROOF OF THEOREM 1.4.8.
(i) $\Rightarrow$ (ii) is just a combination of Lemmas 1.4.1 and 1.4.4:

$$K_n \subset \mathrm{Sk}_1(\mathcal{H}) \subset M \subset \mathbb{E}^{n-1}.$$

(ii) $\Rightarrow$ (i) : We start with an embedding $K_m \subset M$, where $m \leq n$ is maximal with respect to the inequality (iii) in Theorem 1.4.7. Then we extend it to a triangulation of M with those m vertices and some extra vertices. Finally the m vertices can be put into general position in $\mathbb{E}^{n-1}$, and the extra vertices have to be chosen in the relative interiors of their neighbors. This implies that the edge graph of this triangulation is 0-tight in $\mathbb{E}^{n-1}$. If the surface is embedded (i.e., without self-intersections) then it is 0-tight and tight by Lemma 1.4.1. Here it can be shown that the case of self-intersections can always be avoided by slight changes of the triangulation. $\square$

A triangulated surface (or any simplicial complex) with n vertices can always be regarded as a subcomplex of an $(n-1)$-dimensional simplex $\triangle^{n-1}$. We call this the *canonical embedding* of the triangulation. An abstract triangulation of a surface is called *tight* if its canonical embedding is tight.

COROLLARY 1.4.10 (TIGHT TRIANGULATIONS [Kühnel 1995]). *Let* M *be a triangulated surface of genus* g *with* n *vertices. Then the following conditions are equivalent*:

(i) *The triangulation is tight*.

(ii) *The triangulation is two-neighborly, that is, its edge graph is a complete graph* K_n.
(iii) $\binom{n-3}{2} = 3\big(2-\chi(M)\big) = 6g$.

Conversely, given an abstract surface not topologically equivalent to the Klein bottle, and a number n satisfying (iii), *then there is a tight triangulation of M with n vertices.*

For a two-manifold M, let N_M denote the maximum dimension of Euclidean space admitting a tight and substantial polyhedral embedding of M. Let n_M denote the minimum number of vertices for any simplicial triangulation of M. Then the results of [Ringel 1955b; Jungerman and Ringel 1980; Huneke 1978] in connection with Theorem 1.4.8 can be reformulated as follows:

THEOREM 1.4.11. *For any surface M we have $N_M \le n_M - 1 \le N_M + 2$. Moreover, for any surface distinct from the Klein bottle, from the orientable surface of genus* 2 *and from the surface with $\chi = -1$, the sharper inequality $N_M \le n_M - 1 \le N_M + 1$ holds.*

THEOREM 1.4.12 [Banchoff 1965; Pohl 1981; Kühnel 1995]. *Let $M \subset \mathbb{E}^d$ be a tightly and substantially embedded polyhedral surface. Then*:

(i) $\binom{d-2}{2} \le 3\big(2-\chi(M)\big)$.
(ii) *If $d \ge 4$, equality in* (i) *holds if and only if M is embedded as a subcomplex of a d-simplex $\triangle^d$ (and the induced triangulation is tight by Corollary* 1.4.10).

Equality in (i) is satisfied by the boundary of any convex three-polytope. This shows that (ii) cannot be extended to the case $d = 3$.

The case $\chi \ne 0$ was treated in [Banchoff 1974], the case $\chi = 0$ in [Pohl 1981; Kühnel 1995, § 2.17].

COROLLARY 1.4.13. *The image of a tight polyhedral real projective plane in $\mathbb{E}^5$ is affinely equivalent to the canonical embedding of the* 6*-vertex $\mathbb{RP}^2$, and the image of a tight polyhedral torus in $\mathbb{E}^6$ is affinely equivalent to the image of the* 7*-vertex torus.*

CONJECTURE 1.4.14 [Pohl 1981]. *Let $M \to \mathbb{E}^d$ be a tight and substantial topological immersion of a surface, and assume $d \ge 6$. Then*:

(i) *The convex hull of M in $\mathbb{E}^d$ is a convex polytope.*
(ii) $\binom{d-2}{2} \le 3\big(2-\chi(M)\big)$.
(iii) *Equality in* (ii) *holds if and only if M is embedded as a subcomplex of a d-simplex $\triangle^d$ (and the induced triangulation is tight by Corollary* 1.4.10).

If (i) turns out to be true, (ii) and (iii) follow by the same arguments as in the proof of Theorem 1.4.12. A particular case of this conjecture is the uniqueness of the polyhedral model induced by the 7-vertex torus as the only tight topological torus in $\mathbb{E}^6$.

Conjecture 1.4.15. *Any tight torus that is centrally symmetric lies in a linear subspace of dimension at most four*. (This is true for a certain subclass of polyhedral surfaces [Kühnel 1996].)

1.5. Tight and 0-tight surfaces with boundary. For surfaces with boundary, the 0-tightness condition is much weaker than the condition of tightness. Recall that an object in $\mathbb{E}^3$ is *0-tight* (condition (iv) in Definition 1.1.1) if the intersection with every open or closed half-space is connected. For *tightness* (condition (iii) in Definition 1.1.1), we also require that every 1-chain in the intersection of an object and a half-space, that bounds in the object also bounds in the intersection of the object with the half-space. For example, a closed hemisphere satisfies the 0-tightness condition, but the plane containing the boundary circle bounds a half-space in which the circle is not a boundary of a 2-chain even though the circle bounds the hemisphere itself.

Proposition 1.5.1. (i) *Assume that $f : M \to \mathbb{E}^3$ is a smooth tight immersion of a closed surface. Then for any $r \in \mathbb{N}$ there exists a smooth 0-tight immersion $\tilde{f} : M_r \to \mathbb{E}^3$ where M_r denotes the surface M with r open discs removed.*

(ii) *Assume that $f : M \to \mathbb{E}^N$ is a tight polyhedral immersion of a closed surface. Then for any $r \in \mathbb{N}$ there exists a 0-tight polyhedral immersion $\tilde{f} : M_r \to \mathbb{E}^N$.*

(iii) *The upper bound for the substantial codimension of 0-tight polyhedral surfaces M_r is the same as for the corresponding closed surfaces M.*

Proof. To obtain (i), note that the image of a smooth tight immersion of a surface without boundary into $\mathbb{E}^3$ must have points of strictly positive curvature. By moving the tangent plane at any such point parallel to itself by an arbitrarily small amount, we can cut off a region bounded by a convex plane curve, and the resulting object will still have the TPP. This procedure can be repeated any desired finite number of times. It is not clear whether there are analogous constructions for smooth surfaces in $\mathbb{E}^4$ or higher.

In order to prove (ii), we may start with any tight polyhedral immersion of a surface M without boundary into $\mathbb{E}^N$ and then remove r disjoint convex polygonal open regions from any of the two-dimensional faces of the polyhedron to produce a TPP embedding of M_r into $\mathbb{E}^N$.

The proof of (iii) is already contained in the proof for the case of closed surfaces, Corollary 1.4.10. ☐

L. Rodríguez [1976] constructed a 0-tight embedding in $\mathbb{E}^3$ of a torus with a disc removed by removing a nonconvex topological disc from the negative Gaussian curvature region of a torus of revolution. In order for the surface with boundary to remain 0-tight, it is necessary that the boundary curve consist of asymptotic curves, that is, curves where the tangent vector is directed along an asymptotic direction at the point (where an asymptotic direction is a null direction of the second fundamental form so that the principal curvature direction of the curve lies in the tangent plane of the surface at every point of the boundary curve).

PROPOSITION 1.5.2. *For any closed surface M (except $\mathbb{RP}^2$) that is known to admit a tight and substantial immersion $M \to \mathbb{E}^4$* (see Theorem 1.3.3), *there is a smooth 0-tight substantial immersion $M_r \to \mathbb{E}^4$ for any $r \geq 1$.* We may start with a tightly embedded surface containing a flat regions, and remove a number of disjoint open regions bounded by smooth convex curves.

LEMMA 1.5.3. *For any immersion $f : M \to \mathbb{E}^N$ (smooth or polyhedral) of a compact surface M with boundary $\partial M \neq \varnothing$, the following two conditions are equivalent*:

(i) *f is tight.*
(ii) *f is 0-tight and $\mathcal{H}(fM) = \mathcal{H}\big(f(\partial M)\big)$.*

Moreover, if f is smooth, we have $\mathrm{TA}(f|_{M\setminus\partial M}) + \frac{1}{2}\,\mathrm{TA}(f|_{\partial M}) \geq 2 - \chi(M)$, and condition (i) *above is equivalent to either of the following conditions* [Grossman 1972; Kühnel 1977; Rodríguez 1976]:

(iii) $\mathrm{TA}(f|_{M\setminus\partial M}) + \frac{1}{2}\,\mathrm{TA}(f|_{\partial M}) = 2 - \chi(M)$.
(iv) *For sufficiently small $\varepsilon > 0$, the boundary f_ε of the ε-tube is 0-tight.*

COROLLARY 1.5.4. *The image of a tight immersion of a two-disc into any $\mathbb{E}^N$ is a convex set contained in an affine two-dimensional subspace of $\mathbb{E}^N$.*

EXAMPLES 1.5.5. The first study of smooth tight immersions $f : M \to \mathbb{E}^2$ of surfaces with boundary into the plane was carried out by L. Rodríguez [1973]; compare [Kuiper 1997, Figures 8, 9, 10]. Except for the disc, these surfaces with boundary must have at least two boundary components: the boundary of the convex hull and one or more inner components that are "locally concave", that is, smooth curves such that, at each point, the part of the tangent line lying in a disc neighborhood of the point is contained in the image of the surface.

Smooth orientable tight immersions of surfaces with boundary $f :\to \mathbb{E}^3$ can be obtained by starting with a smooth immersion of a surface without boundary that contains a flat piece, and then removing a finite number of disjoint convex discs.

Nonorientable surfaces of this type can be obtained by cutting two convex holes into Kuiper's tight Klein bottles with a handle or into the tight surface with $\chi = -3$ described in [Kühnel and Pinkall 1985]; compare Examples 1.2.6 and 1.2.7. In any case handles can be attached and convex discs can be removed while still preserving the tightness.

CONJECTURE 1.5.6 [White 1974]. *There is no tight and substantial smooth immersion of any compact orientable surface with exactly one boundary component, except for the disc.*

The key lemma in [White 1974] states that this boundary curve would have to be planar and convex, in contradiction with Lemma 1.5.3. However, it seems that the proof of this lemma has never appeared. The conjecture may be extended to the nonorientable case as well. By a theorem of Kuiper [Kuiper 1971/72] there is no smooth tight Möbius band.

THEOREM 1.5.7 [Rodríguez 1976]. *There is no tight and substantial smooth immersion of any compact surface with nonempty boundary into* $\mathbb{E}^N$, *for* $N \geq 4$.

In contrast with Theorem 1.5.7, it is easy to construct polyhedral examples with boundary. Start with a tight polyhedral surface without boundary. Then cut out a finite number of disjoint convex holes from polyhedral faces in such a way that every vertex is contained in the boundary of one of the holes. Then the remaining surface with boundary is tight, according to Proposition 1.5.2. Another example of a polyhedral surface with the TPP is the torus with disk removed illustrated in Figure 7.

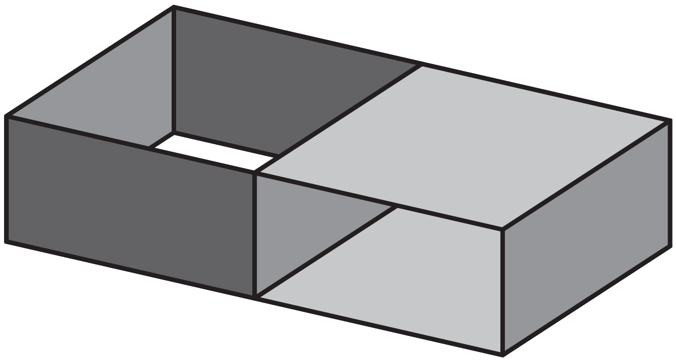

Figure 7. A tight polyhedral embedding of a torus minus a disc.

EXAMPLE 1.5.8. The 5-vertex Möbius band includes all edges joining pairs of vertices, so any simplexwise linear embedding of the 5-vertex Möbius band is tight. Similarly if we remove the open star of a vertex of the 7-vertex torus, we obtain a surface with boundary that contains every edge joining a pair of the remaining six vertices, so any simplexwise linear embedding of this surface with boundary is tight. These are examples of *tight triangulations*.

EXAMPLE 1.5.9. The product of two one-dimensional planar convex polygons γ_1, γ_2 is a tight torus in $\mathbb{E}^4$. Remove the products $\gamma_1 \times \{p\}$, where p ranges over all vertices of γ_2. This leads to a tight polyhedral torus in $\mathbb{E}^4$ with a number of holes.

EXAMPLE 1.5.10. The highest possible codimension for a tight polyhedral surface M without boundary is attained in the cases $N = \frac{1}{2}\big(5 + \sqrt{49 - 24\chi(M)}\,\big)$; see Corollary 1.4.10. By removing sufficiently many disjoint convex holes from one of the 2-faces, we obtain a tight surface in the same highest possible codimension. For the details see [Kühnel 1980].

EXAMPLE 1.5.11. A tight $\mathbb{RP}^2$ with three holes in $\mathbb{E}^5$ can be constructed as follows: Start with the 6-vertex triangulation, regarded as a subcomplex of the 5-simplex. Then cut three convex holes into the faces such that all 6 vertices are covered by the boundaries. Observe that two holes are not sufficient to do this: there is no tight polyhedral Möbius band with one hole, substantial in $\mathbb{E}^5$.

CONJECTURE 1.5.12 (EXTENSION OF POHL'S CONJECTURE). *If for a compact surface M with nonempty boundary $f : M \to \mathbb{E}^N$, with $N \geq 4$, is a tight topological immersion, then its convex hull $\mathcal{H}(fM)$ is a polyhedron.*

By [Kuiper 1971/72; 1980, Theorem 12], this is true for the Möbius band.

1.6. Congruence and rigidity theorems. We now turn to the question of whether or not two isometric tightly immersed surfaces are necessarily congruent.

THEOREM 1.6.1 (CONGRUENCE THEOREM FOR SMOOTH TIGHT TORI). *If two tightly immersed smooth tori in $\mathbb{E}^3$ are isometric, they are congruent if either*

(i) *they are analytic* [Aleksandrov 1938], *or*
(ii) *the immersions are at least five times differentiable*, $\operatorname{grad} K \neq 0$ *at every point for which $K = 0$, and an additional technical condition on asymptotic curves is satisfied* [Nirenberg 1963].

The situation for tight polyhedral surfaces is quite different:

EXAMPLE 1.6.2 (NONCONGRUENCE FOR POLYHEDRAL TIGHT SURFACES [Banchoff 1970b]). *There are pairs of tight polyhedral tori in $\mathbb{E}^3$ that are isometric but not congruent. By attaching handles one can obtain examples of higher genus as well.*

QUESTION 11. *Is it true that any two isometric smooth tight immersions of the torus into $\mathbb{E}^3$ are congruent?*

For smooth immersions, tightness is an intrinsic property: if two surfaces M and $\bar{M}$ are isometric and one of them is tight, the other one must be tight since the Gaussian curvature is intrinsic, so $\int |K| = \int |\bar{K}|$. Tightness of surfaces in higher codimension is definitely not an intrinsic property. Note also that for polyhedral surfaces, tightness is not an intrinsic property; in particular, a convex polyhedron can be isometric to a nonconvex polyhedron.

DEFINITION 1.6.3. A polyhedral surface in $\mathbb{E}^3$ is called *rigid* if it does not allow a globally defined continuous deformation (other than by Euclidean motions) where each edge and each face moves by a Euclidean motion (that is, moves rigidly).

A famous example of R. Connelly [1978/79] disproved the rigidity conjecture for polyhedral spheres in general. However, rigidity does hold for convex polyhedral surfaces [Connelly 1993]. In other words: *A tight polyhedral surface of genus 0 is rigid.*

For the case of higher genus this seems to be an open question:

CONJECTURE 1.6.4 [Kalai 1987]. *Any tight closed polyhedral surface in* $\mathbb{E}^3$ *is rigid.*

It is sufficient to consider only the case of triangulated surfaces because planar n-gons can always be subdivided into triangles. Then Conjecture 1.6.4 just says that the edge graph of any tight triangulated surface is rigid.

It is a trivial consequence of Theorem 1.4.11 that a tight surface is rigid if it is substantial in $\mathbb{E}^N$ with $N = \frac{1}{2}\big(5 + \sqrt{49 - 24\chi(M)}\,\big)$.

Note that a rigidity theorem is weaker than a congruence theorem: Rigidity does not a priori exclude the possibility of two noncongruent positions of the same polyhedron. It just says that there is no continuous one-parameter family of polyhedra of the same type joining them. The examples in [Banchoff 1970b] do allow such a continuous and isometric one-parameter family, but in this case the polyhedral structure is not preserved; faces are creased at continuously varying edges.

The rigidity (even infinitesimal rigidity) of smooth ovaloids in $\mathbb{E}^3$ is a classical result [Liebmann 1900]. For congruence theorems of ovaloids see [Blaschke and Leichtweiß 1973, Section 105] (smooth case) and [Pogorelov 1973] (general convex surfaces).

QUESTION 12. *Are there congruence or rigidity theorems for higher dimensional tight submanifolds, smooth or polyhedral?*

If the rank of the shape operator of a hypersurface is at least 3 everywhere, then a congruence theorem holds even locally. Besides cylinders with rank 1, there are examples of three-dimensional hypersurfaces in $\mathbb{E}^4$ with rank 2 that are isometric but not congruent [Hollard 1991]. For congruence and rigidity of convex hypersurfaces, see [Sen'kin 1972].

1.7. Isotopy, knots, and regular homotopy. Instead of studying immersions for which the minimum of the total absolute curvature is achieved, it is possible to consider lower bounds on the total absolute curvature for smooth immersions in a given isotopy class or regular homotopy class. The first examples of theorems in this area come from knot theory: By Fenchel's Theorem, a closed curve in $\mathbb{E}^N$ has total absolute curvature $\int |\kappa| \geq 2\pi$, with equality only for planar convex curves [Fenchel 1929]. However, for a knotted curve in $\mathbb{E}^3$, the total (absolute) curvature is more than twice that large: $\int |\kappa| > 4\pi$ [Fáry 1949; Fox 1950; Milnor 1950b]. This lower bound is not attained for any knot. More precisely, the infimum of $\frac{1}{2\pi}\int |\kappa|$ within a given isotopy class of embeddings is the *bridge number* of the knot, defined as the minimum number of relative maxima of any height function in this isotopy class. This infimum is attained only for the unknot (the isotopy class containing the circle). The same results hold for polygonal knots, and in fact the method of Milnor [1950b] made essential use of approximation of smooth curves by polygons in the same isotopy class.

These results have been extended to the case of knotted surfaces:

THEOREM 1.7.1 [Langevin and Rosenberg 1976; Meeks 1981; Morton 1979]. *Assume that for a smooth embedded orientable surface* $M \subset \mathbb{E}^3$ *the total absolute curvature satisfies*

$$\frac{1}{2\pi}\int_M |K| < 8 - \chi(M).$$

Then M *is unknotted*, *that is*, M *is isotopic to the 'standard' embedding given in Example* 1.2.3. *The same conclusion holds for polyhedral surfaces satisfying*

$$\frac{1}{2\pi}\sum_v K_*(v) < 8 - \chi(M).$$

The polyhedral case can be derived by the process of smooth approximation [Brehm and Kühnel 1982].

The bound on $\chi(M)$ cannot be improved, since there are surfaces for which the equality is achieved that are not isotopic to the standard embedding. This leads to the following notion [Kuiper and Meeks 1984]:

For a given isotopy class of embeddings, we call an embedded surface *isotopy tight* if the total absolute curvature realizes the infimum in this isotopy class. As mentioned above, a knotted curve is never isotopy tight.

THEOREM 1.7.2 [Kuiper and Meeks 1984; 1987]. (i) *A knotted torus is never isotopy tight*.

(ii) *There exist knotted surfaces of genus* $g \geq 3$ *that are isotopy tight and satisfy*

$$\frac{1}{2\pi}\int_M |K| = 8 - \chi(M).$$

These examples can also be made polyhedral, *satisfying*

$$\frac{1}{2\pi}\sum_v K_*(v) = 8 - \chi(M).$$

We can broaden the question of the existence of tight immersions of a surface by asking if there are tight mappings in a given regular homotopy class of immersions of a surface. Given two immersions f and g of a surface M, we say these immersions are *image homotopic* if there is a homeomorphism ϕ of M such that f and $g \circ \phi$ are regularly homotopic. Pinkall [Pinkall 1986b] classified all surfaces up to image homotopy, and exhibited tight immersions for a given image homotopy class in all but a finite number of cases. His examples are tight polyhedral immersions that can be smoothed preserving tightness by means of a specific algorithm. More recently, Cervone [Cervone 1996] constructed polyhedral immersions for most of the missing cases; only two cases remain unresolved. However, these models do not meet the demands of Pinkall's smoothing algorithm, and some may in fact represent tight polyhedral immersions in an image homotopy class for which there exists no tight smooth immersion. (Compare the case of the projective plane with one handle, for which there is a tight polyhedral immersion but no tight smooth immersion).

The image homotopy class of an immersion of a surface can be described in terms of the number of generators of the one-dimensional homology represented by curves with neighborhoods that are twisted cylinders. For example, if both generators of the first homology of a torus have twisted neighborhoods, the torus is not image homotopic to any embedding; such a torus is called a *twisted torus*. For the Klein bottle, one generator of the first homology has an orientable neighborhood and if this generator is represented by a twisted cylinder, then the immersion is called a *twisted Klein bottle*; such Klein bottles come in both left- and right-handed versions.

THEOREM 1.7.3 (REGULAR HOMOTOPY [Pinkall 1986b; Cervone 1996]). (i) *Apart from the previously mentioned cases of the Klein bottle and the real projective plane, there are no tight immersions of the twisted Klein bottle, the twisted torus, or the connected sum of three projective planes (all of the same handedness).*

(ii) *It is unknown whether there exist smooth or polyhedral tight immersions for the connected sum of three projective planes plus a handle, or for the twisted torus with a handle.*

(iii) *Tight polyhedral immersions exist for all the remaining image homotopy classes of surfaces* [Cervone 1996].

(iv) *There is no tight immersion of the projective plane with one handle, and no smooth examples are known for*

- (i) *the twisted Klein bottle with one, two or three handles,*
- (ii) *the connected sum of three projective planes with one, two or three handles,*
- (iii) *the Klein bottle with one twisted handle or the Klein bottle with one twisted handle and one standard handle, or*
- (iv) *the twisted torus with fewer than four handles.*

In addition to these results about mappings into $\mathbb{E}^3$, we mention another question concerning mappings into Euclidean four-space:

QUESTION 13. *Is there a tight smooth immersion of an orientable surface into $\mathbb{E}^4$ with an odd number of double points?*

The tube around such an example would be $\mathbb{Z}_2$-tight but not $\mathbb{Z}$-tight [Breuer and Kühnel 1997]. There is a polyhedral example of a tight torus in $\mathbb{E}^4$ with exactly one double point; see Example 1.8.6 below.

1.8. Tight mappings of surfaces with singularities. The title of this section has two aspects: On the one hand, it can mean that we consider ordinary nonsingular surfaces and tight mappings on them that fail to be immersions at

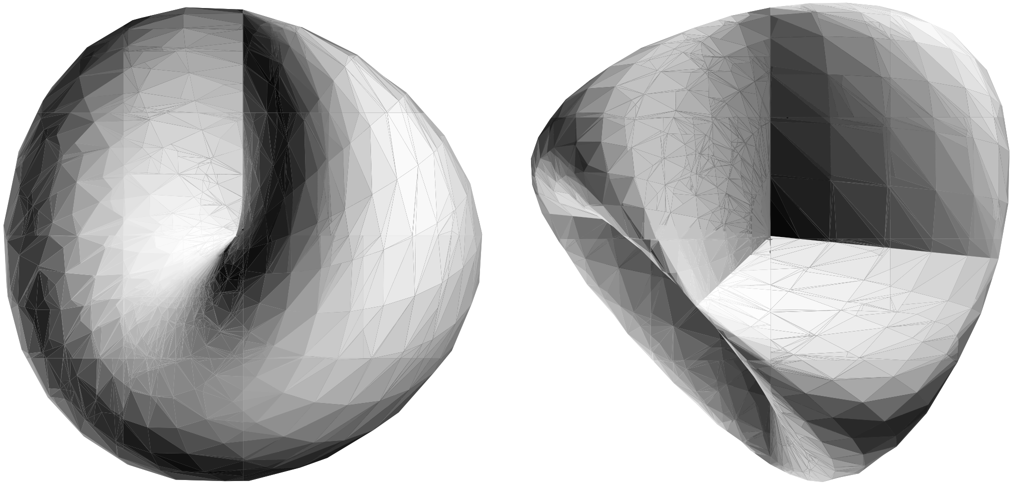

Figure 8. Tight stable mappings of the real projective plane: the cross-cap and Steiner's Roman surface.

certain points. On the other hand, it can mean that we consider surfaces with singular points, like a pinched sphere (a sphere with two points identified), and then consider tight mappings that are immersions, that is, locally one-to-one.

A smooth or polyhedral map $f : M \to \mathbb{E}^N$ from a surface is called *tight* if condition (iii) of Definition 1.1.1 is satisfied, so the preimage of any closed half-space is connected. A trivial example of a tight map on a connected topological space is the mapping that sends the entire space to a single point. More generally, the composition of a tight immersion into $\mathbb{E}^N$ with any linear mapping to a subspace of $\mathbb{E}^N$ produces a tight map.

René Thom proved that almost all differentiable mappings of nonsingular surfaces into $\mathbb{E}^3$ have at most a finite number of singularities, all equivalent to "pinch points", topologically equivalent to a cone over a figure-eight. Such mappings are called *stable*. In particular, any smooth immersion is a stable mapping.

THEOREM 1.8.1 [Kuiper 1975]. *All closed surfaces admit tight stable smooth maps into* $\mathbb{E}^3$.

In particular, Kuiper exhibited tight stable mappings into $\mathbb{E}^3$ for the real projective plane (see Figure 8) and the Klein bottle. Such mappings were analyzed further by Coghlan [1987; 1989], who also proved the following:

THEOREM 1.8.2 [Coghlan 1987; 1989]. *For any surface other than the sphere or the real projective plane, and any integer* $n \geq 2$ *there is a tight stable smooth map into* $\mathbb{E}^3$ *with exactly* n *top-cycles* (in contrast with Theorem 1.3.4).

EXAMPLE 1.8.3 (DUPIN CYCLIDES WITH SINGULARITIES). A torus of revolution obtained by revolving a circle around a disjoint axis in its plane is tight, and as we let the radius of the circle increase, we obtain a family of embedded tori of revolution having as a limit a singular mapping, where the circle becomes tangent to the axis and an entire circle is sent to a single point. As a limit of TPP mappings, this map is also a tight differentiable mapping of the torus into $\mathbb{E}^3$. This is called a "limit torus" in [Cecil and Ryan 1985].

The image of this singular tight mapping is topologically equivalent to a pinched sphere. The natural inclusion of this singular surface into $\mathbb{E}^3$ has the TPP so it is 0-tight. But it is not 1-tight since either topset is a circle that bounds in the object but not in the half-space containing the topset and not containing the rest of the object.

A related important class of tight singular surfaces is given by the *cyclides of Dupin*, obtained as images of a torus of revolution under inversion of $\mathbb{E}^3$ through spheres with centers not lying on the torus. The parallel surfaces of such cyclides also give tight smooth embeddings of the torus into $\mathbb{E}^3$. There are two limiting positions for such parallel surfaces, called *limit horn cyclide* and *limit spindle cyclide* [Cecil and Ryan 1985]. These are described by differentiable mappings of a torus into $\mathbb{E}^3$ that are singular along an entire circle that is mapped to a single point, and such mappings are tight. (They have the TPP, as limits of maps with the TPP).

These singular cyclides as point sets are topologically equivalent to the pinched sphere. The natural inclusions of these singular surface into $\mathbb{E}^3$ are both 0-tight but not 1-tight.

Note that conditions (iii) and (iv) in Definition 1.1.1 are not equivalent for such surfaces with singularities because this equivalence depends on the validity of Poincaré duality, which does not hold in this case. However, if we use the intersection homology [Goresky and MacPherson 1980] instead of ordinary singular homology, Poincaré duality is still valid for this type of surface with singularities, and it is possible to consider a notion of *intersection tightness*, where we consider condition (iii) of Definition 1.1.1 applied to intersection homology groups.

QUESTION 14. *Does any given closed surface with any given number of pinch points admit an embedding into some Euclidean space with intersection tightness*?

With respect to ordinary singular homology, we have the following lemma:

LEMMA 1.8.4. *Let $M \subset \mathbb{E}^N$ be a connected closed surface with a pinch point that is an isolated local maximum for a some height function. Then M is not tight.*

PROOF. For a connected closed surface with a finite number of pinch points the second homology with coefficients mod 2 is always one-dimensional. On the other hand, the maximum pinch point is a critical point of index two and multiplicity at least two, in contradiction to equality in the Morse inequalities. This argument is the same for smooth and polyhedral surfaces. □

Note that it is easy to obtain 0-tight examples of this type, e.g., the limit horn cyclide.

COROLLARY 1.8.5. *Let $M \subset \mathbb{E}^N$ be a connected and tight closed surface with finitely many pinch points. Then each pinch point of M lies in the relative interior of the convex hull of ordinary points of M.*

The situation is altered for three-dimensional objects. There are three-manifolds with isolated singularities that have tight triangulations, and tight mappings into $\mathbb{E}^N$ for which there are height functions with an extremum at a singular point [Kühnel 1995, § 7.16].

EXAMPLE 1.8.6 [Kühnel 1995, Section 2F]. *There are projections of the 7-vertex torus into $\mathbb{E}^4$ with one double point* (by projecting along a line joining the centers of two triangles with disjoint closures). This can also be considered as a tight inclusion into $\mathbb{E}^4$ of a pinched torus, that is, a torus with two points identified.

EXAMPLE 1.8.7 [Kühnel 1992]. *There is a two-dimensional simplicial complex that is a surface with one singular cycle and a polyhedral immersion into $\mathbb{E}^4$ that is $\mathbb{Z}_3$-tight but not $\mathbb{Z}_2$-tight.*

This example is a 12-vertex triangulation containing all $\binom{12}{2}$ edges. However, as in Corollary 1.8.5, the three singularities have to lie in the interior of the 9 nonsingular vertices.

2. Tightness in Higher Dimensions

While the tightness condition and the TPP are identical for surfaces without boundary, and closely related in the case of more general two-dimensional objects like surfaces with boundary and surfaces with singularities, the situation is different when we consider higher-dimensional manifolds. No longer does control over the numbers of maxima and minima of height functions on an object enable us to conclude such strong results about the topology of the object. More subtle conditions have to be brought in, even in the case of smooth hypersurfaces. Although a number of important classes of tight examples have been discovered over the years, there are many areas where little is known about tightness in higher dimensions.

Nonetheless, in certain cases we have a good understanding of tightness, for example for manifolds topologically equivalent to spheres. The first results of this type in higher dimensions are found in [Chern and Lashof 1957; 1958].

2.1. The Chern–Lashof theorem and related results. As a first generalization of surfaces in $\mathbb{E}^3$, consider the case of a smooth hypersurface M embedded in $\mathbb{E}^N$. At each point there is an outward unit normal vector and we may determine a Gauss mapping sending each point to the point of the $(N-1)$-sphere with the same outward unit normal vector. As in the two-dimensional case, the

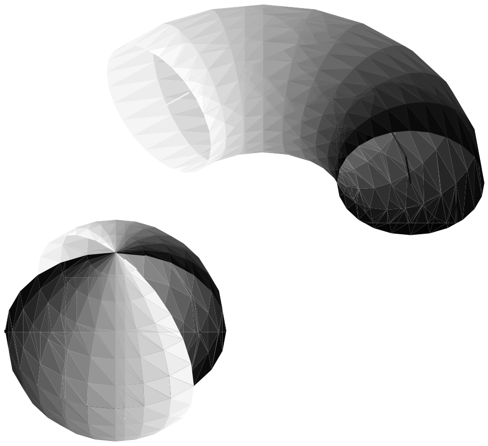

Figure 9. The spherical image of the normal tube around a space curve.

absolute measure of the image of a region U on the hypersurface, divided by the measure of the entire sphere, is called the *total absolute curvature* of the region, denoted $\mathrm{TA}(U)$. The total absolute curvature of the hypersurface itself can be expressed as one half the average number of critical points of height functions restricted to the hypersurface. For a strictly convex hypersurface, there will be exactly two critical points for every height function and $\mathrm{TA}(M) = 2$. The theorem of Chern and Lashof shows that the condition $\mathrm{TA}(M) = 2$ characterizes convex hypersurfaces even in the case where some height functions have degenerate critical points, as for example when the hypersurface contains flat pieces on which the Gauss mapping is constant.

Moreover, the Chern–Lashof results extend to immersions of surfaces of higher codimension. Let $f : M \to \mathbb{E}^N$ be an immersion of an n-dimensional submanifold M, where $n \leq N - 1$. Then, for sufficiently small r, the collection of normal vectors of length r perpendicular to the tangent space at a point forms a sphere of dimension $N - 1 - n$. If r is chosen sufficiently small, the union of all these normal spheres of radius r forms an immersed hypersurface called *the normal tube of radius r about M*. The total absolute curvature of the union of $(N-1-n)$-dimensional spheres at points of a region U is called the *total absolute curvature of U*, again denoted $\mathrm{TA}(U)$.

It is useful to recall the lowest dimension in which we may consider submanifolds that are not hypersurfaces, namely the case of curves in $\mathbb{E}^3$. There the normal tube of radius r about a curve M^1 is a torus of radius r, and a calculation shows that the total absolute curvature of the torus equals the integral of the absolute value of the curvature of the curve with respect to arclength (Figure 9).

Fenchel's Theorem says that, for M one-dimensional, we have $\mathrm{TA}(M) \geq 2$, with equality if and only if M is a plane convex curve. This result is generalized by the famous theorem by S.-S. Chern and R. K. Lashof, in part anticipated by Milnor [1950a]. We state it in its original form as follows:

THEOREM 2.1.1 [Chern and Lashof 1957; 1958]. *Let $f : M^n \to \mathbb{E}^N$ be a smooth immersion of a compact manifold. Then the total absolute curvature*

$$\mathrm{TA}(f) := \frac{1}{c_{N-1}} \int_{\perp f} |K|$$

satisfies:

(i) $\mathrm{TA}(f) = (1/c_{N-1}) \int_z \sum_i \mu_i(zf)$.
(ii) $\mathrm{TA}(f) \geq \sum_i b_i(M;F)$ *for any field F.*
(iii) $\mathrm{TA}(f) < 3$ *implies that M is homeomorphic to the sphere S^n.*
(iv) $\mathrm{TA}(f) = 2$ *if and only if f is an embedding and $f(M)$ is the boundary of a convex body in an $(n+1)$-dimensional Euclidean subspace.*

If one assumes that (i) is given, then (ii) follows directly from the Morse inequalities $\mu_i(\phi) \geq b_i(M;F)$ for the number of critical points of any Morse function ϕ defined on M. In fact, almost all height functions in the sense of the Lebesgue measure satisfy this condition. Assertion (iii) follows from (i) by Reeb's theorem, which says that if a compact n-manifold that admits a Morse function with two critical points is homeomorphic to the sphere S^n.

The proof of (iv) requires more subtle geometric arguments. The main problem is how to deal with the parts of the manifold where the second fundamental form (or the Gauss mapping) degenerates. Assertion (iv) can be regarded as one of the many characterizations of convexity [Mani-Levitska 1993]. In particular, there is no immersion f of an exotic sphere with $\mathrm{TA}(f) = 2$, an observation already mentioned in [Kuiper 1959]. Generalizations for the case of less regular immersions (e.g., topological immersions) can be found in [Kuiper 1980; Lastufka 1981]. A generalization of (iii) can be obtained by using more general versions of Reeb's theorem.

Here is one of the consequences of Theorem 2.1.1:

COROLLARY 2.1.2. *If* $\mathrm{TA}(f) < 4$, *then M is either homeomorphic to S^n or it is a manifold with Morse number* 3, *a "manifold like a projective plane" in the sense of* [Eells and Kuiper 1962].

Such manifolds with Morse number 3 can occur only as the compactification of $\mathbb{E}^2$, $\mathbb{E}^4$, $\mathbb{E}^8$, or $\mathbb{E}^{16}$ by a "sphere at infinity" S^1, S^2, S^4 or S^8.

Even in the case where the total absolute curvature is not at its minimum, restrictions on the total absolute curvature can place conditions on smooth immersions of the sphere. For example, in codimension two, we have the following result of Ferus:

THEOREM 2.1.3 [Ferus 1968]. *Suppose that Σ is an n-manifold homeomorphic to the sphere S^n and admitting an immersion $f : \Sigma \to \mathbb{E}^{n+2}$ with* $\mathrm{TA}(f) < 4$. *Then Σ is diffeomorphic with the standard sphere.*

QUESTION 15. *Does there exist an immersion of an exotic sphere $f : \Sigma^n \to \mathbb{E}^{n+2}$ with* $\mathrm{TA}(f) = 4$? Compare Example 2.7.6.

Note that the restriction on the codimension is essential in the above theorem; if we allow higher codimension, we can always find an immersion with total absolute curvature arbitrarily close to the minimum value. Given any immersion $f : M^n \to \mathbb{E}^N$, and given any smooth function $g : M^n \to \mathbb{E}^1$, we may obtain an immersion $f = cg : M^n \to \mathbb{E}^{N+1}$, where c is a large constant. Then in $\mathbb{E}^{N+1}$, nearly all of the height functions restricted to $f = cg(M^n)$ will have the minimal number of critical points, and all the height functions with a larger number can be clustered in a region with arbitrarily small volume. In particular, if $f : S^n \to \mathbb{E}^N$ is an immersion of an exotic sphere, we can choose a function g on S^n with two critical points, and for sufficiently large c, we can make $\mathrm{TA}(f + cg) \leq 2 + \varepsilon$ for arbitrarily small positive ε.

In the case of immersions of noncompact manifolds, the Chern–Lashof inequality for the total absolute curvature is no longer valid in its original form. However, it is possible to derive an analogue by regarding the geometry and topology of the ends. We assume that $f : M \to \mathbb{E}^N$ is a proper immersion of a noncompact manifold with finitely many ends $\infty_1, \dots \infty_k$ and finitely many limit directions in the sense of [Wintgen 1984]. A *limit direction* is a possible accumulation point of a sequence of normalized position vectors converging to an end. The number of limit directions is finite if the immersion converges to one direction at each end. In this case the Gauss–Bonnet formula remains valid [Wintgen 1984], and the following inequality holds:

THEOREM 2.1.4 [van Gemmeren 1996]. *Let $f : M \to \mathbb{R}^N$ be a proper immersion with finitely many limit directions. Then*

$$\mathrm{TA}(f) \geq \sum_i \mu_i(M) \geq \sum_i \bigl| b_i(M) - \tfrac{1}{2} b_i(\infty) \bigr|,$$

and the equality $\mathrm{TA}(f) = 1$ *for one end is possible only for convex hypersurfaces.*

The case of a cylinder shows that $\mathrm{TA}(f){=}0$ is possible if there are two ends.

2.2. Tightness and k-tightness. The case of equality in Theorem 2.1.1(ii) is very special, because it forces almost all height functions zf to have the minimal number of critical points of index i, namely b_i, for all indices i. In this case every critical point is really necessary from the topological point of view, in that each critical point generates one additional homology class of the manifold. Such functions have been called "perfect functions" (of "linking type" in the sense of [Morse and Cairns 1969].) The condition of equality in Theorem 2.1.1(ii) is what we mean by *tightness* in higher dimensions.

DEFINITION 2.2.1. A smooth (at least C^2) immersion $f : M \to \mathbb{E}^N$ is said to be *tight* with respect to a field F if any of the following equivalent conditions is satisfied:

(i) $\mathrm{TA}(f) = \sum_{i \geq 0} b_i(M; F)$.

(ii) Every nondegenerate height function $zf : x \mapsto \langle fx, z \rangle$ in the direction of a unit vector $z \in \mathbb{R}^N$ has exactly b_i critical points of index i.
(iii) For every open half-space h the induced morphism $H_*(f^{-1}(h)) \to H_*(M)$ is injective, where H_* denotes the singular homology with coefficients in F.

It is also possible to obtain significant, although weaker, information about an immersion when there is a condition on the numbers of critical points of height functions of index less than or equal to some fixed number k. We have already seen that the TPP is equivalent to the condition that almost every height function restricted to the object has exactly one critical point of index 0, and this notion is known as 0-*tightness*. More generally we may define k-*tightness* for other values of k:

An immersion $f : M \to \mathbb{E}^N$ of an n-dimensional manifold is called k-*tight* with respect to F if in (ii) the number of critical points satisfies $\mu_0 = b_0$, $\mu_1 = b_1, \dots,$ $\mu_k = b_k$ or, equivalently, if in (iii) the induced morphism $H_i(f^{-1}(h)) \to H_i(M)$ is injective for $i = 0, 1, \dots, k$.

In terms of critical point theory, tightness means that every critical point of a nondegenerate height function is homology-generating ("linking type"), whereas k-tightness means that this holds for critical points of index less or equal to k. By condition (iii) it follows that tightness (and k-tightness as well) is invariant under projective transformations of the ambient space. Although we have been concentrating on smooth immersions, the definition of k-tightness also makes sense for polyhedral immersions.

Lemma 2.2.2 (duality). *Let $f : M \to \mathbb{E}^N$ be an immersion (smooth or polyhedral) of a compact n-dimensional manifold satisfying Poincaré duality, so that $H_i(M; F) \cong H_{n-i}(M; F)$ for every i. Then the following conditions are equivalent*:

(i) *f is tight with respect to F.*
(ii) *f is k-tight with respect to F for one particular $k \geq \frac{1}{2}(n-2)$.*

The proof relies on the duality for critical points $\mu_i(-zf) = \mu_{n-i}(zf)$ and the Poincaré duality $b_i = b_{n-i}$ in combination with the Euler–Poincaré equation $\sum_i (-1)^i \mu_i = \chi = \sum_i (-1)^i b_i$.

As an example, the Veronese surface is 0-tight for any field, but 1-tight and 2-tight only for fields of characteristic 2.

Theorem 2.2.3. *The space of all smooth tight immersions of compact manifolds into Euclidean spaces is closed under the following operations*:

(i) *composition with projective transformations $P : \mathbb{E}^N \to \mathbb{E}^N$ (not sending any point of the manifold to infinity)*;
(ii) *composition with linear embeddings $j : \mathbb{E}^N \to \mathbb{E}^{N+1}$*;
(iii) *cartesian products $f_1 \times f_2 : M_1 \times M_2 \to \mathbb{E}^{N_1+N_2}$.*

Moreover, for a tight smooth immersion, the tube of sufficiently small radius r gives a tight immersion of the unit normal bundle. More precisely, if $f : M \to \mathbb{E}^N$ *is tight then the* ε*-tube around* $j \circ f : M \to \mathbb{E}^{N+1}$ *is tight for any linear embedding* j. *If* f *itself is a tight embedding then the* ε*-tube around* f *itself is tight* [Pinkall 1986a; Breuer and Kühnel 1997].

Note that $\mathrm{TA}(j \circ f) = \mathrm{TA}(f)$ by the choice of the normalization and that $\mathrm{TA}(f_1 \times f_2) = \mathrm{TA}(f_1)\,\mathrm{TA}(f_2)$. Formally, one follows from the other if j is regarded as the trivial embedding of the one-point space into $\mathbb{E}^1$. The proof of (iii) uses also the Künneth formula $b_k(M_1 \times M_2) = \sum_{i+j=k} b_i(M_1) b_j(M_2)$. For a more general version of (iii) see [Ozawa 1983].

2.3. Smooth examples. If we start with the tight examples in Section 1.2, Theorem 2.2.3 immediately leads to a large variety of higher-dimensional examples, just by taking products and tubes. The argument concerning tubes can also be extended to the case of submanifolds of spheres (connected with the notion of taut submanifolds).

EXAMPLE 2.3.1 (SPHERE PRODUCTS). The Cartesian product of an arbitrary number of tightly embedded spheres (smooth or polyhedral) leads to a tight embedding where the codimension equals the number of factors in the product. We can obtain examples of hypersurfaces topologically equivalent to products of spheres by iterating the tube construction. Special cases are tight n-tori $T^n \cong (S^1)^n$ recursively defined as follows: T^1 is the unit circle in the plane, and T^{n+1} is the ε-tube around $T^n \subset \mathbb{E}^{n+1} \subset \mathbb{E}^{n+2}$ with radius $\varepsilon = 2^{-n}$.

There is also a polyhedral version of the tube construction, where a circle in the normal plane is replaced by a square.

There also exists a smooth tight (and taut) immersion of a twofold quotient of the product $S^{p-1} \times S^{q-1}$ defined as the tensor product $S^{p-1} \otimes S^{q-1}$ in $\mathbb{E}^{pq}$ [Kühnel 1994b].

EXAMPLE 2.3.2 (CONNECTED SUMS OF HANDLES). As in Example 1.2.3, we can easily obtain tight hypersurfaces in $\mathbb{E}^4$ that are diffeomorphic to the standard three-manifold of Heegaard genus g, the connected sum of g copies of $S^1 \times S^2$: start with a convex hypersurface containing two flat regions in parallel hyperplanes. Then attach handles invariant under SO(3)-rotation by rotating the same curve as in Example 1.2.3 under SO(2)-rotation. Such a handle contributes $c_3 = \mathrm{Vol}(S^3)$ to the total absolute curvature. Nonorientable versions can be obtained by starting with a suitable $S^1 \times S^2$ containing two flat regions, one in the outside, the other in the inside. Then a handle joining outside and inside can be attached tightly. This extends Kuiper's construction of a tight Klein bottle with one handle (or more handles) to the case of arbitrary dimensions. Similar constructions are possible in codimension two and for connected sums of k-handles; see also Example 3.2.2.

Question 16. *Is there any smooth tight immersion of a connected sum of at least two handles* $S^k \times S^1$ $(k \neq 1)$ *in codimension at least three*?

Recall that there is a tight substantial embedding $S^1 \times S^{2k-1} \to \mathbb{E}^{4k}$, defined as the complexification of S^{2k-1} or the tensor product $S^1 \otimes S^{2k-1}$. For the case $k = 1$, compare Proposition 3.3.1.

Question 17. *Is there a tight immersion of a lens space* $L(p, 1)$ *for* $p \neq 2$? *Is there any smooth tight immersion of a manifold with* p-*torsion for* $p \neq 2$?

There is no tight hypersurface that is a lens space, according to [Coghlan 1991].

Example 2.3.3 (Isoparametric submanifolds). Any isoparametric hypersurface in the sphere S^{N-1} is tight (and taut), regarded as a submanifold of $\mathbb{E}^N$. It is a tube around a so-called focal manifold, which is also tight (and taut) [Cecil and Ryan 1985]. Any isoparametric submanifold of arbitrary codimension is also tight (and taut) [Terng 1993]. Particular classical examples of isoparametric hypersurfaces are the tubes around the Veronese embeddings of projective planes into S^4, S^7, S^{13}, S^{25}. These are precisely the hypersurfaces of spheres with three distinct constant principal curvatures (compare also Examples 1.2.4 and 3.2.3). These Veronese embeddings are special cases of the following construction:

Example 2.3.4 (Grassmannians). Let F denote either $\mathbb{R}$, $\mathbb{C}$, or the quaternions $\mathbb{H}$. Then the unoriented Grassmann manifold $\mathcal{G}_{p,q}(F)$ is defined as the set of all p-dimensional linear subspaces through the origin in F^{p+q}. Every such subspace A can be identified with the matrix representing the orthogonal projection from F^{p+q} onto A. This leads to the so-called *standard embedding*

$$\mathcal{G}_{p,q}(F) \to \mathbb{E}^{(p+q)^2},$$

which is substantial in a linear subspace of dimension $p + q - 1 + d\binom{p+q}{2}$, where $d = 1, 2, 4$ for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. This standard embedding turns out to be tight with $\mathrm{TA}(f) = \binom{p+q}{p} = \sum_i b_i(\mathcal{G}_{p,q}(F); \mathbb{Z}_2)$. As an exceptional case, there is a standard embedding of the Cayley plane into $\mathbb{E}^{26}$ [Tai 1968; Kuiper 1970; Cecil and Ryan 1985, § 9.4].

Example 2.3.5 (Unitary groups and R-spaces). The unitary group $F\mathrm{U}(n)$ for $F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ can be regarded as a submanifold of Euclidean dn^2-space, just by regarding an element as a quadratic unitary matrix over F. This is also tight with total absolute curvature $\mathrm{TA} = 2^n = \sum_i b_i(F\mathrm{U}(m), \mathbb{Z}_2)$. In the real case $F\mathrm{U}(m) = \mathrm{O}(m)$ is not connected, so one may restrict the construction to $\mathrm{SO}(m)$.

In much more generality, the standard embedding of any symmetric R-space is tight (and taut) [Bott and Samelson 1958; Takeuchi and Kobayashi 1968]. Homogeneous Kähler manifolds can be tightly embedded [Kobayashi 1967]. For tight embeddings of other homogeneous spaces, see [Wilson 1969]. Not all homogeneous spaces can be embedded tautly [Thorbergsson 1988]. Ferus [1982] characterized the case of standard embeddings of symmetric R-spaces as the extrinsically symmetric submanifolds. For the connection between tautness and Dupin submanifolds, see [Cecil 1997] in this volume.

Example 2.3.6 (Intermediate tightness). Let $D^n \subset \mathbb{E}^n$ denote a tightly embedded n-ball that is rotationally symmetric with respect to the SO$(n-1)$-action around a fixed axis. If we rotate it in higher-dimensional space $\mathbb{E}^{n+m}$, we obtain a tight $(n+m)$-ball D^{n+m} and a tight $(n+m-1)$-sphere as boundary. Similarly, if the D^n is only k-tight then D^{n+m} will be $(k+m)$-tight. The boundary in this case is $(k+m-1)$-tight, either by direct calculation or by applying Proposition 2.6.1 below. In particular, if $D^2 \subset \mathbb{E}^2$ is not tight (that is, not convex) and SO(1)-symmetric (that is, congruent to its mirror image), the induced $D^3 \subset \mathbb{E}^3$ is 0-tight but not 1-tight, and its boundary is not 0-tight. The induced $D^4 \subset \mathbb{E}^4$ is 1-tight but not 2-tight, and its boundary is a 0-tight three-sphere that is not tight; compare [Kuiper 1970].

Curtin [1991] found similar examples with intermediate tautness, e.g., 0-taut three-spheres that are not taut. These examples are ellipsoids with rotational symmetry in two orthogonal planes. Inverse stereographic projection to $\mathbb{E}^5$ leads to 0-tight but not 1-tight three-spheres in codimension two, giving a positive answer to a question of Kuiper [1970].

2.4. The substantial codimension and the Little–Pohl theorem. The argument leading to the upper bound of the substantial codimension in Section 1.3 remains valid in arbitrary dimensions: The mapping $\xi \mapsto A_\xi$ is injective.

Theorem 2.4.1 [Kuiper 1959; 1970]. *Let $f : M^n \to \mathbb{E}^N$ be a 0-tight smooth and substantial immersion. Then $N - n \le \binom{n+1}{2}$.*

Observe that equality is realized by the standard embeddings of $\mathcal{G}_{1,n}(\mathbb{R}) = \mathbb{RP}^n$ in Example 2.3.4.

Theorem 2.4.2 [Little and Pohl 1971; 1985, p. 98]. *Let $f : M^n \to \mathbb{E}^N$ be a 0-tight smooth and substantial immersion with $N = n + \binom{n+1}{2}$. Then f is the standard embedding of $\mathbb{RP}^n$ (up to projective transformations of $\mathbb{E}^N$).*

Sketch of proof. One of the main tools is the 2-*jet* of the immersion spanning the so-called *osculating space* of second order. At an extreme point (a maximum of a nondegenerate height function) the TPP implies that the third-order derivatives stay in the same half-space as the first- and second-order derivatives. This has the consequence that $f(M)$ is contained in the osculating space at that point. Another step is to show that the dimension of the osculating space is in fact the special number $N = n + \binom{n+1}{2}$ at every extreme point. A sophisticated combination of this type of arguments with classical and very special properties of the Veronese embeddings of $\mathbb{RP}^n$ ultimately leads to the local (and global) coincidence of $f(M)$ with the Veronese embedding. □

CONJECTURE 2.4.3. *The substantial codimension of any tight topological immersion of FP^n into Euclidean space is bounded by the codimension of the corresponding standard embedding in Example* 2.3.3.

This is true for $n = 1$ (tight spheres [Kuiper 1980]) and $n = 2$ (see Theorem 3.3.2). Compare [Arnoux and Marin 1991] for essentially the same bounds for the number of vertices of triangulations of FP^n.

QUESTION 18. *Are there nonsmooth tight substantial immersions of* $\mathbb{RP}^3$ *into* $\mathbb{E}^9$? Note that any smooth immersion is congruent to the standard embedding by Theorem 2.4.2. The "canonical" polyhedral candidate would be a 10-vertex triangulation, which, however, does not exist [Walkup 1970].

2.5. Tight polyhedra.

DEFINITION 2.5.1. For a compact polyhedron $M \subset \mathbb{E}^N$ *tightness* means that condition (iii) of Definition 2.2.1 is satisfied: For every open half-space h and for any i, the induced morphism $H_i\big(f^{-1}(h)\big) \to H_i(M)$ is injective, where H_* denotes the singular homology with coefficients in F. Naturally, k-tightness means the morphism is injective for $i = 0, 1, \dots k$.

Equivalently, one can define tightness by the equality $\mu_i(zf) = b_i(M; F)$ for any height function in general position. In this case the number of critical points of index i is defined as

$$\mu_i(zf) := \sum_v \dim_F H_i\big(M_v,\, M_v \backslash \{v\}\big),$$

where the sum ranges over all vertices of the polyhedron, and where $M_v := \{p : (zf)(p) \le (zf)(v)\}$ denotes the sublevelset determined by z and v. For details of this type of critical point theory, including Morse relations and duality, see [Kuiper 1971; Banchoff 1967; Kühnel 1990; 1995]. It seems that the Chern–Lashof Theorem 2.1.1 and Corollary 2.1.2 remain valid for polyhedral immersions of manifolds. In particular, we have the following result:

PROPOSITION 2.5.2 (TIGHT SPHERES). *Any tightly embedded polyhedral sphere* $\Sigma \subset \mathbb{E}^N$ *of dimension n is the boundary of a convex polytope in some* $(n+1)$-*space*.

PROOF. The proof of Corollary 1.4.5 can be carried over directly from dimension two to arbitrary dimension.

An alternative proof can be formulated using Corollary 3.1.3 below: Since the sphere is $(n-1)$-connected, the tight polyhedral sphere in $\mathbb{E}^N$ must contain the n-dimensional skeleton of the convex hull $\mathcal{H}$, which is a convex N-polytope. For $N > n + 1$ this n-skeleton contains several distinct n-spheres, each as the boundary of one of the $(n + 1)$-dimensional faces. This is impossible, hence $N = n + 1$, and the image coincides with the boundary of $\mathcal{H}$. ◻

For the case of even dimension $n = 2k$ one may compare this argument with the theorem of van Kampen and Flores [Grünbaum 1967, § 11.2], stating that there is no topological embedding of the underlying set of $\mathrm{Sk}_k(\triangle^{2k+2})$ into $\mathbb{E}^{2k}$ or S^{2k}.

EXAMPLE 2.5.3 (k-TIGHT SPHERES [Banchoff 1971a; Kühnel 1995]). *For any given $N \geq n+2$ and any $k \leq (n-3)/2$, there exists a k-tight substantial polyhedral embedding of the sphere S^n into $\mathbb{E}^N$. This inequality is the best possible, according to the polyhedral version of Lemma* 2.2.2.

Such an example can be chosen as the boundary complex of the cyclic polytope $C(N{+}1, n{+}1)$, regarded as a subcomplex of the N-dimensional simplex. Since the cyclic polytope contains all $\binom{N+1}{k+2}$ $(k+1)$-simplices for $k+1 \leq (n-1)/2$ [Grünbaum 1967], it follows that the embedding into $\mathbb{E}^N$ is k-tight.

We remark also that any given compact three-manifold admits a 0-tight polyhedral embedding into $\mathbb{E}^N$ if N is large enough, just by a two-neighborly triangulation [Walkup 1970; Sarkaria 1983; Kühnel 1995].

EXAMPLE 2.5.4 (TIGHT TRIANGULATIONS). As in Example 1.2.8, one can regard every n-vertex triangulation of a manifold as a subcomplex of the $(n-1)$-dimensional simplex in $\mathbb{E}^{n-1}$. If this embedding into $\mathbb{E}^{n-1}$ is tight, we call the triangulation a *tight triangulation*; compare Corollary 1.4.10 and [Kühnel 1995]. In this case every simplexwise linear embedding into any Euclidean space is tight. Particular cases are $(d+1)$-dimensional 1-handlebodies with $n = 2d+3$ vertices and their boundaries: Regard the n vertices as elements of $\mathbb{Z}_n$ and let this group act on the starting simplex $\langle 0\,1\,2 \ldots d{+}1\rangle$. The union of this $\mathbb{Z}_n$-orbit of simplices is a tight triangulation of a one-handle, its boundary is tight as well [Kühnel 1995, Chapter 5]. The 9-vertex triangulation of the boundary of a (nonorientable) one-handle is the only triangulation of any three-manifold with 9 vertices that is not a sphere [Altshuler and Steinberg 1976].

We mention an interesting 8-vertex triangulation of a three-pseudomanifold, which is also a tight triangulation [Emch 1929; Kühnel 1995, § 7.16]. It contains $\binom{8}{2}$ edges, $\binom{8}{3}$ triangles and 28 tetrahedra, each vertex link is a 7-vertex torus. If we regard it as a subcomplex of the 7-simplex, then any slice by a hyperplane in general position is a tight polyhedral two-manifold substantial in $\mathbb{E}^6$.

EXAMPLE 2.5.5 (TIGHT SUBCOMPLEXES OF THE CUBE). We now present L. Danzer's general construction for getting tight polyhedra as subcomplexes of higher dimensional cubes [McMullen and Schulte 1989]. Let K be a simplicial complex with n vertices $1, 2, \ldots, n$. Each k-simplex of K can be identified with a subset $\triangle = \{i_0, \ldots, i_k\}$ of $\{1, \ldots, n\}$. Set

$$A_j(\triangle) := \begin{cases} [0,1] & \text{if } j \in \{i_0, \ldots, i_k\}, \\ \{0,1\} & \text{otherwise;} \end{cases}$$

furthermore, set $F(\triangle) := A_1(\triangle) \times \cdots \times A_n(\triangle)$ and $2^K := \bigcup_{\triangle \in K} F(\triangle)$. By definition, we may regard each $F(\triangle)$ and therefore the entire 2^K as a subcomplex of the n-dimensional cube $C^n := [0,1]^n$, as well as a subset $2^K \subset C^n \subset \mathbb{E}^n$ of the ambient Euclidean space.

A particular case is $M(n) := 2^{\{n\}}$, where $\{n\}$ denotes the boundary of an n-gon. This is a tight surface of genus $2^{d-3}(d-4)+1$ in the boundary of the n-cube [Coxeter 1937; Banchoff 1965; Ringel 1955a; Beineke and Harary 1965].

An unexpected statement is the following:

THEOREM 2.5.6 [Kühnel 1995]. *The subset 2^K is tight in $\mathbb{E}^n$ for any simplicial complex K with n vertices. This holds for any choice of a field F. If K is a triangulated sphere, 2^K is a topological manifold.*

In particular, this leads to strange examples as follows:

EXAMPLE 2.5.7 (TORSION). Let K be any simplicial sphere such that a certain subset of vertices spans a subcomplex with p-torsion in the homology. Then 2^K is a tightly embedded manifold with p-torsion in the homology.

EXAMPLE 2.5.8 (HOMOLOGY MANIFOLDS). Let K be a triangulated homology sphere that is not a sphere. Then 2^K is a tight homology manifold that is not a manifold.

EXAMPLE 2.5.9 (TOPOLOGICAL MANIFOLDS THAT ARE NOT PL). The double suspension K of a certain homology 3-sphere is a triangulated 5-sphere [Edwards 1975] and it is not PL with respect to this simplicial decomposition, since the link of some edge is the original homology sphere. Then 2^K is a tightly embedded topological 6-manifold that is not PL with respect to this induced polyhedral structure.

2.6. Manifolds with boundary and tubes around submanifolds. For manifolds with boundary M, *tightness* is again defined by condition (iii) of 2.2.1: For every open half-space h the induced morphism $H_*\big(f^{-1}(h)\big) \to H_*(M)$ is injective, where H_* denotes the singular homology with coefficients in F.

This is equivalent to the equality $\mathrm{TA}(f) = \sum_i b_i(M;F)$. We remark that the equation $\mathrm{TA}(M) = \mathrm{TA}(M \setminus \partial M) + \frac{1}{2}\,\mathrm{TA}(\partial M)$ remains valid in general.

One of the basic cases to be considered in both the smooth and the polyhedral situations is the case of n-manifolds with boundary in $\mathbb{E}^n$.

PROPOSITION 2.6.1. *Let $M \subset \mathbb{E}^n$ be a compact n-manifold with boundary ∂M. Then M is tightly embedded if and only if ∂M is tightly embedded.*

In the case $n=2$, this proposition states that a compact tightly embedded two-manifold with boundary in $\mathbb{E}^2$ must be a closed convex set with a collection of open convex sets removed, so that the closures of these sets are disjoint from one another and from the boundary of the original convex set.

The condition that the mapping be an embedding is essential. For smooth immersions with self-intersections the proposition is not true, even for $n=2$ (see the immersed surfaces with boundary in the plane 1.5.5, where the inner boundary curves are locally convex but not necessarily globally convex).

The proof of Proposition 2.6.1 follows from the equation $\mathrm{TA}(M) = \mathrm{TA}(M \setminus \partial M) + \frac{1}{2}\,\mathrm{TA}(\partial M)$ on the one hand and the equation $\sum_i b_i(\partial M) = 2\sum_i b_i(M)$ on the other hand. The latter one follows from the Alexander duality for M and its complement.

Proposition 2.6.2. *For any tight immersion f of a manifold without boundary into $\mathbb{E}^N$ for $N \geq 3$, the Euclidean solid $f_{\leq\varepsilon}$ of radius ε-tube is a tight immersion. If there are no self-intersections, or if $\sum_i b_i(\partial M) = 2\sum_i b_i(M)$, the same holds also for the boundary of the tube.*

The analogous result holds for tight subcomplexes of the cube (see Section 2.5.5) *if we replace the Euclidean tube by the polyhedral tube, that is, the tube with regard to the maximum norm.*

Theorem 2.6.3 [Breuer and Kühnel 1997]. *The boundary of the ε-tube around any smooth tight immersion of a compact two- or three-manifold is again tight if the codimension is at least two.*

For four-manifolds, compare Question 21 in Section 3.4 below.

Example 2.6.4 (the ladder construction). Let C^k denote the boundary of the $(k+1)$-dimensional unit cube in $\mathbb{E}^{k+1}$ with one vertex at the origin and with edges parallel to the coordinate axes. Take an arbitrary number of congruent copies of C^k in $\mathbb{E}^{k+1}$, attached to one another along facets in the form of a ladder. Specifically we translate each cube-boundary a certain number of units along the first coordinate axis so that the union of these cube-boundaries then consists of the boundary of a rectangular parallelepiped together with a certain number of interior copies of the k-dimensional unit cube; this is what we call a ladder. Regard $\mathbb{E}^{k+1}$ as a linear subspace of $\mathbb{E}^N$ for $N > (k+1)$. Then, for small ε, the polyhedral ε-tube around the ladder is a tightly embedded k-handlebody, and its boundary is a tightly embedded connected sum of sphere products $S^k \times S^{N-k-1}$. Similar polyhedral examples are given by 2^K for triangulated balls K.

Proposition 2.6.5 [Rodríguez 1977; Kühnel 1978; Banchoff 1971b]. *For an immersion* (*smooth or polyhedral*) $f : M^n \to \mathbb{E}^N$ *with* $\partial M \neq \varnothing$, *the following conditions are equivalent*:

(i) *f is tight.*
(ii) *f is $(n-2)$-tight and $\mathcal{H}(fM) = \mathcal{H}(f(\partial M))$.*

Corollary 2.6.6. *A tight embedding* (*smooth or polyhedral*) *of a ball B^n into $\mathbb{E}^N$ is a convex embedding into some $(n+1)$-dimensional subspace.*

Proof. Any nondegenerate height function has only one critical point on B^n, the absolute minimum. Hence it has exactly two critical points on the boundary. The tight boundary sphere is convex by Theorem 2.1.1 or Proposition 2.5.2, as the case may be. Then Theorem 2.6.3 implies the convexity of the embedding of the ball. □

The same argument leads to this result:

COROLLARY 2.6.7. *Assume that M is a compact manifold with boundary satisfying $\sum_i b_i(\partial M) = 2\sum_i b_i(M)$. Then the tightness of a substantial smooth immersion $f : M \to \mathbb{E}^N$ implies that*

(i) *the restriction of f to ∂M is tight and substantial*, *and*
(ii) *the Lipschitz–Killing curvature vanishes identically in $M \setminus \partial M$.*

If the boundary has several components, the single components do not have to be substantial in $\mathbb{E}^N$.

2.7. Higher-dimensional knots. An *n-knot* is defined as a smooth or polyhedral embedding $f : S^n \to \mathbb{E}^{n+2}$ (or $f : S^n \to S^{n+2}$). Here S^n indicates the topological sphere with no particular differentiable structure, so we will also consider embeddings of spheres with exotic smooth structures.

THEOREM 2.7.1. *Let f be a smooth n-knot. If* $\mathrm{TA}(f) < 4$ *and n is odd*, *or if* $\mathrm{TA}(f) < 6$ *and n is even*, *then f is unknotted*, *that is*, *isotopic to the standard embedding.*

Note that the number of critical points of a Morse function on the sphere must be even, so if $\mathrm{TA}(f) < 4$ there must be a height function with exactly two critical points. Thus the proof in the odd-dimensional situation is analogous to the Fáry–Milnor theorem (see Section 1.7). For even n, this theorem is mentioned in [Kuiper 1984] as a consequence of hard results in topology; see [Scharlemann 1985] for the case $n = 2$.

CONJECTURE 2.7.2 [Kuiper 1984]. *The assumption in Theorem* 2.7.1 *can be replaced by* $\mathrm{TA}(f) \leq 4$ *for n odd or* $\mathrm{TA}(f) \leq 6$ *for n even.*

This is true for $n = 1$; see Section 1.7.

CONJECTURE 2.7.3. *Proposition* 2.7.1 *is true for polyhedral n-knots that are locally unknotted.*

EXAMPLE 2.7.4 [Wintgen 1980]. *For any $\varepsilon > 0$, there is a suspension of a polyhedral* (*locally unknotted*) *n-knot that is a polyhedral $(n + 1)$-knot with* $\mathrm{TA}(f) < 2 + \varepsilon$. *However*, *this $(n + 1)$-knot is locally knotted at the two additional vertices of the suspension.*

The Veronese surface $\mathbb{RP}^2 \to \mathbb{E}^4$ is unknotted in the sense that it is isotopic to the cone over an unknotted Möbius band in a 3-hyperplane. We mention the following:

THEOREM 2.7.5 [Bleiler and Scharlemann 1988]. *If $f : \mathbb{RP}^2 \to \mathbb{E}^4$ is a smooth embedding with* $\mathrm{TA}(f) < 5$ *then it is isotopic to the Veronese surface.*

Note that under this assumption there is a Morse height function with three critical points because the number of critical points must be odd.

Exotic spheres in codimension two can also be regarded as n-knots.

Example 2.7.6 [Ferus 1968]. *For any $\varepsilon > 0$ and any $n = 4m + 1$ there is a smooth embedding of an exotic sphere* (*Brieskorn's sphere*) $f : \Sigma^n \to \mathbb{E}^{n+2}$ *with* $\mathrm{TA}(f) < 4 + \varepsilon$.

A tube around an ordinary knot in $\mathbb{E}^3$ produces a knotted torus (see Section 1.7), and there are analogous constructions for hypersurfaces arising from knots or containing knots. This tube construction leads to an unexpected phenomenon:

Example 2.7.7 [Kuiper and Meeks 1984]. *There is a compact hypersurface embedded into four-space that satisfies* $\mathrm{TA}(f) = \gamma(M) > b(M; F)$ *for any* F *where* $\gamma(M)$ *denotes the Morse number*, *i.e.*, *the minimum number of critical points of any Morse function defined on* M. *Thus it attains the minimal total absolute curvature but it is not tight with respect to any field.*

This example starts with the isotopy tight surface M of genus 3 in $\mathbb{E}^3$ given by Theorem 1.7.2(ii). Assume that M is contained in a large ball B; then M decomposes B into an interior component B_I and an exterior component B_E. Denote the closure of the exterior component by $B^\#$. The total absolute curvature of $B^\#$ is 7, and the sum of the Betti numbers is $b(B^\#) = 5$. By the knottedness, the fundamental group of $B^\#$ requires at least four generators, although the homology $H_1(M^\#)$ is only three-dimensional. Now consider the manifold $M^\#$ defined as the ε-tube around $B^\# \subset \mathbb{E}^3 \subset \mathbb{E}^4$ in four-space. This is an embedded hypersurface with $\mathrm{TA} = 14$ and $b = 10$, so it is not tight. This hypersurface is only of class C^1 along the parts arising from the boundary of $B^\#$, but it can be made smooth preserving tightness. By the same argument used for $B^\#$, we can prove that every Morse function on $M^\#$ must have at least 14 critical points.

3. Highly Connected Manifolds

An even-dimensional manifold is called *highly connected* if it has the highest degree of connectivity in the sense of homotopy theory. More precisely, a $2k$-dimensional manifold M is called highly connected if it is $(k-1)$-connected, that is, if the homotopy groups $\pi_1(M), \dots, \pi_{k-1}(M)$ all vanish. In particular, a surface is highly connected if and only if it is simply connected, since $k = 1$ in this case. Thus any connected and highly connected surface is topologically equivalent to a two-sphere.

The theory of highly connected manifolds has a very interesting history, with some surprising developments. A classical reference is [Whitehead 1949], which proves that in dimension 4 the homotopy type is uniquely determined by a quadratic form (or the cup-product) on the two-dimensional homology (or cohomology) with integer coefficients. Another important reference is [Wall 1962] on the classification of $(k-1)$-connected $2k$-manifolds for $k > 2$. The case of simply connected four-manifolds remained quite mysterious until the spectacular results by Donaldson, Freedman and others in the 1980's [Kirby 1989].

From the geometric point of view, it seems to be natural to begin by studying the following standard examples:

(i) $S^k \times S^k$;
(ii) a nontrivial S^k-bundle over S^k [Steenrod 1944];
(iii) the projective planes FP^2 for $F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, Ca (the Cayley numbers) in dimensions 2, 4, 8, 16;
(iv) "manifolds like projective planes" in dimension 8 and 16 [Eells and Kuiper 1962] (by definition, they are manifolds admitting a Morse function with 3 critical points);
(v) exotic spheres (compare Theorem 2.1.3 and Example 2.7.6).

If $k = 1$, the first example is the torus, the second is the Klein bottle, and the third is the real projective plane. Taking the connected sum of a surface with one of these examples amounts to adding a handle, adding a twisted handle, or adding a cross-cap.

The unimodular quadratic forms corresponding to the manifolds in the list above are respectively $(+1)$, (-1) (only possible for $k = 1, 2, 4, 8$) and the matrix

$$\begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}.$$

This last unimodular quadratic form represents the only indecomposable case, for k odd. The connected sum of α copies of the first, β of the second and γ of the third will then have quadratic form

$$\alpha(+1) \oplus \beta(-1) \oplus \gamma \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}.$$

The Euler characteristic of such a connected sum is $\chi(M) = 2 + (-1)^k b_k(M) = 2 + (-1)^k(\alpha + \beta + 2\gamma)$. For even k, the *signature* is defined to be $\sigma(M) = \alpha - \beta$. Note, however, that other definite quadratic forms may correspond to manifolds. For example, the form E_8 occurs as part of the quadratic form of a K3 surface.

Unless stated otherwise, in this section a *manifold* will always be compact, connected, without boundary, and of dimension $2k \geq 4$.

3.1. Tightness and the highly connected two-piece property.

LEMMA 3.1.1. *Let M be a highly connected manifold of dimension $2k$ and let $f : M \to \mathbb{E}^N$ be a smooth or polyhedral immersion. Then the following conditions are equivalent*:

(i) *f is tight.*
(ii) *f is $(k-1)$-tight.*
(iii) *Every nondegenerate height function* (*or height function in general position*) *has exactly one minimum and one maximum and $(-1)^k(\chi(M) - 2)$ critical points of index k* (*counted with multiplicity in the polyhedral case*), *and no critical points of any other index.*

(iv) *For any hyperplane H of $\mathbb{E}^N$, the preimage $f^{-1}(\mathbb{E}^N \setminus H)$ has at most two connected components, each of them being $(k-1)$-connected.*

We call (iv) the highly connected two-piece property (HTTP).

PROOF. (i) $\Leftrightarrow$ (ii) holds for manifolds in general by Lemma 2.2.2.

(i) $\Leftrightarrow$ (iii) follows directly from the Morse relations.

In order to see (ii) $\Leftrightarrow$ (iv), consider an open half-space $h \subset \mathbb{E}^N$ and the inclusion $j : f^{-1}(h) \to M$. The $(k-1)$-tightness means that $\tilde{H}_i(j) : \tilde{H}_i\big(f^{-1}(h)\big) \to \tilde{H}_i(M) = 0$ is injective for $i = 1, \ldots, k-1$, where $\tilde{H}$ denotes the i-th reduced homology. By the Hurewicz isomorphism theorem [Dold 1972], this is equivalent to the injectivity of $\pi_i(j) : \pi_i\big(f^{-1}(h)\big) \to \pi_i(M) = 0$. This in turn just says that $f^{-1}(h)$ is $(k-1)$-connected. $\square$

COROLLARY 3.1.2. *Let A be a k-topset of the convex hull of a tightly embedded highly connected $2k$-manifold $M \subset \mathbb{E}^N$. Then A is convex and contained in M. The same holds for i-topsets for $0 \le i \le k$.*

PROOF. Induction on i. $\square$

COROLLARY 3.1.3. *For a polyhedral highly connected $2k$-manifold M the tightness condition implies that M contains the k-dimensional skeleton of its convex hull. Conversely, for a subcomplex of a convex polytope, this necessary condition for tightness is also sufficient.*

For the sufficiency we observe that on the one hand every $(k-1)$-cycle in the manifold can be deformed into the k-skeleton, and that on the other hand the k-skeleton of a simplex and its intersections with arbitrary half spaces is $(k-1)$-connected. This implies the HTPP.

COROLLARY 3.1.4. *Let B be a $(k+1)$-topset of a tightly embedded highly connected $2k$-manifold M. Then either B is convex or B is a convex set minus a number of convex open sets in its interior, in any case the boundary ∂B^* of the induced topset B^* of the convex hull is contained in M.*

If B itself is not convex then a generator of $H_k(B)$ is called a *top-cycle* or a *convex cycle* [Thorbergsson 1983]. By the tightness it certainly represents a nonvanishing element of $H_k(M)$; compare Proposition 1.3.4.

3.2. Examples, smooth and polyhedral.

EXAMPLE 3.2.1 (THE SPHERE S^n). The boundary of any convex body in $(n+1)$-space is a tightly embedded sphere S^n. Such convex hypersurfaces can have any degree of differentiability, including C^∞ and C^ω. Examples of tight polyhedral spheres are given by boundaries of convex polytopes. With respect to the topsets, the main difference is that a polyhedral example necessarily has i-topsets for any $i = 0, 1, \ldots, n$, while in the differentiable case there may be gaps. For example, a strictly convex body of class C^2 or any convex body of class C^ω has only 0-topsets.

EXAMPLE 3.2.2 (CONNECTED SUM OF "HANDLES" $S^k \times S^k$). Examples of tight embeddings of $S^k \times S^k$ into $\mathbb{E}^{2k+1}$ and $\mathbb{E}^{2k}$ can be obtained by constructions analogous to the ones in Example 1.2.2: The Cartesian product of two standard spheres $S^k(1) \times S^k(1) \subset S^{2k+1}(\sqrt{2}) \subset \mathbb{E}^{2k+2}$ is tight. Stereographic projection to $\mathbb{E}^{2k+1}$ gives examples similar to the torus of revolution in $\mathbb{E}^3$, and its conformal images (generalized Dupin cyclides). Polyhedral examples are given by the product of two $(k+1)$-cubes as a subcomplex of the $(2k+2)$-cube. A Schlegel diagram of a hypercube in $(2k+1)$-space includes cubical versions of the Dupin cyclides.

In order to attach handles tightly to these examples, we observe first that we have to replace an $S^{k-1} \times B^{k+1}$ by $B^k \times S^k$, an instance of ordinary surgery. In the case of $k = 1$ (Example 1.2.3) we replace an $S^0 \times B^2$ (where B^2 lies in a flat region) by a rotationally symmetric cylinder $B^1 \times S^1$ and smooth this out along the boundary $S^0 \times S^1$. This can be done by a suitable choice of a concave radius function r of the (warped product) cylinder $B^1 \times_r S^1$ depending on the radius ρ in $B^1 = [-1, +1]$. We use the same function $r(\rho)$ in general where ρ is the polar radius of B^k (in ordinary polar coordinates) and r is the scaling factor of the fibre S^k in the warped product $B^k \times_r S^k$. To get started we have to find a product $S^{k-1} \times B^{k+1}$ where S^{k-1} is a standard round sphere and B^{k+1} lies in a flat region. There are certainly convex hypersurfaces in $\mathbb{E}^{2k+1}$ containing such regions $S^{k-1} \times B^{k+1}$ (rotationally symmetric ovaloids containing flat regions).

To obtain examples in $\mathbb{E}^{2k+2}$ we may start with the Cartesian product of an S^{k-1}-rotationally symmetric ovaloid A^k containing a 1-flat with another ovaloid B^k containing a k-flat. We can then attach a handle $S^k \times S^k$ by surgery. Without loss of generality, we can assume that the original example contains arbitrarily many regions of this type, so we can attach arbitrarily many handles tightly. The tightness can be seen from the HTPP because the handles are rotationally symmetric warped products with a concave radius function and because the fibre S^k is $(k-1)$-connected. These smooth examples are essentially due to J. Hebda [1984].

Polyhedral examples $(S^k \times S^k) \# \ldots \# (S^k \times S^k)$ in $\mathbb{E}^{2k+1}$ can be obtained by the polyhedral ε-tube around the ladder construction in Section 2.6, just by setting $N = 2k + 1$.

In high codimension we can apply the 2^K-construction from Section 2.5.5: Let K denote the boundary complex of a cyclic polytope $C(N+1, 2k)$. This is a simplicial sphere containing all $\binom{N+1}{k}$ simplices of dimension $k-1$. Therefore $2^{\partial C(N+1, 2k)}$ is a $2k$-manifold containing every k-dimensional face of the N-dimensional cube. It is tight and substantial in $\mathbb{E}^N$ by 2.5.5, and it is homeomorphic to a connected sum of copies of $S^k \times S^k$ [Kühnel and Schulz 1991].

EXAMPLE 3.2.3 (PROJECTIVE PLANES). For $F = \mathbb{C}$ or $F = \mathbb{H}$, the projective plane FP^2 has a standard embedding, sending F^3 to $R^3 \oplus F^3$ in such a way that

(x, y, z) and (ux, uy, uz) have the same image for any nonzero element u of F:

$$(x, y, z) \mapsto (x\bar{x}, y\bar{y}, z\bar{z}, \sqrt{2}x\bar{y}, \sqrt{2}y\bar{z}, \sqrt{2}z\bar{x}.)$$

If we restrict to the unit sphere, where $x\bar{x} + y\bar{y} + z\bar{z} = 1$, this defines a mapping $S^5 \to S^7 \subset \mathbb{E}^8$ or $S^{11} \to S^{13} \subset \mathbb{E}^{14}$ so that (x, y, z) and (ux, uy, uz) have the same image for any unit element u. It follows that this mapping gives an embedding $\mathbb{CP}^2 \to S^7 \subset \mathbb{E}^8$ or $\mathbb{HP}^2 \to S^{13} \subset \mathbb{E}^{14}$. These embeddings are tight because any nondegenerate height function has exactly three critical points (it is sufficient to show here that the index of each critical point is even). For the Cayley plane $\mathrm{Ca}\mathbb{P}^2$ there is a similar tight embedding into $S^{25} \subset \mathbb{E}^{26}$ but the formulas are different [Tai 1968; Kuiper 1970; Kuiper 1980].

Under this embedding, the images of each of these projective planes lies on a sphere of the appropriate dimension. Stereographic projection of these three embeddings from the north pole on the sphere leads to tight embeddings of these projective planes into $\mathbb{E}^7$, $\mathbb{E}^{13}$, and $\mathbb{E}^{25}$ respectively. By an argument involving characteristic classes, there are no topological embeddings of these manifolds into lower dimensional Euclidean spaces [Cecil and Ryan 1985].

As in the case of the real Veronese surface, we may ask:

QUESTION 19. *Is it possible to attach a handle $S^2 \times S^2$ tightly to the projected Veronese type embedding $\mathbb{CP}^2 \to \mathbb{E}^7$?*

A remarkable polyhedral analogue of the standard smooth tight embedding of $\mathbb{CP}^2$ is given by the unique 9-vertex triangulation $\mathbb{CP}^2_9$ [Kühnel and Banchoff 1983] regarded as a subcomplex of the 8-dimensional simplex $\triangle^8$:

$$\mathrm{Sk}_2(\triangle^8) \subset \mathbb{CP}^2_9 \subset \mathrm{Sk}_4(\triangle^8) \subset \mathbb{E}^8.$$

This triangulation contains every edge determined by a pair of vertices, and moreover it contains every two-simplex determined by any triple of vertices (so the triangulation is *three-neighborly* [Kühnel and Lassmann 1983]). From this the HTPP follows directly. Condition (iv) of Lemma 3.1.1 is satisfied because any 1-cycle can be deformed homotopically into the 2-skeleton, and because $\mathrm{Sk}_2(\triangle^8) \cap h$ is simply connected for any half-space $h \subset \mathbb{E}^8$.

This situation is analogous to the tight polyhedral embedding of the real projective plane of Example 1.2.4:

$$\mathrm{Sk}_1(\triangle^5) \subset \mathbb{RP}^2_6 \subset \mathrm{Sk}_2(\triangle^5) \subset \mathbb{E}^5,$$

where the TPP is satisfied because $\mathrm{Sk}_1(\triangle^5) \cap h$ is connected for any half-space h.

EXAMPLE 3.2.4 (POLYHEDRAL MANIFOLDS WITH ODD INTERSECTION FORM). Examples 3.2.2 and 3.2.3 leave open the question whether we can combine "handles" of type $S^2 \times S^2$ with the complex projective plane. From the point of view of intersection forms, a connected sum of handles represents the case of an even intersection form and signature $\sigma = 0$, whereas the complex projective plane represents an odd intersection form.

A tight polyhedral embedding $\mathbb{CP}^2 \# (-\mathbb{CP}^2) \to \mathbb{E}^8$ can be constructed from the 9-vertex triangulation of $\mathbb{CP}^2$, just as a tight Klein bottle was constructed from the 6-vertex triangulation of $\mathbb{RP}^2$ in Example 1.2.5: Cut out the open star of one vertex, take two parallel copies of the remaining part in a 7-simplex (or 4-simplex, respectively), and then join the two boundaries by a straight cylinder. The tightness is easily verified since the TPP and HTPP are satisfied. By this cylinder construction, the two copies of $\mathbb{CP}^2$ have opposite orientations. The case of the same orientation seems to be open:

QUESTION 20. *Does there exist a tight polyhedral* (*or topological*) *embedding of* $\mathbb{CP}^2 \# \mathbb{CP}^2$ *into any* $\mathbb{E}^N$?

Tight polyhedral connected sums $\mathbb{CP}^2 \# (-\mathbb{CP}^2) \# \cdots \# (-\mathbb{CP}^2)$ can be constructed by truncation of the tight $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$; see Theorem 3.5.7.

EXAMPLE 3.2.5 (COMBINATORIAL). As a generalization of Example 1.2.8 and $\mathbb{CP}^2_q$, we observe the following: when an n-vertex triangulation of a $2k$-manifold M that contains all $\binom{n}{k+1}$ k-dimensional simplices, M is highly connected and the natural inclusion of M as a subcomplex of the $(n-1)$-dimensional simple is a tight embedding into $\mathbb{E}^{n-1}$:

$$\mathrm{Sk}_k(\triangle^{n-1}) \subset M \subset \mathrm{Sk}_{2k}(\triangle^{n-1}) \subset \mathbb{E}^{n-1}.$$

Again, the tightness follows from the HTPP, condition (iv) in Lemma 3.1.1.

3.3. The substantial codimension of a tight immersion. For any tight smooth immersion $f : M \to \mathbb{E}^N$ of a $2k$-dimensional manifold M, the substantial codimension $N-2k$ is less than or equal to $\binom{2k+1}{2}$ (see 2.4.1). If M is highly connected, this upper bound can be improved by using the fact that only quadratic forms of indices $0, k$ and $2k$ can occur in the image space of the mapping

$$\xi \mapsto A_\xi.$$

We have the following generalization of 1.3.1:

PROPOSITION 3.3.1 [Kuiper 1970]. *The substantial codimension of a smooth tight immersion of a highly connected* $2k$*-manifold in* $\mathbb{E}^N$ *satisfies*

$$N - 2k - 2 \le \begin{cases} k & \text{if } k \in \{1, 2, 4, 8\}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that equality $N = 3k + 2$ is attained for the Veronese embeddings of the projective planes FP^2; see Example 3.2.3. In these exceptional cases the same upper bound is valid even without the assumption of smoothness:

THEOREM 3.3.2 [Kuiper 1980]. *If* $f : M \to \mathbb{E}^N$ *is a tight substantial continuous embedding of a* $2k$*-dimensional "manifold like a projective plane", then* $N \le 3k + 2$.

Sketch of proof. We first must show that there is a k-dimensional top-cycle given by the boundary of a $(k+1)$-topset; see Corollary 3.1.4. This is a convex hypersurface in some $\mathbb{E}^{k+1}$. The orthogonal projection onto the subspace perpendicular to this space leads to a mapping $f^* : M \to \mathbb{E}^{N-k-1}$. Since now the k-th homology is killed, the image is a homotopy sphere of dimension $2k$, therefore the boundary of a convex set in $(2k+1)$-space. This implies $N-k-1 \leq 2k+1$. $\square$

A sharp upper bound for the substantial codimension of a tight $S^k \times S^k$ does not seem to be known in general. In the smooth case, the upper bound is 2, but for polyhedra it may be considerably larger, as indicated by the case $k = 1$, where the tight 7-vertex embedding of the torus in $\mathbb{E}^6$ has substantial codimension 4.

Conjecture 3.3.3. *Any continuous and substantial tight embedding $S^k \times S^k \to \mathbb{E}^N$ that is centrally-symmetric satisfies $N \leq 2k+2$.*

This extends Conjecture 1.4.15.

Conjecture 3.3.4 [Kühnel 1995]. *Let $M \subset \mathbb{E}^N$ be a tight and substantial polyhedral embedding of a $(k-1)$-connected $2k$-manifold. Then*

$$\binom{N-k-1}{k+1} \leq (-1)^k \binom{2k+1}{k+1} \big(\chi(M) - 2\big) = \binom{2k+1}{k+1} b_k(M),$$

with equality for $N \geq 2k+2$ only for subcomplexes of the N-simplex containing the k-skeleton of the N-simplex.

This would generalize Theorem 1.4.11. The inequality can be regarded as a generalization of the classical Heawood inequality of Theorem 1.4.7 in the case $k = 1$. In the case of $M = S^k \times S^k$, Conjecture 3.3.4 states that $N-k-1 \leq 2k+2$.

Theorem 3.3.5 [Kühnel 1994a]. *Conjecture 3.3.4 is true under the additional assumption that M is a subcomplex of the boundary complex of a simplicial convex n-polytope P that contains all vertices of P.*

For centrally symmetric versions, see [Sparla 1997a]. In this case the upper bound for N in terms of $\chi(M)$ can be improved; compare Conjectures 3.3.3 and 1.4.15. There is an example of a 12-vertex triangulation of $S^2 \times S^2$ as a tightly embedded subcomplex of the 6-dimensional cross-polytope. See [Sparla 1997b].

3.4. The smooth case. According to Proposition 3.3.1 tight smooth and substantial immersions of simply connected 4-manifolds can exist only in $\mathbb{E}^5$, $\mathbb{E}^6$, $\mathbb{E}^7$, and $\mathbb{E}^8$. Unfortunately, in codimension greater than two, no construction principle seems to be known for smooth tight immersions. Therefore this Section contains more negative results on restrictions and obstructions than positive results and examples.

Theorem 3.4.1 [Kuiper 1980]. *Let $f : \mathbb{CP}^2 \to \mathbb{E}^8$ be a smooth tight and substantial immersion. Then, up to projective transformations of $\mathbb{E}^8$, the image is* *congruent to the image of the standard* (*Veronese type*) *embedding in Example* 3.2.2.

For tight embeddings of the complex projective plane into $\mathbb{E}^7$ no geometric uniqueness result can be expected, just as there is no uniqueness in the case of tight embeddings of the real projective plane into $\mathbb{E}^4$.

THEOREM 3.4.2 [Thorbergsson 1983]. *Let f be a substantial tight and smooth immersion of a simply connected four-manifold M into $\mathbb{E}^N$, for $N = 6$ or $N = 7$. Then, for a suitable choice of an orientation*:

(i) *If $N = 6$ then M splits diffeomorphically as a connected sum $(S^2 \times S^2) \# M^*$ and the middle Betti number $b_2(M)$ is even. Moreover, if the intersection form is odd, then $b_2(M) \geq 4$.*
(ii) *If $N = 7$ then M splits diffeomorphically as a connected sum $\mathbb{CP}^2 \# M^*$. In particular, the intersection form is odd.*

The proof is quite involved and relies on a careful study of the intersections of various top-cycles (see the end of Section 3.1). This can be considered as an obstruction to the existence of tight immersions:

COROLLARY 3.4.3 [Thorbergsson 1983]. *Infinitely many distinct simply connected differentiable four-manifolds do not admit a tight immersion into any $\mathbb{E}^N$.*

Particular examples are the K3-surfaces and algebraic surfaces in $\mathbb{CP}^3$ of even degree $d \geq 4$.

QUESTION 21 [Thorbergsson 1983]. *Is there a smooth tight immersion $f : M \to \mathbb{E}^6$ of a simply connected four-manifold with odd intersection form*?

According to Theorem 3.4.2, candidates would be $\mathbb{CP}^2 \# \mathbb{CP}^2 \# (-\mathbb{CP}^2) \# (-\mathbb{CP}^2)$ or connected sums with more copies of $\pm\mathbb{CP}^2$. Such a tight example would have quite unexpected behavior with respect to tubes: The ε-tube around it (regarded as an immersion of an S^1-bundle over the manifold) would not be tight [Breuer and Kühnel 1997]. So far there does not seem to be any example of a tight smooth immersion in codimension at least two for which ε-tube is not tight. (Compare Proposition 2.6.2 and Question 13 at the end of Section 1.7.)

THEOREM 3.4.4 [Thorbergsson 1983]. *Let $f : M \to \mathbb{E}^N$ be a substantial tight and smooth immersion of a highly connected $2k$-manifold with $k = 4$ or $k = 8$. Then*:

(i) *If the intersection form of M is even then $N \leq 2k + 2$.*
(ii) *If the intersection form of M is odd then $N = 3k + 1$ or $N = 3k + 2$.*

For the case $k \notin \{1, 2, 4, 8\}$ see Proposition 3.3.1.

Any such substantial immersion with $N = 3k+2$ is projectively equivalent to the Veronese-type embedding of the projective plane over the complex, quaternion, or Cayley numbers [Niebergall and Thorbergsson 1996].

THEOREM 3.4.5. *For any given natural number $m \geq 1$ there exists a tight smooth embedding of a connected sum of m copies of $S^k \times S^k$ into $\mathbb{E}^{2k+1}$ and $\mathbb{E}^{2k+2}$* [Hebda 1984], *but there is no tight analytic embedding into $\mathbb{E}^{2k+2}$ for $m \geq 2$* [Niebergall 1994].

See Example 3.2.2.

QUESTION 22. *Is there a smooth tight immersion of any simply connected four-manifold with even intersection form that is not diffeomorphic to a connected sum of copies of $S^2 \times S^2$?*

3.5. The polyhedral case. In the polyhedral case we have the opposite situation to the smooth case in Section 3.4. There are many examples and various construction principles but only a few restrictions. Recall first the construction principle mentioned in Corollary 3.1.3: If a $2k$-dimensional M is a subcomplex of the boundary complex of a convex polytope, tightness is satisfied if M contains the whole k-skeleton of this polytope.

THEOREM 3.5.1 [Kühnel 1995]. *For arbitrary given numbers k, N satisfying $N \geq 2k+1$ there is a tight and substantial polyhedral embedding of a $(k-1)$-connected $2k$-manifold into $\mathbb{E}^N$. Particular examples are PL homeomorphic to a connected sum of copies of $S^k \times S^k$.*

PROOF. Let K be a k-neighborly triangulation of S^{2k-1} with N vertices, and define M to be $2^K \subset C^N \subset \mathbb{E}^N$, as in Example 3.2.2. Then M is $(k-1)$-connected and tight. In the particular case of the cyclic polytope $K = \partial C(d, 2k)$, the manifold 2^K is PL homeomorphic to a connected sum of $(-1)^k \frac{1}{2}(\chi(d,k)-2)$ copies of $S^k \times S^k$ where, by definition,

$$\chi(d,k) = 2\,\chi\bigl(\mathrm{Sk}_k(C^{d-k-1})\bigr).$$

The number $\chi(d,k)$ is the Euler characteristic of any k-Hamiltonian submanifold of C^d, that is, a submanifold containing k-dimensional faces of C^d [Kühnel and Schulz 1991]. For any such K the intersection form of 2^K on $H_k(2^K)$ is a sum of copies of

$$\begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}. \qquad \square$$

It does not seem to be known whether there are distinct topological types of such examples in the skeleton of C^N for $k \geq 2$.

THEOREM 3.5.2 [Kühnel and Banchoff 1983; Morin and Yoshida 1991]. *There exists a unique tight 9-vertex triangulation of the complex projective plane $\mathbb{CP}^2$. The canonical embedding of this complex into the 8-simplex determines a tight polyhedral embedding $\mathbb{CP}^2 \to \mathbb{E}^8$ and following this embedding by projection into almost any 7-dimensional linear subspace gives a tight embedding $\mathbb{CP}^2 \to \mathbb{E}^7$.*

PROOF. Since all vertices, edges, and two-dimensional faces of $\triangle^8$ must be contained in any such tight embedding of a simply connected four-manifold, we may determine the number f_i of i-dimensional simplices as $f_0 = 9$, $f_1 = \binom{9}{2} = 36$, $f_2 = \binom{9}{3} = 84$. From the Dehn–Sommerville relations, it follows that $f_3 = 90$ and $f_4 = 36$. To construct the 9-vertex triangulation, called $\mathbb{CP}^2_9$, we denote the nine vertices by 1, 2, 3, ..., 9 and we take the union of the two orbits of the four-dimensional simplices $\langle 12456\rangle$ and $\langle 12459\rangle$ under the action of a group H_{54} on $\{1, 2, \ldots, 9\}$ generated by

$$\alpha = (147)(258)(369), \quad \beta = (123)(465), \quad \gamma = (12)(45)(78).$$

The generator γ corresponds to the action of complex conjugation; in fact its fixed point set is combinatorially isomorphic to an $\mathbb{RP}^2_6$. The triangulation is unique (up to relabelling of the vertices) [Kühnel and Lassmann 1983; Arnoux and Marin 1991; Bagchi and Datta 1994]. The link of each vertex is combinatorially isomorphic to the so-called *Brückner–Grünbaum sphere* $\mathcal{M}$ [Grünbaum and Sreedharan 1967], a triangulation of the three-sphere with unusual properties. $\square$

QUESTION 23 [Kuiper 1980]. *Are there tight and substantial topological embeddings* $\mathbb{CP}^2 \to \mathbb{E}^8$ *other than the standard algebraic embedding and the canonical polyhedral embedding of* $\mathbb{CP}^2_9$ (*up to projective transformations*)?

By Theorem 3.3.5 there is no such example in the boundary complex of a simplicial polytope. Conjecture 3.3.4 together with Theorem 3.5.2 would imply the uniqueness in the polyhedral case.

We remark that the combinatorial formula for the first Pontrjagin number of a four-manifold has been explicitly evaluated for $\mathbb{CP}^2_9$ by L. Milin [1994]. The flattenings of the (nonpolytopal) Brückner–Grünbaum sphere play a particular role in Milin's work.

EXAMPLE 3.5.3. *For any* m, $1 \le m \le 256$, *there is a tight polyhedral embedding into* $\mathbb{E}^8$ *of a simply connected four-manifold with* $\operatorname{rank}(H_2) = 62 + m$ *whose intersection form on* H_2 *is odd*.

To construct this example, we take $2^{\mathcal{M}}$, where $\mathcal{M}$ is the Brückner–Grünbaum sphere mentioned above. The link of each vertex is combinatorially equivalent to $\mathcal{M}$. Therefore we can truncate at each vertex by a hyperplane section and attach in this hyperplane a copy of $\mathbb{CP}^2_9$ minus an open vertex star. The tightness follows from Corollary 3.1.3. The intersection form of this manifold is the one of $2^{\mathcal{M}}$ plus m direct summands (± 1).

PROPOSITION 3.5.4 [Banchoff and Kühnel 1992]. *There is a tight polyhedral embedding* $\mathbb{CP}^2 \to \mathbb{E}^7$ *that is essentially different from a linear projection of the one in Theorem* 3.5.2. *It is a simplexwise linear embedding of a* 10-*vertex triangulation*, *denoted by* $\mathbb{CP}^2_{10}$.

This triangulation is based on the decomposition of the complex projective plane into three 4-balls as "zones of influence" of three points $X = [1, 0, 0], Y = [0, 1, 0]$ and $Z = [0, 0, 1]$, given in homogeneous coordinates. The *equilibrium torus* is the set of points $[z_0, z_1, z_2]$ with the same absolute value of each coordinate z_i. We take the 7-vertex triangulation of this equilibrium torus and then introduce X, Y, Z as extra vertices. Each of the three 4-balls is triangulated as a cone over the boundary complex of the cyclic polytope $C(7,4)$, which occurs in three different, and combinatorially equivalent, versions. For the tight embedding the 7 vertices of the torus are chosen in general position in an $\mathbb{E}^6$, then X and Y as vertices of a double cone over the seven ones, and finally Z at the centre.

THEOREM 3.5.5 [Casella and Kühnel 1996]. *There is a tight* 16-*vertex triangulation of a* K3 *surface*, *leading to a tight polyhedral embedding into* $\mathbb{E}^{15}$.

The construction is algebraic and combinatorial. The triangulation has 16 vertices and 288 four-simplices. One can regard the vertices as the elements of a field with 16 elements. Then the triangulation is invariant under the group of all invertible affine transformations $x \mapsto ax + b$ of this field.

THEOREM 3.5.6 [Brehm and Kühnel 1992]. *There are at least three combinatorially distinct tight* 15-*vertex triangulations of an* 8-*manifold "like the quaternionic projective plane". These triangulations induce tight polyhedral embeddings into* $\mathbb{E}^{14}$ *and* $\mathbb{E}^{13}$.

The construction of such a triangulated 8-manifold M^8_{15}, which is quite complicated, generalizes the construction of $\mathbb{CP}^2_9$ above, according to Example 3.2.5:

$$\mathrm{Sk}_4(\triangle^{14}) \subset M^8_{15} \subset \mathrm{Sk}_8(\triangle^{14}) \subset \mathbb{E}^{14}.$$

By a straightforward computation the numbers f_i of i-dimensional simplices are $f_0 = n = 15$, $f_1 = \binom{15}{2} = 105$, $f_2 = \binom{15}{3} = 455$, $f_3 = \binom{15}{4} = 1365$, $f_4 = \binom{15}{5} = 3003$, $f_5 = 4515$, $f_6 = 4230$, $f_7 = 2205$, $f_8 = 490$. The actual example M^8_{15} is presumably a triangulated quaternionic projective plane; for some evidence of this conjecture see [Brehm and Kühnel 1992].

QUESTION 24. *Is there a tight polyhedral embedding of a* 16-*dimensional manifold "like the Cayley plane" into* $\mathbb{E}^{26}$, *possibly as a tight* 27-*vertex triangulation*?

A tight 27-vertex triangulation would have exactly 100386 16-dimensional simplices and would contain all $\binom{27}{9}$ 8-dimensional subsimplices:

$$\mathrm{Sk}_8(\triangle^{26}) \subset M^{16}_{27} \subset \mathrm{Sk}_{16}(\triangle^{26}) \subset \mathbb{E}^{26}.$$

THEOREM 3.5.7 [Kühnel 1995]. *Let M be a tight triangulation of a* $(k-1)$-*connected* $2k$-*manifold with n vertices. Then for an arbitrary integer* $m \geq 0$ *there is a tight and substantial polyhedral embedding* $M \# m(-M) \to \mathbb{E}^{n-1}$.

The proof uses the construction by iterated truncation. Start with the n-vertex triangulation, regarded as a subcomplex of the $(n-1)$-dimensional simplex. Then truncate the simplex at a certain vertex and glue in a small copy of the same triangulation minus an open vertex star. Then repeat this procedure, either at (old) vertices of the $(n-1)$-simplex or at (new) vertices of the truncated simplex.

COROLLARY 3.5.8. *For an arbitrary integer $m \geq 0$ there is a tight polyhedral embedding $\mathbb{CP}^2 \# m(-\mathbb{CP}^2) \to \mathbb{E}^8$, and a tight polyhedral embedding $M^8 \# m(-M^8) \to \mathbb{E}^{14}$, where M^8 is the 8-manifold "like the quaternionic projective plane" from Theorem* 3.5.6.

We mention the following sharper form of Conjecture 3.3.4: For a highly connected $2k$-manifold M let N_M denote the maximum dimension of a Euclidean space admitting a tight and substantial polyhedral embedding of M. Let n_M denote the minimum number of vertices for any simplicial triangulation of M. The approximate size of N_M and n_M should satisfy the relations $N_M \approx n_M - 1$ and

$$\binom{N_M - k - 1}{k+1} \approx (-1)^k \binom{2k+1}{k+1} \big(\chi(M) - 2\big) = \binom{2k+1}{k+1} b_k(M);$$

see Conjecture 3.3.4.

QUESTION 25. *Is there a universal constant C such that $N_M \leq n_M - 1 \leq N_M + C$ for any highly connected manifold M that is a connected sum of the standard examples* 1, 2, *and* 3 *of page* 99?

This is true for $k = 1$ with $C = 2$ ($C = 1$ with only a few exceptions); see Lemma 1.4.11. For higher dimensions it is a very general form of a Heawood problem, compare the generalized Heawood inequalities in [Kühnel 1994b; Kühnel 1995].

PROPOSITION 3.5.9. *If a highly connected manifold M admits a tight polyhedral embedding into $\mathbb{E}^N$, there is an embedding of the k-dimensional skeleton of the N-simplex into M.*

This follows from Corollary 3.1.3 and a lemma of Grünbaum [Grünbaum 1967, § 11.1] saying that the k-skeleton of any convex N-polytope contains the k-skeleton of the N-simplex as a subset. Compare Example 3.2.5 and Conjecture 3.3.4.

The question remains whether the converse of Proposition 3.5.9 is true. More precisely: If M admits an embedding of the k-skeleton of the N-simplex $\triangle^N$ (N sufficiently large, tame embedding in the topological sense), does there exist a tight polyhedral embedding into $\mathbb{E}^N$?

QUESTION 26. *Given a $(k-1)$-connected $2k$-manifold M and a number $N \geq 4k$, are the following conditions equivalent*?

(i) *There exists a tight and substantial polyhedral embedding $M \to \mathbb{E}^N$.*
(ii) *There exists a (topologically tame) embedding $\mathrm{Sk}_k(\triangle^N) \to M$.*

This is true for $k = 1$ by Theorem 1.4.8. The implication (i) $\Rightarrow$ (ii) holds in general by Proposition 3.5.9. One strategy for a proof of the converse could be the construction of a suitable triangulation from the embedding $\mathrm{Sk}_k(\triangle^N) \to M$ as a kind of starting data; compare the construction in Theorem 1.4.8.

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Thomas F. Banchoff
Mathematics Department
Brown University
Providence, RI 02912
USA

Wolfgang Kühnel
Mathematisches Institut B
Universität Stuttgart
70550 Stuttgart
Germany