The Eightfold Way
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Eightfold Way: The Sculpture

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Abstract. This article covers some of my thinking while developing *The Eightfold Way* and some of the physical processes I used in creating it. The sequence of topics followed is:

My View
Ramanujan–Michelangelo
Geometry–Topology
Counting–Philosophy
Geometry Center–MSRI
Two Stones
Athena–Escher
Robot–Stewart platform

The pictures, the text, and the references can all be read independently of each other.

Wheeled wheels of processes and thoughts form a sort of multidimensional torus embedded in our time and space. In this paper we survey a few of these satellites and their orbits about the sculpture called *The Eightfold Way.* This amounts to making explicit part of the mathematical environment when I finished this sculpture. I intend no mysticism here, only some shared furnishings of our minds and hearts — shared cultivation of our neuron and capillary landscapes.

A typical Helaman sculpture has layers of titles, ranging from a colloquial expression such as "eightfold way" at the top to precise mathematical symbols and syntax such as $x^3y + y^3z + z^3x = 0$ deeper down. The last equation describes the algebraic surface of Klein that inspired this sculpture. For some reason, maybe because there was going to be a sculpture at MSRI, maybe not, at about the end of May 1992, a lot of email correspondence on the Klein surface began among Thurston, Asimov, Osserman, Brock, Gross, Sibley, Kuperberg, Bumby, Clemens, Hirbawi, Mess, Grayson, Adler, Elkies, Riera, to list a few. The rich mathematical folklore that exploded via the internet led eventually to the publication of this book.

Dr. John Slorp, President of the Minneapolis College of Art and Design, once observed: "*The Eightfold Way* is the perfect biomorphic form: it is sensuous and intelligent at the same time." But for most people educated in traditional schools, mathematics comes across as anything but sensuous. My sculptures attempt to bridge this gap.

My View

Mathematics is an art form, which need not remain invisible [Hill et al. 1989; Simmons 1991; Cole 1998]. Some Math evokes art, some Art evokes math.

Art is a social event which the artist recognizes and sets up. People frequently ask me how long it took to do a particular sculpture. I answer by recalling my age at the time I finished that piece. Few people are satisfied with this humorous answer, but it really did take that long. Perhaps people ask this question because they wonder how long it would take them. This article answers the question and reveals how I go about doing a mathematical sculpture.

Sculpture has to occupy physical time and space, and aesthetically I consider time just as important as space. My sculpture involves the mathematical content of a timeless discipline. As a first response to its timeless aspect, I work in stone which took hundreds of millions or thousands of millions of years to form. Secondly, I work with endless geometries like tori or surfaces without boundary with no obvious beginning or end.

I have a practical motive for working in rock and stone. Stone, for all its potential beauty and age is common and worthless. These days it has small military value (though this was not always so; compare, for example [Avery 1966; Homer n.d., Book 16, lines 757-780; Holinshed 1587]). Metal, on the other hand, is still essential in warfare. A few thousand years ago the Romans appropriated the bronze sculptures of Athens to make war, and only a few decades ago the Nazis confiscated the bronzes of Paris to support their war. I prefer stone over iron, steel, and bronze. Why not use iron? Today we are iron rich, there is iron everywhere. But metal is still vulnerable — wait until a war in space creates a greater appetite for all metals. A stone sculpture without military value may extend the life of my art as social event.

I don't think that my choice of stone bears on the philosophical question of whether mathematical objects actually exist in some Platonic universe. The process of getting aspects of mathematical objects into our physical universe dramatizes fundamental things about this universe. For example, I carve by subtraction from an quarried piece of some geological formation. Subtraction makes the piece smaller and smaller. The chips and dust I make are not small compared to atoms and wavelengths of visible light. We do not live in a purely mathematical continuum universe; certainly continued subdivision breaks down. Our universe on a smaller scale is undulant with particles and lumpy with waves.

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Figure 1. Top view of the Eightfold Way and the hyperbolic disc, taken from an upper window in the MSRI building.

I haven't thought seriously about doing sculpture on this scale. For now I am sticking with stone.

Mathematical theorems occupy neither physical time nor space but share the characteristics of a communication tool. Mathematics as a conceptual language has its own aesthetic. The language of mathematics has three remarkable features: abstraction, condensation, and prediction.

In mathematics we consciously choose the level of abstraction. Consider for example the idea of the group of symmetries which underlies the *Eightfold Way.* This group of symmetries can be thought of at two levels of abstraction: algebraic or geometric.

For condensation consider that vast tables laboriously computed (for example, [Spenceley et al. 1952; Luke 1977]) have been replaced by a single equation or algorithm encoded in silicon. Kepler replaced Tycho Brahe's tables of planetary orbits with simple equations of ellipses. Newton reduced Kepler's equations to simple derivations from the inverse square law.

Prediction is possible if some correlation can be established between mathematical abstractions and a physical situation. Engineers study hundreds of models for every airplane, bridge, or boat before construction. Physical model construction is expensive and time consuming. Mathematics provides a kind of ghost realm, which coupled with computer graphics, makes modelling quick and inexpensive. As a sculptor I work with this cheap ghost real estate and find the mathematical ghost language very helpful in designing sculpture. It may help that the sculpture has mathematical content, although sometimes this content creates additional difficulties in the form of new problems to solve [Cox et al. 1994; Ferguson and Rockwood 1993; Ferguson et al. 1996].

The usual access to mathematical ghost material is through the imagination or the use of computers. What you see on the two-dimensional computer screen is a very different thing when you have had studio hand eye experience. It is like the difference between watching underwater films without scuba experience and then watching underwater films having had scuba experience. There is no comparision. Computer graphics does not replace studio experience.

There has been a philosophical prejudice in mathematics against the use of pictures to communicate mathematical ideas. Lagrange was not the first to brag that his book contained no deceptive figures or drawings. It has been pointed out, however, [Barwise and Etchemendy 1991] that more people have probably been deceived by specious linear arguments than by two-dimensional pictures. (See also [Barwise and Etchemendy 1996].) Some middle eastern religions were quite explicit in barring images. These anti graven image attitudes also colonized thinking in Puritan America and persist in some quarters even today. Curiously enough there is a historical record of a reversal in conservative religions in which certain properly clothed people forms may be acceptable but abstract visual form is considered unquestionably questionable. Robert Hughes in his treatise on art in America [Hughes 1997] argues these matters compellingly. Such attitudes have been part of our recent past, yet the visual image, suitably qualified, enjoys a rebirth in all the sciences, particularly mathematics.

In the previous century many drawings of functions and dramatic plaster models of mathematical forms were made; see [Jahnke and Emde 1945; Fischer 1986]. Today, computer images replace the dramatic plaster models. Bill Thurston has made the observation that computer graphics enables mathematicians, who typically are not trained to draw well, to draw computer pictures to communicate their ideas visually. Alfred Gray fills his book [1993] with images which took many person years to create before computers. The images of Gray's book come from parametric equations which have been under design by mathematicians for hundreds of years. Now they appear on the computer screen in a few keystrokes by anyone who can type. However, the old plaster models have a three-dimensional immediacy that transcends an image on a computer screen.

Mathematics is timeless, conjectural, and minimalist. How old is a theorem? It seems timeless because once thought and concluded, it appears to have always existed. Conjecture, one of the most creative acts in mathematics, can be stated as simply asking the right question. Conjecture grasps limbs of the complex tree of possible deductions, but runs deeper than that. Intuition becomes vital because there are assertions which are true but not deducible within the ambient system [Adamowicz and Zbierski 1997; Blum et al. 1998]. Mathematics is minimalist to the point of being invisible. Very few people get to see the theorems. Often a lot of hard work is involved in seeing or understanding a theorem. Mathematicians tend to communicate their most sublime creative acts to only a few of the mathematically trained. Part of the reason for this solipsism lies in the inherent character of the discipline. First, it is a discipline, hence the hard work. My algebraic topology professor, Tudor Ganea, used to say that "mathematics progresses by faith and hard work, the former augmented and the latter diminished by what others have done." Second, the mathematician strips away every nonessential idea. This makes her or him a minimalist of ideas. Furthermore, she or he may create a new language to assist in this reduction [MacLane 1971]. Who can speak such ex nihilo language?

I think of one of my sculptures as moving across time and space, an accretion of secondary aesthetics, anatomy, concepts, history, mathematics, philosophy, and process. Suppose someone digs up my sculpture in 1000 years, or 10,000, or even a million years. Can an interesting chapter of our nineteenth to twenty-first century mathematics be derived from it? We live in a golden age of mathematical creativity which does not necessarily have to continue. I prepare my sculpture to evoke deep thought in future 10^t time.

As I travel around the country lecturing and exhibiting my sculpture, I am approached by various mathematicians who shyly confess their artistic side. I enjoy encouraging them. When I began doing mathematical sculpture three decades ago, I had no one to talk to, no guide. Art was art and science was science and the two didn't converse [Snow 1959]. In graduate mathematics classes I knew better than to reveal that I took graduate sculpture classes and vice versa. On the rare occasions when such facts leaked out there was usually some display of hostility from one side or the other. A lot has changed in thirty years for the better, and I have probably helped change the old attitudes, but my work is just a beginning. More theorems in sculptural form would advance public appreciation and understanding of mathematics.

By writing this I hope to expose a path for others by providing a couple of forms of encouragement. First, a description of processes that a budding mathematical sculptor could in principle follow — a guide. Second, to make it seem easy — this is a lie, but one I have believed myself frequently and persistently. I excuse this mind game because I have found that most of what we believe isn't true and that verity doesn't stop us from acting in either creative or destructive ways and then justifying those ways by our beliefs. Most of what any person believes is not regarded as true or even helpful by some others. That, however, doesn't stop us from acting effectively and passionately on our beliefs and thereby accomplishing worthwhile and inspiring things. Mathematics is part of my belief system [Davis 1994].

Ramanujan–Michelangelo

Mathematicians have a highly developed, if solipsistic, aesthetic of their own, which they seldom share. They seem pretty shy or emotional about this [Cannon 1991; 1996]. However, they sometimes express this aesthetic with analogies outside their own field. G. N. Watson offers a particularly striking and mystifying example; he was an analyst after the school of G. H. Hardy, the English mathematician who had a remarkable and close relationship with Srinivasa Ramanujan [Newman 1956, vol. 1, pp. 366–376]. Watson's example appeals to me because in academic life, I was a computational number theorist [Ferguson and Forcade 1979; 1982; Ferguson et al. 1998], and cut mathematical milkteeth in [Whittaker and Watson 1927]. I will describe how Watson introduces a sculptural example, coincidentally close to my early artistic milkteeth [Avery 1966; Beck et al. 1994; Poeschke 1996].

Ramanujan loved to write down well poised specific cases of very general mathematical identities, choosing aesthetically rich examples. He seldom gave proofs of these identities and the way he came up with them seems mysterious to most. Watson spent a good part of his mathematical work proving Ramanujan's identities and confessed that the following integral series identity of Ramanujan thrilled him.

$$\int_{0\le x<\infty} e^{-3\pi x^2}\frac{\sinh \pi x}{\sinh 3\pi x}\,dx = \frac{1}{e^{2\pi/3}\sqrt{3}}\sum_{n\ge 0} e^{-2n(n+1)\pi}\prod_{0\le k\le n}(1+e^{-(2k+1)\pi})^{-2}$$

Do you get a buzz from this? I certainly did; enough so that I had to verify this identity for myself. For starters, the left hand side has a continuous integral, the right had side has a discrete sum. Watson compares his thrill to the time he stepped into the Medici Chapel in Florence and saw the tomb of Giuliano surmounted with Night and Day and the orthogonally placed tomb of Lorenzo surmounted with Dusk and Dawn. There are identities there too: Night and Day — female and male figures, Dusk and Dawn — male and female figures. The angular male summands of the right hand side of Ramanujan's identity are in counterpoint with the rounded female integrand of the left hand side. The integrand drawn as a function over the domain $-\frac{1}{3} < x < \frac{1}{3}$ could well be a template for parts of Night and Dawn.

When Claire and I stepped into the Medici Chapel early one morning in Florence, I did not think of Ramanujan's identities, but I was definitely aroused and stirred in new ways. I reacted to the work of both masters whose seamless work looks deciptively easy with the exhilarating thought that I could do this. But while I could prove or duplicate or participate in either in a constructive way, coming up with either work originally constitutes another matter entirely. Both are priceless gifts to our civilization, but we ourselves choose to prize the priceless. This choice has to be based upon discipline and hard work.

The integral series identity is a coupling of two things, computationally representing very different algorithms, but actually the same. Both sides of this identity give the same real number, an infinitude of decimal digits beginning with

$$0.06532958595187866756820037871441595811555343284066915628840323\ldots$$

The duality, the coupling of dissimilar things which unite in some way is a strong formal theme in many works of art and science. The Night and Day, Dusk and Dawn, first female and male, then male and female nude figures is an explicit duality. Each represents different activities, sleeping and awake, dozing and arousing. These nudes provide a very human and very alive counterpoint to the dessicated corpses inches away under their forms. Yet the nudes are dead stone. Their aliveness centers wholly in the observer, just as Ramanujan's identity lives in an acutely sensitive reader, without whom his work remains dead ink on a page. Watson senses within himself a moving relationship between the creation of a poor Indian clerk from Kumbakonam near Madras and the creation of a semi-orphaned Italian stone cutter from Settignano near Florence. That Watson shared this duality confirms my aesthetic perception of mathematics and sculpture as relatible forms. Chandrasekhar, a mathematical physicist, reported on Watson's feelings in his book on truth and beauty [Chandrasekhar 1987], by devoting a part of a page to the integral-series identity of Ramanujan and two whole pages to the male-female images of Michelangelo.

In *The Eightfold Way* I used a number of formal dualities similar to those in the Ramanujan–Michelangelo relationship. To illustrate the pairing of two areas of mathematics, geometry and topology, I couple a black two-dimensional platform with a white three-dimensional tetrahedroid. In a later work I couple three-dimensional red and black Klein bottles even more explicitly so that they orbit each other over a platform of two-dimensional multiple images of themselves [Blankstein 1998; Cipra 1997]. [Senechal 1996] contains an illustration of another two-dimensional platform and correlated three-dimensional sculpture subsequent to *The Eightfold Way.*

Geometry–Topology

Mathematical elements of topology and geometry, united by group theory comprise environmental influences upon *The Eightfold Way.* The white Carrara marble tetrahedroid is a topological statement. This is not, strictly speaking, a tetrahedron. I carved it in a qualitative free form process known as direct carving, paying attention to the combinatorics and topology but not rigid or measured geometry. By contrast, the black and green serpentine hyperbolic disc tiled platform base is a geometric statement. This is quantitative, everything is measured carefully to preserve the rigid geometry. Indeed, I carved this part using a computer driven water-jet robot, driven by a straight line program foloowing coordinates of explicit numbers. The black serpentine prism creates a connecting homotopy from the regular 120-degree hyperbolic geometric heptagon in the base platform to the topological palm of a hand shaped heptagon supporting the tetrahedral form. This junction prism provides the transition from quantitative accuracy (one-thousandth of an inch in this case) to qualitative flowing elusive forms.

The expressive relationship between the two, the topology and the geometry was very important to me. I was pleased when I installed the heptagonal prism or pedestal upright in the center of the hyperbolic disc and Bill Thurston came out and saw it for the first time, remarking, "that really is Topology".

By the expression *quantitative* in a sculpture, I mean that overall accurate measurements are important: exact angles, exact lengths. I regard the expression of a distance function and the specific point to point length relationships as essential to the reading of the form. Rigid relationships are important, but not necessarily the physical scale, in this context [Apéry n.d.; Sequin n.d.]. Once an object is made we can measure it and it becomes quantitative. But here I am speaking about the original process leading to the design and creation of the final object.

By the expression *qualitative* in a sculpture, I mean that specific measurements were not an important part of the process of creating it. The expression of the topological or combinatorial features, however, are paramount. The form idea is invariant under smooth deformations. I make quantitative choices in smooth deformations for reasons other than the mathematical reading of the form. An observer of sculpture tends to want to touch qualitative but not quantitative. Marble provides a medium for qualitative expression, as it polishes to a reflective sheen which pleases the eye and pleasures the touch. Running a hand over the grooves and surfaces of *The Eightfold Way* provides an unforgettable sensual experience.

One general theme reflected in this sculpture is that rigid geometry has an underlying topology and vice versa: a way to look at topology is to look for an underlying rigid geometry. This theme arose repeatedly in the last couple of centuries in mathematics, from Euler to Poincaré, Klein, and Fricke. It recently has been developed extensively in the work of William Thurston and others; see [Thurston 1997; Ratcliffe 1994], for example.

The Eightfold Way directly addresses the symmetry of surfaces: A sphere with $g > 1$ handles cannot have infinite symmetry, whereas a sphere $g = 0$ and a torus $g = 1$ both have infinitely many symmetry preserving transformations. Hurwitz' Theorem [1893; 1987] gives an upper bound on the symmetry: the group of automorphisms of a surface of genus $g > 1$ is bounded by $84(g - 1)$. The surface offered by the marble has genus $g = 3$ which in this case is a tetrahedral form with four faces, each face penetrated by the ambient space so that all penetrations meet in the middle.

This tetrahedral configuration appears to have four "handles" corresponding to the four edges of the tetrahedron. That these four are really three handles could confuse non-mathematicians. Claire's immediate response was, "Oh, that's easy, it's just like having a baby, you make a great big open smile here and then you see this head with two eyes and a mouth". (Claire knows the topology of having babies by heart, inside out and backwards.)

To make sense out of Claire's response, take a piece of soft clay and deform it into a tetrahedron, press holes into the four sides so they all meet in the middle, then without tearing deform open one of the triangular holes and flatten everything until three holes are visible. Another way to see this is to make three cuts through three limbs until the form becomes a ball with knobs and no loops.

The automorphism or symmetry group of a surface of genus three can have as many as $84 \cdot (3 - 1) = 84 \cdot 2 = 168 = 24 \cdot 7$ elements. I hear a lot of talk in the software world about deadlines being met by a $24 \cdot 7$ effort of twenty-four hours seven days a week, a never ending symmetry in time. Everyone experiences the stretched out symmetries of 24 hours in a day, 7 days in a week, and 168 hours in a week. The choice of the prime factors 2, 3, 7 in our organization of time keeping is ancient and interesting in its origins of oversimplifying of natural events [Neugebauer 1975]. The 2 is structural, diurnal, day and night, but why the 3 and 7?

What does such symmetry mean in a spatial physical or sculptural context? The mathematical definition of symmetry cannot be taken literally because of fundamental physics usually expressed by Heisenberg's uncertainty principle. It is impossible to manufacture a large object with precisely matching parts or exact symmetry. We come mechanically, molecularly, or even visually close; we come dramatically close in the case of cutting diamonds. We don't touch the symmetry of diamonds, we keep them small and wear them instead. If they were measured in pounds and feet instead of carats and millimeters their ever present imperfections would be grossly evident.

Symmetry in our world approximates perfection at best and a deception at worst. In the case of *The Eightfold Way*, the literal group of symmetries of the polished marble surface itself has only one element. The group of symmetries is trivial which means no symmetry at all. By doing sculpture in physical materials invariably all the symmetry gets broken, so symmetry has to be implied. Broken symmetry implies that a symmetry in theory is there to be broken [Morrison 1988]. By implication, *The Eightfold Way* surface articulates all 168 automorphism elements. More handily and literally, the generators can be read out of it.

Two- and threefold symmetries are implied by the tetrahedral form. The heptagon covering implies sevenfold symmetry. Each heptagon vertex forms a triple point or triskelion with the edges of its three neighbor points. The grooves or ridges of the three edges are curved to meet the neighbor point. There are 56 points and 84 edges to make up 56 triskelions in all. In carving this marble, I used a small plexiglass equilateral triangle form as a pattern to keep these triskelions under some equianglular control. This was a loose qualitative 120-degree consideration which rhymes with the more exact quantitative 120-degree triple points of the base platform.

This same symmetry reads more literally in the quantitative two-dimensional base platform. There the triple points are embedded in a system of infinitely many triple points. An infinite discrete group associates with this platform. This infinite group acts by hyperbolic transformations on the hyperbolic plane and has a fundamental domain of exactly 24 heptagons. In this case, there are 23 darker heptagons grouped around the 1 dark polished stone heptagonal prism in the center of the hyperbolic disk. The triskelions have been cut to have metrically accurate angles of 120 degrees. The discrete group transforms the fundamental domain in such a way that certain edges are identified. The transformations sew up the 24 heptagon domain into a surface of genus three, viz., into the marble surface lying above the fundamental domain. The boundaries of the 24 white marble heptagons carved into the tetrahedral form are articulated as either ridges or incisions. The incisions or cuts define the doubled outside boundary of the lower fundamental domain. The ridges on the marble also (compared to the incisions just described) form edges of heptagons. These ridges correspond to the geodesic arcs in the hyperbolic plane which lie inside the fundamental domain or cluster of darker contiguous heptagons. The day I installed this piece, Bill Thurston came out and started pasting notated tape on the white marble heptagons and connecting them with string to the corresponding black serpentine heptagons. The sun got too hot or something interrrupted this project. Some of the photographs taken at the time show these tapes.

If the viewer "reads" the sculpture a certain way the reason for the title "Eightfold Way" becomes quite clear. To "read", select an edge somewhere on

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Plate 1. *The Eightfold Way* seen from the MSRI library (facing East).

Red/green/white circle

Plate 2. Thurston's rendition of the heptagon tiling.

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Plate 3. Water-jet robot cutting out one of the serpentine block for the hyperbolic tiling. This set-up, part-holding, and cutting process had to be done over 232 times.

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Plate 4. Full set of stone tiles as cut by the water-jet robot. Note the dark cluster in the middle and the corona stones on the rim. The cluster "sews up" into the tesselated tetrahedroid.

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Plate 5. The artist doing finishing work during the installation in August 1993. Bill Thurston and Joe Christy stand behind.

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Plate 6. Cathedral view from inside *The Eightfold Way*.

the white marble tetrahedral form. Go along this edge to the fork in the road and take the left fork. Go to the next and take the right fork, then the left fork, then the right fork, left fork, right fork, left fork, right fork. If the viewer counted carefully, she is back on the starting edge. There were eight turns at eight forks in the road, hence the title.

The left right path is a cycle because it returns. Cycles like this are called Petrie nets. In general, a Petrie net in some fixed polyhedron is a skew polygon where every two but no three consecutive sides belong to the same face of the polyhedron. Petrie nets were named by H. S. MacDonald Coxeter after John Flinders Petrie [Coxeter 1973], the only son of the great founding archaeologist Sir William Matthews Flinders Petrie, who studied pyramids in Egypt [Drower 1985]. These Petrie cycles correspond to powers of products of generators (commutators) of the 168 element group of automorphisms. There 21 such cycles possible among the 56 triple points, since each path returns after eight alternating turns to the inital choice. In reading this sculpture, we visualize the full symmetry genus three surface with more than our eyes, we have our fingers touching the stone along these eightfold paths. The human haptic sense of around and through becomes a vital supplement to seeing, perceiving and certainly enjoying a symmetry from more dimensions than we usually experience.

By coincidence, the group of 3×3 invertible and commutator matrices with entries over the two element field F_2 consisting of $\{0, 1\}$ has exactly 168 elements, $(2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 7 \cdot 6 \cdot 4 = 7 \cdot 24 = 168$. This set of elements is a group denoted by $\mathrm{GL}(3, 2)$. There are 21 elements of order two and 56 elements of order three in this group. Does this correspond to the 21 Petrie cycles and the 56 vertices?

There are three abelian groups of order 168, and two nonabelian groups of order 168 of which only one is a simple group, viz., this $\mathrm{GL}(3, 2)$ is given by generators as $\langle a, b \mid a^2 = b^3 = [a, b]^4 = (ab)^7 = 1\rangle$, where $[a, b]$ is defined to be the commutator $[a, b] = aba^{-1}b^{-1}$. The relation $[a, b]^4 = 1$ is precisely the origin of the eightfold way path. Specific generators that satisfy these relations for $\mathrm{GL}(3, 2)$ are

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

This group $\mathrm{GL}(3, 2)$ has the polynomial

$$\eta(t) = 1 + 21t + 56t^2 + 42t^4 + 48t^7,$$

where the coefficient of t^n is the number of elements in the group of order n; moreover $\eta(1) = 168$, $\eta(-1) = -2 \cdot 21$.

Counting–Philosophy

This sculpture involves counting. New environments sometimes make counting difficult. It is easy to lose track along a given eightfold path so concentration is required. Counting was a major cultural and scientific achievement. Our systems of counting are very old — we have no real idea just how old. "No class of words, not even those denoting family relationship, has been so persistent as the numerals in retaining the inherited words" [Buck 1949, Chapter 13; Pappas 1994, pp. 145, 191]. Once one has learned to count, it seems the integers were there all along. Are they infinitely old?

As I carve these counting opportunities in stone, I wonder if maybe the stone is older than our counting. In fact, the age of the stone I carve is some fraction of a billion years. My process of direct carving seems to be moving back in time, as I reveal layer after layer of stone, deeper into our earth or solar system past, time and space coagulate and congeal. Mathematics seems timeless, and stone seems timeless. This is one reason I think stone is a natural material to express mathematical ideas.

Cultures that count cover the surface of the earth. World languages have been organized by linguists along continental and island structures where people have settled, isolated, and preserved their old counting ways. I believe it is possible to count orally to eight in exactly twenty one language groups, a different language group for each Petrie cycle and at the same time pretty much cover our globe. There seem to be 84 fairly different languages available for this purpose. On the other hand our world collection of visual number symbols does not seem to be as rich as our phonetic symbols. However, there is a set of Mayan heads that represent the first eight counting numbers. These were carved in stone so it seems I am not alone in carving numbers in stone.

The expression "Eightfold Way" has been recycled from similar titles. This may be appropriate given a sermon by the Buddha to potential disciples in a deer park near Benares or Varanasi [Kitagawa 1971; Levenson 1996; Powell 1995]. He summarized the practice of the Eightfold Path as the Three Learnings. This invites a confluence with my sculpture. The edges of each eightfold path on the sculpture are joined at each stage by a vertex of valence three. Each triple point in three dimensions above and in two dimensions below can be thought of as the Three Learnings organizing the path: moral precepts (sīla) encompassing speech, action, livelihood; meditative practive (dhyaña) encompassing effort, mindfulness, concentration; initially faith and ultimately attainment of wisdom (prajñā) encompassing understanding (view) and thought.

The triple points, and maybe old style triskelions, themselves are anatomical especially among peoples who recycle the bones of their dead; for example, skull offering bowls [Nomachi 1997, p. 147], kapala, from life in arid, rocky mountainous plateau regions (to dig is hard and unprofitable) include various anatomical references. Consider the sagittal and coronal sutures, anterior and posterior fontanelles. See [Levenson 1996, pp. 54-55] for eightfold path pictures as well as the triskelion images of buddhist skull bowls showing the common anatomical triple points in the skull, anterior: sagittal suture and two coronal sutures, triple junction; posterior: sagittal and two lambdoidal sutures. These type triple points also occur in the wheel of the law images [Netter 1996; Gray 1901; Richer 1890]. The human skull, cleaned, is a natural visual source of the triskelion form [Chumbawamba 1997].

The Eightfold Path was a paradigm designed for novitiates, a learners' course for beginners. At a certain stage of development there are ten: right concentration leads to perfect insight and perfect deliverance (two more), the end of the noble path. A sculptural analogy would be that at the end of eight edges is a triple point: after right concentration or meditation, one is faced with two new edges bordering on an entire heptagon. This leaves the linear or one-dimensional path to a field of two dimensions or infinitely many paths and perhaps insight and deliverance.

Sand paintings of the Eightfold Path involve arcs of circles in a disc arrangement suggesting a connection with the hyperbolic disc [Levenson 1996, p. 22]. Indeed there are mirrors that I have seen at the Art Institute of Chicago where the back of the mirrors have hyperbolic-like arrangements of arcs of circles. They are made with Dragon arabesques, Eastern Zhou Dynasty, Warring States period or early Western Han Dynasty, 3rd/2nd century B.C. Could these have Eightfold Path origins?

Religious or theological systems of thought tend to be very abstract systems of thought. Perhaps they represent some of our earliest mathematical forms. Much of history, especially the history of conflict reflects systems of thought. Even though they are based on very different, in some cases mutually inconsistent axioms, they have provided ample opportunity to go to war. On the other hand, I myself commit considerable violence in the process of carving stone. My abstract systems of mathematical thinking clash with the geology and mineralogy of the stone as I reform it in my own images. I war on my stone with hammers, chisels, diamond saws, grinders, and I have made the time honored excuses and impose my abstract thoughts with compelling violence.

The phrase "The Eightfold Way" is also the title of a book by Gell-Mann and Ne'eman [Gell-Mann and Ne'eman 1964], referring to an earlier paper by Gell-Mann where quarks where introduced theoretically by assuming three base states considered to transform according to the eight-dimensional group SU(3). (See [Lichtenberg 1978, pp. 166–171; Schensted 1976, pp. 218-228] for an exposition.) One of the systems of weight points for an irreducible representation of SU(3) corresponds to an octet of baryons including the proton and neutron. The dynamics of elementary particle interaction (scattering) is not well understood, so even approximate symmetries are vital to making predictions. Unitary transformations are associated with conservation laws, and the matrix group SU(3) provides approximate symmetries. The eight comes from SU(3) and the representation dimensions beginning with $1, 8, 10, 27$, where the 8 cooresponds to the most frequent higher mass of the vector baryons or mesons. Just to have some idea of how approximate this symmetry is from a mass perspective, consider the variation in the masses of the baryon octet, $(p, n, \Lambda, \Sigma^+, \Sigma^-, \Sigma^0, \Xi^0, \Xi^-)$ with masses $(938, 940, 1116, 1189, 1192, 1314, 1321)$. Other than the presence of the two 3×3 matrix groups $\mathrm{GL}(3, 2)$ and $\mathrm{SU}(3)$ any relationship between the physics eightfold way and the sculpture eightfold way remains unexplored.

Geometry Center–MSRI

A key fact behind the existence of any larger sculpture is funding. Elwyn Berlekamp had facilitated some funding for an unspecified MSRI sculpture from the Mitsubishi Electric Research Laboratories in Cambridge, Massachusetts. This vaguely had something to do with Kaplansky's retirement. Kaplansky was then Director of MSRI. The sculpture was originally going to be a development of the circle of theorems around the $(2, 3, 7)$ pretzel knot. This knot had begun its mathematical life with Seifert [1934, Satz 6, p. 589] computing its Alexander polynomial, then later came number theory connections with Lehmer [1933] and much since [Reid 1991; Riley 1975]. My granite $(2, 3, 7)$ pretzel knot sculpture has yet to see the light of day. Kaplansky's mathematical work has resisted sculptural expression so far.

I have a note from September 1990 about a chat with Bill Thurston, who said he would bring up the idea of one of my sculptures at MSRI. Then at the AMS–MAA meeting in Baltimore in January 1992 I talked with Lenore Blum, Deputy Director at MSRI, about the suggestion and she like the idea. The 18-month gap between these two events is typical of developing my sculpture. One has to be patient.

In March 1992 I FedExed a video and some posters to Bob Osserman, also Deputy Director at MSRI. We discussed some of Kaplansky's work and also the Lehmer conjecture. Later in the month Arlene Baxter, manager at MSRI, designer of the MSRI brochure, sent some photographs of possible sites for a sculpture. Bill was gone when I visited MSRI a few weeks later, but I sketched a brief idea on Bob's blackboard. This was a concept based on my findings at the Geometry Center in Minnesota during the great Halloween blizzard of a year or so before.

After Al Marden, director of the Geometry Center, saw *Knotted Wye I* at my exhibition at Ohio State University, in August 1990, he said they really needed a sculpture like it at the Geometry Center at the University of Minnesota. He felt that many people at the University there did not understand mathematicians. He had observed that people thought they were just computer hackers because the Geometry Center used heavy computer graphics as a research tool for gaining insight into geometry. He thought if they had a sculpture like *Knotted Wye I* that people would understand through it that they were mathematicians — artists of a certain kind. Creative mathematicians tend to think of their science as an art form, perhaps the ultimate conceptual art form (even if canonized academically in some ways).

How did I get involved with Bill Thurston and the topology and geometry themes of *The Eightfold Way*? In 1991 Don Davis at Lehigh University invited me to give a math sculpture talk. This was followed by another talk at the Five Colleges Geometry seminar at the University of Massachusetts at Amherst sponsored by Donal O'Shea and Lester Senechal of Mount Holyoke College. When Claire and I give such talks, I usually haul along some examples of smaller sculptures. This time I included the first Knotted Wye. (This knotted wye hyperbolic theme had been mentioned to me by Bill Thurston at the AMS-MAA meeting in Boulder, he did a clay sketch which Gary Lawlor, a post-doc at Princeton, brought down to me while visiting us in Maryland.) On the way back we stopped in Princeton and I showed Bill the bronze *Figureight Knot Complements, a Wild Sphere*, as well as the Carrara marble *Knotted Wye* (it didn't have a number then; cf. [Ferguson 1994]). At that time Bill mentioned a $\mathrm{PSL}(2,7)$ symmetry group surface problem and suggested that perhaps there was a sculpture there. This was the first hint of what eventually developed into *The Eightfold Way*.

The primes 2, 3, 7 of the pretzel knot reappear as the only prime factors of the order of the group $\mathrm{PSL}(2,7)$. The connection of $\mathrm{PSL}(2,7)$ with the Klein surface became more interesting to me sculpturally on the occasion of the dedication of one of my *Knotted Wye II* [Ferguson 1994]. This 1500-pound Carrara marble was preceded by the smaller *Knotted Wye I* mentioned above. Both are direct carvings. Their configuration can be decoded from a verbal description of the planar knotted graph presentation. The first link goes over, under, over, under, the second link under, over, and the third link over, under, over, under, going in each sequence from the first vertex to the second vertex. This knotted graph fits into the family of Kinoshita–Wolcott knotted graphs of k, m, n full twists; see [Farmer and Stanford 1996]. It appears in open ended but equivalent form in Ashley's Book of Knots [1944] as the wall knot and the further development of Matthew Walker's knot. There is an associated yarn of how the knot saved this sailor Walker from certain hanging. The judge was a former sailor and said he would let Walker off if Walker could tie a knot the judge had never seen. Matt tied his knot like a small fist in the middle of six fathoms of rope. The judge was impressed enough to give the sailor his freedom.

The dedication for *Knotted Wye II* illustrates the desire people have to experience mathematics in a direct way. As soon as the dedication was over there was a collective breath and the audience rushed forward to touch the marble carving. They climbed all around it, and held hands through the sculpture's limbs. These were adults, their spontaneity, lack of selfconsciousness, and involvement made it a delicious moment for me.

Knotted Wye II will communicate mathematics for many generations, sitting as it does on four oak cuboids. This is another instance of rigid geometry underlying the fluid topology, a precursor of the "Eightfold Way" concept described above. This work was first installed in the Geometry Center but has since been moved to the Mathematics Library at the University of Minnesota. It is the first 1500-pound theorem in the Frederick Weisman Museum of Fine Art Collection.

One of the interesting discussions that week at the Geometry Center was about the Klein surface and $PSL(2,7)$. John Horton Conway showed me an amazing $PSL(2,7)$ contiguity relationship for the Klein surface. He grabbed a scrap of paper and scribbled down the $PSL(2,7)$ relations, group elements, the appropriate conjugacy classes and what I called the eightfold way relationship. It took me several years to convert this two-dimensional scribble, its implications, and some of its mathematical context into the eightfold way sculpture.

For most of this week Margaret Thurston was sewing up a patchwork of regular heptagons into John Conway's incidence scheme. Margaret's stuffed heptagons were still at the Geometry Center when we visited in the Fall. We were scheduled to give a slide lecture to a large group of high school math teachers and math students from all over Minnesota (The Humpty-Up program of Harvey Keynes). The Great Halloween blizzard of 1991 closed the airport and marooned us in a hotel, but I managed to wade to the Center through four feet of drifting snow. There I found Margaret's extraordinary stuffed object and started thinking about it. I found the heptagons were somehow wrapped around a tetrahedral skeleton which suggested two- and threefold symmetry. I made a foam version and carved some figureight knot complements in styrofoam. I left knot complements there, but brought the tetrahedral foam with its tesselation of heptagons back.

In May 1992 Bob Osserman showed my blackboard drawing to Bill Thurston who talked to John Horton Conway about $PSL(2,7)$ as an MSRI sculpture. Bob came out in visited us in early May 1992. By this time I had a full scale size tetrahedroid carved out of white styrofoam with incisions to indicate the tesselation. By May eighth I settled on a hyperbolic prism to support the tetrahedroid. I wanted this piece to be approachable by an average size person and easily touched. Meanwhile, in our communications, Bob was pressing for a circular area maybe filled with sand in which to stand the sculpture. I had no plans to include a hyperbolic disc, this was where the Gauss problem came in.

The problem was Gauss, Gauss was the name of MSRI's resident cat. If the circle were filled with sand or raked white gravel Gauss might chose to appropriate the site as his personal cat box. This would discourage people from stepping close to the piece. I wanted to encourage people getting close enough to reach in and around the sculpture to follow the tesellation ridges and grooves. What to do? Gauss the Cat had to be respected. Gauss the Cat's namesake, Carl Frederick Gauss, had actually invented hyperbolic geometry perhaps even before Bolyai or Lobachevsky. It eventually occurred to us that if the Poincaré disc model in stone replaced the proposed sand or gravel then the Gauss problem would be solved. Gauss the Cat showed no proprietary interest in the Poincaré disc model of hyperbolic plane geometry and there would be no problem with

Slide #1

Figure 2. John Horton Conway's page of notes describing his $\mathrm{PSL}(2,7)$ action on the Klein surface.

Slide #2

Figure 3. Conway's sketch of heptagon contiguity associated with the $\mathrm{PSL}(2,7)$ action. The heptagon ∞B became the joint between the white marble and black serpentine.

Slide #3

Figure 4. Michael Ferguson (our youngest son) wearing Margaret Thurston's stuffed version of the Eightfold Way tetrahedron. She had made a multicolored version in 1991, which Helaman studied during the Halloween Blizzard in Minneapolis.

people standing on the hyperbolic tiling. The hyperbolic platform required some pretty extensive logistics. The heptagonal prism of 120-degree angles had to fit the real size of the conformal Poincaré disc that would mathematically scale with the rigid central hyperbolic regular heptagon. The next difficulty was cutting the hundreds of heptagonal tiles in stone to a few thousands of an inch precision; I solved that problem by cutting with a water-jet.

Serpentine–Marble

The black stone in the hyperbolic platform base of *The Eightfold Way* is serpentine, a magnesium silicate mineral related to granite, a compacted mineral talc with a very small rhomboid crystal size. Also called steatite, serpentine comes in a wide spectrum of quality and hardness. The softer steatites, which may contain asbestos type fibers, are called soapstones. Some of the oldest artifacts known were carved from soapstone. Because this mineral is impervious to heat and chemicals, it is used to line steel furnaces, build efficient wood and coal stoves, and make laboratory table tops. Some varieties are as hard as granite but with a finer grain. I wanted one of these hard types a vein of which occurs in the Blue Ridge mountain area of Albemarle County in Virginia. Early May Claire and I brought a thirty four hundred block from a stone yard there. I cut the fluted heptagonal prism out of this block. The age of this serpentine has been estimated to be between 400 and 500 million years old.

The white stone for the tetrahedroid from posed an interesting size problem. I needed enough stone to rough carve a tetrahedron $2\sqrt{2}$ feet on a side. Did this need to be a block of white marble $2\sqrt{2}$ feet thick? I could not find among my stone suppliers any cube that thick. Fortunately a two foot thick block would suffice! A very convenient feature of tetrahedrons is that they are not as thick as they seem from the edge length. I was first impressed by this listening to a talk in 1966 by Tracy Hall, the first person to synthesize diamonds in the laboratory [Hall 1986; Nassau 1980]. His technique, standard production process now, was to use a tetrahedral press with high pressure rams focussed on a regular tetrahedron. In his talk he showed how he tightly pinched a cylinder or straw, each successive pinch orthogonal to the next, giving a string of tetrahedrons. A seemingly too big tetrahedron slides through a seemingly too small straw. Baby heads are sort of tetrahedral and they get through an impressively small birth canal, in similar fashion. A tetrahedron of edge length $2\sqrt{2}$ can be carved out of a $2 \times 2 \times 2$ cube of marble, so I really only needed a block two feet thick. After checking on availability up and down the east coast I did find a suitably thick block of white marble from importer Harold Vogel of Manassas, Virginia.[1] I went down, split out my $2 \times 2 \times 2$ foot cube and brought that back in my

[1] It is an Italian Carrara marble. I incorrectly described in [Ferguson 1994] as Imperial Danby Vermont marble because of its similarity in carving to the latter stone.

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Figure 5. Silvio Levy used Mathematica to make a 6-page collage of the heptagonal tiling of the Poincaré disc, containing about 600 tiles. Much annotated by the artist, this drawing served as a model for the stone cutting. About half the tiles — all but the outer layer — made it into the sculpture; they are represented by 18 classes, each of a slightly different Euclidean shape. (See Plate 3 for the cutting of the tiles.)

4×4 truck with augmented undercarriage. The age of this marble from Italy is around 200 million years.

In October 1992, I visited MSRI and worked with Silvio Levy on finalizing the tile data for input to the completely different PC system which controlled the water-jet I was planning to use to cut the hyperbolic tiles. I had worked out my own Mathematica programs for tiling the Poincaré disc with regular 120-degree heptagons. Silvio had been through something like this before when he generated an automatic version of the type of Escher's *Circle Limit III* [Levy 1994; Escher 1989, p. 43; Escher 1982, pp. 97, 320]. He quickly adapted the Geometry Center word generation programs to extract the Postscript form data I needed for the water-jet programs. After the conference I went to supervise the water-jet cutting.

Architect Bill Blass produced the final concrete patio drawings in July 1992. Since this sculpture was being installed over the Hayward fault zone I worked out the structural issues with engineer Nellie Ingraham. We agreed finally on five internal solid steel rods. This required an extensive system of holes to be drilled in the fluted heptagonal serpentine prism and the matching marble. Physically matching the heptagonal hand of the white marble to the corresponding heptagonal hand of the black serpentine was a problem that had to be solved before these holes could be drilled.

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Figure 6. To debug the water-jet program a full set of wooden blocks was cut and then assembled.

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Figure 7. Hyperbolic "circle limit" of heptagons tiling installed and curing at MSRI. The earthquake stabilization hole in the middle is not yet drilled into the concrete pad and footings.

Athena–Escher

The base platform of *The Eightfold Way* makes a direct visual connection with the circle limit woodcuts of Escher [Escher 1989; 1995; 1992], as mentioned above. Remarkably enough, Escher had solved the problem of having limit tilings converge to a boundary triangle or a boundary square, but was stopped at a circle boundary. His dilemma was solved when he discovered hyperbolic geometry by making the acquaintance of H. S. MacDonald Coxeter and his work [Coxeter 1957; 1979]. Escher's wonderful circle limit woodcuts came after that and an unthinkable amount of painstaking labor. See also [Coxeter 1998; Emmer 1980].

Developing the circle limit of heptagons of *The Eightfold Way* was very different from Escher's technique of creating woodblocks for his prints. We have computer technology today that Escher would I believe have been delighted to use. I did use a computer directed water-jet to cut wood blocks of the heptagonal disc tiling and I did make canvas prints of the hyperbolic tesselation. It is certainly possible to take rubbings suitable for framing off the stone hyperbolic platform of *The Eightfold Way*. I personally encourage people to do those rubbings, they are easy to do. The stone tiling itself was so difficult to create that visitors lifting off versions of it to take home will share the joy of the thing. Escher drilled holes in his circle limit wood blocks to prevent more impressions from being made. There are no limits to the number of impressions to be taken from the circle limit of heptagons of *The Eightfold Way*.

It was important to specify exactly in computer form the hyperbolic tiling blocks. Since the 231 precision water-jet cut stone blocks or tiles were to form a tesselation of the Poincaré conformal disk model of the hyperbolic plane, Mathematica and other programs were written to develop inputs for the controller computer of the water-jet. A robot, the water-jet system responds only to a meticulously prepared set of instructions. To accomodate the circle at infinity or boundary of the disc there are 14 disc rim or corona stones. Interior to that are 217 hyperbolic heptagon stone blocks. Each heptagon has seven interior angles, each of 120 degrees. The 217 tile ensemble has 23 dark and 194 light serpentine heptagons surrounding a center prism with an exact conformal heptagon base. The tiles were set in May 1993 by Lajos Biczo. (Given the role of Janos Bolyai in the early history of non-Euclidean geometry [Greenberg 1993], it was appropriate to have someone of Hungarian heritage set this non-Euclidean geometry disc.) Joe Christy of MSRI facilitated the setting of the tiles and protected them while the grout cured. The rest of the sculpture could not be installed until the curing process was complete.

One of the tiles, the center tile, is actually a prism. It relates not to Escher but to Athena. καλὸς κ'ἀγαθός (read "kalós kagathós"), the beautiful and the good, a Greek saying applied to people whose outer beauty reflected internal moral goodness. I want my sculpture to outwardly reflect the internal integrity and consistency of mathematical theorems. A reflexion of this theme appears more

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Figure 8. Styrofoam maquette showing ∞B heptagon matching by SP-2, tetrahedron ∞B input, prism base ∞B output. This was to check the digitizer vs. inverse digitizer software of the SP-2 written in C by Sam Ferguson.

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Figure 9. The 200 million year old white marble cubical block at a stage of being carved into its tetrahedral form.

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Figure 10. SP-2 fitted maquette. A cloud of points for the surface of ∞B in the previously carved stryofoam tetrahedroid was rotated and became a virtual image target to create the other side of ∞B in the stryofoam base. This was to check the procedure before carving the stone version.

literally, in that I have included in the heptagonal prism a three-dimensional quotation from a fifth century Greek work which occurs in a series of twelve high relief metope over entrances to the temple of Zeus at Olympia. These sculptures, dated about 460 B.C., feature the labors of Herakles, the legendary founder of the Olympic Games, known to the Romans as Hercules. The specific metope, now in the Archaeological Museum at Olympia, was taken from the west end of the temple at Olympia and featured Herakles receiving the golden apples of the Hesperides from Atlas [Buitron-Oliver 1992, Plate 9, p. 96]. Athena attends Herakles, helping him hold up the skies while Atlas fetches the four golden apples from the tree of life, this being the last of the twelve labors of Herakles. Athena ruled wisdom and literature, arts and crafts, a war goddess; see [Buitron-Oliver 1992, Plate 7, pp. 92–3] for the Statue of Athena, Acropolis Museum, Athens, a marble from 480 B.C. Athena wears a peplos, a thick woolen garment belted at the waist with vertical parallel folds, right leg showing through front, left leg back. Unlike men, women were not represented nude.

The white marble of *The Eightfold Way* is open to the California sky, upheld by a serpentine prism of vertical parallel folds, echoing the traditional form of Athena in her peplos engaged in the task of helping Herakles support the heavens. One of the curves rising from the heptagon is a quotation from a fold in Athena's peplos near her neck. Plate 9 of [Buitron-Oliver 1992] is not so clear; I made a

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Figure 11. Left: The 450 million year old black serpentine prism block with base end marked as a regular heptagon. The other end will be the ∞B space heptagon homotopic to this heptagon. This stone is an unremarkable grey before polishing. Right: The white marble and black serpentine are finally together in the studio after the various matching holes have been drilled for the steel rod reinforcements for stabilization during an earthquake.

sketch from the original. This quote I felt appropriate to *The Eightfold Way* in the way it involves rigid verticals emphasizing the weight of the two-, three-, and sevenfold symmetry of the tetrahedral form, a quotation from the geometrical period of those early historical times.

Robot–Stewart Platform

The precise serpentine geometry counterpoints the free marble topology in more than form, also in process, a robot water-jet for the former, a Stewart Platform computer system for the coupling. The top of the serpentine prism exactly matches one of the 24 topological hexagons carved into the surface of the tetrahedral form. How could this matching be done? This kind of matching of two stones has been done before, the Incas and the Italians each have their own tricks, we have a new one.

The abrasive Colorado River carved the Grand Canyon out of solid rock. I mused about capturing that sort of power in my studio. Looking over the south rim at the tiny filament of water glistening in the sun far below was about the

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Figure 12. One of the three pairs of sensors of the Stewart Platform SP-2. A computer monitors the lengths of the six high tensile strength aircraft cable emerging from the semi-toroids coupled to the string potentiometers.

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Figure 13. Regular heptagon base end of the prism being prepared to fit in the hyperbolic platform disc. The end matches the missing central heptagon of the already installed platform.

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Figure 14. Left: Bill Thurston and Joe Christy attaching tapes to the spatial heptagons above for connecting strings to the regular heptagons below. Right: Close-up of the marble cloud and supporting serpentine prism with a view of the Athena quote.

filament size I saw close up in a water jet. The noise of the water jet seems to compress millions of years of erosion into a few seconds of roaring tornado sound, churning the catch chamber below into white water. This violent roar comes from a filament of water issuing from a diamond orifice under 55,000 pounds per square inch pressure. When I used a water-jet to cut the stones for *The Eightfold Way*, the water-jet was still somewhat of an experimental device. Since that time it is a common industrial tool, used to cut all manner of materials from textiles to five inch thick steel. These devices are not suitable for carving, they are through cut devices which explode from one side of the material to the other. They are also robots in the strict sense that they respond to a predetermined straight line program which allows no variation. All motions have to be calculated in advance. A complete set of 232 blocks were "dry run" out of $\frac{3}{4}$-inch plywood. This set of hyperbolic tiles was cut first and assumbled before cutting the serpentine stone. The final stone, counting the prism, was 24 black and 208 green serpentines. The Virginia green in this case was actually a bit harder stone than the black. (Greens from other parts of the country tend to be softer.) The greens tended to be the smaller stones, all the heptagons were cut to great accuracy with seven circular arc geodesic edges and seven 120-degree

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Figure 15. Close up of some of the incision and excision boundaries of the spatial heptagonal tiling of the thetrahedroid. Note the identification tapes of Thurston and Christy.

interior angles. The cutting itself took about a week for both the debugging run with plywood and the actual cutting of the stones. One of the most difficult parts of the cutting process was how to hold the part to be cut. Part holding became more challenging for the smaller heptagons in both the wood and the stone.

By contrast to the water-jet, the Stewart Platform system in my studio, SP-2, is not a robot but an information machine [Albus et al. 1993]. The cutting tool can be moved freely to any accessible point where information is provided as to the location of a virtual image. There is no straight line program relative to the cutting process, work is interruptible at any time and point and can be continued. The operator solves all the trajectory problems as they arise, they do not need to be computed in advance.

The mating of the white and black heptagons was accomplished by the second generation of the Stewart Platform virtual image projection system, SP-2, was used in the creation of *The Eightfold Way*. SP-2 has six instead of three cables with all six lengths monitored by sensors arranged in Stewart platform format [Albus et al. 1990; 1993]. The operator interactively flies the triangle (much as if flying a helicopter). Tool tip position (x, y, z) coordinates and tool orientation (pitch, roll, yaw) are computed from the six cable lengths. Carving the *Eightfold Way* included matching two stone parts, a hand shaped heptagon in the serpentine with a matching rounded heptagon on the tetrahedral marble

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Figure 16. Left: Spatial heptagon tesselation boundaries, part of a Petrie cycle, for tracing left, right, left, right, left, right, left, right and returning — or for that matter right, left, right, left, right, left, right, left and returning. Right: Matching white and black heptagons with whimsically mirror imaged pair of Helaman Ferguson signatures.

form. The SP-2 helped. First the concave heptagon was carved in the marble. This heptagon was then touched with the tip of the inactive air drill to input a cloud of points in no particular order close enough together. The three registration points were relocated in reversed order to carve the convex hand in the serpentine to hold the marble at its concave heptagon.

The SP-2 or Stewart Platform Number Two, or the VIP or Virtual Image Projection refers to one *inverse digitization* process which I have developed jointly in a CRADA (Cooperative Research and Development Agreement) between my studio and NIST (National Institute of Standards and Technology). This inverse digitization, goes from either parametric equations or a data base in the computer, into physical materials. My aesthetic choice is direct carving in the final material, e.g., subtractive carving of natural stone. The present form of this computer instrument has been strongly influenced by that aesthetic choice. The concepts are simple and powerful and can be adapted to other forms, as was the case with my series of minimal surface sculptures Costa II and Costa III.

The SP-2 itself is mathematical engineering based on a theorem of Cauchy from over a century and a half ago. Cauchy discovered many theorems referred to nowadays as Cauchy's Theorem. This one states that a convex polyhedron is

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Figure 17. Spatial heptagons crowding together to tesselate the inside the tetrahedron.

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Figure 18. Triple point or triskelion, this one all incision edges. This corresponds to a boundary point of the cluster. Which one?

determined if the lengths of its edges are known. Cauchy applies to the polyhedron being an octahedron of eight triangular faces, twelve edges and six vertices, the dual of a cube. The SP-2 which hangs in my studio includes two rigid equilateral triangles, one on the ceiling 13 feet on a side and one triangle suspended in midair 3 feet on a side. The other six edges are made of high tensile strength fine cable of variable length feeding under tension into six length sensors. These six lengths are then available to the computer (a MacII soon to be replaced by a G3) through an analog to digital interface. Since the six edges of the two rigid triangles are known exactly, the other six variable lengths, when known at any instant completely determine the octahedron. They determine implicitly the position and orientation of the suspended and moveable triangle, in particular the position and orientation of any tool fixed to that triangle.

A complex mathematical model originally developed for NASA for the space shuttle has been adapted for this engineering setup. The current software includes a C language implementation of this model which takes the six lengths imput and computes six coordinates which are three for the location of the tool tip and three for the orientation of the tool. This computation is done in real time on the Mac II.

It is helpful to compare the Stewart Platform system SP-2 with the traditional pointing machine. An accurate but not helpful comparison would be that a pointing machine is to the SP-2 as a hand cart is to the helicopter. Pointing machines for sculpture have been around for hundreds of years. Pointing machines, whatever their variety, refer to an existing object, a solid model or maquette, which is to be copied or enlarged. These pointing machines are slow and laborious to use, but quite effective. On the other hand, the SP-2 does not need a physical model to work from, the image can be in the computer as a data base or as equations. Digitization is a process for getting physical image coordinates into a computer data base. The SP-2 can be run in reverse, as a digitizer. A helpful description of the SP-2 is that it is an inverse digitizer. The heptagonal hand of the white marble was digitized, once the data was in in the computer, the heptagonal surface image in three dimensions was rotated around (in the computer) and then that virtual image was projected back into three dimensions, this time cut directly in the black serpentine.

Explicit quantitative sculpture includes a quantitative creation (mathematical) prior to the physical creation. The physical artifact then partakes in various ways of the original quantitative creation, but tends to be convolved with geologically or physically interesting natural materials. Technology is just emerging to make such sculpture possible in person hours instead of months or years. It should be kept in mind, *even possible at all*, due to the inhumanly huge numbers of calculations involved, once impossible, now possible.

Location

The Eightfold Way is permanently installed on the southeast patio of the Mathematical Sciences Research Institute (MSRI), at 1000 Centennial Drive, approximately 1300 feet (400 meters) above sea level (see http://www.msri.org). This land, part of the upper Berkeley hills, belongs to the University of California, although MSRI is an independent entity. Centennial Drive winds up from the Berkeley campus past the Lawrence Berkeley National Laboratory and the Lawrence Hall of Science to the Space Science Laboratory and MSRI. On the slope from the Lawrence Hall to MSRI there are parking lots; the sculpture patio, however, faces the other way, onto a fold of the hills, with a lovely view of mountainside, Oakland, and part of the San Francisco Bay (see Plate 1 and Figure 7). A wide trail, popular with joggers and walkers, leads from MSRI along a level curve of the hills; narrower trails crisscross the hillside through the scrub and scree.

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[Blankstein 1998] A. Blankstein, "Helaman Ferguson, Four Canoes, 1997, St. Paul, Minnesota", *Sculpture,* a publication of the International Sculpture Center (May/June 1998), 27.

[Blum et al. 1998] L. Blum, F. Cucker, M. Shub, and S. Smale, *Complexity and real computation*, Springer, New York, 1998. With a foreword by Richard M. Karp.

[Buck 1949] C. D. Buck, *A dictionary of selected synonyms in the principal Indo-European languages*, U. Chicago Press, Chicago, 1949. Reprinted in paperback, 1988.

[Buitron-Oliver 1992] D. Buitron-Oliver, *The Greek miracle: classical sculpture from the dawn of democracy, the fifth century B.C.*, National Gallery of Art, Washington, DC; distributed by Harry N. Abrams, New York, 1992. Companion book to the exhibition presented at The National Gallery of Art, Washington, DC, 22 November 1992 to 7 February 1993; The Metropolitan Museum of Art, New York, 11 March to 23 May 1993.

[Cannon 1991] J. W. Cannon, "Mathematics in marble and bronze: the sculpture of Helaman Rolfe Pratt Ferguson", *Math. Intelligencer* **13**:1 (1991), 30–39. The issue's cover shows *Alexander's Horned Sphere*. Cannon refers to Carl Faith's theory: ". . . it is because mathematicians are so emotional that they can become mathematicians."

[Cannon 1996] J. W. Cannon, *Math. Intelligencer* **18**:2 (1996), 73–75. A review of [Ferguson 1994].

[Chandrasekhar 1987] S. Chandrasekhar, *Truth and beauty: aesthetics and motivations in science*, U. Chicago Press, Chicago and London, 1987. See Chapter 4, "Beauty and the quest for beauty in science", the opening pages of which bring up Watson's comparison of an identity of Ramanujan to the Medici Chapel. Dick Askey brought Chandrasekhar to a small exhibition (AMS-MAA Meeting in Cincinnati, 1994) of my sculptures. C's remark was "There is a lot more to this than twisting Möbius bands out of strips of paper."

[Chumbawamba 1997] Chumbawamba, *Tubthumper* (disc), Republic, Universal Records Inc., New York, 1997. "Amnesia", track 2, 3:22-3:35, before "Drip, Drip, Drip".

[Cipra 1997] B. A. Cipra, "Quod granite demonstrandum", *SIAM News* **30**:10 (December 1997), 1.

[Cole 1998] K. C. Cole, *The universe and the teacup: the mathematics of truth and beauty*, Harcourt Brace, 1998. In Chapter 14, "Emmy and Albert", the author cites Hermann Weyl on truth vs. beauty and then has this: "Beauty in the mathematical sense is a lot more than a pretty face. It is a way of distilling the essence of things out of the messy mix that nature presents us."

[Cox et al. 1994] J. J. Cox, Y. Takezake, H. R. P. Ferguson, K. E. Kohkonen, and E. L. Mulkay, "Space-filling curves in tool path applications", *Computer Aided Design* **26**:3 (1994), 215–224.

[Coxeter 1957] H. S. M. Coxeter, "Crystal symmetry and its generalizations", *Trans. Royal Soc. Canada* (3) **51** (1957), 1–11. Contains the solution of the problem Escher was having and is the mathematical source of Escher's *Circle Limit* series.

[Coxeter 1973] H. S. M. Coxeter, *Regular polytopes*, Dover, New York, 1973.

[Coxeter 1979] H. M. S. Coxeter, "The non-Euclidean symmetry of Escher's picture 'Circle Limit III"', *Leonardo* **12** (1979), 19–25.

[Coxeter 1998] H. S. M. Coxeter, *Non-Euclidean geometry*, 6th ed., Mathematical Association of America, Washington, DC, 1998.

[Davis 1994] P. J. Davis, "Mathematics and art: the work of Helaman Ferguson", *SIAM News* **27**:5 (May/June 1994), 6–7. "A person viewing [Helaman's] pieces with no knowledge of their mathematical iconography might be compared to someone totally ignorant of western history and culture viewing a Raphael Madonna."

[Drower 1985] M. S. Drower, *Flinders Petrie: a life in archaeology*, Gollancz, London, 1985. Second edition, 1995, University of Wisconsin Press, Madison, WI. William Matthews Flinders-Petrie, a founding father of archaeology, developed field excavations (stratification). He developed measures into inductive metrology, using sequence dating methods which are now standard. He applied his system of metrology to the great pyramid at Giza and at Stonehenge. He actually spent much time penetrating the sides of the pyramid of Hawara to find the tomb of the pharoah Amenemhet III. He discovered how (other) grave robbers got through the labyrinth surrounding the tomb. An typical Egyptian pyramid has a square base and four triangular sides. The tetrahedral form we have been considering are topologically equivalent to an ordinary Egyptian pyramid with the four triangular sides tunneled into and meeting in the center (burial chamber somewhere inside). Again three cuts open up the handles and sunder the pierced solid pyramid into a ball. Petrie's books include *Inductive Metrology, or the Recovery of Ancient Measures from the Monuments*, written at age 24, *Stonehenge: Plans, Description, and Theories* (1880), and *The Formation of the Alphabet* (1912).

[Emmer 1980] M. Emmer, *The fantastic world of M. C. Escher* (VHS Video, 50 minutes), Film 7 International and M. C. Escher Foundation, 1980. Rereleased by Atlas Video, Inc., 1994. There is a sequence of H. S. M. Coxeter discussing his interaction with Escher on the Circle Limit series; Escher had done a square limit series but until he saw Coxeter's paper had not been able to do a circle limit image; I would not fault a draftsman for not discovering non-Euclidean geometry in the course of doing a woodcut.

[Escher 1982] *M. C. Escher, his life and complete graphic work*, H. N. Abrams, New York, 1982. ". . . with a fully illustrated catalogue . . . by F. H. Bool, J. R. Kist, J. L. Locher, F. Wierda; with essays by Bruno Ernst, M. C. Escher; edited by J. L. Locher; translated from the Dutch by Tony Langham and Plym Peters." Reprinted by Abradale Press and H. N. Abrams, New York, 1992. *Circle Limit III*, woodcut unidirectional fish in five colors, yellow, green blue, green, black, diameter 415 mm, December 1959, Cat. No. 434, pp. 97, 320; *Circle Limit IV*, heaven and hell, woodcut in two colors, black and ochre, July 1960, Cat. No. 436, pp. 98, 322; *Circle Limit I*, black and white fish, Cat. No. 429, diameter 418 mm, p. 319; *Circle Limit II*, March 1959, red and black $\times$'s, diameter 417 mm, p. 320.

[Escher 1989] M. C. Escher, *Escher on Escher: exploring the infinite*, Harry N. Abrams, New York, 1989. With a contribution by J. W. Vermeulen; translated from the Dutch by Karin Ford. The plates for *Circle Limit I* (fishes), *Circle Limit III* (unidirectional fishes), and *Circle Limit IV* (angels and devils) can be found on pages 126, 43, and 42.

[Escher 1992] M. C. Escher, *The graphic work: introduced and explained by the artist*, Benedikt Taschen Verlag, 1992. *Circle Limit I*, white and black fish, 1958, page 22; *Circle Limit III*, five color fish, 1959, page 24; *Circle Limit IV*, white angels and black devils, 1960, page 25.

[Escher 1995] M. C. Escher, *The M. C. Escher sticker book*, Harry N. Abrams, New York, 1995. 79 imaginative stickers, including *Circle Limit III* (unidirectional fishes) and *Circle Limit IV* (angles and devils).

[Farmer and Stanford 1996] D. W. Farmer and T. B. Stanford, *Knots and surfaces: a guide to discovering mathematics*, Mathematical World **6**, Amer. Math. Soc., Providence, RI, 1996.

[Ferguson 1994] C. Ferguson, *Helaman Ferguson: mathematics in stone and bronze*, Meridian Creative Group (1-800-530-2355), Erie, PA, 1994.

[Ferguson and Forcade 1979] H. R. P. Ferguson and R. W. Forcade, "Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two", *Bull. Amer. Math. Soc. (N.S.)* **1**:6 (1979), 912–914.

[Ferguson and Forcade 1982] H. R. P. Ferguson and R. W. Forcade, "Multidimensional Euclidean algorithms", *J. Reine Angew. Math.* **334** (1982), 171–181.

[Ferguson and Rockwood 1993] H. Ferguson and A. Rockwood, "Multiperiodic functions for surface design", *Comput. Aided Geom. Design* **10** (1993), 315–328.

[Ferguson et al. 1996] H. Ferguson, A. Gray, and S. Markvorsen, "Costa's minimal surface via Mathematica", *Mathematica in Education and Research* **5**:1 (1996), 5–10. The Weierstrass ζ function is used to provide an effective implementation of Costa's minimal surface.

[Ferguson et al. 1998] H. R. P. Ferguson, D. H. Bailey, and S. Arno, "Analysis of PSLQ, an integer relation finding algorithm", *Math. Comp.* (1998). To appear.

[Fischer 1986] G. Fischer, *Mathematische Modelle: aus den Sammlungen von Universitäten und Museen / Mathematical models: from the collections of universities and museums*, Vieweg, Braunschweig, 1986. Dual German-English text.

[Gell-Mann and Ne'eman 1964] M. Gell-Mann and Y. Ne'eman, *The eightfold way*, Frontiers in physics, W. A. Benjamin, New York, 1964. On pages 11–57 is reprinted the paper "The eightfold way: a theory of strong interaction symmetry", by Murray Gell-Mann, California Institute of Technology Laboratory Report CTSL-20 (1961). In this paper Gell-Mann and Ne'eman predicted quarks and a heavy particle Ω_-. Quarks were introduced theoretically by assuming three base states considered to transform according to the eight-dimensional group SU(3). One of the systems of weight points for an irreducible representation of SU(3) corresponds to an octet of baryons including the proton and neutron. Since this paper, quarks have been regarded as structural elements of baryons.

[Gray 1901] H. Gray, *Anatomy, descriptive and surgical*, 1901. Reprinted 1974 by Running Press, Philadelphia, as *Gray's Anatomy*.

[Gray 1993] A. Gray, *Modern differential geometry of curves and surfaces*, CRC Press, Boca Raton, FL, 1993.

[Greenberg 1993] M. J. Greenberg, *Euclidean and non-Euclidean geometries: Development and history*, Third ed., W. H. Freeman and Company, New York, 1993. The cover image is M. C. Escher's *Circle Limit IV*. The accompanying text describes how this was modelled on a tesellation of the hyperbolic plane by $45° - 45° - 90°$ triangles which are pairings of the alternately colored $30° - 45° - 90°$ triangles in Figure 7 of [Coxeter 1957].

[Gross and Tucker 1987] J. L. Gross and T. W. Tucker, *Topological graph theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley, New York, 1987.

[Hall 1986] H. T. Hall, "The transformation of graphite into diamond", *Journal of Crystal Growth* **16**:1 (March 1986), 1–8. Hall was the first to make synthetic diamonds, first with the 'Belt' press at General Electric in 16 December 1954, (U.S. Pat. Nos. 2,941,248 of June 21, 1965 and 2,947,608 of 2 Aug 1960), then with the tetrahedral press at Brigham Young University in 1956 (U.S. Pat. No. 2,918,699 of 29 December 1959). His later cubic preeses became the standard for industrial diamond manufacture. These are the diamonds I use for carving stone. Now that I think about it, it is curious that the tetrahedroid of *The Eightfold Way* is not unlike the negative space of Hall's tetrahedral press. As a sculptor I am heavily dependent upon the synthetic diamond industry of which Hall is the father. Diamonds, industrial diamonds, made synthetically are very important to me. I use an impressive array of diamond impregnated tool faces to subtract volumes of stone. Every stone worker nowadays does; these diamond tools are expensive but nowhere as expensive as they would be without synthesis. (The Tracy Hall Foundation home page is http://www.novatekonline.com/hthfound.htm; it includes his articles and patents.)

[Hill et al. 1989] S. A. Hill, P. A. Griffiths, and J. F. Bucy, "Our invisible culture: mathematics is the invisible culture of our ages", pp. 32–33 in *Everybody counts: a report to the nation on the future of mathematics education*, National Academy Press, Washington, DC, 1989.

[Holinshed 1587] R. Holinshed, *Chronicles of England, Scotland, and Ireland*, 1587. Reprinted in The Signet Classic Shakespeare Series, Sylvan Barnet, general editor; The Life of Henry V, edited by John Russel Brown, pages 172–210 (The New American Library, New York, 1965). The physical qualifications of Henry V for war as described by Holinshed include being strong and nimble, and a wrestler, leaper, and runner without compare; "in casting of great iron bars and heavy stones he excelled commonly all men".

[Homer n.d.] Homer, *The Iliad*, See, for example, the translation by Stanley Lombardo, with an introduction by Sheila Murnaghan, Hackett Pub. Co., Indianapolis, 1997. The lines referred to are summarized as follows: Patroclus is consumed with Achilles' refusal to fight after his squabble with Agamemnon over the prize girl Briseis. Patroclus decides to enter the fray with the Trojans in Achilles' stead. The Trojan prince Hector and his charioteer Cebriones drove straight through the other Greeks to get to Patroclus, who scooped up a jagged piece of granite with one hand and threw it at Cebriones, smashing his brows together and shattering his skull so his eyeballs spurted out. In final convulsion Cebriones flipped backwards out of his chariot. Nice dive, comments Patroclus. Rocks picked up off the ground and thrown figure in several episodes in the Iliad.

[Hughes 1997] R. Hughes, *American visions: the epic history of art in America*, Alfred A. Knopf, New York, 1997. On page 574, he points out that for the Puritan the word was law and the image was a delusion and snare. Of extreme puritan heritage myself, emphasis on ex now, I myself responded to the linear word made three-dimensional flesh.

[Hurwitz 1893] A. Hurwitz, "Über algebraische Gebilde mit eindeutigen Transformationen in sich", *Math. Annalen* **41** (1893), 403–442.

[Jahnke and Emde 1945] E. Jahnke and F. Emde, *Tables of functions with formulae and curves*, 4th ed., Dover, New York, 1945. See Addendum VI, "The exponential function $e^{1/z}$", p. 39.

[Kitagawa 1971] J. M. Kitagawa, "The Eightfold Path to Nirvana", pp. pages 88–101 in *Great religions of the world*, edited by M. Severy, National Geographics Society, 1971. The expression is aṣṭāṅgika-marga in Sanskrit and aṭṭhaṅgika-magga in the more recent Pali. The Eightfold Path is the way shown by the Four Noble Truths summarized in practice by the Three Learnings. The eight unfolded ways briefly summarized are right understanding (view), right thought, right speech, right action (conduct), right livelihood, right effort, right mindfulness, right concentration (meditation). By following the Eightfold Way one is offered the hope of being reborn more suitably ultimately achieving nīrvāṇa. Perhaps for the context of the sculpture, one can think of the classic eight as right understanding, left thought, right speech, left action, right livelihood, left effort, right mindfulness, left concentration as one recites around the sculpture. The adjective *right* connotes that they are all in accord with reality as it is (yathā-bhūtaṃ). Left here may be more in accord with "reality as it is" than a right turn, especially if one is to appreciate the underlying mathematical reality of the symmetry. To understand the eightfold way you must follow it as in ἔρχου καὶ ἴδε or "come and see".

[Lehmer 1933] D. H. Lehmer, "Factorization of certain cyclotomic functions", *Ann. Math.* (2) **34** (1933), 461–479.

[Levenson 1996] C. B. Levenson, *Symbols of Tibetan Buddhism*, Éditions Assouline, Paris, 1996. Photographs by Laziz Namani; Translated by Nessim Marshall. See particularly Chapter 3, "The wheel of the law".

[Levy 1994] S. Levy, "Mathematics in stone and bronze", *The Mathematica Journal* **4**:3 (1994), 14–16. A review of [Ferguson 1994].

[Lichtenberg 1978] D. B. Lichtenberg, *Unitary symmetry and elementary particles*, 2nd ed., Academic Press, 1978. Chapter 9, "The eightfold way", describes the situation: Strongly (not electromagnetic) interacting elementary particles are called hadrons. The dynamics of elementary particle interaction (scattering) is complex, so even approximate symmetries are vital to making predictions. An acceptable aprroximation is on the order of the fine structure constant ($\sim 1/137$). Unitary transformations are associated with conservation laws, and the matrix groups SU(2), SU(3) and SU(4) all provide approximate symmetries. The "eight" in "eightfold way" comes from the sequence beginning with $1, 8, 10, 27$ where the 8 is the most frequent, higher mass of the vector mesons or baryons. 1 is trivial, 10 and 27 refer to higher excited mass (only 10 has been seen experimentally so far). Note that the Clebsch–Gordan series for the Kronecker product of an eight-dimensional zero-triality representation of SU(3) is $\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}$; thus $8^2 = 1 + 2 \cdot 8 + 2 \cdot 10 + 27$. Baryon number B and charm C are defined for hadrons. Baryons have $B = 1$, mesons have $B = 0$; baryons are composed of three quarks, mesons of a quark-antiquark pair. Both form a charm octet, but not quite a mass octet. The eight baryons are denoted by $(p, n, \Lambda, \Sigma^+, \Sigma^-, \Sigma^0, \Xi^0, \Xi^-)$ with masses (in MeV, with $c = 1$) $(938, 940, 1116, 1189, 1192, 1314, 1321)$, with isospin I $(\frac{1}{2}, \frac{1}{2}, 0, 1, 1, 1, \frac{1}{2}, \frac{1}{2})$, giving one isospin singlet, one isospin triplet and two isospin doublets; with third component of isospin I_3 $(\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, -1, \frac{1}{2}, -\frac{1}{2})$, with hypercharge Y $(1, 1, 0, 0, 0, 0, -1, -1)$ where the last six ($Y \neq 1$) are called hyperons. The isospin I_3 and hypercharge Y together give the hexagonal weight diagram. The eight mesons are denoted by (π^0, π^+, π^-, K^+, K^-, K^0, $\bar{K}^0$, η), with masses (135, 140, 140, 494, 494, 498, 498, 549); there is an associated singlet η' with mass 957. Mesons are messier than baryons for a number of reasons; note the symmetry broken masses relative to the charm octet of $(0, 0, 0, 0, 0, 0, 0, 0)$.

[Luke 1977] Y. L. Luke, *Algorithms for the computation of mathematical functions*, Academic Press, New York, 1977.

[MacLane 1971] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5**, Springer, New York, 1971.

[Morrison 1988] P. Morrison, "On broken symmetries", pp. 54–70 in *On aesthetics in science*, edited by J. Wechsler, Birkhäuser, Boston, 1988.

[Nassau 1980] K. Nassau, *Gems made by man*, Chilton Book Co., Radnor, PA, 1980.

[Netter 1996] F. H. Netter, *Atlas of human anatomy*, 2nd ed., Ciba-Geigy Corporation, Summit, NJ, 1996. See particularly Plate 4 for bregma (anterior) and lambda (posterior); Plate 2 for other triple points and pterion; and Plate 48.

[Neugebauer 1975] O. Neugebauer, *A history of ancient mathematical astronomy*, Studies in the history of mathematics and physical sciences **1**, Springer, Berlin, 1975.

[Newman 1956] J. R. Newman, *The world of mathematics*, Simon and Schuster, New York, 1956.

[Nomachi 1997] K. Nomachi, *Tibet*, Shambhala Publishers, Boston, 1997.

[Pappas 1994] T. Pappas, *The magic of mathematics*, World Wide Publishing/Tetra, 1994. This is the only source I have seen for visual counting systems such as the Chinese rods.

[Poeschke 1996] J. Poeschke, *Michelangelo and his world: sculpture of the Italian Renaissance*, Harry N. Abrams, New York, 1996. Photographs by Albert Hirmer and Irmgard Ernstmeier-Hirmer. Plates 3, 4, 5, 6 show Michelangelo's *The Battle of the Centaurs*, c. 1490–1492, Casa Buonarroti, Florence.

[Powell 1995] A. Powell, *Living Buddhism*, University of California Press, Berkeley, 1995. Photography by Graham Harrison. See particularly p. 125, the facade on a Tibetan temple (gompa) near Dharmsala.

[Ratcliffe 1994] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Math. **149**, Springer, New York, 1994.

[Reid 1991] A. W. Reid, "Arithmeticity of knot complements", *J. London Math. Soc.* (2) **43**:1 (1991), 171–184.

[Richer 1890] P. M. L. P. Richer, *Anatomie artistique: description des formes extérieures du corps humain au repos et dans les principaux mouvements*, Plon, Paris, 1890. Translated by Robert Beverly Hale as *Artistic anatomy*, Watson-Gupthill, New York, 1971.

[Riley 1975] R. Riley, "A quadratic parabolic group", *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 281–288.

[Schensted 1976] I. V. Schensted, *A course on the application of group theory to quantum mechanics*, NEO Press, Peaks Island, ME, 1976. Section 6.4: Applications to some physical problems. Spin functions again. Something about the Gell-mann Ne'eman elementary particle scheme.

[Seifert 1934] H. Seifert, "Über des Geschlecht von Knoten", *Math. Annalen* **110** (1934), 571–592.

[Senechal 1996] M. Senechal, *Quasicrystals and Geometry*, paperback ed., Cambridge University Press, New York, 1996. The frontispiece shows the sculpture "Aperiodic Penrose Torus, α".

[Sequin n.d.] C. Sequin, "TetraThing, computer generation of genus-3 tetrahedroids", personal communication (letter and photographs).

[Simmons 1991] G. L. Simmons, *Differential equations with applications and historical notes*, 2nd ed., McGraw-Hill, New York, 1991. From page xix: "Mathematics is one of the supreme arts of civilization; like the other arts, it derives its grandeur from the fact of being a human creation."

[Snow 1959] C. P. Snow, *The two cultures and the scientific revolution*, Cambridge University Press, New York, 1959.

[Spenceley et al. 1952] G. W. Spenceley, R. M. Spenceley, and E. R. Epperson, *Smithsonian logarithmic tables to base e and base 10*, Smithsonian Institution, Washington, 1952. Reprinted by The Lord Baltimore Press, Baltimore, 1960.

[Thurston 1997] W. P. Thurston, *Three-dimensional geometry and topology, vol. 1*, edited by S. Levy, Princeton Mathematical Series **35**, Princeton University Press, Princeton, NJ, 1997.

[Whittaker and Watson 1927] E. T. Whittaker and G. N. Watson, *A course of modern analysis; an introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions*, 4th ed., The University Press, Cambridge, 1927. Reprinted in 1963.

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