Tight and Taut Submanifolds
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# Null-Homotopic Embedded Spheres of Codimension One

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Abstract. Let $S$ be an $(n-1)$-sphere smoothly embedded in a closed, orientable, smooth $n$-manifold $M$, and let the embedding be null-homotopic. We show that, if $S$ does not bound a ball, then $M$ is a rational homology sphere, the fundamental groups of both components of $M \setminus S$ are finite, and at least one of them is trivial.

Let $M$ be a closed, oriented $n$-manifold, and suppose that $\iota : S^{n-1} \to M$ is a smooth embedding that is null-homotopic. It follows easily that the image $\iota(S^{n-1}) = S$ separates $M$ into two pieces: $M = X_0 \cup_S Y_0$, or $M = X \# Y$ with $X = X_0 \cup B^n$ and $Y = Y_0 \cup B^n$. An obvious instance is when $X_0$ or $Y_0$ is diffeomorphic to $B^n$; we then say that $S$ bounds a ball on one side. The question as to whether this is the only possibility arises in [Terng and Thorbergsson 1997]. The following theorem describes what can happen; there are examples in every dimension to show that this is (more or less) the best possible. The only qualification is that it is perhaps possible to show that both $X$ and $Y$ must be simply connected; all of the examples constructed at the end of this article have this property.

Theorem 1. *Suppose that $\iota : S^{n-1} \to M^n$ is a null-homotopic smooth embedding. Then either $S$ bounds a homotopy ball on one side, or the following statements hold*:

(i) *$M$ is a rational homology sphere, and therefore $X$ and $Y$ are as well.*
(ii) *The fundamental groups of both $X$ and $Y$ are finite, and at least one of them is trivial.*

*For $n > 4$, if $S$ bounds a homotopy ball then it bounds a (smooth) ball, while if $n = 4$ it bounds a topological ball.*

The basic ingredient in the proof is the well-known principle that a manifold admitting a map from a sphere of nonzero degree must be a rational homology sphere:

Lemma 2. *Suppose that $M$ is an $n$-dimensional oriented manifold, and that $f : S^n \to M$ has degree $k > 0$. Then $M$ is a rational homology sphere, and $H_*(M;\mathbb{Z})$ has no $m$-torsion if* $\gcd(m,k) = 1$. *Moreover*, $\pi_1(M)$ *is finite, and its order divides $k$.*

The first part follows by Poincaré duality with rational or $\mathbb{Z}/m$ coefficients. The second part follows by considering the lift of $f$ to the universal cover of $M$.

To apply the lemma, note that there are maps $\pi_X : M \to X$ and $\pi_Y : M \to Y$ collapsing $Y_0$ and $X_0$, respectively, to a point. These maps induce an isomorphism from $H_n(M,S)$ to the direct sum $H_n(X) \oplus H_n(Y)$. Here $X$ and $Y$ are oriented by the image of $H_n(M)$ in $H_n(M,S)$, and also $X_0$ and $Y_0$ acquire orientations as manifolds with boundary. The inverse of the isomorphism $(\pi_X)_* \oplus (\pi_Y)_*$ is then induced by the inclusions $\iota_X, \iota_Y$ of $X_0, Y_0$ into $M$.

Suppose now that $F : B^n \to M$ is an extension of $\iota$ coming from the null-homotopy of $\iota$. Composing with the projections $\pi_X$ and $\pi_Y$ gives maps $F_X : S^n \to X$ and $F_Y : S^n \to Y$.

Lemma 3. *The degrees of $F_X$ and $F_Y$ satisfy* $\deg F_X - \deg F_Y = \pm 1$.

Proof. This is a small diagram chase. The point is that (with suitable orientation conventions) the boundary map $\partial : H_n(M,S) \to H_{n-1}(S)$, takes the class $\iota_*([X_0])$ to $+1$ and $\iota_*([Y_0])$ to $-1$. □

Proof of Theorem 1. Suppose that $S$ is null-homotopic, and that neither $X$ nor $Y$ is a homotopy ball. The fact that one of $X$ and $Y$ must be simply connected follows from the van Kampen theorem, which implies that $\pi_1(M)$ is the free product $\pi_1(X) * \pi_1(Y)$. It is easily seen that a lift of $S$ to the universal cover $\tilde{M}$ intersects a properly embedded line, and is thus essential (in homology). But the covering homotopy theorem implies that any lift of $S$ is null-homotopic. In dimension $n = 3$, a standard argument implies that a simply connected manifold with boundary $S^2$ is a homotopy ball; hence one of $X_0$ or $Y_0$ is a homotopy ball.

Suppose now that $n > 3$, and that one of the degrees, say $\deg(F_Y)$, is zero. By the preceding lemma, the other one must be $\pm 1$. By the first lemma, $X$ must be a homotopy sphere, i.e., $X_0$ is a homotopy ball. In all dimensions except 4, $X_0$ is then known to be diffeomorphic to a ball [Milnor 1965]; in dimension 4, all one can say at present is that $X_0$ is homeomorphic to a ball [Freedman and Quinn 1990].

If neither degree is zero, both $X$ and $Y$ are rational homology spheres, by the first lemma. □

We remark that a simply connected four-manifold has no torsion in its homology, so a simply connected rational homology four-sphere must be homotopy equivalent to, and thus homeomorphic to, a sphere. In dimension four, therefore, a null-homotopic sphere must bound a ball, and the new phenomena must be in higher dimensions.

We now construct examples that show that in some sense the theorem gives as much information as possible. Clearly, by the theorem, one needs a source of simply connected manifolds that arise as the target of a map of nonzero degree from a sphere. We use the following two lemmas to put such manifolds together to give examples of manifolds $M$ containing a null-homotopic sphere.

LEMMA 4. *Suppose that $X$ is a simply connected $n$-manifold whose homology in dimensions $0 < m < n$ is all $k$-torsion, for some integer $k$. Then the image of the Hurewicz map $\pi_n(M) \to H_n(M)$ is given by $k^r\mathbb{Z}$ for some $r$. In particular, there is a map $S^n \to M$ of degree $k^r$.*

PROOF. This follows from the mod $\mathcal{C}$ Hurewicz theorem [Serre 1953], where $\mathcal{C}$ is the class of finite abelian groups. ☐

LEMMA 5. *Suppose that $X$ and $Y$ are oriented simply connected manifolds, admitting maps from $S^n$ of degrees $k$ and $l$, respectively. Then the connected sum $M = X \# Y$ admits a map from $B^n$ such that the restriction takes $S^{n-1}$ to the sphere separating $X$ from $Y$, and the induced map has degree $k + l$.*

PROOF. Choose regular values $x \in X$ and $y \in Y$. By a homotopy of the maps, if necessary, we can assume that the local degree at some point $p$ in the preimage of $x$ is positive, and that the local degree at some point $q$ in the preimage of $y$ is negative. Remove small ball neighborhoods of $x$ and $y$, and form the connected sum $X \# Y$ using an orientation reversing diffeomorphism of $S^{n-1}$. There is an obvious map of a punctured sphere to $X_0$, and another one to $Y_0$, that fit together (near $p$ and $q$) to give a map of a punctured sphere to $M$. All of the boundary $S^{n-1}$'s map to $S$, and the total degree of all the maps is clearly $k + l$.

Choose one of the boundary components $S_0$ of the punctured sphere, and for each of the other boundary components, choose an arc joining it to $S_0$. The arcs become loops in $M$, which can be contracted to lie in a neighborhood of $S$. (This is where the simple connectivity gets used.) Remove a neighborhood of each of the arcs, to get a map of $B^n$, with boundary lying in $S \times I$. The map on the boundary can be homotoped to lie in S; the homotopy extension theorem says that this homotopy extends to a homotopy of the map of the ball as well. ☐

REMARK 6. The simple connectivity of at least one of $X$ and $Y$ is essential, as the proof of the theorem shows. It is not known if $X$ and $Y$ both have to be one-connected. There is also some possible confusion about orientations: the sphere $S$ gets its orientation as the boundary of the submanifold $X_0$ of $M$.

To apply these lemmas, suppose $X$ and $Y$ admit degree-$k$ and degree-$l$ maps from the sphere. By precomposing with maps of the sphere to itself, of degrees $a$ and $b$, we can get a map from the ball to $X \# Y$ sending $S^{n-1}$ to $S$ with degree $ak + bl$. If $\gcd(k, l) = 1$, we can choose $ak + bl = 1$, so the map is homotopic to the embedding of $S$ in $M$. So all we need is a collection of rational homology spheres, in each dimension $n \geq 5$, with only $k$-torsion in their homology.

EXAMPLE 7. For $n \geq 5$, start with the manifold $S^2 \times B^{n-2}$. Add a three-handle to $S^2 \times B^{n-2}$ where the attaching two-sphere in the boundary $S^2 \times S^{n-3}$ represents $k$ times the generator of $H_2(S^2 \times S^{n-3})$. (When $n = 5$, some care must be taken, as not every homology class is represented by an embedded sphere. But in the case at hand, this is not a problem; tube together $k$ parallel copies of the obvious sphere $S^2 \times \mathrm{pt.}$) Double the resulting manifold with boundary, to obtain a simply connected manifold $X_k$. If $n > 5$, the only homology in $X_k$ (apart from dimensions 0 and $n$) is $\mathbb{Z}/k$ in dimensions 2 and $n-2$. For dimension 5, the homology is $\mathbb{Z}/k \oplus \mathbb{Z}/k$ in dimension 2.

The 5-manifolds $X_k$ were constructed by D. Barden [1965] by a somewhat different method. As an alternative to the previous paragraph, one could obtain higher-dimensional examples inductively, starting from Barden's manifolds, as follows: From an $n$-dimensional $X_k$, form the product $X_k \times S^1$, and then surger the circle (this is called spinning $X$) to get an $(n+1)$-manifold $X_k$ with nontrivial homology $(\mathbb{Z}/k \oplus \mathbb{Z}/k)$ only in dimensions 2 and $n-2$.

EXAMPLE 8. Start with the Hopf map $p : S^7 \to S^4$. As a (linear) $S^3$ bundle over $S^4$, it has an Euler class that is easily seen to be a generator of $H^4(S^4)$. Now let $g : S^4 \to S^4$ have nonzero degree, say $k$, and let $p_k : X_k \to S^4$ be the pull back bundle $g^*(p)$. By naturality, $p_k$ has Euler class $k$; it is easy to compute (with a Gysin sequence) that the homology of $X_k$ is $\mathbb{Z}/k$ in dimension 3, $\mathbb{Z}$ in dimensions 0 and 7, and trivial otherwise. Using the naturality of the Gysin sequence, or a geometric argument, it is easy to see that the degree of the map $X_k \to S^7$ covering $g$ is exactly $k$. From properties of the Hopf invariant, it is not hard to check that there is a map $S^7 \to X_k$ of degree exactly $k$. By spinning as in the previous example, one gets examples in every dimension $\geq 7$.

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