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Hirzebruch's Curves F_1, F_2, F_4, F_{14}, F_{28} for $\mathbb{Q}(\sqrt{7}\,)$

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Abstract. We give a detailed proof of Hirzebruch's remarkable result that the symmetric Hilbert modular surface of level $\sqrt{7}$ for $\mathbb{Q}(\sqrt{7})$ is $\mathrm{PSL}_2(\mathbb{F}_7)$-equivariantly isomorphic to the complex projective plane. We identify the curves F_1, F_2, F_4, F_{28} explicitly as plane curves defined by invariants of degrees 4,12,18,21 for a three-dimensional representation of $\mathrm{PSL}_2(\mathbb{F}_7)$, and we explain their geometry. For example, F_1 is the Klein curve, F_{12} is the Steinerian of the Klein curve and F_{18} is essentially the Caylean of the Klein curve. The curves F_{12} and F_{18} are birationally equivalent to the Hessian of the Klein curve, which was shown to be defined by a cocompact arithmetic group by Fricke; Hirzebruch's theory gives another uniformization using subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We compute the group of invariant line bundles on the Hessian and offer the Hessian as a challenge to extending Doglachev's recent work on the invariant vector bundles on modular curves to the case of triangle groups $\{p, q, r\}$ in which p, q, r are not pairwise relatively prime. The curve F_{14} maps to the 21-point orbit in $\mathbb{P}^2$. Using our explicit identification of $F_1, F_2, F_4, F_{14}, F_{28}$, we are able to complete Hirzebruch's identification of the nonsymmetric Hilbert modular surface.

Contents

Introduction

Hirzebruch [1977] proved the remarkable result that the complex projective plane $\mathbb{P}^2$ is a minimal model of the symmetric Hilbert modular surface of level $\sqrt{7}$ for the extended Hilbert modular group of $\mathbb{Q}(\sqrt{7})$. Furthermore, the identification is equivariant for natural actions of $\mathrm{PSL}_2(\mathbb{F}_7)$ on the two surfaces. This result also enabled him to identify the (nonsymmetric) Hilbert modular surface associated to this group: namely, it is obtained from $\mathbb{P}^2$ by a certain sequence of blowings up and then passing to a two-sheeted covering branched along a certain curve on the resulting surface.

The curve in question and the sequence of blowings up are given in terms of some $\mathrm{PSL}_2(\mathbb{F}_7)$ orbits and certain $\mathrm{PSL}_2(\mathbb{F}_7)$ invariant curves. These curves are defined from the point of view of the theory of Hilbert modular surfaces as the images of the so-called modular curves F_N in $\mathbb{P}^2$ for $N = 1, 2, 4$. However, the construction of the Hilbert modular surface starting with $\mathbb{P}^2$ fails to be completely explicit since the invariant curves involved in the construction were not completely identified in terms of the geometry of $\mathbb{P}^2$ and $\mathrm{PSL}_2(\mathbb{F}_7)$ alone. The identification of modular curves F_N on a Hilbert modular surface is a matter of independent interest.

The purpose of the present article is twofold. First, we show how to identify the images of the curves F_N in $\mathbb{P}^2$ for $N = 1, 2, 4, 14, 28$. Second, since the details of Hirzebruch's theorem were never published, we provide those details here as a public service. In this latter endeavor, I relied on some unpublished notes and private communications [Hirzebruch 1995; 1979]. Hirzebruch also encouraged me to provide details of his unpublished determination of the curves F_{14} and F_{28}, which were also discussed in [Hirzebruch 1995].

In 1979 or so, I learned of this work of Hirzebruch from a letter of Serre and began looking at [Hirzebruch 1977]. From the sketches there, I was motivated to study the images in $\mathbb{P}^2$ of the curves F_1, F_2 and F_4 knowing only that they were $\mathrm{PSL}_2(\mathbb{F}_7)$ invariant plane curves of degrees 4, 12 and 18 respectively and that their genera were 3, 10 and 10 respectively. The first curve, as Hirzebruch already pointed out, must be Klein's curve

$$x^3y + y^3z + z^3x = 0,$$

but the nature or identity of the other two is not so easy to determine.

I was able to identify the curve F_2 explicitly as the Steinerian of the Klein curve and to write down its equation. (See the beginning of Section 15 for the definition of the Steinerian.) I also showed that there were essentially two possibilities for the singularities of the curve F_4: either it had double points on the 21-point orbit and the 42-point orbit or it had quadruple points on the 21-point orbit. Through a careless error, I incorrectly concluded that the latter

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possibility did not occur and only noticed the error after sending the manuscript [Adler 1979] to Hirzebruch. When I pointed out my error to Hirzebruch, he replied that [Berzolari 1903–15] has a footnote describing a way to get a covariant of degree 18 and genus 10 with 21 quadruple points and suggested that this might be the way to obtain F_4 from the Klein curve, since he knew from his study of the Hilbert modular surface that the image of the curve F_4 in $\mathbb{P}^2$ must have 21 quadruple points.

There the matter stood until recently when I again took up the task of completing this article and of providing details of Hirzebruch's results. This effort was partly motivated by my conviction that his striking result was perfect for inclusion in the present volume on the Klein curve.

The effort of providing details given my limited experience with Hilbert modular surfaces turned out to be considerable, partly because of the difficulty of reading and using the relevant literature. Accordingly, I am preparing a set of lecture notes [Adler $\geq$ 1998], which I hope will make it easier for others to gain access to this beautiful subject. The notes will also contain a much more detailed and general examination of the Hilbert modular surface of level $\sqrt{7}$ for $\mathbb{Q}(\sqrt{7})$ than is possible in this article.

The first part of the article is devoted to a proof of Hirzebruch's published results, including the fact that the projective plane is a minimal model of the symmetric Hilbert modular surface of level $\sqrt{7}$ for $\mathbb{Q}(\sqrt{7})$. The second part is devoted to a determination of the curves F_1, F_2, F_4. Some of the computations were carried out using the algebra package REDUCE 3.4 on a personal computer. There are also results regarding the nonsingular models of the curves F_2 and F_4. It turns out that they are isomorphic to the Hessian of the Klein curve and that this is the unique curve of genus 10 with $\mathrm{PSL}_2(\mathbb{F}_7)$ acting on it. We also study the group of invariant line bundles on this curve.

As this article was nearing publication, I ran across [Fricke 1893a] and learned that some of these results on the various models of the Hessian of the Klein curve were anticipated by Fricke a century ago. In particular, Fricke knew that the Hessian of the Klein curve is characterized by its genus and automorphism group. He also wrote down the equations of the unique invariant curve of degree 12 and genus 10 and considered the pencil of all invariant curves of degree 12. Thus, in effect, he found the equations of Hirzebruch's curve F_2. He also considered the pencil of all invariant curves of degree 14 and, as we do, the net of all invariant curves of degree 18 but stopped short of the extensive computations of that net which we have carried out. It will be pleasant to learn more from our late colleague about his old work in this field which for us is so new.

1. Some Hilbert Modular Surfaces for $\mathbb{Q}(\sqrt{7})$

Throughout this article, k will denote the real quadratic field $\mathbb{Q}(\sqrt{7})$ of discriminant $D = 28$ and $\mathcal{O}_k$ will denote the ring of integers of k. The conjugate over $\mathbb{Q}$ of an element x of k or of a matrix M with entries in k will be denoted x' and M' respectively. Denote by $\widehat{\Gamma}$ the group of all 2×2 matrices

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries in $\mathcal{O}_k$ such that the determinant of γ is a totally positive unit of k. Denote by $\widehat{\Gamma}(\sqrt{7})$ the subgroup of $\widehat{\Gamma}$ consisting of matrices which are congruent to the identity matrix modulo $\sqrt{7}$. We will refer to $\widehat{\Gamma}$ as the *extended Hilbert modular group.* (The adjective "extended" refers to the fact that $\widehat{\Gamma}$ contains the usual Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_k)$.) It is known [van der Geer 1988, § I.4, pp. 11–14] that $\widehat{\Gamma}$ is a maximal discrete subgroup of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. We will refer to $\widehat{\Gamma}(\sqrt{7})$ as the *congruence subgroup of level* $\sqrt{7}$ of $\widehat{\Gamma}$. If Γ is a subgroup of $GL_2^+(k)$ (the group of 2×2 matrices with entries in k and totally positive determinant) commensurable with $\mathrm{SL}_2(\mathcal{O}_k)$, we will call Γ a *group of Hilbert modular type.*

A group Γ of Hilbert modular type acts on the product $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ of two copies of the projective line $\mathbb{P}^1(\mathbb{C})$ by the rule

$$\gamma \cdot (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right).$$

We denote by $\mathcal{H}$ the upper half plane in $\mathbb{C}$ and by $\mathcal{H}^2$ the product of two copies of $\mathcal{H}$. If Γ is of Hilbert modular type, we denote the orbit space for Γ acting on $\mathcal{H}^2$ by $\Gamma \backslash \mathcal{H}^2$ and we will refer to any complex analytic surface bimeromorphic to $\Gamma \backslash \mathcal{H}^2$ as a *surface of Hilbert modular type.*

The group $\widehat{\Gamma}(\sqrt{7})$ acts without fixed points on $\mathcal{H}^2$ and the orbit space is a complex manifold. The group $\widehat{\Gamma}$ has some isolated fixed points arising from elliptic elements and its orbit space is a complex orbifold but is not a complex manifold due to the singularities arising from the elliptic fixed points. (For the notion of an orbifold see [Satake 1956; Weil 1962], where the terminology "V-manifold" is used.)

One can prove that both orbit spaces are isomorphic to quasiprojective algebraic varieties. The proof depends on showing that both spaces have natural compactifications which are isomorphic to projective varieties. We will not present the proof of this result, but the compactifications themselves [Satake 1960; Baily and Borel 1966] are of interest to us. They are defined in [van der Geer 1988, § I.4], and we refer the reader there for details. The compactification and its structure of normal complex analytic variety were constructed in a general setting by Satake [1960]. That complex analytic variety is called the *Satake compactification.* Baily and Borel [1966] proved that the Satake compactification is isomorphic to a projectively normal (hence normal) projective variety.

That variety is called the *Baily–Borel compactification*, but since we don't want Satake's role to be forgotten, we will refer to it as the *SBB compactification* throughout this article.

We will denote the SBB compactifications of $\widehat{\Gamma}\backslash\mathcal{H}^2$ and $\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2$ respectively by $\widehat{X}$ and $\widehat{X}(\sqrt{7})$. Write Γ to denote either $\widehat{\Gamma}$ or $\widehat{\Gamma}(\sqrt{7})$ and X to denote the SBB compactification of $\Gamma\backslash\mathcal{H}^2$. Then the complement of $\Gamma\backslash\mathcal{H}^2$ in X is a finite set, namely the orbit space for Γ acting on the projective line $\mathbb{P}^1(k)$ over k. This projective line sits in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ via the embedding $\xi\mapsto(\xi,\xi')$. The orbits for Γ on $\mathbb{P}^1(k)$ will be called the *cusps* of Γ or of X. The point (∞,∞) is a cusp. It and the points corresponding to it in surfaces of Hilbert modular type will be denoted ∞.

Since the class number of k is 1, the number of cusps of $\widehat{\Gamma}$ is 1 [van der Geer 1988, Prop. I.1.1, p. 6], and the number of cusps of $\widehat{\Gamma}(\sqrt{7})$ is $(7^2-1)/2=24$.

The natural mapping of $\widehat{X}(\sqrt{7})$ onto $\widehat{X}$ has degree equal to the order of $\mathrm{PSL}_2(\mathbb{F}_7)$, which is 168. For the rest of this article, we will denote the group $\mathrm{PSL}_2(\mathbb{F}_7)$ by G.

2. Some Congruence Subgroups of Unit Groups of Orders in Quaternion Algebras

Recall that $k=\mathbb{Q}(\sqrt{7})$, $D=28$ is the discriminant of k and $\mathcal{O}_k$ is its ring of integers. For the rest of this article, $\eta=3-\sqrt{7}$ is an element of norm 2 and $\varepsilon=\eta/\eta'=8-3\sqrt{7}$ is a fundamental unit of k.

Definition 2.1. By a *skew-hermitian matrix* we will mean a matrix of the form

$$B=\begin{pmatrix} a\sqrt{D} & \lambda \\ -\lambda' & b\sqrt{D}\end{pmatrix},$$

where a,b are rational numbers and λ is an element of k. We will say such a matrix is *integral* if a,b are rational integers and λ is an integer of k. We will say an integral skew-hermitian matrix B is *primitive* if there is no rational integer $n>1$ such that $\frac{1}{n}B$ is also integral.

The proofs of the next two lemmas are left to the reader.

Lemma 2.2. *If ν is a nonnegative integer, the skew-hermitian matrix B_ν given by*

$$\begin{pmatrix} 0 & \eta^\nu \\ -\eta'^\nu & \sqrt{D}\end{pmatrix}$$

is primitive and has determinant 2^ν.

Lemma 2.3. *If ν is a nonnegative integer, the matrix B_ν given by*

$$\begin{pmatrix} 2^\nu\sqrt{D} & 0 \\ 0 & \sqrt{D}\end{pmatrix}$$

is a primitive skew-hermitian matrix of determinant $2^\nu D$.

LEMMA 2.4. *There is no integral skew-hermitian form B with determinant 7 but there is one of determinant 14.*

PROOF. The skew-hermitian matrix B given by

$$\begin{pmatrix} \sqrt{D} & 7-\sqrt{7} \\ -7-\sqrt{7} & -\sqrt{D} \end{pmatrix} \tag{2.5}$$

has determinant 14. We have $\mathcal{O}_k = \mathbb{Z}[\sqrt{7}]$. The general integral skew-hermitian matrix

$$\begin{pmatrix} a\sqrt{D} & \lambda \\ -\lambda' & b\sqrt{D} \end{pmatrix}$$

has determinant $28ab + \nu(\lambda)$, where ν denotes the norm of the quadratic field k. If this determinant equals 7 then 7 divides the norm $\lambda\lambda'$ of λ, whence $\sqrt{7}$ divides λ. Writing

$$\lambda = (c + d\sqrt{7})\sqrt{7},$$

we have

$$4ab - c^2 + 7d^2 = 1,$$

where a, b, c, d are rational integers. Modulo 4 this becomes

$$-c^2 - d^2 \equiv 1,$$

which clearly has no solution. □

PROPOSITION 2.6. *Let B be a primitive integral skew-hermitian matrix with entries in k. Denote by Q_B the set of all 2×2 matrices*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries in k such that

$$^tM'B = BM^*, \tag{2.7}$$

where

$$M^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then Q_B is an indefinite quaternion algebra over $\mathbb{Q}$ generated by elements i, j such that

$$i^2 = D, \quad j^2 = -N/D, \quad ij = -ji,$$

where N is the determinant of B. The intersection of Q_B with the ring of 2×2 matrices with entries in $\mathcal{O}_k$ is an order, denoted $\mathcal{O}_B$, of Q_B. The discriminant of the order $\mathcal{O}_B$ is N^2.

(The discriminant of the order $\mathcal{O}_B$ is defined to be the index of $\mathcal{O}_B$ in its dual lattice in Q_B with respect to the pairing defined by the reduced trace of Q_B. This is not the same as the discriminant of the quaternion algebra Q_B, which is the product of the primes q such that Q_B is ramified at q. The discriminant of a maximal order of Q_B is the square of the discriminant of Q_B.)

Proof. See [van der Geer 1988, Prop. V.1.5, p. 90]. □

Lemma 2.8. *If B is as in Lemma 2.2 then the quaternion algebra Q_B is a matrix algebra over $\mathbb{Q}$. If B is as in Lemma 2.3 or Equation 2.5 then Q_B is a division algebra with discriminant 14.*

Proof. Since k has discriminant $D = 28$, it follows from Lemma 2.6 that the reduced norm for Q_B is of the form

$$w^2 - 28x^2 + \tfrac{1}{7}\cdot 2^{\nu-2}y^2 - 2^\nu z^2$$

with $\nu \geq 0$ or, after replacing x by $x/2$ and y by $14y$,

$$\xi\xi' - 2^\nu \zeta\zeta',$$

where $\xi = w + x\sqrt{7}$ and $\zeta = z + y\sqrt{7}$. Since $\eta\eta' = 2$ and the norm from k to $\mathbb{Q}$ is multiplicative, the form is equivalent to

$$\xi\xi' - \zeta\zeta',$$

which obviously represents 0. This proves the first assertion.

If B is B_ν as in Lemma 2.3 or B as in Equation 2.5 then by Lemma 2.7, the norm form is

$$w^2 - 28x^2 + 2^{\nu-2}y^2 - 2^\nu\cdot 7z^2$$

with $\nu \geq -1$, which is equivalent to

$$\xi\xi' + \zeta\zeta'$$

with $\xi = w + x\sqrt{7}$ and $\zeta = y + z\sqrt{7}$, by a reduction similar to the preceding one, and hence to

$$w^2 - 7x^2 + y^2 - 7z^2.$$

It is easy to see that this form cannot represent zero nontrivially. Indeed, one can assume that w, x, y, z are all integers without common factor. Reducing modulo 7, one concludes that 7 must divide w and y. Replacing w by $7w$, y by $7y$, dividing through by 7 and again reducing modulo 7, one concludes that x, z are divisible by 7, which contradicts the assumption that w, x, y, z are integers without common factor. □

Lemma 2.9. *Let B be as in Lemma 2.8. With the notation of Lemma 2.6, denote by $\mathcal{O}_B^1$ the group of units of $\mathcal{O}_B$ of reduced norm 1. Denote by $\mathcal{O}_B^1(\sqrt{7})$ the intersection of the group of $\mathcal{O}_B^1$ consisting of elements which are congruent to 1 with the group $\widehat{\Gamma}(\sqrt{7})$. If B is B_ν as in Lemma 2.2 then the factor group $\mathcal{O}_B^1/\{\pm 1\}\mathcal{O}_B^1(\sqrt{7})$ is isomorphic to G. If B is B_ν as in Lemma 2.3 or B as in Equation 2.5, then the factor group is a cyclic group of order 4.*

Proof. By the approximation theorem, the factor group $\mathcal{O}_B^1/\mathcal{O}_B^1(\sqrt{7})$ is the product of the analogous groups for the q-adic completions of the algebra Q_B, as q runs over all primes. In particular, it must contain the analogous group for the 7-adic completion. On the other hand, the factor group $\mathcal{O}_B^1/\mathcal{O}_B^1(\sqrt{7})$ clearly injects into the factor group $\widehat{\Gamma}/\widehat{\Gamma}(\sqrt{7})$, which is isomorphic to $\mathrm{SL}_2(\mathbb{F}_7)$. Now suppose we are in the case of $B = B_\nu$ as in Lemma 2.2. Then Q_B is a matrix algebra isomorphic to $M_2(\mathbb{Q})$. According to Lemma 2.2, the discriminant of the order $\mathcal{O}_B$ is a power of 2, hence its 7-adic completion is a maximal order of $M_2(\mathbb{Q}_7)$ and the 7-adic completion of $\mathcal{O}_B^1$ is $\mathrm{SL}_2(\mathbb{Z}_7)$. Similarly, since $i^2 = D = 28$, the 7-adic completion of the congruence subgroup $\mathcal{O}_B^1(\sqrt{7})$ is the congruence subgroup of level 7 of $\mathrm{SL}_2(\mathbb{Z}_7)$ and the factor group is $\mathrm{SL}_2(\mathbb{F}_7)$. Therefore, the contribution at 7 to the factor group $\mathcal{O}_B^1/\mathcal{O}_B^1(\sqrt{7})$ is $\mathrm{SL}_2(\mathbb{F}_7)$ and to the factor group $\mathcal{O}_B^1/\{\pm 1\}\mathcal{O}_B^1(\sqrt{7})$ is G, which implies that $\mathcal{O}_B^1/\{\pm 1\}\mathcal{O}_B^1(\sqrt{7})$ must be all of G in this case. If B is B_ν as in Lemma 2.3 or B as in Equation 2.5, the quaternion algebra Q_B is ramified at 7 by Lemma 2.6. The discriminant of the order $\mathcal{O}_B$ is a power of 2 times the square of the discriminant of the algebra Q_B. Therefore, the 7-adic completion of $\mathcal{O}_B$ is a maximal order of the 7-adic completion of Q_B. The 7-adic completion of $\mathcal{O}_B^1$ is then the group of units of reduced norm 1 in that maximal order and the 7-adic completion of $\mathcal{O}_B^1(\sqrt{7})$ is the subgroup of elements congruent to 1 modulo the maximal ideal of that maximal order. The maximal order modulo its maximal ideal is isomorphic to the field $\mathbb{F}_{49}$ with $7^2 = 49$ elements [Weil 1974, Prop. I.4.5, pp. 20-21]. The factor group $\mathcal{O}_B^1/\mathcal{O}_B^1(\sqrt{7})$ is therefore isomorphic to the group of elements of norm 1 in $\mathbb{F}_{49}$. Since the norm map is a surjective homomorphism from $\mathbb{F}_{49}^\times$ to $\mathbb{F}_7^\times$, the elements of norm 1 form a group of order $(49-1)/(7-1) = 8$, in fact a cyclic group, and that cyclic group is therefore contained in the factor group $\mathcal{O}_B^1/\mathcal{O}_B^1(\sqrt{7})$. When we pass to the factor group $\mathcal{O}_B^1/\{\pm 1\}\mathcal{O}_B^1(\sqrt{7})$, the contribution from the 7-adic completion is therefore a cyclic group of order 4. This cyclic group is a maximal commutative subgroup of the group G. In particular, it is its own centralizer in G. Since the group $\mathcal{O}_B^1/\mathcal{O}_B^1(\sqrt{7})$ is the product of this cyclic group with the analogous groups at primes other than 7, the latter must commute with the cyclic group. It follows that the groups coming from the other primes must be trivial. $\square$

If B is a primitive skew-Hermtitian matrix with positive determinant, we denote by $\widehat{\Gamma}_B$ the subgroup of $\widehat{\Gamma}$ consisting of similitudes of B, i.e.,

$$\widehat{\Gamma}_B = \{M \in \widehat{\Gamma} \mid (\exists \xi \in k)\, {}^t M' B M = \xi B\}. \tag{2.10}$$

Using the notation of equation (2.7), for invertible matrices M we have $M^* = M^{-1} \cdot \det(M)$. Replacing ξ by $\xi \cdot \det(M)$ in (2.10), we can therefore rewrite (2.10) as

$$\widehat{\Gamma}_B = \{M \in \widehat{\Gamma} \mid (\exists \xi \in k)\, {}^t M' B = \xi B M^*\}, \tag{2.11}$$

which is a natural generalization of (2.7). It is easy to verify that $\widehat{\Gamma}_B$ is a subgroup of $\widehat{\Gamma}$.

LEMMA 2.12. *Let B be as in Lemma* 2.3 *or equation* (2.5). *Then the image of $\widehat{\Gamma}_B$ in G is a Sylow* 2*-subgroup of G and therefore has order* 8.

PROOF. We will refer to B in Lemma 2.3 as Case 1 and B as in equation (2.5) as Case 2. Let $C = \frac{1}{\sqrt{7}}B$. Then we have

$$C = \begin{pmatrix} 2^{\nu+1} & 0 \\ 0 & 2 \end{pmatrix}$$

if B in Case 1 and

$$C = \begin{pmatrix} 2 & -1+\sqrt{7} \\ -1-\sqrt{7} & -2 \end{pmatrix}$$

in Case 2. Clearly, in the definition of $\widehat{\Gamma}_B$ we can replace B by C. Let $\overline{C}$ denote the reduction of C modulo $\sqrt{7}$. Since $x' \equiv x$ modulo $\sqrt{7}$ for all x in $\mathcal{O}_k$, the matrix $\overline{C}$ will be a symmetric matrix with entries in $\mathbb{F}_7$. Explicitly,

$$\overline{C} = \begin{pmatrix} 2^{\nu+1} & 0 \\ 0 & 2 \end{pmatrix}$$

in Case 1 and

$$\overline{C} = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$$

in Case 2. Therefore, the image of $\widehat{\Gamma}_B$ in $\widehat{\Gamma}/\widehat{\Gamma}(\sqrt{7}) = \mathrm{SL}_2(\mathbb{F}_7)$ lies in the group of similitudes of determinant 1 of the quadratic form determined by $\overline{C}$. That quadratic form is $2^{\nu+1}x^2 + 2y^2$ in Case 1 and is $2x^2 - 2xy - 2y^2$ in Case 2. Using the fact that 2 is a square in $\mathbb{F}_7$ it is easy to verify that in either case the quadratic form is anisotropic and therefore equivalent, up to scalar multiple, to the quadratic form x^2+y^2 whose group of orthogonal similitudes of determinant 1 is easily seen to be of order 16 and is a Sylow 2-subgroup of $\mathrm{SL}_2(\mathbb{F}_7)$. Its image in $\mathrm{PSL}_2(\mathbb{F}_7)$ is therefore a Sylow 2-subgroup of of $\mathrm{PSL}_2(\mathbb{F}_7)$. It remains to show that the group $\widehat{\Gamma}_B$ maps onto this subgroup. From Lemma 2.9, we know that the subgroup $\mathcal{O}_B^1$ maps onto a cyclic subgroup of $\mathrm{PSL}_2(\mathbb{F}_7)$ of order 4. Since the elements of $\mathcal{O}_B^1$ consist of elements of $\widehat{\Gamma}_B$ which satisfy (2.10) with $\xi \equiv 1$ modulo $\sqrt{7}$, it is enough to show that $\widehat{\Gamma}_B$ contains an element which satisfies (2.10) with $\xi = -1$. But this is in fact a special case of [Hausmann 1980, Cor. 2.9, pp. 14–15]. □

3. Modular Curves on a Hilbert Modular Surface

If B is a primitive integral skew-hermitian matrix over k, we will denote by $\mathcal{H}_B$ the locus in $\mathcal{H}^2$ consisting of all $z = (z_1, z_2)$ in $\mathcal{H}^2$ such that

$$(z_2\ 1)\ B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0. \tag{3.1}$$

If S is a complex surface with a mapping ϕ (usually understood from the context) of $\mathcal{H}^2$ into S, then the closure of the image $\phi(\mathcal{H}_B)$ of $\mathcal{H}_B$ in S under ϕ will be denoted $F_B(S)$. For example, $F_B(\mathcal{H}^2) = \mathcal{H}_B$.

Let N be a positive integer. The union of all $F_B(S)$ as B runs over all primitive integral skew-hermitian forms of determinant N will be denoted $F_N(S)$. The union of all F_{N/d^2}, where d runs over all positive integers such that d^2 divides N, will be denoted $T_N(S)$. It is the same to say that one obtains T_N by omitting the primitivity condition from the definition of F_N.

In most of the cases of interest to us (though not all, since some or all of the curves may collapse to points), the loci $F_B(S)$, $F_N(S)$ and $T_N(S)$ will be complex analytic curves on the surface S. We will be particularly interested in cases where S is an algebraic surface and in that case, these loci will actually be algebraic.

We need to know the number of components of the curves $F_N(S)$ in certain cases. For this it is helpful to know the subgroup of $\widehat{\Gamma}$ leaving $\mathcal{H}_B$ invariant. It turns out to be a group with which we are already familiar.

Lemma 3.2. *The subgroup of $\widehat{\Gamma}$ leaving $F_B(\mathcal{H}^2)$ invariant is $\widehat{\Gamma}_B$* (see Equation 2.10).

Proof. Suppose $\gamma \in \widehat{\Gamma}$. Then γ^{-1} maps the locus of of all $z = (z_1, z_2)$ in $\mathcal{H}^2$ such that

$$0 = (z_2\ 1)\ B \begin{pmatrix} z_1 \\ 1 \end{pmatrix}$$

to the locus of all z such that

$$0 = (z_2\ 1)\ {}^t\gamma' B\gamma \begin{pmatrix} z_1 \\ 1 \end{pmatrix}.$$

Since $F_B(\mathcal{H}^2)$, when nonempty, determines B up to a scalar factor, the lemma follows at once. $\square$

Lemma 3.3. *Let $\phi : \widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2 \to S$ be a holomorphic mapping with dense image. Then the curve $F_7(S)$ is empty. Let $r \geq 1$ and $N = 2^r\cdot 7$. Then the curve $F_N(\widehat{X}(\mathcal{O}_k))$ has exactly one component. For $r \geq 0$ and $N = 2^r$, the curve $F_N(\widehat{X}(\mathcal{O}_k))$ has just 1 component for $r \leq 3$ and exactly 2 components for $r \geq 4$.*

Proof. The assertion about F_7 follows at once from the definition and from Lemma 2.4. By Lemmas 2.2, 2.3, 2.4, there do exist primitive skew-hermitian forms over k with determinant $N = 2^r\cdot 7$ with $r \geq 1$ and determinant $N = 2^r$ with $r \geq 1$. Therefore the loci $F_N(S)$ are nonempty for such N. The number of components is now determined using the table in [Hausmann 1980, p. 20] (cf. [van der Geer 1988, § V.3, pp. 93–100]). $\square$

Corollary 3.4. *The number of components of $F_N(\widehat{X}(\sqrt{7}))$, with $N = 1, 2, 4$ is 1. The number of components of $F_N(\widehat{X}(\sqrt{7}))$ with $N = 2^r\cdot 7$, $r \geq 1$, is 21.*

Proof. By Lemma 3.3, $F_N(\widehat{X})$ has only one component. Therefore, by Lemma 3.2 the set of components of $F_N(\mathcal{H}^2)$ can be identified $\widehat{\Gamma}$-equivariantly with $\widehat{\Gamma}/\widehat{\Gamma}_B$, where B is as in Lemma 2.2, Lemma 2.3 or equation (2.5) according to the value of N. It follows that the set of components of $F_N(\widehat{X}(\sqrt{7}))$ can be identified with the double coset space $\{\pm 1\}\widehat{\Gamma}(\sqrt{7})\backslash\widehat{\Gamma}/\widehat{\Gamma}_B$. Since $\{\pm 1\}\widehat{\Gamma}(\sqrt{7})$ is a normal subgroup of $\widehat{\Gamma}$ with quotient G, it follows that the number of components of $F_N(\widehat{X}(\sqrt{7}))$ is the index in G of the image of $\widehat{\Gamma}_B$ in G. By Lemma 2.9, the image of $\widehat{\Gamma}_B$ in G is all of G if $N = 1, 2, 4$, which proves the first assertion of the lemma. If $N = 2^r \cdot 7$ with $r \geq 1$, then by Lemma 2.12, the image of $\widehat{\Gamma}_B$ has order 8, so $F_N(\widehat{X}(\sqrt{7}))$ has $168/8 = 21$ components. $\square$

Lemma 3.5. *Let*

$$B = \begin{pmatrix} a\sqrt{D} & \lambda \\ -\lambda' & b\sqrt{D} \end{pmatrix}$$

be a primitive integral skew-hermitian matrix over k with positive determinant. Then the analytic disc $\mathcal{H}_B$ has ∞ as a limit point if and only if $a = 0$.

Proof. The equation (3.1) defining $\mathcal{H}_B$ when expanded becomes

$$az_1z_2\sqrt{D} + \lambda z_2 - \lambda' z_1 + b\sqrt{D} = 0. \tag{3.6}$$

Dividing through by z_1z_2 we see that if (z_1, z_2) can approach ∞ along $\mathcal{H}_B$, in the limit we have $a = 0$. Coversely, if $a = 0$ then $\lambda \neq 0$ and z_2 will approach ∞ along with z_1. $\square$

4. Volumes and Genera of Modular Curves on $\widehat{X}(\sqrt{7})$

In the following lemma, the characters χ_{-7} and χ_{-4} are Dirichlet characters modulo 7 and modulo 4 respectively and are defined as follows. Letting t denote either 7 or 4, let ζ_t denote a primitive t-th root of unity and $\mathbb{Q}(\zeta_t)$ the cyclotomic field of t-th roots of unity. The value of $\chi_{-t}(n)$ is 0 if t divides n. To compute $\chi_{-t}(n)$ for n prime to t, consider the automorphism σ_n of $\mathbb{Q}(\zeta_t)$ given by $\zeta_t \mapsto \zeta_t^n$. Then the value of $\chi_{-t}(n)$ is 1 or -1 according to whether the automorphism σ_n does or does not induce a trivial automorphism of the quadratic subfield $\mathbb{Q}(\sqrt{-t})$ of $\mathbb{Q}(\zeta_t)$. The Legendre symbol $\left(\frac{D}{q}\right)$ and the Hilbert symbol $\left(\frac{N,D}{q}\right)$ appear in the definition of α_q.

Lemma 4.1. *The volume of $F_N(\widehat{X})$ with respect to the volume form*

$$-\frac{1}{2\pi}\frac{dx \wedge dy}{y^2}$$

is

$$\mathrm{vol}(F_N(\widehat{X})) = -\frac{1}{24}N(1 + \chi_{-p}(N))(1 + \chi_{-4}(N))\prod_{q|N}\alpha_q,$$

where α_q *is defined for primes* q *dividing* N *by*

$$\alpha_q = \begin{cases} 1 + \left(\dfrac{D}{q}\right)/q & \text{if } q \nmid D \\ 1 + \left(\dfrac{N, D}{q}\right)/q & \text{if } q|D \text{ and } q||N \\ 1 - \dfrac{1}{q^2} & \text{if } q|D \text{ and } q^2|N. \end{cases}$$

Proof. In [Hausmann 1980, p. 49] (cf. [van der Geer 1988, § V.5, pp. 101–102]) one finds a volume formula which specializes to the one given above except for having the fraction on the right-hand side equal to $-1/12$ instead of $-1/24$. However, the formula of Hausmann was for the volume of the curve $F_N(X)$, where X is a compactification of $\mathrm{SL}_2(\mathcal{O}_k)\backslash\mathcal{H}^2$. Therefore, $F_N(X)$ is a two-sheeted cover of the curve $F_N(\widehat{X})$, because $\mathrm{PSL}_2(\mathcal{O}_k)$ is isomorphic to a subgroup of index 2 in $\widehat{\Gamma}/\mathcal{O}_k^\times$. Therefore our curve has half the volume given by Hausmann's formula. □

Corollary 4.2. *Let* $k = \mathbb{Q}(\sqrt{7})$ *and let* J *be the two sided ideal* $(\sqrt{7}) = \sqrt{7}\mathcal{O}_k$ *of* $\mathcal{O}_k$. *Then the volumes of the curves* $F_N(\widehat{X}(\sqrt{7}))$ *for* $N = 1, 2, 4, 7, 14, 2^r\cdot 7$ *are given by*:

N	1	2	$2^r,\ r \geq 2$	7	14	$2^r\cdot 7,\ r \geq 2$
$\mathrm{vol}(F_N(\widehat{X}(\mathcal{O}_k)))$	-28	-42	$-2^{r-1}\cdot 21$	0	-42	$-2^{r-1}\cdot 63$

Proof. Since $\widehat{X}(\sqrt{7})$ is a 168-sheeted branched cover of $\widehat{X}$, with branching only at a finite set of points, the volume of the curve $F_N(\widehat{X}(\sqrt{7}))$ is 168 times the volume of $F_N(\widehat{X})$. The latter is computed directly from Lemma 4.1. We leave the arithmetic to the reader. □

Corollary 4.3. *For* $N = 2^r\cdot 7$, *with* $r \geq 1$, *the curve* $F_N(\widehat{X}(\sqrt{7}))$ *consists of exactly* 21 *nonsingular irreducible components. Each component of* $F_{14}(\widehat{X}(\sqrt{7}))$ *has volume* -2 *and genus* 2. *If* $r > 1$, *each component of* $F_N(\widehat{X}(\sqrt{7}))$ *has volume* $-2^{r-1}\cdot 3$ *and genus* $1 + 2^{r-2}\cdot 3$.

Proof. The volume of $F_N(\widehat{X}(\sqrt{7}))$ for $N = 2^r\cdot 7$, $r \geq 1$, is given in the preceding corollary. By Corollary 3.4 the number of components of $F_N(\widehat{X}(\sqrt{7}))$, for such N is 21. Hence, each component of $F_N(\widehat{X}(\sqrt{7}))$ has volume -2 if $r = 1$ and $-2^{r-1}\cdot 3$ if $r > 1$. Since these volumes coincide with the Euler numbers of the components, we conclude that the genus of each component of $F_N(\widehat{X}(\sqrt{7}))$ is 2 if $r = 1$ and is $1 + 2^{r-2}\cdot 3$ if $r > 1$. □

Remark 4.4. In Lemma 7.15, we will prove that the components of $F_{14}(\widehat{X}(\sqrt{7}))$ and $F_{28}(\widehat{X}(\sqrt{7}))$ are hyperelliptic curves.

5. Intersections of Modular Curves on $\widehat{X}(\sqrt{7})$

In this section, we will compute the intersection numbers of certain of the modular curves $F_N(\sqrt{7})$ on the complete normal singular surface $\widehat{X}(\sqrt{7})$. The theory of intersections on complete normal singular surfaces is due to Mumford [1961] (cf. [Fulton 1984, Ex. 7.1.16, p. 125]). We will have more to say about it in Section 7. Our main tool for computing these intersection numbers is a formula due to Hausmann. However, Hausmann's formula is for intersections of modular curves on the normal singular surface $\widehat{X}$, so we need some preliminary results (Lemmas 5.1 and 5.3), which seem to be implicit in the literature on Hilbert modular surfaces. I am indebted to Angelo Vistoli and Torsten Ekedahl for their help in explaining to me how to prove it and in criticizing my earlier drafts of the proofs. The relevant principles will be discussed in greater detail. in [Adler $\geq$ 1998].

LEMMA 5.1. *If $g : X \to Y$ is a surjective morphism of degree n of complete normal surfaces, with X smooth, and if C is a Weil divisor on Y, then there is a unique Weil divisor C' on X with rational coefficients with the following properties*:

(1) *Let Y_0 be the smooth locus of Y, let $X_0 = g^{-1}(Y_0)$ and let $g_0 : X_0 \to Y_0$ be the restriction of g. Let $C_0 = C \cap Y_0$ and $C' = C' \cap X_0$. Then $C'_0 = g_0^* C_0$.*
(2) *If D is any curve component of the complement of X_0 in X then $(C' \cdot D)_X =$ 0.*

PROOF. For $n = 1$, this is due to Mumford [1961] and is the key to his intersection product. In the general case, use the Stein factorization of g:

$$X \to Y' \to Y$$

to first pull C back to C'' on Y' and then to use Mumford's result for $n = 1$ for the morphism $X \to Y'$. The condition (2) follows from the fact that the complement of X_0 in X maps to a finite set in Y'. $\square$

REMARK 5.2. We will call C' the *Mumford pullback* of C and denote it $g^M C$. In the proof of Lemma 5.3, we also need to use another notion of pullback. Let $g : X \to Y$ be a surjective quasifinite morphism of complete normal surfaces and let C be a Weil divisor on Y. Let Y_0 be the smooth locus of Y, let $X_0 = f^{-1}(Y_0)$, let $C_0 = C \cap Y_0$ and let $g_0 : X_0 \to Y_0$ be the restriction of g. Then C_0 is a Cartier divisor on Y_0 whose pullback to X_0 is denoted $g^*(C_0)$. The scheme theoretic closure of $g^*(C_0)$ in X is then associated to a Weil divisor on X which we denote $g^*(C)$. We remark that on any surface, a Weil divisor is uniquely determined by its restriction to the complement of a finite set. This notion of pull-back is a special case of the notion of pullback of codimension-1 cycles presented in [Grothendieck 1967, § 21.10] and will be denoted f^*. In the case of Cartier divisors, it coincides with the usual notion of pullback.

Lemma 5.3. *Let $f : U \to V$ be a finite surjective morphism of complete normal surfaces. Let n be the degree of f. Let C_1, C_2 be curves on V. Denote by $(\,\cdot\,,\cdot\,)_U$ and $(\,\cdot\,,\cdot\,)_V$ the Mumford intersection products on U and V respectively. Then if C_i' is the pullback to U of the divisor C_i in U, we have*

$$(C_1', C_2')_U = n\cdot(C_1, C_2)_V.$$

Proof. Suppose that $\pi_1 : Z \to V$ is a desingularization and that $\pi_2 : W \to U$ is a desingularization factoring through the fibre product of f and π_1. Denote by $h : W \to Y$ the natural mapping from W to Z. Then we have $f\pi_1 = \pi_2 h$. I claim that we also have

$$(f\pi_1)^M = \pi_1^M f^* = h^*\pi_2^M.$$

Indeed, our assumptions imply that $W \to U \to V$ is the Stein factorization of $f\pi_1$, so the first equality is an immediate consequence of our construction of $(f\pi_1)^M$. As for the second, we just have to prove that it satisfies the conditions (1) and (2) above. Let C be a Weil divisor on V. Clearly $h^*\pi_2^M C$ satisfies condition (1). As for condition (2), let D be a curve component of the preimage in W of the singular locus of V. We have

$$(h^*\pi_2^M C\cdot D)_W = (\pi_2^M C\cdot h_*(D))_Z.$$

If h maps D to a point, this last expression is already 0. If h maps D to a curve, that curve is mapped by π_2 to a singular point of V and by definition of π_2^M the right side is again 0. We can now prove the lemma. Let C_1, C_2 be Weil divisors on V. Then

$$\begin{aligned}(f^*C_1\cdot f^*C_2)_U &= (\pi_1^M f^*C_1\cdot\pi_1^M f^*C_2)_W\\ &= (h^*\pi_2^M C_1\cdot h^*\pi_2^M C_2)_W\\ &= n(\pi_2^M C_1\cdot\pi_2^M C_2)_Z\\ &= n(C_1\cdot C_2)_V.\end{aligned}$$

Here the penultimate equality follows from the fact that the lemma is already well known for morphisms between smooth varieties. □

As for Hausmann's formula, it is given below. It is derived in [Hausmann 1980, Satz 5.13, p. 102] (cf. [van der Geer 1988, Cor. VI.5.3, p. 144]) in greater generality than we state it here. The notation appearing in the formula, other than $P_\rho(M, N)$, requires more extensive explanation, which will be given after the statement of the formula. We will discuss Hausmann's formula in greater detail in [Adler $\geq$ 1998].

Intersection Formula 5.4. *Let k be a real quadratic field of discriminant D. Let M and N be positive integers. Then the intersection number of T_M and T_N*

on $\widehat{X}$ (in the sense of Mumford's theory) is given by

$$(T_M\cdot T_N)_{\widehat{X}} = \sum_{n|(M,N)}\left(n\Big(H_D\Big(\frac{MN}{n^2}\Big)+I_D\Big(\frac{MN}{n^2}\Big)\Big)\prod_{\rho|D}\frac{\chi_{D(\rho)}(n)+\chi_{D(\rho)}(P_\rho(M,N)/n)}{2}\right),$$

where the product runs over all rational primes dividing D and where

$$P_\rho(M,N)=\rho^{\min(\mu,\nu)},$$

with $\mu=\operatorname{ord}_\rho M$ and $\nu=\operatorname{ord}_\rho N$.

We now explain the notation.

(1) The Dirichlet characters χ_{-4} and χ_{-7} were defined at the beginning of Section 4.

(2) In order to explain the notation H_D, we first have to explain the number theoretic functions h, h' and H.

(a) If $\mathcal{A}$ is an order in an imaginary quadratic field K, we denote by $h(\mathcal{A})$ the class number of $\mathcal{A}$. Since an order is determined up to isomorphism by its discriminant, we will also define $h(d)$ to be $h(\mathcal{A})$ if d is the discriminant of $\mathcal{A}$. We also denote by $\mathcal{A}_d$ the unique order of discriminant d. If d is a rational number which is not the discriminant of an order in an imaginary quadratic field, we define $h(d)$ to be 0.

(b) We define the number theoretic function h' as follows. For all rational numbers d, we define $h'(d)$ by

$$h'(d)=\begin{cases}-1/12 & \text{if } d=0,\\ 1/2 & \text{if } d=-4,\\ 1/3 & \text{if } d=-6,\\ h(d) & \text{otherwise.}\end{cases}$$

(c) We define the Hurwitz–Kronecker class number H by

$$H(d)=\sum_{c^2|d}h'(-d/c^2)$$

for all rational numbers d.

(d) We define the function H_D by

$$H_D(n)=\sum H\Big(\frac{4n-x^2}{D}\Big),$$

where the summation on the right runs over all rational numbers x.

(3) Let k be a real quadratic field with discriminant D and let $\mathrm{Id}(k)$ denote the group of fractional ideals of k. Then we define the function $f:\mathrm{Id}(k)\to\mathbb{R}$ by

$$f(M)=\frac{1}{\sqrt{D}}\sum min(\lambda,\lambda'),$$

where the summation runs over all totally positive elements λ of k such that $M = \mathcal{O}_k\lambda$. We define the number theoretic function I_D on the positive rational integers by the rule

$$I_D(N) = \sum f(M)$$

where the summation runs over integral ideals k of norm N.

We now provide some values of H_D and of I_D.

Corollary 5.5. *We have the following values of H_D when $D = 28$.*

N	1	2	4	8	16	49	98	196	392	784
$H_D(N)$	$-\frac{1}{6}$	0	$-\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{5}{6}$	2	$\frac{11}{6}$	8	$\frac{71}{6}$

Proof. Since $h(7) = 1$ and $h(56) = h(84) = 4$ (see any table of class numbers of imaginary quadratic fields), this follows at once from the following lemma, which is used for computing the class numbers of nonmaximal orders. $\square$

Lemma 5.6. *Let $\mathcal{A}$ be an order in a quadratic field K. Let h be the class number of K and let $\mathcal{O}_K$ be the maximal order of K. Then the class number of $\mathcal{A}$ is given by*

$$h(\mathcal{A}) = h(\mathcal{O}_K)\frac{|(\mathcal{O}_K/f\mathcal{O}_K)^\times|}{|\mathcal{O}_K^\times/\mathcal{A}^\times|\cdot\varphi(f)},$$

where φ is the Euler φ function.

Proof. See [Borevich and Shafarevich 1966, pp. 152–153]. $\square$

Lemma 5.7. *$I_D(2^r\cdot 7) = 0$ for all $r \geq 0$.*

Proof. There is only one ideal of k with norm $2^r\cdot 7$, namely the one generated by $\eta^r\sqrt{7}$. Since this element has negative norm and since k has no unit of negative norm, the ideal is not generated by a totally positive element and therefore the function f vanishes on this ideal. $\square$

Lemma 5.8. *Let r be a nonnegative integer and let $s = [r/2]$. Then*

$$I_D(2^r) = \begin{cases} \dfrac{2^{s-1}\cdot 3}{7} & \text{if } r = 2s, \\ \dfrac{2^s}{7} & \text{if } r = 2s+1. \end{cases}$$

Proof. The fundamental unit of k is given by $\varepsilon = \eta/\eta' = 3 - \sqrt{7}$ and we have $\varepsilon < 1$. According to [van der Geer 1988, § V.8, p. 112], if M is a fractional ideal generated by a totally positive element μ satisfying

$$\varepsilon^2 \leq \mu/\mu' \leq 1,$$

then the value of $f(M)$ is also given by

$$f(M) = \frac{1}{\sqrt{D}}\left(\frac{\mu}{1-\varepsilon} + \frac{\mu'\varepsilon}{1-\varepsilon}\right) = \operatorname{tr}\left(\frac{\mu}{(1-\varepsilon)\sqrt{D}}\right).$$

Since 2 is ramified in k, there is only one ideal of k of norm 2^r. If $r = 2s$ the ideal is generated by $\mu = 2^s$. If $r = 2s+1$ the ideal is generated by $\mu = 2^s \cdot \eta$. In either case, $\varepsilon^2 \leq \mu/\mu' \leq 1$. Therefore,

$$I_D(2^r) = \begin{cases} \operatorname{tr}\left(\dfrac{2^s}{(3\sqrt{7}-7)\sqrt{28}}\right) & \text{if } r = 2s, \\ \operatorname{tr}\left(\dfrac{2^s \cdot \eta}{(3\sqrt{7}-7)\sqrt{28}}\right) & \text{if } r = 2s+1, \end{cases} = \begin{cases} \dfrac{2^{s-1}\cdot 3}{7} & \text{if } r = 2s, \\ \dfrac{2^s}{7} & \text{if } r = 2s+1. \end{cases}$$

We now use Hausmann's formula to compute some intersection numbers on $\widehat{X}$ and $\widehat{X}(\sqrt{7})$. This is entirely straightforward, using the numerical values we have provided for the various number theoretic functions that appear in it and we leave the details to the reader. However, there is one detail that we wish to point out. Hausmann's formula is for intersections of the modular curves $T_M(\widehat{X})$ and $T_N(\widehat{X})$, whereas the table in the following lemma is for intersections of the modular curves $F_M(\widehat{X})$ and $F_N(\widehat{X})$. For $M = 1, 2, 14, 28$, we have $F_M(\widehat{X}) = T_M(\widehat{X})$ and for $M = 4$ we have

$$F_4(\widehat{X}) = T_4(\widehat{X}) - T_1(\widehat{X}).$$

Therefore, Hausmann's formula is perfectly adequate for our purposes. In general, one can always write a curve $F_M(\widehat{X})$ as a linear combination of curves $T_N(\widehat{X})$. $\square$

Lemma 5.9. *We have the following table of intersection numbers on* $\widehat{X}(\sqrt{7})$.

	$F_1(\widehat{X}(\sqrt{7}))$	$F_2(\widehat{X}(\sqrt{7}))$	$F_4(\widehat{X}(\sqrt{7}))$	$F_{14}(\widehat{X}(\sqrt{7}))$	$F_{28}(\widehat{X}(\sqrt{7}))$
$F_1(\widehat{X}(\sqrt{7}))$	8	24	36	0	84
$F_2(\widehat{X}(\sqrt{7}))$	24	30	24	42	84
$F_4(\widehat{X}(\sqrt{7}))$	36	24	-6	84	42
$F_{14}(\widehat{X}(\sqrt{7}))$	0	42	84	-42	168
$F_{28}(\widehat{X}(\sqrt{7}))$	84	84	42	168	210

The curves $F_N(\widehat{X}(\sqrt{7}))$ are irreducible for $N = 1, 2, 4$ but are reducible for $N = 14$ and $N = 28$. We also need to know the intersections of the individual components of the curves $F_N(\widehat{X}(\sqrt{7}))$ with the curves $F_M(\widehat{X}(\sqrt{7}))$ for $N = 14, 28$ and $M = 1, 2, 4, 14, 28$. Just from knowing the number of components of the curves $F_N(\widehat{X}(\sqrt{7}))$ with $N = 14, 28$ and the above table of intersection numbers, we can obtain some partial information, which we summarize in the following lemma.

COROLLARY 5.10. *Let $Z_N(\widehat{X}(\sqrt{7}))$ denote a component of $F_N(\widehat{X}(\sqrt{7}))$, with $N = 14$ or $N = 28$. Then we have the following intersection numbers*:

	$F_1(\widehat{X}(\sqrt{7}))$	$F_2(\widehat{X}(\sqrt{7}))$	$F_4(\widehat{X}(\sqrt{7}))$	$F_{14}(\widehat{X}(\sqrt{7}))$	$F_{28}(\widehat{X}(\sqrt{7}))$
$Z_{14}(\widehat{X}(\sqrt{7}))$	0	2	4	−2	8
$Z_{28}(\widehat{X}(\sqrt{7}))$	4	4	2	−8	10

The more detailed problem of the intersection of the individual components of $F_{14}(\widehat{X}(\sqrt{7}))$ with themselves, with each other and with those of $F_{28}(\widehat{X}(\sqrt{7}))$ will be taken up in Section 7.

6. The Switching Involution τ

Denote by τ the mapping of $\mathcal{H}^2$ to itself defined by

$$\tau(z_1, z_2) = (z_2, z_1).$$

We will call τ the *switching involution*. One can show that if γ belongs to $\widehat{\Gamma}$ then as mappings of $\mathcal{H}^2$ to itself we have

$$\gamma\tau = \tau\gamma',$$

where the prime indicates that the nontrivial automorphism of k is to be applied to all of the entries of γ. Since the ideal $\sqrt{7}\mathcal{O}_k$ is invariant under the Galois group of k over $\mathbb{Q}$, the group $\widehat{\Gamma}(\sqrt{7})$ is invariant under conjugation by τ.

The proof of the following lemma is straightforward and is left to the reader.

LEMMA 6.1. *Let γ be a matrix with entries in k and totally positive determinant. Write*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then for $z_1, z_2 \in \mathcal{H}$, the relation

$$z_2 = \gamma \cdot z_1$$

is equivalent to the relation

$$(z_2\ 1)\ B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0,$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \gamma.$$

It is important for us to know the fixed point set of τ acting on $\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2$.

LEMMA 6.2. *The fixed point set of τ on the surface $\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2$ is the union of $F_1(\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2)$, $F_2(\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2)$ and $F_4(\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2)$.*

PROOF. Let $z = (z_1, z_2)$ be a point of $\mathcal{H}^2$. Then z represents a point x of X fixed by τ if and only if there is an element γ of $\Gamma(\sqrt{7})$ such that $\tau z = \gamma z$ or, what is the same,

$$z_2 = \gamma z_1, \quad z_1 = \gamma' z_2.$$

Then we have

$$z_2 = \gamma\gamma' z_2,$$

which implies that $\gamma\gamma'$ has z as a fixed point on $\mathcal{H}^2$. This implies that $\gamma\gamma'$ acts as the identity transformation on $\mathcal{H}^2$. We may assume that the determinant of γ is either 1 or ε. Therefore we can write

$$\gamma = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the second matrix on the right-hand side, which we will denote δ, has determinant 1 and entries in $\mathcal{O}_k$, and where ξ is the determinant of γ. Note that both matrices on the right-hand side lie in Γ. Since $\xi\xi' = 1$, it follows that

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \gamma\gamma' = \begin{pmatrix} a & \xi b \\ \xi' c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

or, what is the same,

$$\begin{pmatrix} a & \xi b \\ \xi' c & d \end{pmatrix} = \pm \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix}.$$

We first remark that the sign on the right-hand side must be $+$. For if it were $-$, we would have $d = -a'$ and therefore modulo $\sqrt{7}$ we would have (since $\varepsilon \equiv 1$ modulo $\sqrt{7}$):

$$1 \equiv \det(\gamma) \equiv ad - bc \equiv -N(a) - bc.$$

This is a contradiction since b, c are congruent to 0 modulo $\sqrt{7}$ and a, a' are congruent to 1 modulo $\sqrt{7}$, hence $N(a)$ is congruent to 1 modulo 7. There are therefore two cases:

(1) the sign is $+$ and $\xi = 1$;
(2) the sign is $+$ and $\xi = \varepsilon$.

In case (1), we have

$$a' = d, \quad b' = -b, \quad c' = -c,$$

and therefore

$$\gamma = \begin{pmatrix} a & b_0\sqrt{7} \\ c_0\sqrt{7} & a' \end{pmatrix}$$

with b_0, c_0 rational integers and a in $\mathcal{O}_k$. The relation

$$z_2 = \gamma z_1 = \frac{az_1 + b}{cz_1 + d} = \frac{az_1 + b_0\sqrt{7}}{c_0\sqrt{7}z_1 + a}$$

then becomes

$$(z_2 \ 1) \ B_0 \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0,$$

where

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \gamma = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = \begin{pmatrix} c_0\sqrt{7} & a' \\ -a & -b_0\sqrt{7} \end{pmatrix}.$$

Thus, the matrix B_0 is skew-hermitian. If b_0, c_0 are even, it is actually an integral skew-hermitian matrix B. If not, we can multiply B_0 by 2 to obtain an integral skew-hermitian matrix B. Since

$$1 = \det(\gamma) = N(a) - 7b_0c_0,$$

there can be no natural number $n > 1$ dividing a, b_0 and c_0. Therefore, the resulting skew-hermitian form will be primitive. If b_0, c_0 are both odd, the determinant of B will be 1. Otherwise, it will be 4. It follows that in case 1, the point z lies in $F_1 \cup F_4$.

In case (2), we have

$$a' = d, \quad b' = -b\varepsilon, \quad c' = -c\varepsilon'.$$

This implies that we can write

$$b = b_0\sqrt{7}(3 + \sqrt{7}) = b_0\eta'\sqrt{7},$$
$$c = c_0\sqrt{7}(3 - \sqrt{7}) = c_0\eta\sqrt{7},$$

where b_0, c_0 are rational integers and where we recall that $\eta = 3-\sqrt{7}$. Therefore, $z = (z_1, z_2)$ lies in the locus defined by

$$(z_2 \ 1) \ B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0,$$

where

$$B = (3 + \sqrt{7}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \gamma = \begin{pmatrix} c_0\sqrt{28} & a'\eta' \\ -a\eta & -b_0\sqrt{28} \end{pmatrix}.$$

The matrix B is integral and skew-hermitian. In fact, B is primitive as well. For if $n > 1$ is a positive integer dividing b_0, c_0, and $a\eta$, then n divides b, c and the identity $\det(\gamma) = \varepsilon$ implies n doesn't divide a. Therefore, since n divides $a\eta$ and η has norm 2, we must have $n = 2$. But then

$$2 = \det(B) = -28b_0c_0 + 2N(a) \equiv 0 \pmod 4,$$

which is a contradiction. It follows that z lies in $F_B \subset F_2$.

We have therefore proved that the fixed point set of τ lies in $F_1 \cup F_2 \cup F_4$. Next we show that the fixed point set contains a component of each of F_1, F_2, F_4. Indeed, the 3 loci

$$z_1 = z_2, \quad z_1 = \varepsilon z_2, \quad z_1 = z_2 + \sqrt{7} \tag{6.3}$$

are easily seen to be invariant under τ modulo $\widehat{\Gamma}(\sqrt{7})$. They are are defined by

$$(z_2\ 1)\ B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0$$

with B given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \eta' \\ \eta & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ -2 & \sqrt{7} \end{pmatrix},$$

respectively, and these three matrices are primitive, integral skew-hermitian matrices of determinants 1,2, and 4 respectively. The proof is now completed by the observation that the loci $F_1(\widehat{X}(\sqrt{7}))$, $F_2(\widehat{X}(\sqrt{7}))$, $F_4(\widehat{X}(\sqrt{7}))$ are all irreducible by Lemma 3.3. □

LEMMA 6.4. *The involution τ of $\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2$ extends to $\widehat{X}(\sqrt{7})$. The fixed point set of τ on the surface $\widehat{X}(\sqrt{7})$ is the union of $F_1(\widehat{X}(\sqrt{7}))$, $F_2(\widehat{X}(\sqrt{7}))$ and $F_4(\widehat{X}(\sqrt{7}))$.*

PROOF. The first assertion is obvious from the definitions. As for the second, it follows from Lemma 6.2, from Lemma 3.6 and from the fact that all of the cusps of $\widehat{X}(\sqrt{7})$ are rational. □

7. Intersections on the Nonsingular Model $\widehat{Z}(\sqrt{7})$ of $\widehat{X}(\sqrt{7})$

Hirzebruch [1971] showed how to resolve the singularities of Hilbert modular surfaces. His construction inspired the more general work of Mumford and others [Ash et al. 1975, § I.5, pp. 39–53] on compactification and desingularization of quotients of Hermitian symmetric spaces by arithmetic groups. We will discuss some aspects of this more general work in [Adler $\geq$ 1998].

The only singularities of $\widehat{X}(\sqrt{7})$ are at the cusps. Applying Hirzebruch's construction to the surface $\widehat{X}(\sqrt{7})$, we obtain a surface which we will denote $\widehat{Z}(\sqrt{7})$. The preimage in $\widehat{Z}(\sqrt{7})$ of each cusp is a pentagonal cycle of 5 rational curves with self intersection numbers -2, -2, -3, -3, -2 respectively. This is described in [Hirzebruch 1977]. It can be justified from the general theory of such resolutions by means of the table in [van der Geer 1988, p. 41], where we take $\alpha = 3$ in the second row, second column of the table. We illustrate the resolution cycle in Figure 1.

If x is a cusp of $\widehat{X}(\sqrt{7})$, we will denote by S_i^x, for $0 \leq i \leq 4$, the 5 curves of the resolution cycle. We number them with subscripts modulo 5 such that

(1) The curve S_i^x has self-intersection number -2 for $i = 0$, 1, 4 and self-intersection number -3 for $i = 2$, 3.
(2) The curves S_i^x and S_j^x meet transversely in one point if $i - j$ is congruent to ± 1 modulo 5 and otherwise do not meet.

Our main goal in this section is to study the intersection numbers of some of the curves $F_N(\widehat{Z}(\sqrt{7}))$ on $\widehat{Z}(\sqrt{7})$. The key to computing these numbers is the

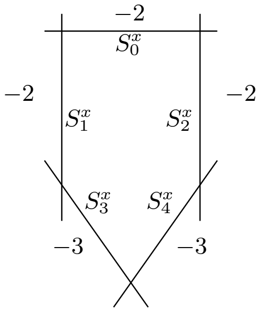

Figure 1. Resolution cycle of cusp x in $\widehat{Z}(\sqrt{7})$.

computation of the corresponding numbers on $\widehat{X}(\sqrt{7})$, which was carried out in Section 5, combined with Mumford's definition [Mu1] of intersection numbers on complete normal singular surfaces. Therefore, although we merely alluded to that notion at the beginning of Section 5, here we need to consider it explicitly and we will now do so.

Let S be a complete surface with isolated normal singularities. Denote by S' a surface obtained by resolving the singularities of S and denote by $f : S' \to S$ the natural mapping. If C is a curve on S, we will denote by C' the preimage of C on S'. For each point p of S such that $f^{-1}(p)$ is a curve, denote the components of $f^{-1}(p)$ by $K_1^p, \dots, K_r^p$, where $r = r(p)$ depends on p. Denote by M_p the matrix $K_i^p \cdot K_j^p$ of intersection numbers of the components of $f^{-1}(p)$. Since $f^{-1}(p)$ can be blown down to a point, namely p, the matrix M_p is negative definite and, in particular, invertible as a matrix with rational entries. If C is a curve on S, denote by $\overline{C} = \overline{C}(f)$ the cycle with rational coefficients in S' given by

$$\overline{C} = C' + \sum_p \sum_{i=1}^{r(p)} a_{ip} K_i^p,$$

where the outer summation runs over all points p of S such that $f^{-1}(p)$ is a curve and where, for each such p, the a_{ip} are the unique rational numbers such that

$$\overline{C} \cdot K_i^p = 0$$

for $1 \leq i \leq r(p)$. One then defines the intersection number of two curves C_1, C_2 on S to be the intersection number on S' of the curves $\overline{C}_1$, $\overline{C}_2$:

$$(C_1 \cdot C_2)_S = (\overline{C}_1 \cdot \overline{C}_2)_{S'}.$$

One can show that the intersection number so defined is independent of the choice of the resolution $f : S' \to S$. We also note that since the preimage of a curve on S differs from the proper transform of the curve only by an integral linear combination of curves of the form K_i^p, one could have taken C' in the above definition to be be the proper transform of C or, indeed, any curve in S' mapping onto C.

If $S' \to S$ is a morphism of surfaces of Hilbert modular type, with S' nonsingular and $C = F_N(S)$, we will also write $F_N(S' \to S)$ to denote the curve $\bar{C}$ on S'.

LEMMA 7.1. *We have*

$$F_1(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7})) = F_1(\widehat{Z}(\sqrt{7})) + \frac{1}{2}\sum_x (3S_0^x + 2S_1^x + S_2^x + S_3^x + 2S_4^x)$$

$$F_2(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7})) = F_2(\widehat{Z}(\sqrt{7})) + \sum_x (S_0^x + S_1^x + S_2^x + S_3^x + S_4^x)$$

$$F_4(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7})) = F_4(\widehat{Z}(\sqrt{7})) + \frac{1}{2}\sum_x (3S_0^x + 2S_1^x + S_2^x + S_3^x + 2S_4^x).$$

PROOF. To prove this, one verifies that the intersection of a component S_i^x with the expression on the right side of each of these equations is 0. Since we know the intersection numbers $S_i^x \cdot S_j^y$, one merely needs to know the intersection numbers $F_N(\widehat{Z}(\sqrt{7})) \cdot S_i^x$. They are given by

$$F_N(Z(\sqrt{7})) \cdot S_i^x = \begin{cases} 1 & \text{if } (N, i) = (1, 0),\ (2, 2),\ (2, 3),\ (4, 0), \\ 0 & \text{otherwise.} \end{cases}$$

That these are the correct intersection numbers between the $F_N(Z(\sqrt{7}))$ and the S_i^x will be shown in Lemma 7.7. □

In order to prepare for the proof of Lemma 7.7, we need to look more closely at Hirzebruch's desingularization of $\widehat{X}(\sqrt{7})$.

A sequence indexed by the set of all integers will be called a $\mathbb{Z}$-*sequence*. If $\boldsymbol{b} = (b_n)_{n \in \mathbb{Z}}$ is a $\mathbb{Z}$-sequence of positive integers, one can associate to $\boldsymbol{b}$ a complex manifold which we will denote $\mathcal{M}_0(\boldsymbol{b})$. The explicit construction is described in [Hirzebruch 1973; 1971; van der Geer 1988]. For certain purposes, one assumes that all of the integers b_n are ≥ 2 with at least one of them ≥ 3, and that is the practice of the authors we have cited. One constructs $\mathcal{M}_0(\boldsymbol{b})$ by forming the disjoint union of a $\mathbb{Z}$-sequence of copies of $\mathbb{C}^2$ and identifying the point (u_n, v_n) in the n-th copy $\mathbb{C}_n^2$ with the point $(u_n^{b_n} v^n, u_n^{-1})$ in the $(n+1)$-th copy $\mathbb{C}_{n+1}^2$. Each copy of $\mathbb{C}^2$ injects into $\mathcal{M}_0(\boldsymbol{b})$ and the injection defines a coordinate chart $(\mathbb{C}_n^2, (u_n, v_n))$. The closure in $\mathcal{M}_0(\boldsymbol{b})$ of the axis $v_n = 0$ of that chart will be a projective line which we will denote S_n and which has self-intersection number $-b_n$ (cf. [van der Geer 1988, p. 33]).

In the following lemma, it is convenient to denote by $\mathcal{I}(\mathbb{Z})$ the isometry group of the the set of integers with their usual metric inherited from the real numbers. As is well-known, every such isometry is of the form $n \mapsto tn + r$, where $r \in \mathbb{Z}$ and $t = \pm 1$. In order to conserve subscripts, in the proof of the lemma we will write $(u, v)_n$ for all $n \in \mathbb{Z}$ to denote a point whose coordinates are (u, v) in the coordinate chart σ_n.

Lemma 7.2. *Denote by ι the automorphism of $\mathbb{C}^2$ given by $(u,v) \mapsto (v,u)$. Then the isometry group $\mathcal{I}(\mathbb{Z})$ of $\mathbb{Z}$ acts holomorphically on the disjoint union $\mathbb{C} \times \mathbb{Z}$ of copies of $\mathbb{C}^2$ indexed by $\mathbb{Z}$ by the rule*

$$\alpha_{(t,r)} \cdot (w,n) = \left(\iota^{(1-t)/2} w,\ t(n - \tfrac{1}{2}) + \tfrac{\pm}{r}\right),$$

with $t = \pm 1$, where $\alpha_{(t,r)}$ is the automorphism of $\mathbb{Z}$ given by $n \mapsto tn + r$. Let $\boldsymbol{b} : \mathbb{Z} \to \mathbb{Z}_{\geq 1}$ be a $\mathbb{Z}$-sequence of positive integers. Let $\mathcal{G}$ be the subgroup of $\mathcal{I}(\mathbb{Z})$ consisting of all automorphisms g of $\mathbb{Z}$ which preserve $\boldsymbol{b}$ in the sense that $\boldsymbol{b} \circ g = \boldsymbol{b}$. Then the action of $\mathcal{G}$ on the complex manifold $\mathbb{C}^2 \times \mathbb{Z}$ descends to an action of $\mathcal{G}$ on the complex manifold $\mathcal{M}_0(\boldsymbol{b})$.

Proof. That we have indeed defined an action of $\mathcal{I}(\mathbb{Z})$ on $\mathbb{C}^2 \times \mathbb{Z}$ as a group of holomorphic automorphisms is straightforward and left to the reader. As for the assertion that $\mathcal{G}$ acts on $\mathcal{M}_0(\boldsymbol{b})$, it suffices to show that for $g \in \mathcal{G}$, the action of g on $\mathbb{C}^2 \times \mathbb{Z}$ is compatible with the identifications which give rise to the complex manifold $\mathcal{M}_0(\boldsymbol{b})$. To see this, let $p = (u,v)_n$ be a point of $\mathbb{C}^2_n$. This point is identified with the point $q = (u^{b_n} v, 1/u)_{n+1}$ of $\mathbb{C}^2_{n+1}$. Let $\alpha_{(t,r)}$ be an element of $\mathcal{G}$. Let $p' = \alpha_{(t,r)}(p)$ and $q' = \alpha_{(t,r)}(q)$. If $t = 1$, we have

$$p' = (u,v)_{n+r}, \quad q' = (u^{b_n} v,\ 1/u)_{n+r+1},$$

and p' is identified with

$$(u^{b_{n+r}} v, 1/u)_{n+r+1} = (u^{b_n} v, 1/u)_{n+r+1} = q'$$

since $b_{n+r} = b_n$. This proves that the translations in $\mathcal{G}$ descend to $\mathcal{M}_0(\boldsymbol{b})$. Therefore, it suffices to show that if $\alpha_{(-1,0)}$ belongs to $\mathcal{G}$ then it descends to $\mathcal{M}_0(\boldsymbol{b})$. Hence we may assume that $b_{-n} = b_n$ for all n. Then as above, we have

$$p' = \alpha_{(-1,0)}(p) = (v,u)_{1-n}, \quad q' = \alpha_{(-1,0)}(q) = (1/u,\ u^{b_n} v)_{-n},$$

and q' is identified with

$$((1/u)^{b_{-n}} (u^{b_n} v), 1/(1/u))_{1-n} = (v,u)_{1-n} = p'$$

since $b_{-n} = b_n$. This proves the lemma. □

In most of the cases of interest in this theory, the sequence is periodic and in many cases it is also symmetric under $n \mapsto -n$. If $\boldsymbol{b}$ is periodic with period r, where r is not necessarily the smallest period of $\boldsymbol{b}$, then $r\mathbb{Z}$ fixes $\boldsymbol{b}$ and therefore acts on $\mathcal{M}_0(\boldsymbol{b})$. The quotient of $\mathcal{M}_0(\boldsymbol{b})$ for the action of $r\mathbb{Z}$ will be denoted $\mathcal{M}_r(\boldsymbol{b})$, and $\mathcal{M}_0(\boldsymbol{b})$ may be regarded as the special case $r = 0$. The images in $\mathcal{M}_r(\boldsymbol{b})$ of the curves S_n will form a closed cycle of rational curves. In case $\boldsymbol{b}$ is not only periodic but also invariant under $n \mapsto -n$, the latter induces an automorphism of $\mathcal{M}_r(\boldsymbol{b})$ which we will denote $\tau_{r,\boldsymbol{b}}$. This choice of notation derives from the relation between the automorphism $\tau_{r,\boldsymbol{b}}$ and the switching map τ, which will be elucidated below.

Corollary 7.3. *Assume that* $\boldsymbol{b}$ *is periodic with period* r *and symmetric under* $n \mapsto -n$. *Then the involution* $\tau_{r,\boldsymbol{b}}$ *interchanges the the projective lines* $S_n(\boldsymbol{b})$ *and* $S_{-n}(\boldsymbol{b})$ *for all* n.

Proof. The component $S_n(\boldsymbol{b})$ is the closure of the axis $v_n = 0$ of the n-th coordinate system. It is mapped to the closure of the axis $u_{1-n} = 0$ in the $(1-n)$-th coordinate system. But that is the same as the closure of the axis $v_{-n} = 0$ of the n-th coordinate system, which is $S_{-n}(\boldsymbol{b})$, and we are done. □

The manifolds $\mathcal{M}_r(\boldsymbol{b})$ are the key to Hirzebruch's resolution of the cusp singularities of Hilbert modular surfaces. We refer the reader to the works cited above for details. We will merely summarize the facts that are pertinent to our work here.

Let Γ be a group of Hilbert modular type acting on $\mathcal{H}^2$ and let X_Γ denote the SBB compactification of $\Gamma\backslash\mathcal{H}^2$. Modulo ± 1, the subgroup fixing the cusp ∞ will be the semidirect product of a subgroup V of finite index in the group of totally positive units of $\mathcal{O}_k$ with a lattice M in k. It is convenient to denote the subgroup fixing ∞ by (V, M). The norm form of the lattice gives rise to a rational binary quadratic form, a root of which can be expanded in a continued fraction of Hirzebruch's type,

$$c_0 - \cfrac{1}{c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \dots}}}.$$

Since the root is a quadratic irrationality, the coefficients c_n are eventually periodic in n. If the smallest period is $(b_1 \dots b_r)$ then one forms the $\mathbb{Z}$-sequence $\boldsymbol{b} = (b_n)_{n\in\mathbb{Z}}$ by extending b_n to a periodic function of n with period r on all of $\mathbb{Z}$. The resolution of the singularity at the cusp ∞ in the SBB compactification of $\Gamma\backslash\mathcal{H}^2$ is then obtained [Hirzebruch 1971, Theorem of § 3] by replacing a neighborhood of the singularity by a neighborhood of the union of the curves curves S_n in $\mathcal{M}_{rs}(\boldsymbol{b})$, where s is the index of V in the full unit group of $\mathcal{O}_k$ modulo ± 1. (Some care has to be taken in the case where $rs \le 2$; see [Hirzebruch 1971] for details.) The components of the resolution cycle at a cusp x will be denoted S_0^x, S_1^x, ..., if they are known from the context, or by $S_0^x(\delta)$, $S_1^x(\delta)$, etc., where δ is data such as $\mathcal{O}_k$, $\sqrt{7}\mathcal{O}_k$ or $\boldsymbol{b}$ describing the group Γ and the resolution in more or less detail.

One can be quite explicit about this construction. Indeed, if one embeds the lattice M into $\mathbb{R}^2$ via the two embeddings of k into $\mathbb{R}$, the image of the set of totally positive elements of M will be denoted M_+. The convex hull of M_+ will be an infinite polygon whose vertices form a $\mathbb{Z}$-sequence A_n in k. If we regard k as a subfield of $\mathbb{R}$ by fixing one of the embeddings in advance, then we can determine the $\mathbb{Z}$-sequence A_n up to a translation of the index n by requiring that

A_n be monotonically decreasing in n. The $\mathbb{Z}$-sequence A_n then has the following interesting properties for all n:

(1) A_{n-1} and A_n form a basis for M;
(2) $A_{n-1} + A_{n+1} = b_n A_n$.

Denoting by B_n, C_n the dual basis to A_{n-1}, A_n with respect to the trace form $(x, y) \mapsto \operatorname{tr}(xy)$ of k, we can then define a map

$$\phi_n : \mathcal{H}^2 \to \mathbb{C}_n^2 \subseteq \mathcal{M}_0(\boldsymbol{b})$$

by the rule

$$\phi_n(z_1, z_2) = \bigl(e(B_n z_1 + B_n' z_2), e(C_n z_1 + C_n' z_2)\bigr) = (u_n, v_n),$$

where $e(t) = e^{2\pi i t}$ for all t. One can then verify that for z, w in $\mathcal{H}^2$, we have $\phi_n(z) = \phi_n(w)$ if and only if for some $\mu \in M$ we have

$$w = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} z = (z_1 + \mu, z_2 + \mu'),$$

where $z = (z_1, z_2)$. Furthermore, $\eta = \varepsilon^s$ is a generator of V and we have

$$\phi_{n+rs} \circ \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix} \cdot z = \phi_{n+rs}(\eta z_1, \eta' z_2) = \phi_n(z)$$

for all n and all $z = (z_1, z_2)$. Therefore, ϕ_n induces a map

$$(V, M)\backslash \mathcal{H}^2 \to \mathbb{C}_n^2 \subseteq \mathcal{M}_{rs}(\boldsymbol{b})$$

which is injective in a neighborhood of ∞ and maps that neighborhood isomorphically onto a neighborhood in $\mathcal{M}_{rs}(\boldsymbol{b})$ of the form $W - \cup S_n$, where W is a neighborhood of $\cup S_n$ in $\mathcal{M}_{rs}(\boldsymbol{b})$. The inverse mapping extends to a mapping from W to a neighborhood of ∞ in the SBB compactification and that mapping resolves the singularity at ∞. The preimage of ∞ will be called the *resolution cycle* at ∞. In the special case where $rs \leq 2$, some care must be taken, as described in [Hirzebruch 1971, § 3], but we will not worry about such details here. Using elements of $GL_2^+(k)$, one can move any cusp to ∞ and back. Therefore, one can give a similar description of the resolution of the cusps of the SBB compactification of $\Gamma\backslash\mathcal{H}^2$ for any group Γ of Hilbert modular type. In the particular case we are considering, where $k = \mathbb{Q}(\sqrt{7})$, the field k has class number 1, and $\Gamma = \widehat{\Gamma}(\sqrt{7})$, the cusps are all equivalent under the action of $\widehat{\Gamma}$ and the description of the resolution is simpler. In particular, all of the resolution cycles for X will look the same. The surface we have denoted by $\widehat{Z}(\sqrt{7})$ has been obtained by this process.

Thanks to the system of bases A_{n-1}, A_n and the associated maps ϕ_n, we have a very clear picture of what $\widehat{Z}(\sqrt{7})$ looks like in a neighborhood of the resolution cycles of the cusps. In this case, we can compute the system of bases A_{n-1}, A_n quite explicitly, thanks to the table in [van der Geer 1988, p. 41], which gives for the the cycle associated to the lattice $\sqrt{28}\mathcal{O}_k$ (the same as the one associated with $\sqrt{7}\mathcal{O}_k$, since the two lattices are related by multiplication by a rational number) as $(3,3,2,2,2)$. However, we find it more convenient to describe the cycle as $(2,2,3,3,2)$.

LEMMA 7.4. *The $\mathbb{Z}$-sequence $A_n = A_n(\sqrt{7}\mathcal{O}_k)$ is given by*

$$A_n(\sqrt{p}\mathcal{O}_k) = \varepsilon^m(7 - j\sqrt{p})$$

if $n = j+5m$, with $|j| \le 2$, where $m \in \mathbb{Z}$ and where the corresponding $\mathbb{Z}$-sequence $\{b_n\}$ is given by

$$b_n = \begin{cases} 3 & \text{if } n \equiv \pm 2 \pmod 5, \\ 2 & \text{otherwise.} \end{cases} \tag{7.5}$$

PROOF. It is straightforward to verify that with $M = \sqrt{7}\mathcal{O}_k$ and $A_n = A_n(M)$ and b_n as given above, we have

$$A_{n-1} + A_{n+1} = b_n A_n$$

for all n in $\mathbb{Z}$. Furthermore, the $\mathbb{Z}$-sequence b_n is cyclic with period $(3,3,2,2,2)$ if $M = \sqrt{7}\mathcal{O}_k$. Furthermore, we have $A_{-n} = A_n'$ and $b_{-n} = b_n$ for all $n \in \mathbb{Z}$. Furthermore, A_0, A_1 is a basis for M, namely $7, 7-\sqrt{7}$ if $M = \sqrt{7}\mathcal{O}_k$. Therefore the A_n are precisely the vertices of the convex hull of the set M_+ of totally positive elements of M [van der Geer 1988, pp. 31–33]. ☐

LEMMA 7.6. *Let x be a cusp of $\widehat{X}(\sqrt{7})$. The only fixed points of τ on the components S_i^x of the resolution cycles of the cusps are its fixed points on the components S_0^x and the point where the two components S_2^x and S_3^x meet. More precisely, since τ induces an involution of the nonsingular rational curve S_0^x, it will have two fixed points on S_0^x and therefore 3 fixed points in all among the cusps.*

PROOF. Since the period of $(2,2,3,3,2)$ is 5 and since the order of M is the maximal order of k, the value of s is 1 in this case. Therefore, a neighborhood of each cusp x of $\widehat{X}(\sqrt{7})$ is replaced by a neighborhood of the cycle of curves $S_i^x(M)$, $0 \le i \le 4$, in the complex manifold $\mathcal{M}_5\overline{(2,2,3,3,2)}$. Furthermore, it is clear from the construction of $\mathcal{M}_5\overline{(2,2,3,3,2)}$, and from the fact that G acts transitively on the set of cusps and the fact that all of the cusps of $\widehat{\Gamma}(\sqrt{7})$ are rational, that the involution τ of $\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2$ extends to an automorphism, also denoted τ, of $\widehat{Z}(\sqrt{7})$ inducing the automorphism $\tau_{5,\boldsymbol{b}}$, where $\boldsymbol{b} = \overline{(2,2,3,3,2)}$ on the copy of $\mathcal{M}_5(\boldsymbol{b})$ at each cusp. It therefore follows from Corollary 7.3 that τ interchanges the curves S_i^x and S_{-i}^x. From this, the lemma follows at once. ☐

We know from Lemma 6.4 that the involution τ acting on $\widehat{X}(\sqrt{7})$ has for its fixed point set the union of the three curves $F_N(\widehat{X}(\sqrt{7}))$, with $N = 1,2,4$. Therefore, the fixed point set of τ acting on $\widehat{Z}(\sqrt{7})$ will contain the union of the three curves $F_N(\widehat{Z}(\sqrt{7}))$. The following lemma sharpens this observation and also determines the intersection number of each $F_N(\widehat{X}(\sqrt{7}))$, for $N = 1, 2, 4$ with the components S_i^x.

Lemma 7.7. *Let x be a cusp of $\widehat{X}(\sqrt{7})$. Each of the three fixed points of τ among the five components S_i^x, $0 \le i \le 4$, of the resolution cycle of x lies in one and only one of the curves $F_N(\widehat{Z}(\sqrt{7}))$, $N = 1, 2, 4$. The point of intersection of S_2^x and S_3^x lies on $F_2(\widehat{Z}(\sqrt{7}))$. The other two, which are found on S_0^x, lie on $F_1(\widehat{Z}(\sqrt{7}))$ and $F_4(\widehat{Z}(\sqrt{7}))$ respectively. The curves $F_N(\widehat{Z}(\sqrt{7}))$, with $N = 1, 2, 4$, do not have any other points of intersection with the components S_i^x, $0 \le i \le 4$. Furthermore, the intersections of $F_1(\widehat{Z}(\sqrt{7}))$ and $F_4(\widehat{Z}(\sqrt{7}))$ with S_0^x and of $F_2(\widehat{Z}(\sqrt{7}))$ with S_2^x and S_3^x are transverse.*

Proof. We prove this by looking at the equations of $F_N(\widehat{X}(\sqrt{7}))$, for $N = 1, 2, 4$, in local coordinate charts σ_i containing the fixed points of τ acting on the components S_i^x. Since G acts transitively on the cusps, we can assume that x is the cusp $\infty = (\infty, \infty)$. The coordinate chart σ_1 does not contain the entire component S_0^x, but the point it does not contain lies on the component S_4^x therefore can't be a fixed point. Therefore, we can use the coordinate chart σ_1 to study the fixed points lying on S_0^x. Since the dual basis B_1, C_1 to A_0, A_1 with respect to the pairing $xy \mapsto \mathrm{tr}(xy)$ is given by $B_1 = (1 + \sqrt{7})/2$, $C_1 = -\sqrt{7}/2$, the coordinates (z_1, z_2) of $\mathcal{H}^2$ are related to the coordinates (u_1, v_1) of σ_1 by

$$(u_1, v_1) = \left(e\Big(\frac{1+\sqrt{7}}{2} z_1 + \frac{1-\sqrt{7}}{2} z_2\Big),\ e\Big(-\frac{z_1+z_2}{2}\Big)\right).$$

Since the curves $F_N(\widehat{Z}(\sqrt{7}))$, for $N = 1, 2, 4$, have only one component each, we obtain them as the images of the curves of the curves described in (6.3), namely:

$$z_1 = z_2, \quad z_1 = \varepsilon z_2, \quad z_1 = z_2 + \sqrt{7}.$$

These three curves are respectively mapped to the following curves in σ_1:

$$v_1 = 1, \quad u_1^7 v_1^4 = 1, \quad v_1 = -1.$$

For example, if we put $z_1 = z_2$, we obtain $(u_1, v_1) = (e(z_2), 1)$; the other cases are handled similarly. From this it is clear that the curves $F_1(\widehat{Z}(\sqrt{7}))$ and $F_4(\widehat{Z}(\sqrt{7}))$ meet the line S_0^x (i.e., $u_1 = 0$) transversely in the two distinct points $(0, 1)$ and $(0, -1)$, respectively, and the curve $F_2(\widehat{Z}(\sqrt{7}))$ does not meet S_0^x. To complete the proof, we work in the coordinate system σ_3, whose intersections with the curves S_2^x, S_3^x are the coordinate axes of the coordinate system σ_3. The points of S_2^x, S_3^x that do not lie on σ_3 lie on the curves S_1^x and S_4^x respectively and therefore are not fixed points of τ. Therefore it suffices to work in the coordinate system σ_3. By Lemma 7.4, we have

$$A_2 = 7 - 2\sqrt{7}, \quad A_3 = 14 - 5\sqrt{7},$$

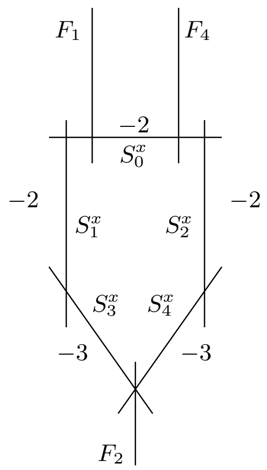

Figure 2. How the curves $F_N(\widehat{Z}(\sqrt{7}))$, for $N = 1, 2, 4$, meet the resolution cycle.

and the dual basis B_3, C_3 to this basis for $\sqrt{7}\mathcal{O}_k$ is given by

$$B_3 = \frac{5+2\sqrt{7}}{14}, \quad C_3 = -\frac{2+\sqrt{7}}{14}.$$

Therefore the coordinates (z_1, z_2) of $\mathcal{H}^2$ are related to the coordinates (u_3, v_3) of σ_3 by

$$(u_3, v_3) = \left(e\Big(z_1\frac{5+2\sqrt{7}}{14} + z_2\frac{5-2\sqrt{7}}{14}\Big),\ e\Big(-z_1\frac{2+\sqrt{7}}{14} - z_2\frac{2-\sqrt{7}}{14}\Big)\right).$$

The curves (6.3) are respectively mapped to the following curves in σ_3:

$$u_3^2 v_3^5 = 1, \quad u_3 = v_3, \quad u_3^2 v_3^5 = -1.$$

Thus, $F_1(\widehat{Z}(\sqrt{7}))$ and $F_4(\widehat{Z}(\sqrt{7}))$ do not meet the axes of σ_3 and $F_2(\widehat{Z}(\sqrt{7}))$ passes through the origin of σ_3 (the point of intersection of S_2^x and S_3^x), transversely to both axes. This completes the proof of the lemma. $\square$

The configuration of curves in the resolution cycle of a cusp x and their intersections with the curves $F_N(\widehat{\Gamma}(\sqrt{7}))$ for $N = 1, 2, 4$ are shown in Figure 2.

Having computed the divisors with rational coefficients $F_N(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7}))$ for $N = 1, 2, 4$, we can now compute the intersection numbers of the modular curves on the surface $\widehat{Z}(\sqrt{7})$. Before doing so, we note that according to Lemma 3.6, the curves $F_{14}(\widehat{X}(\sqrt{7}))$ and $F_{28}(\widehat{X}(\sqrt{7}))$ do not pass through the cusps of $\widehat{X}(\sqrt{7})$. This is also consistent with the fact that by Lemma 2.5 the components of $F_N(\widehat{X}(\sqrt{7}))$ for $N = 14, 28$ are quotients of the upper half plane by arithmetic groups arising from quaternion division algebras over $\mathbb{Q}$ and are therefore compact subvarieties of $\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2$. Therefore we have

$$F_N(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7})) = F_N(\widehat{Z}(\sqrt{7}))$$

for $N = 14, 28$. That implies that the intersection numbers of $F_N(\widehat{Z}(\sqrt{7}))$ on $\widehat{Z}(\sqrt{7})$ will be the same as the corresponding intersection numbers on $\widehat{X}(\sqrt{7})$. That allows us to fill in the last two rows and columns of the table in the following lemma simply by copying them from the corresponding entries of the table in Lemma 5.9.

Lemma 7.8. *We have the following table of intersection numbers on $\widehat{Z}(\sqrt{7})$.*

	$F_1(\widehat{Z}(\sqrt{7}))$	$F_2(\widehat{Z}(\sqrt{7}))$	$F_4(\widehat{Z}(\sqrt{7}))$	$F_{14}(\widehat{Z}(\sqrt{7}))$	$F_{28}(\widehat{Z}(\sqrt{7}))$
$F_1(\widehat{Z}(\sqrt{7}))$	28	0	0	0	84
$F_2(\widehat{Z}(\sqrt{7}))$	0	-18	0	42	84
$F_4(\widehat{Z}(\sqrt{7}))$	0	0	-42	84	42
$F_{14}(\widehat{Z}(\sqrt{7}))$	0	42	84	-42	168
$F_{28}(\widehat{Z}(\sqrt{7}))$	84	84	42	168	210

Proof. According to Mumford's theory, we have

(7.9)
$$F_M(\widehat{X}(\sqrt{7}))\cdot F_N(\widehat{X}(\sqrt{7})) = F_M(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7}))\cdot F_N(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7})).$$

The values of the left side are given by the table in Lemma 5.9 for $M, N = 1, 2, 4, 14, 28$. The factors of the intersection product on the right side of (7.9) can be written in the form

$$F_M(\widehat{Z}(\sqrt{7})) + \sum_x \sum_{i=0}^{4} a_{ixM} S_i^x,$$
$$F_N(\widehat{Z}(\sqrt{7})) + \sum_x \sum_{i=0}^{4} b_{ixN} S_i^x,$$

where the coefficients a_{ixM}, a_{ixN} are known to us from Lemma 7.1. Furthermore, since

$$F_M(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7}))\cdot S_i^x = 0,$$

the intersection product on the right side of (7.9) is equal to

$$\begin{aligned}
&F_M(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7}))\cdot F_N(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7})) \\
&\quad = F_M(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7}))\cdot (F_N(\widehat{Z}(\sqrt{7}) + \sum_x \sum_{i=0}^{4} a_{ixN} S_i^x) \\
&\quad = F_M(\widehat{Z}(\sqrt{7}) \to \widehat{X}(\sqrt{7}))\cdot F_N(\widehat{Z}(\sqrt{7})) \\
&\quad = F_M(\widehat{Z}(\sqrt{7}))\cdot F_N(\widehat{Z}(\sqrt{7})) + \sum_x \sum_{i=0}^{4} a_{ixM} S_i^x \cdot F_N(\widehat{Z}(\sqrt{7})).
\end{aligned}$$

Knowing the intersection numbers of the modular curve $F_N(\widehat{Z}(\sqrt{7}))$ with the cuspidal components S_i^x, we can use (7.9) to solve for $F_M(\widehat{Z}(\sqrt{7}))\cdot F_N(\widehat{Z}(\sqrt{7}))$ in terms of quantities we know how to compute. In fact, we have

$$\begin{aligned}(7.10)\quad & F_M(\widehat{Z}(\sqrt{7}))\cdot F_N(\widehat{Z}(\sqrt{7}))\\ &= F_M(\widehat{X}(\sqrt{7}))\cdot F_N(\widehat{X}(\sqrt{7})) - \sum_x \sum_{i=0}^{4} a_{ixM} S_i^x \cdot F_N(\widehat{Z}(\sqrt{7})).\end{aligned}$$

We will leave the detailed verification of the table to the reader, but we will illustrate the computation of one of its entries. According to Lemma 5.9, the self-intersection number of $F_1(\widehat{X}(\sqrt{7}))$ is 8. Using Lemma 7.1 and equation (7.10), we therefore have

$$\begin{aligned}F_1(\widehat{Z}(\sqrt{7}))^2 &= 8 - \frac{1}{2}\sum_x F_1(\widehat{Z}(\sqrt{7}))\cdot(3S_0^x + 2S_1^x + S_2^x + S_3^x + 2S_4^x)\\ &= 8 - \frac{1}{2}\cdot 24\cdot 3 = -28.\end{aligned}$$ □

Like their counterparts on $\widehat{X}(\sqrt{7})$, the curves $F_N(\widehat{Z}(\sqrt{7}))$ are irreducible for $N = 1, 2, 4$ but reducible for $N = 14$ and $N = 28$. As noted above, the two curves for $N = 14, 28$ do not pass through the cusps and there intersection properties are therefore unaffected by the resolution of the cuspidal singularities. The same applies to the irreducible components of these curves. From Corollary 5.10, we therefore immediately have the following result.

Corollary 7.11. *Let $Z_N(\widehat{Z}(\sqrt{7}))$ denote a component of $F_N(\widehat{Z}(\sqrt{7}))$, with $N = 14$ or $N = 28$. Then we have the following intersection numbers*:

	$F_1(\widehat{Z}(\sqrt{7}))$	$F_2(\widehat{Z}(\sqrt{7}))$	$F_4(\widehat{Z}(\sqrt{7}))$	$F_{14}(\widehat{Z}(\sqrt{7}))$	$F_{28}(\widehat{Z}(\sqrt{7}))$
$Z_{14}(\widehat{Z}(\sqrt{7}))$	0	2	4	-2	8
$Z_{28}(\widehat{Z}(\sqrt{7}))$	4	4	2	8	10

In order to compute the self-intersection numbers of $Z_N(\widehat{Z}(\sqrt{7}))$ for $N = 14, 28$, we need to recall the adjunction formula and its interpretation for Hilbert modular surfaces.

Lemma 7.12 (Adjunction Formula). *Let W be a complete nonsingular surface and let V be a curve on W. Then we have*

$$2 - 2p_a(V) = c_1(W)\cdot V - V\cdot V,$$

where $p_a(V)$ is the arithmetic genus of V and $c_1(W)$ is the first Chern class of W.

Proof. See [van der Geer 1988, p. 162]. Here $p_a(V)$ is just the genus of V if V is nonsingular. □

Using the adjunction formula, one can compute self-intersection numbers provided one can compute the other terms in the formula. The next two tools provide formulas for computing $c_1(\widehat{Z}(\sqrt{7}))\cdot F_N(\widehat{Z}(\sqrt{7}))$ and $p_a(F_N(\widehat{Z}(\sqrt{7})))$ respectively.

LEMMA 7.13. *Let B be a primitive integral skew hermitian matrix over k. Then the intersection number of $F_B(\widehat{Z}(\sqrt{7}))$ with the cohomology class c_1 on $\widehat{Z}(\sqrt{7})$ is given by*

$$c_1\cdot F_B(\widehat{Z}(\sqrt{7})) = 2\int \omega + \sum_x Z_x\cdot F_B,$$

where the integral is over $\Gamma_B\backslash\mathcal{H}^2$, the integrand is the volume form

$$\omega = -\frac{1}{2\pi}\frac{dx\wedge dy}{y^2},$$

the summation runs runs over all of the cusps x of $\widehat{Z}(\sqrt{7})$ and where for a cusp x, we denote by Z_x the sum of all of the curves S_i^x in the resolution cycle of the cusp x.

PROOF. See [van der Geer 1988, Cor. VII.4.1] and the explicit description of the local Chern cycle at x on pp. 46 and 63 of the same reference. The result is not stated in there in a way that makes it completely clear that it is also valid for congruence subgroups, but that is implicit in its proof. □

LEMMA 7.14. *The self-intersection number of $Z_N(\widehat{Z}(\sqrt{7}))$ for $N = 14, 28$ is*

$$Z_N(\widehat{Z}(\sqrt{7}))^2 = \begin{cases} -2 & \text{if } N = 14, \\ -6 & \text{if } N = 28. \end{cases}$$

PROOF. We already know that the arithmetic genus of $Z_N(\widehat{Z}(\sqrt{7}))$ for $N = 14, 28$ is equal to the volume of $Z_N(\widehat{Z}(\sqrt{7}))$. It therefore follows from the adjunction formula and the preceding lemma that the self-intersection number of $Z_N(\widehat{Z}(\sqrt{7}))$ is equal to its volume of $Z_N(\widehat{Z}(\sqrt{7}))$. We are therefore done by Corollary 4.2. □

We now verify that the curves $Z_N(\widehat{Z}(\sqrt{7}))$, for $N = 14, 28$, are hyperelliptic.

LEMMA 7.15. *Let $N = 14, 28$. Then the intersection number of $Z_N(\widehat{Z}(\sqrt{7}))$ with*

$$F_1(\widehat{Z}(\sqrt{7})) + F_2(\widehat{Z}(\sqrt{7})) + F_4(\widehat{Z}(\sqrt{7}))$$

is equal to 6 if $N = 14$ and is equal to 10 if $N = 28$. The curve $Z_N(\widehat{Z}(\sqrt{7}))$ is hyperelliptic, with the involution τ inducing the hyperelliptic involution on $Z_N(\widehat{Z}(\sqrt{7}))$.

PROOF. The computation of the intersection number follows at once from the table in Lemma 7.9. Since $F_N(\widehat{Z}(\sqrt{7}))$ has 21 components, an odd number, and since these components must be permuted by the involution τ of $\widehat{Z}(\sqrt{7})$, there must be at least 1 component invariant under τ. Since τ commutes with the action of $\mathrm{PSL}_2(\mathbb{F}_7)$, it follows that every component $Z_N(\widehat{Z}(\sqrt{7}))$ is invariant under τ. The number of fixed points of τ on $Z_N(\widehat{Z}(\sqrt{7}))$ is equal to the intersection number of $Z_N(\widehat{Z}(\sqrt{7}))$ with the fixed point set of τ, which by Lemma 7.7 is

$$F_1(\widehat{Z}(\sqrt{7})) + F_2(\widehat{Z}(\sqrt{7})) + F_4(\widehat{Z}(\sqrt{7})).$$

The intersection number computed in the first part of this lemma therefore gives the number of fixed points of τ on each component. Since $Z_N(\widehat{Z}(\sqrt{7}))$ has genus 2 and 6 fixed points for τ, when $N = 14$, and genus 4 and 10 fixed points for τ when $N = 28$, it follows that $Z_N(\widehat{Z}(\sqrt{7}))$ is a hyperelliptic curve and τ is its hyperelliptic involution. $\square$

LEMMA 7.16. *The curve $F_N(\widehat{Z}(\sqrt{7}))$ has genus 3 for $N = 1$ and genus 10 for $N = 2, 4$.*

PROOF. By Corollary 4.2, the volume of $F_N(\widehat{\Gamma}(\sqrt{7}))$ is -28 for $N = 1$ and -42 for $N = 2, 4$. Therefore these numbers are also their Euler numbers. The curve $F_N(\widehat{Z}(\sqrt{7}))$ for $N = 1, 2, 4$ is obtained by adding the points where it meets the resolution cycles at the cusps. According to Lemma 7.7 (cf. Figure 2), there is one such point for each cusp, so there are 24 points in all. Therefore the Euler number of $F_N(\widehat{Z}(\sqrt{7}))$ is $-28 + 24 = -4$ for $N = 1$ and is $-42 + 24 = 18$ for $N = 2, 4$. Writing the Euler number as $2 - 2g$, we have $g = 3$ for $N = 1$ and $g = 10$ for $N = 2, 4$. $\square$

8. The Symmetric Hilbert Modular Surface $W = \widehat{Z}(\sqrt{7})/\tau$

We denote by W the orbit space for the action of the switching involution τ on $\widehat{Z}(\sqrt{7})$. It follows from Lemma 6.3 and Lemma 7.6 that the natural mapping of $\widehat{Z}(\sqrt{7})$ onto W is a two-sheeted covering branched along $F_1(W) \cup F_2(W) \cup F_4(W)$. Furthermore, since τ commutes with the elements of G acting on $\widehat{Z}(\sqrt{7})$), the group G acts on W.

LEMMA 8.1. *The surface W is an algebraic surface whose Betti numbers are given by*:

$$b_i = \begin{cases} 1 & \textit{if } i = 0, 4, \\ 0 & \textit{if } i = 1, 3, \\ 94 & \textit{if } i = 2. \end{cases}$$

PROOF. First of all, a surface of Hilbert modular type is always an algebraic surface, by the results of Baily and Borel [1966] (cf. [van der Geer 1988, Prop. II.7.1, p. 44]). It follows that the surface W is also an algebraic surface. Furthermore, Hilbert modular surfaces have vanishing first Betti number [van der Geer 1988, comments following Cor. IV.6.2, p. 82]. So $b_1 = 0$ and therefore, by Poincaré duality, $b_3 = 0$. Since $\widehat{W}(\sqrt{7})$ is connected, we have $b_0 = 1$ and then $b_4 = 1$ by

Poincaré duality. Therefore, the Euler number $e(W)$ satisfies

$$e(W) = b_2 + 2,$$

so we will prove b_2 has the value we claim by proving that $e(W) = 96$. Since $\widehat{Z}(\sqrt{7})$ is a two-sheeted covering branched of W branched along $F_1(W) \cup F_2(W) \cup F_4(W)$, we have

$$e(\widehat{Z}(\sqrt{7})) = 2e(W) - e(F_1(\widehat{Z}(\sqrt{7}))) - e(F_2(\widehat{Z}(\sqrt{7}))) - e(F_4(\widehat{Z}(\sqrt{7}))).$$

By [van der Geer 1988, Theorem IV.1.2, p. 60], the Euler numbers of the open curves $F_N(\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2)$, for $N = 1, 2, 4$, are equal to their volumes, which we have computed in Corollary 4.2. We obtain these open curves by deleting the 24 points where $F_N(\widehat{Z}(\sqrt{7}))$ meets the resolution cycles of the cusps, $N = 1, 2, 4$. Therefore

$$e(F_N(\widehat{Z}(\sqrt{7}))) = \begin{cases} -4 & \text{if } N = 1, \\ -18 & \text{if } N = 2, 4; \end{cases}$$

hence

$$e(\widehat{Z}(\sqrt{7})) = 2e(W) + 4 + 18 + 18 = 2e(W) + 40.$$

Therefore, we just have to prove that $e(\widehat{Z}(\sqrt{7})) = 232$. By [van der Geer 1988, Theorem IV.2.5, p. 64], we have

$$e(\widehat{Z}(\sqrt{7})) = \mathrm{vol}(\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2) + 5\cdot 24,$$

since there are 5 components in each of the 24 resolution cycles. Since the Hilbert modular group for $k = \mathbb{Q}(\sqrt{7})$ has index 2 in the extended Hilbert modular group and since $\widehat{\Gamma}(\sqrt{7})$ has index 168 in the extended Hilbert modular group $\widehat{\Gamma}$, it follows from [van der Geer 1988, Theorem IV.1.1, p. 59] that

$$\mathrm{vol}(\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2) = 168\zeta_k(-1),$$

where ζ_k denotes the Dedekind zeta function of the quadratic field k. Using [van der Geer 1988, Theorem I.6.5, p. 20], we have

$$\zeta_k(-1) = \frac{1}{60}\sum_{x\in\mathbb{Z}} \sigma_1\left(\frac{28 - x^2}{4}\right) = \frac{2}{3},$$

where $\sigma_1(n)$ is the sum of the divisors of n if n is a positive integer and otherwise is 0. Therefore

$$\mathrm{vol}(\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2) = 168\cdot\frac{2}{3} = 112,$$

hence

$$e(\widehat{Z}(\sqrt{7})) = \mathrm{vol}(\widehat{\Gamma}(\sqrt{7})\backslash\mathcal{H}^2) + 5\cdot 24 = 232.$$

This completes the proof of the lemma. □

We now compute the intersection numbers of the modular curves and their components on W.

LEMMA 8.2. *We have the following table of intersection numbers on W.*

	$F_1(W)$	$F_2(W)$	$F_4(W)$	$F_{14}(W)$	$F_{28}(W)$
$F_1(W)$	-56	0	0	0	84
$F_2(W)$	0	-36	0	42	84
$F_4(W)$	0	0	-84	84	42
$F_{14}(W)$	0	42	84	-21	84
$F_{28}(W)$	84	84	42	84	105

PROOF. The intersection product of two divisors on W is half the intersection product of their pullbacks to $\widehat{Z}(\sqrt{7})$. Since the curves $F_N(W)$ for $N = 1, 2, 4$ form the branch locus of the natural mapping of $\widehat{Z}(\sqrt{7})$ onto W, we have

$$F_M(W)\cdot F_N(W) = c_{MN} F_M(\widehat{Z}(\sqrt{7}))\cdot F_N(\widehat{Z}(\sqrt{7})),$$

where

$$c_{MN} = \begin{cases} 2 & \text{if } M, N = 1, 2, 4, \\ 1 & \text{if } M = 1, 2, 4 \text{ and } N = 14, 28, \\ 1 & \text{if } M = 14, 28 \text{ and } N = 1, 2, 4, \\ \frac{1}{2} & \text{if } M, N = 14, 28. \end{cases}$$

Using this, the lemma follows at once from Lemma 7.8. We leave it to the reader to verify the computation. □

LEMMA 8.3. *Each component of $F_{14}(W)$ has self-intersection number -1. Each component of $F_{28}(W)$ has self-intersection number -3. For $N = 14, 28$, each component meets exactly 4 components of $F_{42-N}(W)$. Each such intersection is a transverse at a single point. Each component $F_N(W)$ is a rational curve for $N = 14, 28$.*

PROOF. Since the components of $F_N(\widehat{Z}(\sqrt{7})$ are not ramified in the covering $\widehat{Z}(\sqrt{7}) \to W$, the intersection numbers of these components with themselves, with each other and with components of $F_{42-N}(\widehat{Z}(\sqrt{7}))$ are one half of the corresponding intersection numbers for components of $F_N(\widehat{X}(\sqrt{7}))$ and $F_{42-N}(\widehat{X}(\sqrt{7}))$. Therefore we are done by the preceding lemma. □

The next lemma describes the image of the resolution cycles of $\widehat{Z}(\sqrt{7})$ in W.

LEMMA 8.4. *The resolution cycle Z_x on $\widehat{Z}(\sqrt{7})$ of a cusp x is mapped to a reducible curve with 3 components $\mathcal{C}_0^x$, $\mathcal{C}_1^x$, $\mathcal{C}_4^x$. For s modulo 5, the image of the component C_s^x is the curve $\mathcal{C}_t^x$ with $t = s^2$. The intersection numbers of the these*

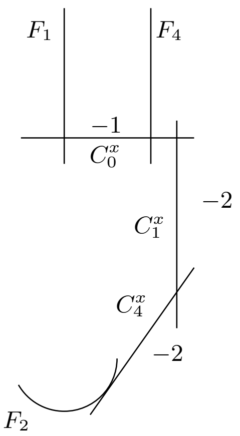

Figure 3. How the curves $F_N(W)$, for $N = 1, 2, 4$, meet the image of a resolution cycle in W.

curves are given by

$$\mathcal{C}_i \cdot \mathcal{C}_j = \begin{cases} -1 & \textit{if } i = j = 0, \\ -2 & \textit{if } i = j \neq 0, \\ 0 & \textit{if } i = 0 \textit{ and } j = 4, \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. The first assertion follows from the fact (arising from Lemma 7.6 and its proof) that τ interchanges the components C_s^x and C_{-s}^x. As for the intersection numbers, we have

$$\begin{aligned} \mathcal{C}_0^x \cdot \mathcal{C}_0^x &= \tfrac{1}{2} S_0^x \cdot S_0^x = \tfrac{1}{2}(-2) = -1, \\ \mathcal{C}_0^x \cdot \mathcal{C}_1^x &= \tfrac{1}{2} S_0^x \cdot (S_1^x + S_4^x) = \tfrac{1}{2}(1+1) = 1, \\ \mathcal{C}_0^x \cdot \mathcal{C}_4^x &= \tfrac{1}{2} S_0^x \cdot (S_2^x + S_3^x) = \tfrac{1}{2}(0+0) = 0, \\ \mathcal{C}_1^x \cdot \mathcal{C}_1^x &= \tfrac{1}{2}(S_1^x + S_4^x) \cdot (S_1^x + S_4^x) = \tfrac{1}{2}(-2-2) = -2, \\ \mathcal{C}_1^x \cdot \mathcal{C}_4^x &= \tfrac{1}{2}(S_1^x + S_4^x) \cdot (S_2^x + S_3^x) = \tfrac{1}{2}(2) = 1, \\ \mathcal{C}_4^x \cdot \mathcal{C}_4^x &= \tfrac{1}{2}(S_2^x + S_3^x) \cdot (S_2^x + S_3^x) = \tfrac{1}{2}(-3-3+2) = -2, \end{aligned}$$

and we are done. □

Figure 3 summarizes the intersection properties of the curves $\mathcal{C}_i^x$ and the curves $F_N(W)$, for $N = 1, 2, 4$.

Proposition 8.5. *The curve F_{28}, the curves $\mathcal{C}_i^x$ and the components of the curve F_{14} represent a basis for the second homology group $H_2(W; \mathbb{C})$.*

Proof. The number of curves is $1 + 21 + 3 \cdot 24 = 94$, which is the same as the second Betti number of W. We can blow down the components of $F_{14}(W)$ one at a time (they have self-intersection number -1 by Lemma 8.3). Independently of that, we can blow down the curves $\mathcal{C}_0^x$ one at a time. Having done so, the curves $\mathcal{C}_1^x$ are mapped to curves with self-intersection number -1, which can then be blown down one at a time. After that is done, the curves $\mathcal{C}_i^x$ are mapped to curves with self-intersection number -1, which can then be blown down one at a time. Each time one of these curves is blown down, the Betti number of W is decreased by 1 and when all have been blown down, the resulting surface has second Betti number $94 - 21 - 3{\cdot}24 = 1$. This proves that the curves $\mathcal{B}_i^x$ and the components of F_{14} are linearly independent in the homology of W. Suppose we can write F_{28} as a linear combination of these curves. Since the curve F_{28} is invariant, components $\mathcal{B}_i^x$ and $Z_{14}(W)$ equivalent under the action of G have the same coefficient in this linear combination. Therefore we can write

$$F_{28}(W) = c_{14}F_{14}(W) + c_0 \sum_x \mathcal{C}_0^x + c_1 \sum_x \mathcal{C}_1^x + c_4 \sum_x \mathcal{C}_4^x,$$

where c_{14}, c_0, c_1, c_4 are rational numbers. However, if we intersect both sides with the cycle

$$F_{28}(W) + 4F_{14}(W),$$

we obtain a contradiction. Indeed, we have

$$F_{28}(W){\cdot}(F_{28}(W) + 4F_{14}(W)) = 105 + 4{\cdot}84 = 441$$

but since $F_{14}(W)$ and $F_{28}(W)$ are disjoint from the cuspidal components and

$$F_{14}(W){\cdot}(F_{28}(W) + 4F_{14}(W)) = 84 + 4{\cdot} - 21 = 0,$$

we have

$$(F_{28}(W) + 4F_{14}(W)){\cdot}(c_{14}F_{14}(W) + c_0 \sum_x \mathcal{C}_0^x + c_1 \sum_x \mathcal{C}_1^x + c_4 \sum_x \mathcal{C}_4^x) = 0. \quad \square$$

9. The Projective Plane as a Hilbert Modular Surface

As noted in the proof of Lemma 8.5, we can blow down all of the components of the curve $F_{14}(W)$ as well as all of the cycles $\mathcal{C}_0^x + \mathcal{C}_1^x + \mathcal{C}_4^x$. The resulting surface will be denoted $\mathbb{P}$. Since the components being blown down are permuted among themselves by the action of G on W, the group G also acts on $\mathbb{P}$. We will prove in this section that W is isomorphic to the complex projective plane.

First we need to recall a rationality criterion [van der Geer 1988, VII.2.2, p. 161]:

LEMMA 9.1. *If S is a nonsingular algebraic surface with $b_1 = 0$ and if S contains either two intersecting exceptional curves or an irreducible curve C with $C^2 \geq 0$ and $K{\cdot}C < 0$, then S is rational.*

In order to apply this formula, we will need to compute some intersection numbers on $\mathbb{P}$.

LEMMA 9.2. *We have the following intersection numbers on* $\mathbb{P}$.

	$F_1(\mathbb{P})$	$F_2(\mathbb{P})$	$F_4(\mathbb{P})$	$F_{28}(\mathbb{P})$
$F_1(\mathbb{P})$	16	48	72	84
$F_2(\mathbb{P})$	48	144	216	252
$F_4(\mathbb{P})$	72	216	324	378
$F_{28}(\mathbb{P})$	84	252	378	441

PROOF. To compute the self-intersection number of $F_1(\mathbb{P})$, we can use Mumford's definition of intersection numbers. After all, the surface on which one is computing intersections doesn't *have* to be singular for it to work. An easy computation shows that

$$F_1(W \to \mathbb{P}) = F_1(W) + \sum_x (3\mathcal{C}_0^x + 2\mathcal{C}_1^x + \mathcal{C}_4^x),$$

since its intersection with any of the curves that were blown down to make $\mathbb{P}$ is 0. Therefore the self-intersection number of $F_1(\mathbb{P})$ on $\mathbb{P}$ is equal to that of $F_1(W \to \mathbb{P})$, which is

$$\begin{aligned} F_1(W \to \mathbb{P}) \cdot F_1(W \to \mathbb{P}) &= F_1(W) + \sum_x (3\mathcal{C}_0^x + 2\mathcal{C}_1^x + \mathcal{C}_4^x) \cdot F_1(W \to \mathbb{P}) \\ &= F_1(W) \cdot (F_1(W) + \sum_x (3\mathcal{C}_0^x + 2\mathcal{C}_1^x + \mathcal{C}_4^x)) \\ &= -56 + 24 \cdot 3 \cdot 1 = 16. \end{aligned}$$

Similarly, it is straightforward to verify that

$$\begin{aligned} F_2(W \to \mathbb{P}) &= F_2(W) + 2 \cdot F_{14}(W) + \sum_x (2\mathcal{C}_0^x + 2\mathcal{C}_1^x + 2\mathcal{C}_4^x)), \\ F_4(W \to \mathbb{P}) &= F_4(W) + 4 \cdot F_{14}(W) + \sum_x (3\mathcal{C}_0^x + 2\mathcal{C}_1^x + \mathcal{C}_4^x)), \\ F_{28}(W \to \mathbb{P}) &= F_{28}(W) + 4 \cdot F_{14}(W). \end{aligned}$$

From this it is easy to complete the table using the table of Lemma 8.2. □

THEOREM 9.3. *The surface* $\mathbb{P}$ *is isomorphic to* $\mathbb{P}^2(\mathbb{C})$. *In particular*, W *is a rational surface*.

PROOF. Denote by K the canonical class of $\mathbb{P}$. By the adjunction formula and the fact that $c_1 = -K$, we have

$$K \cdot F_1(\mathbb{P}) = -e(F_1(\mathbb{P})) - F_1(\mathbb{P}) \cdot F_1(\mathbb{P}) = -(-4) - 4.4 = -12 < 0.$$

On the other hand, we already know that $F_1 \cdot F_1 = 16 > 0$, so by the rationality criterion, it follows that $\mathbb{P}$ is rational. Since $b_2(\mathbb{P}) = 1$ and the self-intersection number of $F_1(\mathbb{P})$ is > 0, the same will be true for the self-intersection number of any curve on $\mathbb{P}$. In particular, $\mathbb{P}$ has no exceptional curves and is therefore a minimal model. Since $b_2(\mathbb{P}) = 1$, we can find a rational number c such that K is homologous to cF_1. Then we have

$$-12 = K \cdot F_1(\mathbb{P}) = cF_1(\mathbb{P})^2 = 16c,$$

so $c = -3/4$. Therefore we have

$$K^2 = c^2 F_1(\mathbb{P})^2 = \frac{9}{16} \cdot 16 = 9.$$

It now follows from the classification of minimal models of algebraic surfaces [van der Geer 1988, § VII.2, p. 160] that $\mathbb{P}$ is actually $\mathbb{P}^2(\mathbb{C})$. The rationality of W now follows from the fact that it is birationally equivalent to $\mathbb{P}$. □

Corollary 9.4. *The curves $F_N(P)$ for $N = 1, 2, 4, 28$ are plane curves of degrees* 4,12,18,21 *respectively.*

Proof. This follows at once from Bezout's theorem and from the table of intersection numbers for these curves on $\mathbb{P}$. □

Corollary 9.5. *The action of G on $\mathbb{P}$ arises from a irreducible complex linear representation of degree* 3.

Proof. The reader is referred to [Conway et al. 1985] for the character table and group of Schur multipliers of G, which provides the basis for the following argument. Any projective representation of degree 3 of G arises from a linear representation of $\mathrm{SL}_2(\mathbb{F}_7)$. Since the action of G on $\mathbb{P}$ is nontrivial and since $\mathrm{SL}_2(\mathbb{F}_7)$ has no nontrivial representation of degree < 3, the representation is irreducible. Furthermore, the irreducible linear representations of degree 3 of $\mathrm{SL}_2(\mathbb{F}_7)$ arise from linear representations of G, so we are done. □

Corollary 9.6. *The locus $F_{14}(\mathbb{P})$ is a* 21*-point orbit for G acting on $\mathbb{P}$. The curves $F_N(\mathbb{P})$ for $N = 1, 2, 4, 28$ are defined by polynomials invariant under the linear* 3*-dimensional representation of G. The locus $F_{28}(\mathbb{P})$ is the union of* 21 *lines permuted transitively by G.*

Proof. The first assertion follows from the fact that G acts transitively on the components of $F_{14}(\widehat{Z}(\sqrt{7}))$ and the corresponding facts on W and $\mathbb{P}$. The last assertion follows from the fact that $F_{28}(\mathbb{P})$ has 21 components permuted transitively by G and the fact that it is a plane curve of degree 21. Finally, suppose $f = 0$ is a polynomial defining a curve in $\mathbb{P}$ invariant under G. Then for all $g \in G$, the polynomial f is mapped to a multiple of itself by g, say to $c_g f$. The function $g \mapsto c_g$ is then easily seen to be a homomorphism from G to the multiplicative group of $\mathbb{C}^\times$. Since G is a simple group, that character is trivial, which proves the second assertion. □

Now that we have identified $\mathbb{P}$, we can identify the surfaces W, $\widehat{Z}(\sqrt{7})$ and $\widehat{X}(\sqrt{7})$. The orbits in P mentioned in Lemma 9.7 and Lemma 9.8 are discussed in more detail in Section 11. (We thank Igor Dolgachev, who pointed out an error in the original statement of Lemma 9.7 and explained how to correct it.)

LEMMA 9.7. *The surface* $\widehat{Z}(\sqrt{7})$ *is obtained from the complex projective plane through the following steps*:

(1) *Blow up the unique* 21-*point orbit* O_{21} *and the unique* 24-*point orbit* O_{24} *for* G *acting on* $\mathbb{P}$. *Call the resulting surface* P'.
(2) *For each point* x *of* O_{24}, *let* $E(x)$ *denote the exceptional curve in* P' *obtained by blowing up* x. *Let* F_1' *denote the proper transform of the Klein curve* $x^3y+y^3z+z^3x=0$ *in* P' *and for each* x *of* O_{24} *let* x' *be the point of* $E(x)$ *where* F_1' *meets* $E(x)$. *The points* x' *form a* 24-*point orbit* $\overline{O}_{24}$ *in* P'. *We blow up this orbit and call the resulting surface* P''.
(3) *For each point* x' *of* $\overline{O}_{24}$, *denote by* $E'(x')$ *the line in* P'' *obtained by blowing up* x'. *Let* F_1'' *denote the proper transform of* F_1' *in* P'' *and for each point* x' *of* $\overline{O}_{24}$ *denote by* x'' *the point where* F_1'' *meets* $E'(x')$ *Then the points* x'' *form a* 24-*point orbit* $\overline{\overline{O}}_{24}$ *on* P''. *Blow up the orbit* $\overline{\overline{O}}_{24}$ *and call the resulting surface* P'''. *The surface* P''' *is* G *equivariantly isomorphic to* W.
(4) *Denote by* $\mathcal{D}_N$ *the proper transform of* $F_N(\mathbb{P})$ *in* P''' *for* $N=1,2,4$. *Let* $\mathcal{Z}$ *denote the two-sheeted cover* P''' *branched along* $\mathcal{D}_1\cup\mathcal{D}_2\cup\mathcal{D}_4$. *Then* $\mathcal{Z}$ *is equivariantly isomorphic to* $\widehat{Z}(\sqrt{7})$.
(5) *The preimage in* $\mathcal{Z}$ *of each point* x *of* $\mathcal{O}_{24}$ *is a cycle* S^x *of curves as shown in Figure* 2. *The cycle* S^x *can be blown down to a double point*. *Blowing down all of the cycles* S^x *results in a surface* $\mathcal{X}$ *equivariantly isomorphic to* $\widehat{X}(\sqrt{7})$.

What prevents Lemma 9.7 from containing a complete characterization of the surfaces $\widehat{X}(\sqrt{7})$ and $\widehat{Z}(\sqrt{7})$ is that we have not yet identified the plane curves $F_N(\mathbb{P})$ for $N=1,2,4$. The rest of this article is devoted to the solution of this problem. The following lemma will be of fundamental importance for that purpose.

LEMMA 9.8. *The curve* $F_4(\mathbb{P})$ *has singularities of order* ≥ 4 *on the* 21-*point orbit* O_{21}. *The curve* $F_2(\mathbb{P})$ *is singular along the* 21-*point orbit* O_{24} *and the* 24-*point orbit* O_{24}.

PROOF. Since each component $Z_{14}(W)$ of $F_{14}(W)$ and each curve $\mathcal{C}_i^x$ is blown down to a point under the natural mapping of W onto $\mathbb{P}$, it will suffice to verify the following intersection numbers on W:

$$Z_{14}\cdot F_2(W)=2,\quad Z_{14}\cdot F_4(W)=4,\quad \mathcal{C}_2^x\cdot F_2(W)=2.$$

The first and second of these follow immediately from the table in Lemma 8.2 if one notes that multiplying these intersection numbers by 21 must give the corresponding intersection numbers for F_{14}. As for the last, the left side must equal the intersection number on $\widehat{Z}(\sqrt{7})$ given by

$$(S_2^x+S_3^x)\cdot F_2(\widehat{Z}(\sqrt{7}))=2. \qquad \square$$

10. The Ring of Invariants of G on $\mathbb{C}^3$

In this section, we recall some of the classical results from [Klein and Fricke 1890–92, vol. 1, § III.7, pp. 732 ff.] on the ring of invariants for a three-dimensional irreducible complex representation ρ of G. An account of some of these results may also be found in [Weber 1896, §§ 122–124]. However, since some of our computations depend essentially on the precise forms for these invariants, we have also computed them ourselves using the algebra program REDUCE 3.4 on a personal computer.

The first invariant is the invariant f of degree 4 given by

$$f = x^3y + y^3z + z^3x. \tag{10.1}$$

The curve $f = 0$ is denoted $\mathcal{C}$ and is referred to as the *Klein curve*. The next invariant, of degree 6, is denoted ∇ and is given, up to a constant factor, by the determinant of the matrix of second partials of f. Explicitly,

$$\nabla = \frac{1}{54}\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y} & \dfrac{\partial^2 f}{\partial x\partial z} \\ \dfrac{\partial^2 f}{\partial y\partial x} & \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 f}{\partial y\partial z} \\ \dfrac{\partial^2 f}{\partial z\partial x} & \dfrac{\partial^2 f}{\partial z\partial y} & \dfrac{\partial^2 f}{\partial z^2} \end{vmatrix} \tag{10.2}$$

$$= 5x^2y^2z^2 - xy^5 - yz^5 - zx^5 = 5x^2y^2z^2 - \sigma(xy^5),$$

where in general we write $\sigma(x^ay^bz^c)$ to denote $x^ay^bz^c + x^by^cz^a + x^cy^az^b$. We will refer to the curve $\nabla = 0$ as the *Hessian* of $\mathcal{C}$ and denote it $\mathcal{H}$. It is the locus of all points in the plane whose polar conics with respect to the Klein curve are line pairs.

The next invariant, of degree 14, is, up to a constant factor, the determinant of the 4×4 symmetric matrix obtained by bordering the matrix of second partials of f with the first partials of ∇. Denoting the matrix by C, we have

$$C = \frac{1}{9}\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y} & \dfrac{\partial^2 f}{\partial x\partial z} & \dfrac{\partial \nabla}{\partial x} \\ \dfrac{\partial^2 f}{\partial y\partial x} & \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 f}{\partial y\partial z} & \dfrac{\partial \nabla}{\partial y} \\ \dfrac{\partial^2 f}{\partial z\partial x} & \dfrac{\partial^2 f}{\partial z\partial y} & \dfrac{\partial^2 f}{\partial z^2} & \dfrac{\partial \nabla}{\partial z} \\ \dfrac{\partial \nabla}{\partial x} & \dfrac{\partial \nabla}{\partial y} & \dfrac{\partial \nabla}{\partial z} & 0 \end{vmatrix}$$

$$= \sigma(x^{14} - 34x^{11}y^2z - 250x^9yz^4 + 375x^8y^4z^2 + 18x^7y^7 + 126x^6y^3z^5).$$

(Klein incorrectly gives the coefficient -126 as 126. The invariant is written correctly in Weber's presentation.) We will denote the curve $C = 0$ by Σ. It is the locus of all points in the plane whose polar lines with respect to $\mathcal{H}$ are tangent to their polar conics with respect to $\mathcal{C}$.

Finally, there is the invariant K of degree 21, which is, up to a constant factor, the functional determinant of f, ∇ and C:

$$K = \frac{1}{14}\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial \nabla}{\partial x} & \frac{\partial \nabla}{\partial y} & \frac{\partial \nabla}{\partial z} \\ \frac{\partial C}{\partial x} & \frac{\partial C}{\partial y} & \frac{\partial C}{\partial z} \end{vmatrix}$$

$$\begin{aligned} &= \sigma(x^{21} - 7x^{18}y^2z + 217x^{16}yz^4 - 308x^{15}y^4z^2 - 57x^{14}y^7 - 289x^{14}z^7 + 4018x^{13}y^3z^5 \\ &\qquad + 637x^{12}y^6z^3 + 1638x^{11}y^9z - 6279x^{11}y^2z^8 + 7007x^{10}y^5z^6 - 10010x^9y^8z^4) \\ &\quad + 10296x^7y^7z^7. \end{aligned}$$

(Weber doesn't give the full expression for this invariant; instead he cites [Gordan 1880] and [Klein and Fricke 1890–92]. In the former work, p. 372, our K is denoted by Ω and is listed with the wrong coefficient 3472 for $x^7y^7z^7$; this error also occurs in the latter work, p. 734. It is easy to guess its origin: since a notation Σ similar to our σ was used, the term $3432x^7y^7z^7$ appearing inside a σ would equal $10296x^7y^7z^7$.) We will denote by Λ the curve $K = 0$. It is the locus of all points in the plane whose polar lines with respect to the curves $\mathcal{C}$, $\mathcal{H}$ and Σ are concurrent.

Klein's generators f, ∇, C, K of the ring of invariants are connected by the relation

$$\begin{aligned} K^2 = C^3 - 88C^2f^2\nabla - 256Cf^7 + 1088Cf^4\nabla^2 + 1008Cf\nabla^4 \\ + 1728\nabla^7 - 60032f^3\nabla^5 + 22016f^6\nabla^3 - 2048f^9\nabla. \end{aligned}$$

The generating function of the ring of invariants is given by

$$\sum_{n=0}^{\infty} a_nt^n = \frac{t^{21}}{(1-t^4)(1-t^6)(1-t^{14})},$$

where a_n is the dimension of the space of G-invariant forms of degree n. The series begins

$$1 + t^4 + t^6 + t^8 + t^{10} + 2t^{12} + 2t^{14} + 2t^{16} + 3t^{18} + 2t^{20} + t^{21} + \dots \tag{10.3}$$

11. Orbits of G Acting on $\mathbb{P}^2(\mathbb{C})$

In this section, we determine the orbit structure of the group G acting on $\mathbb{P}^2(\mathbb{C})$. Our approach is based on the enumeration of the subgroups of G and the decomposition of ρ restricted to each subgroup. For that purpose, it is useful to have a list of all of the conjugacy classes of subgroups of G. That information is provided in the following proposition, due to Klein [1879, § 1] (as corrected in his collected works). In [Adler $\geq$ 1998], we will make a more systematic and general study of orbits. A discussion of the orbits of G on $\mathbb{P}^2(\mathbb{C})$ can also be found in [Weil 1974, §§ 116–121].

LEMMA 11.1. *The group G has subgroups of the following orders*: 1, 2, 3, 4, 6, 7, 8, 12, 21, 24, 168. *All are unique up to conjugacy except for*:

(1) *the groups of order* 24, *which form two conjugacy classes of subgroups interchanged by the outer automorphisms of G*;
(2) *the subgroups of order* 12 *which occur as normal subgroups of these groups of order* 24 *and which fall into two conjugacy classes*;
(3) *the three conjugacy classes of subgroups of order* 4, *which consist of a conjugacy class of cyclic subgroups of order* 4 *and two conjugacy classes of Klein* 4-*groups*.

G has no other subgroups.

For each element γ of G, the element $\rho(\gamma)$ has three eigenvalues, not necessarily distinct, forming a set with repetitions. That set with repetitions is denoted $\eta(\gamma)$.

In the next lemma and the following material, we follow the notation of [Conway et al. 1985] for the conjugacy classes.

LEMMA 11.2. *The eigenvalues of the elements of $\rho(G)$ are as follows*:

γ	*order*	$\eta(\gamma)$
$1A$	1	$\{1,1,1\}$
$2A$	2	$\{1,-1,-1\}$
$3A$	3	$\{1,\omega,\omega^2\}$
$4A$	4	$\{1,i,-i\}$
$7A$	7	$\{\zeta_7,\zeta_7^2,\zeta_7^4\}$
$7B$	7	$\{\zeta_7^3,\zeta_7^5,\zeta_7^6\}$

COROLLARY 11.3. *The identity element of G is the only element fixing all of $\mathbb{P}^2(\mathbb{C})$.*

Elements from the following conjugacy classes $3A$, $4A$, $7A$, *and* $7B$ *have only isolated fixed points on $\mathbb{P}^2(\mathbb{C})$.*

Elements from the remaining conjugacy class S^2 have a fixed line and a fixed point on $\mathbb{P}^2(\mathbb{C})$.

Proof. This is obvious from Lemma 11.2. □

Corollary 11.4. *If a subgroup H of G has a fixed point in $\mathbb{P}^2(\mathbb{C})$, the order of H must be* 1, 2, 3, 4, 6, 7, *or* 8. *A subgroup of order* 6 *has a unique fixed point. A noncyclic subgroup of order* 4 *has* 3 *fixed points. A subgroup of order* 8 *has a unique fixed point. A subgroup of order* 4 *has* 3 *isolated fixed points. A subgroup of order* 7 *has* 3 *isolated fixed points. A subgroup of order* 2 *has a fixed line and a fixed point.*

Proof. A subgroup H of G has a fixed point if and only if the three-dimensional representation of G restricts to a reducible representation of H. The cases where the restriction is irreducible can be found by a straightforward character computation. The remainder are as described in the statement of this corollary. To see that the fixed point is unique in the case of the subgroup of order 6, note that an element of order 2 normalizing an element $3A$ of order 3 sends $3A$ to its inverse and therefore interchanges the two fixed points of $3A$ corresponding to the eigenvalues ω, ω^2. It must leave the entire fixed point set of $3A$ invariant and therefore must fix the fixed point with eigenvalue 1. Therefore, a subgroup of order 6 has a unique fixed point. To see that a noncyclic subgroup H of order 4 has exactly 3 fixed points, note that an element a of order 2 has a fixed point corresponding to the eigenvalue 1 and a fixed line corresponding to the eigenvalue -1. If b is another element of order 2 of H, then b must leave the fixed point set of a invariant. Therefore it fixes the isolated fixed point of a and also has two fixed points of its own on the fixed line of a (it can't fix the fixed line of a since then ab would act as the identity). This proves the assertion about the noncyclic subgroups of order 4. Finally, a subgroup H of order 8 is the normalizer of a cyclic subgroup of order 4. Since an element of order 2 in H not lying in the cyclic subgroup of order 4 sends an element $4A$ of order 4 in H to its inverse, it will interchange the two fixed points of $4A$ corresponding to the eigenvalues $\pm i$. It will also necessarily fix the remaining fixed point of $4A$ corresponding to the eigenvalue 1. This proves the corollary. □

Corollary 11.5. *The group G has orbits of the following orders in its action of $\mathbb{P}^2(\mathbb{C})$*: 21, 24, 28, 42, 56, 84, 168. *The orbits of orders* 21, 24, 28, 42 *and* 56 *are unique. The* 42*-point orbit arises from the conjugacy class of cyclic subgroups of order* 4 *of G. There are ∞^1 orbits of order* 84 *and ∞^2 orbits of order* 168. *The closure of the union of the* 84*-point orbits coincides with the locus $K = 0$, where K is the invariant of degree* 21 *of Klein.*

Proof. From the orders of the subgroups of G having fixed points, we know that the possible orders of orbits are 21, 24, 28, 42, 56, 84, 168. The uniqueness of the orbit of order 21 follows from the fact that a subgroup of order 8 is unique up to conjugacy and has a unique fixed point. A similar argument shows the uniqueness of the orbit of order 28. The uniqueness of the orbit of order 24 follows from that fact that an element $7A$ of order 7 is sent to $7A^2$ and $7A^4$ by the normalizer of the cyclic group it generates. Therefore, the normalizer acts transitively on the three fixed points of $7A$, which correspond to the eigenvalues $\zeta_7, \zeta_7^2, \zeta_7^4$. If H is a cyclic group of order 4, it has 3 fixed points. The fixed point belonging to the eigenvalue 1 is actually fixed by the normalizer of H, as we noted above, and therefore belongs to a 21-point orbit, not a 42-point orbit. The other two fixed points are interchanged by the normalizer of H and give rise to a single 42-point orbit. If H is a Klein 4-group then H has 3 fixed points. Each involution in H has a fixed line and an isolated fixed point. The isolated fixed points of the three involutions are fixed points of H as a whole. However, none of them can lie in a 42-point orbit since the isolated fixed point of an involution is, as we have already noted, in a 21-point orbit. Therefore, the Klein 4-groups contribute no 42-point orbits. That the union of the 84-point orbits has dimension 1 follows from the fact that an element a of order 2 has a fixed line and an isolated fixed point. Furthermore, that isolated fixed point lies on the fixed line of another element $b \neq a$ of order 2 provided b commutes with a. Therefore the union of the 84-point orbits is the same as the union of the fixed lines of the elements of order 2. Since there are 21 elements of order 2 in G, the union of their fixed lines will be the union of 21 lines and invariant under the action of G. The product of the linear forms defining these 21 lines will then be an invariant of degree 21 and must therefore coincide, up to a nonzero scalar factor, with Klein's invariant of degree 21. A cyclic subgroup H of order 3 has 3 fixed points, two of which are exchanged by the normalizer of H and which therefore give rise to a single orbit of order 56. The remaining fixed point is also a fixed point of the normalizer of H and gives rise to a 28-point orbit. Since the orbits of order < 168 consist of a finite number of points and a finite number of lines, there must remain ∞^2 orbits of order 168. ☐

NOTATION 11.6. We will denote by O_d an orbit of order d. When such an orbit is unique, there is no ambiguity in this notation. In order to resolve the ambiguity in the case of the orbits of orders 84 and 168, we can denote an orbit by $O_d(p)$ where p is a point of the orbit.

Our next goal is to give a list of explicit orbit representatives for the orbits of orders 21, 24, 28, 42 and 56. This computation was carried out using REDUCE 3.4 on a personal computer.

LEMMA 11.7. *For $d = 21, 24, 28, 42$, the point p_d given below is a representative of the orbit O_d.*

$$\begin{aligned} p_{21} = \big[&\sqrt{-7}(-\zeta_7^2 - 3\zeta_7 - 3) + 7(\zeta^2 + \zeta_7 + 1), \\ &2\sqrt{-7}(\zeta_7^2 - 1), \\ &\zeta_7(\sqrt{-7}\zeta_7 - \sqrt{-7} + 7\zeta_7 + 7)\big] \end{aligned}$$

$$\begin{aligned}
p_{24} &= [1,0,0]\\
p_{28} &= [1,1,1]\\
p_{42} &= \big[\sqrt{-7}(-i\zeta_7^2-4i\zeta_7+\zeta_7-1-2i)+7(-1+\zeta_7-2i+i\zeta_7^2),\\
&\quad 2\sqrt{-7}(i\zeta_7^2-i\zeta_7+\zeta_7-1),\\
&\quad i\sqrt{-7}(\zeta_7^2+2\zeta_7+4)+\sqrt{-7}(-3\zeta_7-4)+7\zeta(-i\zeta_7+2\zeta_7+1)\big]\\
p_{56} &= [1,\omega,\omega^2]
\end{aligned}$$

PROOF. We will merely explain the method by which these values were obtained. The matrices $\rho(2A)$, $\rho(3A)$, $\rho(4A)$ and $\rho(7A)$ are given in an appendix to this article, where ρ is the three-dimensional representation of G we are considering. As noted above, one obtains a point of O_{21} by taking an eigenvector of $\rho(4A)$ corresponding to the eigenvalue 1. Since the eigenvalues of $\rho(4A)$ are $1, \pm i$, the operator $\rho(4A)^2+1$ maps $\mathbb{C}^3$ into the 1 eigenspace. So we can take

$$p_{21} = (\rho(4A)^2+1)\begin{pmatrix}1\\0\\0\end{pmatrix},$$

which leads to the value for p_{21} obtained above. The 24-point orbit consists of the fixed points of $\rho(7A)$, of which $[1,0,0]$ is clearly one, so we have p_{24}. The 28-point orbit consists of the points belong to the eigenvalue 1 of the elements of order 3 of G. Since cyclic permutation of the coordinates is such an element and since $[1,1,1]$ clearly arises from an eigenvector with eigenvalue 1, we have p_{28}. To obtain p_{42}, we need to find an eigenvector of $\rho(4A)$ with eigenvalue i. The operator $(\rho(4A)-1)(\rho(4A)+i)$ clearly projects onto the i-eigenspace and we obtain p_{42} by applying this operator to the point $[1,0,0]$. The orbit O_{56} arises from the ω- and ω^2-eigenvectors of $\rho(3A)$, a cyclic permutation of the coordinates. So we can take p_{56} to be $[1,\omega,\omega^2]$. □

12. Characterization of the Hessian of Klein's Quartic

Denote by $\mathcal{C}$ the plane curve $x^3y+y^3z+z^3x=0$ and by $\mathcal{H}$ its Hessian.

PROPOSITION 12.1. *The plane curve $\mathcal{H}$ is defined by*

$$5x^2y^2z^2-xy^5-yz^5-zx^5=0,$$

is irreducible, nonsingular, has genus 10 *and admits G as a group of automorphisms.*

PROOF. This is the expression given in (10.1) for the invariant ∇ of degree 6 for G. Since G has no permutation representation of degree < 7, any factor of ∇ is also an invariant. We see from the generating function (10.3) that ∇ is therefore irreducible. For if ∇ were reducible, the only factor it could have would be the invariant of degree 4, whereas 6 is not a multiple of 4. If $\mathcal{H}$ were singular, its singular locus would contain a G-orbit. Since every orbit of G on $\mathcal{P}^2(\mathbb{C})$ has at least 21 points, $\mathcal{H}$ would have at least 21 points of the same multiplicity ≥ 2. By the Plücker formula, an irreducible sextic curve can't have more than 10 singular points. Therefore $\mathcal{H}$ is nonsingular and it follows from the Plücker formula that $\mathcal{H}$ ias genus 10. $\square$

PROPOSITION 12.2. *Let Γ denote the group whose presentation is*

$$\langle a, b, c \mid a^2 = b^4 = c^7 = abc = 1\rangle.$$

Let ϕ be any surjective homomorphism from Γ to G. Then there is an automorphism α of G such that $\alpha \circ \phi(a)$, $\beta \circ \phi(b)$ and $\gamma \circ \phi(c)$ are the elements of G represented respectively by

$$\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 5 \\ 4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

PROOF. We know that $\phi(c)$ must satisfy

$$\phi(c)^7 = 1.$$

If $\phi(c) = 1$ then

$$1 = \phi(abc) = \phi(a)\phi(b),$$

which would imply that the image of ϕ is commutative and that ϕ is not surjective. Therefore $\phi(c)$ has order 7. Since the group of automorphisms of G acts transitively on the set of elements of order 7 in G, we can assume without loss of generality that $\phi(c)$ is the element of G represented by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The same reasoning as before shows that $\phi(a)$ must have order 2. There are 21 elements of order 2 in G and they are all conjugate under the upper triangular subgroup of G. Therefore, $\phi(a)$ is of the form $g\left(\begin{smallmatrix} 0 & 3 \\ 2 & 0 \end{smallmatrix}\right)g^{-1}$, where g is an upper triangular matrix of determinant 1. We can write g in the form

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ commutes with $\phi(c)$, we may after composing ϕ with the inner automorphism determined by $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ suppose without loss of generality that x is 0. Therefore,

$$\phi(a) = \begin{pmatrix} 0 & y^2 \\ y^{-2} & 0 \end{pmatrix}$$

and

$$\phi(b) = \phi(a^{-1}c^{-1}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & y^2 \\ y^{-2} & 0 \end{pmatrix} = \begin{pmatrix} -y^{-2} & y^2 \\ y^{-2} & 0 \end{pmatrix}.$$

If $\phi(b)^2 = 1$, then $\phi(b)$ has trace 0, which is impossible since the trace is $-y^{-2}$. Therefore, $\phi(b)$ has order 4 and its trace is ± 4, whence $y = \pm 3$. This proves the proposition. $\square$

Corollary 12.3. *There is one and only one normal subgroup Δ of Γ such that Γ/Δ is isomorphic to G.*

Proof. Let $A = \left(\begin{smallmatrix} 0 & 3 \\ 2 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 3 & 5 \\ 4 & 0 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ viewed as elements of G. Then $A^2 = B^4 = C^7 = ABC = 1$, so there is a homomorphism $\phi_0 : \Gamma \to G$ such that $\phi_0(a) = A$, $\phi_0(b) = B$ and $\phi_0(c) = C$. Let Δ_0 denote the kernel of ϕ_0. Since A, B, C generate G, the group Γ/Δ_0 is isomorphic to G. This proves the existence. As for the uniqueness, let Δ be any normal subgroup of Γ such that Γ/Δ is isomorphic to G. Then there is a surjective homomorphism $\phi : \Gamma \to G$ whose kernel is Δ. By Proposition 12.2, there is an automorphism α of G such that $\alpha \circ \phi = \phi_0$. Consequently the kernel of ϕ equals the kernel of ϕ_0, so $\Delta = \Delta_0$. $\square$

Theorem 12.4. *There is one and up to isomorphism only one smooth curve of genus* 10 *whose automorphism group contains a group isomorphic to G. Any such curve is equivariantly isomorphic to the plane curve $\mathcal{H}$.*

Proof. The last assertion follows from the first combined with Proposition 12.1. The same proposition shows that such curves exist, so we only have to prove that any two such curves are isomorphic. Let S be a compact Riemann surface of genus 10 on which G acts nontrivially. Let S' denote the orbit space for this action of G on S. Then S' naturally has the structure of a compact Riemann surface. Let g denote the genus of S'. For every point $\mathcal{P}$ of S let $e_{\mathcal{P}}$ denote the order of the stabilizer of $\mathcal{P}$ in G. If γ is any element of G then the stabilizer of $\mathcal{P}$ is conjugate via γ to the stabilizer of $\gamma \cdot \mathcal{P}$. Therefore $e_{\mathcal{P}}$ depends only on the orbit of $\mathcal{P}$ under G. If p is the point of S' representing that orbit, we may also write e_p for $e_{\mathcal{P}}$.

Using the Riemann–Hurwitz relation between the Euler characteristic of S and that of S', we have

$$-18 = 2 - 2(10) = 168(2 - 2g) - \sum_{\mathcal{P} \in S} (e_{\mathcal{P}} - 1).$$

If we group together all the terms in the summation which belong to the same orbit and instead sum over S', we obtain

$$-18 = 168(2 - 2g) - \sum_{p \in S'} \frac{168}{e_p}(e_p - 1) = 168(2 - 2g) - 168 \sum_{p \in S'} \left(1 - \frac{1}{e_p}\right).$$

Dividing through by -168, we get

$$\frac{3}{28} = 2g - 2 + \sum_{p \in S'} \left(1 - \frac{1}{e_p}\right).$$

Since the left-hand side is not an integer, the set of $p \in S'$ such that $e_p > 1$ must be nonempty and, of course, finite. Denote by n the number of points p of S'

with $e_p > 1$ and call these points $p_1, \ldots, p_n$. Also, we will write e_i for e_p when $p = p_i$. Then we have

$$\frac{3}{28} = 2g - 2 + \sum_{i=1}^{n} \left(1 - \frac{1}{e_i}\right).$$

If $g \geq 1$, the right-hand side of this equation is at least $\frac{1}{2}$, and so greater than $\frac{3}{28}$. Therefore $g = 0$. Furthermore, 7 divides the denominator of the left-hand side, so one of the e_i must be a multiple of 7. However, the stabilizer of any point of S must be cyclic, so actually one of the e_i equals 7, say, $e_1 = 7$. Then we have

$$\frac{5}{4} = \sum_{i=2}^{n} \left(1 - \frac{1}{e_i}\right).$$

If $n > 3$, the right-hand side is at least $\frac{3}{2}$, and so greater than $\frac{5}{4}$; hence $n \leq 3$. If $n < 3$, the right-hand side is less than 1, and so less than $\frac{5}{4}$; hence $n = 3$. Therefore

$$\frac{1}{e_2} + \frac{1}{e_3} = \frac{3}{4}.$$

If e_2, e_3 are both > 2, the left-hand side is at most $\frac{2}{3}$, and so less than $\frac{3}{4}$; hence one of e_2, e_3 is 2, say, $e_2 = 2$, and then $e_3 = 4$.

This proves that S is isomorphic to a Galois covering of $\mathbb{P}^1(\mathbb{C})$ with Galois group G and branched at exactly 3 points, the orders of branching being 2,4 and 7. The three points may, after applying a suitable projective transformation of $\mathbb{P}^1(\mathbb{C})$, be taken to be 0, 1 and ∞ respectively. The fundamental group Π of $\mathbb{P}^1(\mathbb{C})$ with these 3 points removed may be presented as a free group on the letters a, b, c with the relation $abc = 1$. Any branched cover of $\mathbb{P}^1(\mathbb{C})$ branched only at $0, 1, \infty$ to orders 2,4,7 respectively corresponds uniquely to a subgroup Δ of the group Π modulo the relations $a^2 = b^4 = c^7 = abc = 1$, that is to say, of the group Γ defined in Propostion 12.2. Furthermore, if such a cover is Galois then Δ is a normal subgroup and the Galois group is isomorphic to Γ/Δ. In our case, let Δ correspond to the branched cover $S \to S' = \mathbb{P}^1(\mathbb{C})$, so that Γ/Δ is isomorphic to G. Then Δ is the kernel of a surjective homomorphism from Γ to G. By Corollary 12.3, there is one and only one normal subgroup Δ' of Γ such that Γ/Δ' is isomorphic to G. It folows that there is only one possibility for Δ and, since Δ determines the branched cover, only one possibility for S. This completes the proof of the theorem. $\square$

Corollary 12.5. *Hirzebruch's curves F_2 and F_4 are birationally equivalent to the Hessian of Klein's quartic.*

Corollary 12.6. *The curve $\mathcal{H} : xy^5 + yz^5 + zx^5 - 5x^2y^2z^2$ arises from an arithmetic subgroup of* $\mathrm{SL}_2(\mathbb{R})$.

Proof. We know that $\mathcal{H}$ arises from a subgroup of the $\{2, 4, 7\}$ triangle group, which was shown to be arithmetic by Fricke [1893b]. $\square$

Remark 12.7. The fact that $\mathcal{H}$ is the normalization of F_2 and F_4 and the explicit definition of these curves in terms of congruence subgroups of unit groups of rational quaternion algebras also implies that the curve $\mathcal{H}$ arises from arithmetic groups. However, by construction, the curves F_2 and F_4 also arise from congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Remark 12.8. While browsing through old journals recently, we found the article [Fricke 1893a]. From it, we learned that not only did he know that the $\{2,4,7\}$ triangle group is arithmetic but he also anticipated most of the results of this section one century ago. Time and space do not allow a detailed discussion of Fricke's papers here, but we will return to them in [Adler $\geq$ 1998].

Remark 12.9. Dolgachev has kindly pointed out that Theorem 12.4 also follows from Corollary 14.7 and Lemma 14.1.

13. The Jacobian Variety of the Hessian

Lemma 13.1. *The Jacobian variety of $\mathcal{H}$ is G-equivariantly isogenous to the product of an abelian variety A of dimension 3 and an abelian variety B of dimension 7, both of which admit G as an automorphism group. Furthermore, A is isomorphic to the Jacobian variety of the Klein curve $\mathcal{C}$ and therefore to a product of 3 copies of the elliptic curve $\mathbb{C}/L$, where L is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-7})$, and B is isogenous to the sum of 7 copies of the elliptic curve*

$$PQR = S^3, \quad P+Q+R = 5S.$$

Proof. The plane cubics give the adjoint system for $\mathcal{H}$. Explicitly, every holomorphic differential on $\mathcal{H}$ is obtained by taking the residue along $\mathcal{H}$ of a rational differential of the form

$$\frac{E}{5x^2y^2z^2 - xy^5 - yz^5 - zx^5}\left(\frac{dy}{y}\wedge\frac{dz}{z}+\frac{dz}{z}\wedge\frac{dz}{x}+\frac{dx}{x}\wedge\frac{dy}{y}\right),$$

where E runs over the space of all cubic forms in x, y, z. Consequently, the action of G on the space of holomorphic differentials of $\mathcal{H}$ may be identified with the action of G on the space of ternary cubics. Since the polars of points of $\mathbb{P}^2(\mathbb{C})$ with respect to $\mathcal{C}$ form a covariant system of cubics, it follows that there is a 3-dimensional irreducible space V of cubic forms generated by the partials of $x^3y + y^3z + z^3x$. The ten-dimensional representation is therefore the sum of an irreducible representation of degree 3 and a representation of degree 7. Since the degrees of the irreducible representations of G are 1, 3, 6, 7, 8, the seven-dimensional space W of cubics can be reducible only if it contains the trivial representation. Since there is no G-invariant cubic, it follows that the invariant seven-dimensional space of differentials is irreducible. This implies at once that the Jacobian variety $J(\mathcal{H})$ of $\mathcal{H}$ contains an abelian variety A of dimension 3 invariant under G and an abelian variety B of dimension 7 invariant under G.

It is well known that A must be the Jacobian of $\mathcal{C}$. In [Adler 1981] one can find a proof of this and of the fact that $J(\mathcal{C})$ is isomorphic to the product of three copies of $\mathbb{C}/L$. Next consider the mapping $\lambda : \mathcal{H} \to \mathbb{P}^3$ given by

$$\lambda([x, y, z]) = [xy^5, yz^5, zx^5, x^2y^2z^2] = [P, Q, R, S].$$

Then $\lambda \circ g = \lambda$, where g is the collineation

$$[x, y, z] \mapsto [\zeta x, \zeta^4 y, \zeta^2 z]$$

of $\mathbb{P}^2(\mathbb{C})$. The image of λ is the elliptic curve

$$PQR = S^3, \quad P + Q + R = 5D,$$

which we will denote $\mathcal{E}$. We therefore have a surjective mapping of $J(\mathcal{H})$ onto $J(\mathcal{E})$ invariant under g, and an elliptic curve $\mathcal{E}'$ in $J(\mathcal{H})$ which maps onto $J(\mathcal{E})$. Durthermore, g must leave $\mathcal{E}'$ pointwise fixed, so $\mathcal{E}' \subseteq B$. Since the action of G on the tangent space of B is irreducible, it follows that B is isogenous to the sum of 7 copies of $\mathcal{E}'$, or what is the same, of $\mathcal{E}$, since $\mathcal{E}'$ and $\mathcal{E}$ are isogenous. □

It is natural to wonder whether $J(\mathcal{H})$ is actually G-equivariantly isomorphic to $A \oplus B$, not merely isogenous. That will be investigated in a sequence of lemmas. If r is a prime number, we will denote by I_r the group of points of order r in $A \cap B$. Then I_r is invariant under G and is a vector space over $\mathbb{F}_r$, so I_r gives a modular representation of G. The representations of G on the fundamental groups $\pi_1(A)$ and $\pi_1(B)$ are integral representations of degrees 6 and 14 respectively. Their extensions to $\mathbb{C}$ decompose as

$$\pi_1(A) \otimes \mathbb{C} \cong V \oplus V^*, \quad \pi_1(B) \otimes \mathbb{C} \cong W \oplus W^* \cong W \oplus W,$$

since $W \cong W^*$. We can reduce the integral represenations on $\pi_1(A)$ and $\pi_1(B)$ modulo r and then obtain modular representations each of which has a submodule isomorphic to I_r. The set of irreducible representations of G over the algebraic closure $\bar{\mathbb{F}}_r$ which occur as composition factors of $\pi_1(A) \otimes \bar{\mathbb{F}}_r$ will be denoted $\mathcal{A}_r$, while the set of those occurring as composition factors of $\pi_1(B) \otimes \bar{\mathbb{F}}_r$ will be denoted $\mathcal{B}_r$. Denote by $\mathcal{C}_r$ the intersection of $\mathcal{A}_r$ and $\mathcal{B}_r$. The composition factors of G acting on $I_r \otimes \bar{\mathbb{F}}_r$ must lie in $\mathcal{C}_r$.

Lemma 13.2. *If r does not divide the order of G then $I_r = (0)$.*

Proof. If r does not divide the order of G, the representations V and W remain irreducible modulo r. Therefore the sets $\mathcal{A}_r$ and $\mathcal{B}_r$ are disjoint, which implies $\mathcal{C}_r$ is empty and $I_r \otimes \bar{\mathbb{F}}_r$ has no composition factors. Therefore $I_r = 0$. □

Lemma 13.3. $I_7 = (0)$.

Proof. It is well known that that both the seven-dimensional representation W and the three-dimensional representations V, V^* remain irreducible modulo 7. Therefore, $\mathcal{C}_7$ is empty and $I_7 = (0)$. □

Lemma 13.4. $I_3 = (0)$.

Proof. We will show that the irreducible representation of degree 7 remains irreducible modulo 3. Since 7 is prime, this will imply that the representation is absolutely irreducible and therefore that $\mathcal{C}_3$ is empty and $I_3 = (0)$. Since 3 is a primitive root modulo 7, G has no nontrivial representation of degree < 6 over $\mathbb{F}_3$. Therefore, since W is nontrivial modulo 3, it is either irreducible modulo 3 or else has composition factors of degrees 1 and 6. Since W is self-dual, the reduction of W, if reducible, would contain the trivial representation. An explicit model of the representation is given by the functions on $\mathbb{P}^1(\mathbb{F}_7)$ with values in $\mathbb{F}_3$ and whose sum over $\mathbb{P}^1(\mathbb{F}_7)$ is 0. Such a function cannot be G-invariant, so the representation of dimension 7 is irreducible modulo 3 and we are done. □

Lemma 13.5. *We have* $\mathcal{B}_2 = \{V, V^*, 1\}$ *and* $\mathcal{A}_2 = \mathcal{C}_2 = \{V, V^*\}$.

Proof. The representation V is irreducible in all characteristics. Therefore the second statement follows from the first. To prove the first, we use an explicit model for the seven-dimensional representation modulo 2, namely the space X of all functions on $\mathbb{P}^1(\mathbb{F}_7)$ with values in $\mathbb{F}_2$ and whose sum over $\mathbb{P}^1(\mathbb{F}_7)$ is 0. The constant functions lie in X and form an invariant subspace Y of dimension 1. We identify $\mathbb{F}_2$-valued functions with subsets of $\mathbb{P}^1(\mathbb{F}_7)$. Modulo constant functions, this means that every subset is identified with its complement. It is not difficult to show that Y is reducible. Indeed, the orbit of the subset $\{\infty, 3, 5, 6\}$ under G consists of 7 subsets up to complements and these together with the empty set form a three-dimensional subspace of Y. □

We summarize the results of these lemmas in the following corollary.

Corollary 13.6. *The kernel of the natural* G*-equivariant homomorphism of* $A \oplus B$ *onto* $J(\mathcal{H})$ *induced by addition is a finite* 2*-group. The elements of order* 2 *in that group form a group* G*-equivariantly isomorphic to* I_2*, which is of order* 1, 8 *or* 64 *and whose composition factors lie in the set consisting of the natural three-dimensional representation of* $GL_3(\mathbb{F}_2)$ *on* $\mathbb{F}_2^3$ *and its contragredient representation.*

Remark 13.7. The discussion so far does not settle the question of the existence of nontrivial elements of order 2 fixed by G. We will show in Lemma 14.1 that there is a G-invariant element of order 2 in B. We also note that we have left open the precise determination of the group I_2 as well as the question of the existence of points of order 2^n in $A \cap B$ with $n \geq 2$. I am informed by Fred Diamond that one can use the methods of Ribet [1983] to determine intersections of invariant abelian subvarieties of the Jacobian varieties of modular curves. It seems reasonable to expect that these methods could also adapt to the case of curves arising from arithmetic groups with compact quotient. Since the curve $\mathcal{H}$ is such a curve, as well as a curve arising from arithmetic subgroups of finite index in $\mathrm{SL}_2(\mathbb{Z})$, we can perhaps expect a precise determination of $A \cap B$ from these methods.

For later use, we also need the following result.

LEMMA 13.8. *Let X be a curve of genus* 10 *which admits G as an automorphism group. Then any equivariant rational mapping of X onto another curve on which G acts is either birational or else maps X onto a rational curve on which G acts trivially.*

PROOF. By Theorem 12.4 and Lemma 13.1, the representation of G on the holomorphic differentials of X is the sum of an irreducible representation of degree 3 and an irreducible representation of degree 7. It follows that if Y is a smooth irreducible complete curve on which G acts and if $\phi : X \to Y$ is a G-equivariant morphism then the genus g of Y can only be 0, 3, 7 or 10. If $g = 10$ then f is necessarily birational. If $g = 0$ then G necessarily acts trivially on Y. It remains to show that the cases $g = 3, 7$ don't occur.

Suppose $g = 3$. It is well known (see [Hecke 1935], for example) that Y must be isomorphic to the Klein curve. Let p be a point of X fixed by an element γ of order 4 of G. Then $\phi(p)$ must be a point of Y which is also fixed by γ. However, an element of order 4 has no fixed points on the Klein curve. So we cannot have $g = 3$.

To show that $g \neq 7$, it suffices to observe that in general there can be no nonconstant mapping of a curve of genus 10 onto one of genus 7. Indeed, the degree n of such a mapping would have to be at least 2 and by the Riemann–Hurwitz relation we would then have

$$-18 = 2 - 2\cdot 10 \leq n\cdot(2 - 2\cdot 7) \leq -24,$$

which is impossible. (I am indebted to Noam Elkies for this observation, which greatly simplified the argument.) □

14. Invariant Line Bundles on the Hessian

Denote by $\mathcal{L}$ the group of isomorphism classes of G-invariant line bundles on the Hessian curve $\mathcal{H}$ and denote by $\mathcal{L}_0$ the subgroup of $\mathcal{L}$ represented by invariant line bundles of degree 0. The quotient group $\mathcal{L}/\mathcal{L}_0$ is the group of all integers which occur as the degrees of G-invariant line bundles and will be denoted $\partial_{\mathcal{L}}$. Similarly, we denote by $\mathcal{M}$ the subgroup of $\mathcal{L}$ whose elements correspond to G-invariant divisors on the curve $\mathcal{H}$. We denote by $\mathcal{M}_0$ the subgroup of $\mathcal{M}$ consisting of elements of degree 0, so that

$$\mathcal{M}_0 = \mathcal{M} \cap \mathcal{L}_0.$$

The subgroup of $\partial_{\mathcal{L}}$ represented by elements of $\mathcal{M}$ is denoted $\partial_{\mathcal{M}}$.

LEMMA 14.1. *The group $\mathcal{M}_0$ is cyclic of order* 2 *and lies in the G-invariant abelian subvariety B of $J(\mathcal{H})$. In particular B (and a fortiori $J(\mathcal{H})$) has a G-invariant element of order* 2. *The group $\partial_{\mathcal{M}}$ is equal to* $6\mathbb{Z}$. *The group $\mathcal{M}$ is the product of $\mathcal{M}_0$ and an infinite cyclic group.*

Proof. Denote by (p_2), (p_4) and (p_7) the G orbits on $\mathcal{H}$ of orders 84, 42 and 24 respectively. Identifying these orbits with the divisors they determine, it is easy to see, using the fact that the orbit space for G acting on $\mathcal{H}$ is $\mathbb{P}^1(\mathbb{C})$, that $2(p_2)$, $4(p_4)$ and $7(p_7)$ are linearly equivalent to each other and to any 168-point orbit. Therefore, $\mathcal{M}$ is generated by the line bundles associated to (p_2), (p_4) and (p_7). Those line bundles will be denoted ξ_2, ξ_4 and ξ_7 respectively. A line bundle represents an element of $\mathcal{M}_0$ if and only if it is of the form

$$\xi_2^a \xi_4^b \xi_7^c$$

where

$$84a + 54b + 24c = 0.$$

By solving this diophantine equation for a, b, c, we see that $\mathcal{M}_0$ is generated by $\xi_2\xi_4^{-2}$ and $\xi_4^4\xi_7^{-7}$. However, the line bundle $\xi_4^4\xi_7^{-7}$ is trivial since $7(p_7)$ and $4(p_4)$ are linearly equivalent. Therefore, $\mathcal{M}_0$ is a cyclic group generated by $\xi_2\xi_4^{-2}$. Since $2(p_2)$ is linearly equivalent to $4(p_4)$, the square of the line bundle $\xi_2\xi_4^{-2}$ is trivial. Therefore, $\mathcal{M}_0$ has order 1 or 2. Suppose the order is 1. Then (p_2) is linearly equivalent to $2(p_4)$. Let f be a rational function on $\mathcal{H}$ whose divisor is $2(p_4) - (p_2)$. Since the divisor $2(p_4) - (p_2)$ is G-invariant, the scalar multiples of f form a one-dimensional representation space for G. Since G is a simple group, that one-dimensional representation must be trivial. Therefore, f is fixed by every element of G and therefore is really a rational function on the orbit space $\mathbb{P}^1(\mathbb{C})$ for G acting on $\mathcal{H}$. In particular, the divisor of f is the preimage under the quotient mapping $\mathcal{H} \to \mathcal{H}/G = \mathbb{P}^1(\mathbb{C})$ of a divisor on $\mathbb{P}^1(\mathbb{C})$. Since the divisor is supported on the 42-point orbit and the 84-point orbit, it would then have to be of the form

$$2s(p_2) + 4t(p_4),$$

and that is a contradiction. Therefore $\mathcal{M}_0$ is cyclic of order 2. As for the group ∂_M, it is clearly generated by the greatest common divisor of the orders of the possible G-orbits on $\mathcal{H}$, i.e. of 24, 42, 84 and 168, which is 6. This proves the lemma. □

Lemma 14.2. *Let H be a finite group acting on an algebraic curve X over the field $\mathbb{C}$ of complex numbers. Let $Y = X/H$ be the orbit space for the action of H on X and assume that Y is of genus 0. Denote by $\mathcal{L}_X$ the group of isomorphism classes of invariant line bundles on X and by $\mathcal{M}_X$ the subgroup of $\mathcal{L}_X$ consisting of elements represented by line bundles associated to H-invariant divisors. Then the quotient group $\mathcal{L}_X/\mathcal{M}_X$ is isomorphic to a subgroup of the group of Schur multipliers of H. In particular, the index of $\mathcal{M}_X$ in $\mathcal{L}_X$ divides the order of the group of Schur multipliers of H.*

Proof. Denote by K_X the function field of X and by $K_X^\times$ the multiplicative group of K_X. Let $A = K_X^\times/\mathbb{C}^\times$, let B denote the group of all divisors on X and let C denote the group of all isomorphism classes of line bundles on X. Then we have the exact sequence

$$1 \to A \to B \to C \to 1.$$

From the associated long exact cohomology sequence of G, we have

$$1 \to A^H \to B^H \to C^H \to H^1(H, A).$$

From the exact sequence

$$1 \to \mathbb{C}^\times \to K^\times \to A \to 1,$$

we deduce that the sequence

$$H^1(H, K^\times) \to H^1(H, A) \to H^2(H, \mathbb{C}^\times)$$

is exact. However, $H^1(H, K^\times)$ is trivial by Hilbert's Theorem 90 and $H^2(H, \mathbb{C}^\times)$ is the group of Schur multipliers of H. Since $\mathcal{L}_X = C^H$ and since $\mathcal{M}_X$ is the image of B^H in $\mathcal{L}_X$, it follows that $\mathcal{L}_X/\mathcal{M}_X$ is isomorphic to a subgroup of the group of Schur multipliers of H. □

Originally, the following lemma merely asserted that $\mathcal{M}$ has index at most 2 in $\mathcal{L}$. I am greatly indebted to Dolgachev for communicating the proof of this stronger and more satisfactory form of the lemma.

Lemma 14.3. *The index of* $\mathcal{M}$ *in* $\mathcal{L}$ *is equal to* 2. *Furthermore, any torsion element of* $\mathcal{L}$ *lies in* $\mathcal{M}$. *In other words,* $\mathcal{L}_0 = \mathcal{M}_0$.

Proof. That the index is at most 2 follows at once from the preceding lemma and from the well known fact that the group of Schur multipliers of $\mathrm{PSL}(2, 7)$ is cyclic of order 2. Dolgachev pointed out that in the preceding lemma the homomorphism $\mathrm{Pic}(X(p))^G \to H^2(G, \mathbb{C}^\times)$ is in fact surjective. We will present his argument in Lemma 14.4 below. Assuming this result, it follows that $\mathcal{L}/\mathcal{M}$ is a cyclic group of order 2. Let ξ be an invariant line bundle of degree 0 on $\mathcal{H}$. If ξ doesn't represent an element of $\mathcal{M}_0$ then the group $\mathrm{PSL}_2(\mathbb{F}_7)$ does not act on the bundle ξ. Instead the group $\mathrm{SL}_2(\mathbb{F}_7)$ will act and, in particular, the nontrivial element of the center of $\mathrm{SL}_2(\mathbb{F}_7)$ will act as -1 on the space of sections of ξ. On the other hand, since ξ is a nontrivial line bundle of degree 0, the Riemann–Roch theorem implies that the space of sections of ξ has dimension 9. However, all irreducible representations of $\mathrm{SL}_2(\mathbb{F}_7)$ in which the center acts nontrivially are of even dimension, which contradicts the fact that the space of sections must decompose into a direct sum of such representations. □

We now present Dolgachev's proof of the following lemma.

Lemma 14.4. *Let* X *be a curve and* $G \subseteq \mathrm{Aut}(X)$ *be a perfect group of automorphisms of* X. *Then the natural mapping of* $\mathrm{Pic}(X)^G$ *to* $H^2(G, \mathbb{C}^\times)$ *is surjective.*

Proof. The argument depends on two spectral sequences given in [Grothendieck 1957], namely

$$'E_2^{p,q} = H^p(G, H^q(X, \mathcal{O}_X^\times)) \Rightarrow H^{p+q}(G; X, \mathcal{O}_X^\times)$$
$$''E_2^{p,q} = H^p(G\backslash X, R^q\pi_*(\mathcal{O}_X^\times)) \Rightarrow H^{p+q}(G; X, \mathcal{O}_X^\times)$$

Here, as A runs over the category of G-sheaves of abelian groups, the functor $H^n(X; G, A)$ is the n-th right derived functor of the functor which associates to such a sheaf A its group of G-invariant sections. Also, π denotes the natural mapping of X onto $G\backslash X$. From these spectral sequences, one derives the following exact sequences [Grothendieck 1957, p. 201]:

$$0 \to H^1(Y, \mathcal{O}_X^{\times G}) \to H^1(X; G, \mathcal{O}_X^\times) \to H^0(Y, H^1(G, \mathcal{O}_X^\times)) \to H^2(Y, \mathcal{O}_X^{\times G}) \to H^2(X; G, \mathcal{O}_X^\times)$$
$$0 \to H^1(G, \mathbb{C}^\times) \to H^1(X; G, \mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times)^G \to H^2(G, \mathbb{C}^\times) \to H^2(X; G, \mathcal{O}_X^\times)$$

The first spectral sequence has $E_2^{2,0} = E_2^{1,1} = E_2^{0,2} = 0$, which implies

$$H^2(G; X, \mathcal{O}_X^\times) = 0.$$

From this and the fact that $H^1(G, \mathbb{C}^\times) = 0$, we derive from the first exact sequence:

$$0 \to H^1(G; X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times)^G \to H^2(G, \mathbb{C}^\times) \to 1.$$

The group $H^1(X, \mathcal{O}_X^\times)^G$ is the group $\mathcal{L}$ and the group $H^1(G; X, \mathcal{O}_X^\times)$ is the group of line bundles on X with G-action. Such line bundles are precisely those arising from G-invariant divisors, so this group may be identified with the group $\mathcal{M}$. This proves the lemma. □

Corollary 14.5.

$$\partial_{\mathcal{L}} = 3\mathbb{Z}, \quad \mathcal{L}_0 = \mathbb{Z}/2\mathbb{Z}.$$

Proof. We know that $\mathcal{M}_0 = \mathbb{Z}/2\mathbb{Z}$ and $\partial_{\mathcal{M}} = 6\mathbb{Z}$. We also know that

$$\mathcal{L}/\mathcal{M} = \partial_{\mathcal{L}}/\partial_{\mathcal{M}} \oplus \mathcal{L}_0/\mathcal{M}_0 = \mathbb{Z}/2\mathbb{Z}.$$

We also know from the preceding lemma that $\mathcal{L}_0 = \mathcal{M}_0$. Therefore

$$\partial_{\mathcal{L}} = 3\mathbb{Z}, \quad \mathcal{L}_0 = \mathbb{Z}/2\mathbb{Z}.$$ □

Remark 14.6. It would be interesting to know something about the geometry of an embedding of $\mathcal{H}$ associated to a line bundle lying in $\mathcal{L}$ but not in $\mathcal{M}$. For example, by using contact quintics[1] of $\mathcal{H}$, one can map it to a curve of degree 15 in $\mathbb{P}^5$, where the action of $\mathrm{SL}_2(\mathbb{F}_7)$ on $\mathbb{P}^5$ is derived from an irreducible representation of degree 6 of $\mathrm{SL}_2(\mathbb{F}_7)$ in which the center acts nontrivially. It would probably not be difficult to write down the equations of the curve of degree 15 explicitly.

[1] A contact quintic of the sextic is one which is tangent to the sextic at every point where it meets the sextic. More precisely, it is a quintic which cuts out a divisor divisible by 2 in the group of all divisors on the sextic.

Corollary 14.7. *There are precisely two G-invariant curves of degree* 18 *and genus* 10 *in* $\mathbb{P}^2$.

Proof. Let $\mathcal{C}$ be such a curve and let $\phi : \mathcal{H} \to \mathcal{C}$ be the unique G-invariant mapping of $\mathcal{H}$ to $\mathcal{C}$. Let $\xi = \phi^*\mathcal{O}(1)$, where we denote by $\mathcal{O}(1)$ here the restriction to $\mathcal{C}$ of the ample generator of the Picard group of $\mathbb{P}^2(\mathbb{C})$. Then ξ is an invariant line bundle of degree 18 on $\mathcal{H}$. It follows from Corollary 14.5 and its proof that there are precisely two possibilities for ξ: either $\xi = K$ or $\xi = K \otimes \xi_0$, where K is the canonical line bundle of $\mathcal{H}$ and ξ_0 is the unique G-invariant line bundle of order 2 and degree 0. By the Wood's Hole Fixed Point Formula, we can write the Euler character of G on the cohomology of ξ as

$$\chi(\gamma; \xi) = \operatorname{tr}(\gamma \,|\, H^0(\xi)) - \operatorname{tr}(\gamma \,|\, H^1(\xi)) \sum \frac{\operatorname{tr}(\gamma \,|\, \xi_x)}{1 - d\gamma_x}$$

for γ in G, where the summation on the right runs over the fixed points x of γ on $\mathcal{H}$ and where $d\gamma_x$ and $\operatorname{tr}(\gamma \,|\, \xi_x)$ denote the scalar by which γ acts on the fibres K_x, ξ_x of K, ξ respectively at x. If $\xi = K$, we already know this Euler character and its decomposition from the proof of Lemma 13.1. It is $\chi_3' + \chi_7 - 1$. Now, the line bundle $\xi_0 = \xi_2\xi_4^{-2}$ is associated to the G-invariant divisor $(p_2) - 2(p_4)$. Therefore the scalar $\operatorname{tr}(\gamma \,|\, \xi_x)$ at a fixed point x of γ will be the same for both K and $K\xi_0$ except when γ has order i and x belongs to (p_i), with $i = 2, 4$. Leaving these traces undetermined for the moment, let $e_1 = \chi(\gamma; K\xi_0)$ where γ has order 4 and let $e_2 = \chi(\gamma; K\xi_0)$ where γ has order 2. For the remaining values we have $\chi(\gamma; K) = \chi(\gamma; K\xi_0)$. Therefore

$$K(\gamma; K\xi_0) = \begin{cases} 9 & \text{if } \gamma = 1A, \\ e_2 & \text{if } \gamma = 2A, \\ 0 & \text{if } \gamma = 3A, \\ e_1 & \text{if } \gamma = 4A, \\ \frac{1}{2}(-3 + \sqrt{-7}) & \text{if } \gamma = 7A, \\ \frac{1}{2}(-3 - \sqrt{-7}) & \text{if } \gamma = 7B. \end{cases}$$

On the other hand, we know from the proof of Lemma 14.3 that $H^1(K\xi_0) = 0$, so the Euler character of $K\xi_0$ is actually the character of G on sections of $K\xi_0$. Since an element of order 2 of G has at most 4 fixed points on $\mathcal{H}$, it follows that $|e_2| \leq 2$. The multiplicity of χ_6 in the Euler character is then $(e_2 + 3)/4$. Since the multiplicity must be an integer, we must have $e_2 = 1$ and the multiplicity of χ_6 is 1. Similarly, we have $|e_1| \leq 2$ and the multiplicity of χ_3' is $(e_1 + 3)/4$. Therefore, $e_1 = 1$ and the multiplicity of χ_3' is 1. Hence, the Euler character of $K\xi_0$ decomposes as $\chi_3' + \chi_6$. The mapping ϕ determines an embedding ϕ_x of the representation space of χ_3' into $H^0(\xi)$. Since χ_3' occurs with multiplicity 1 in both $H^0(K)$ and $H^0(K\xi_0)$, the embedding ϕ_ξ is uniquely determined up to a scalar factor for each line bundle ξ. Converely, ϕ_ξ determines the mapping ϕ and its image $\mathcal{C}$. This proves the lemma. $\square$

REMARK 14.8. The techniques of [Dolgachev $\geq$ 1998] apparently do not apply directly to the study of G-invariant vector bundles on $\mathcal{H}$ since $\mathcal{H}$ is associated to a triangle group (2,4,7) and 2,4 are not relatively prime. But it is reasonable to hope that Dolgachev's methods can be extended to handle this case.

15. Identification of the Curve F_2 of Degree 12

The *Hessian* $\mathcal{H}$ is the locus of points in the plane whose polar conics with respect to the Klein curve $\mathcal{C}$ are pairs of lines. These lines are always distinct since $\mathcal{H}$ is nonsingular. The *Steinerian* of $\mathcal{C}$ is the locus of the point where these two lines meet. Denote the Steinerian by $\mathcal{S}$. There is a morphism ι from $\mathcal{H}$ to $\mathcal{S}$ which associates to the point p of $\mathcal{H}$ the singular point $\iota(p)$ of the polar conic of p with respect to $\mathcal{C}$.

The following result is due to Fricke [1893a, p. 386, eq. (3)]. It has been rediscovered in modern times by Dolgachev and Kanev [1993, p. 256, Ex. 6.1.1] and independently by the author.

LEMMA 15.1. *The degree of* $\mathcal{S}$ *is* 12 *and its equation is*

$$4f^3 + \nabla^2 = 0. \tag{15.2}$$

Furthermore, S *has* 45 *double points. These consist of the* 24*-point orbit and the* 21*-point orbit, each taken once. The points of the* 24*-point orbit are cusps and those of the* 21*-point orbit are nodes.*

PROOF. The graph of ι consists of all pairs $(p, q) \in \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ such that $M_p \cdot q = 0$, where M_p is the matrix of second partials of f at p. Consequently the graph is defined by 3 bihomogeneous equations of bidegree $(2, 1)$. Denote by h the generator of the cohomology group $H^2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$ and for $i = 1, 2$ denote by h_i the pullback of h via the projection π_i of $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ onto its i-th factor. Then the graph of ι is Poincaré dual to

$$(2H_1 + h_2)^3 = 12h_1^2 h_2 + 6h_1 h_2^2.$$

The Steinerian $\mathcal{S}$ is simply the projection of the graph of ι onto the second factor and is Poincaré dual, at least as a cycle in $\mathbb{P}^2(\mathbb{C})$, to

$$(\pi_2)_*(2H_1 + h_2)^3 = 12h_2,$$

so $\mathcal{S}$ has degree ≤ 12. Since $\mathcal{S}$ is an irreducible invariant curve, we know from Klein's determination of the ring of invariants that $\mathcal{S}$ is either of degree 12 or else is either $\mathcal{H}$ or $\mathcal{C}$. By Lemma 13.7, $\mathcal{H}$ doesn't map onto $\mathcal{C}$, so $\mathcal{S} \neq \mathcal{C}$. If $\mathcal{S} = \mathcal{H}$, then ι would be a G-equivariant automorphism of $\mathcal{H}$, hence the identity. But it is easy to check that

$$\iota([1, 0, 0]) \neq [1, 0, 0].$$

This proves that $\mathcal{S}$ has degree 12. The general invariant of degree 12 is of the form

$$af^3 + b\nabla^2.$$

Since S has genus 10, it must have 45 multiple points, counted with their multiplicities. Since the 24-point orbit is the intersection of $\mathcal{C}$ and $\mathcal{H}$, it is clear that these are double points. The remaining 21 points form the 21-point orbit to which Hirzebruch refers [1977, p. 319]. Since neither $\mathcal{C}$ nor $\mathcal{H}$ passes through the 21-point orbit, there is only one invariant curve of degree 12 pasing through the 21-point orbit and that is $\mathcal{S}$. One of the 21 points is $[-y-z, y, z]$, where y and z are given by

$$y = \frac{\gamma^3 - \gamma^4}{\sqrt{-7}}, \quad z = \frac{\gamma^6 - \gamma}{\sqrt{-7}}. \tag{15.3}$$

By requiring that $\mathcal{S}$ pass through this point, we find that $\mathcal{S}$ is given by

$$4f^3 + \nabla^2 = 0.$$

To see that the 21-point orbit is double on $\mathcal{S}$, note that each cyclic group of order 4 in G has 3 fixed points, 2 of which lie on $\mathcal{H}$. The mapping ι sends the two on $\mathcal{H}$ to the remaining fixed point, which creates a double point on $\mathcal{S}$. More precisely, it has a node there since there are two points of the normalization $\mathcal{H}$ corresponding to it. As for the 24-point orbit, it is clear from the expression $4f^3 + \nabla^2$ that there is only one tangent at a point of the 24-point orbit, namely the tangent to $\mathcal{H}$ at that point, since $4f^3$ vanishes to third order at the point and ∇^2 only to second order. □

16. Identification of the Curve F_4 of Degree 18

In Corollary 14.7, we showed that there are exactly two irreducible G-invariant plane curves of degree 18 and genus 10. In this section, we will describe them in more detail. As a first step in studying such curves, we note that there are 3 linearly independent invariants of degree 18 for G, namely fC, ∇^3 and $f^3\nabla$. Accordingly, there is a net of G-invariant curves of degree 18. We will denote that net by $\mathcal{N}$. We then have the following technical result, where we adopt the notation of § 11 for orbits.

LEMMA 16.1. *Denote by $\mathcal{N}$ the net of invariant curves of degree* 18. *Every element of the net $\mathcal{N}$ passes through the* 24*-point orbit and the* 42*-point orbit. If O_d is an orbit with d elements, then for $d = 21, 28, 56$ the elements of $\mathcal{N}$ passing through O_d form a pencil which we denote $\mathcal{P}_d$ and these three pencils are not concurrent in $\mathcal{N}$. For $d = 21, 28$, the orbit O_d is singular on any element of $\mathcal{N}$ containing it. The pencil $\mathcal{P}_{56}$ consists entirely of reducible curves. The elements of $\mathcal{N}$ singular on the orbit O_d form a pencil $\mathcal{P}_d$ for $d = 24, 42$. The pencil $\mathcal{P}_{24}$ consists entirely of reducible curves containing $\mathcal{H}$ as a component. There is one and only one element of $\mathcal{N}$ having multiplicity at least* 3 *at the points of O_{28}.*

The pencil $\mathcal{P}_{21}$ has a unique element for which the multiplicity at the points of O_{21} is at least 3 *and for that element the multiplicity is actually equal to* 4.

PROOF. These assertions were all verified using REDUCE 3.4 on a personal computer. Some of them are easy to verify by hand. □

To show the existence of two essentially different types of invariant curves of degree 18 and genus 10, we begin with the following result which is based on a classical result for generic quartics [Berzolari 1903–15].

PROPOSITION 16.2. *There is an irreducible invariant of degree* 18 *and genus* 10 *for G with all of the points of* 21*-point orbit as quadruple points and no other singularities. There is another irreducible invariant of degree* 18 *and genus* 10 *for G whose singularities are the points of the* 42*-point orbit and the points of an* 84*-point orbit. The former is the reflex* (*see below*) *of the Caylean of the Klein curve and the latter is the reflex of the Steinerian of the Klein curve.*

PROOF. According to [Berzolari 1903–15, footnote 78 on p. 340 and table on p. 341], the dual of the Steinerian of a plane quartic has class 18, with 84 bitangents and 42 inflectional tangents. This means that the image of the Steinerian in $\mathbb{P}^2(\mathbb{C})^*$ (the dual $\mathbb{P}^2(\mathbb{C})$) under the Gauss map is a curve of degree 18 with 84 nodes and 42 cusps. Furthermore, in case the quartic is Klein's quartic, this image is invariant under G since S is. Now, the action of G on $\mathbb{P}^2(\mathbb{C})^*$ is related to the action of G on $\mathbb{P}^2(\mathbb{C})$ by an outer automorphism of G. Therefore, whenever an invariant curve U can be found in $\mathbb{P}^2(\mathbb{C})^*$ having certain properties, there will be a uniquely determined invariant curve, which we call the *reflex*[2] of U, in the original $\mathbb{P}^2(\mathbb{C})$ with the same properties as U. (Naturally, one cannot take this statement too literally. For example, the action of G on the reflex of U differs from the action of G on U by the outer automorphism. But the degree, the genus, the number of singularities, etc. will be the same for the curve and its reflex.) This observation allows us, in effect, to gloss over the distinction between invariant curves in $\mathbb{P}^2(\mathbb{C})$ and in $\mathbb{P}^2(\mathbb{C})^*$.

We know that the Steinerian of Klein's quartic is birationally equivalent to the Hessian, which has genus 10, and therefore any rational mapping from it onto another curve on which G acts nontrivially must be birational. In particular, the dual of the Steinerian will have genus 10 as well. Therefore the reflex of the dual of the Steinerian will be an invariant curve of degree 18 and genus 10 in $\mathbb{P}^2(\mathbb{C})$, which proves the second assertion of the proposition. As for the first, one works with the *Caylean* of a plane quartic. This is defined to be the curve in the dual $\mathbb{P}^2(\mathbb{C})$ consisting of of the lines joining points of the Hessian to the corresponding points of the Steinerian. The citation in [Be] also shows that the Caylean of a plane quartic has class 18 and has 21 quadruple tangent lines. In the case of Klein's quartic, one concludes that there is an invariant plane curve

[2]We don't refer to it as the dual of U because the dual of a curve already has a meaning which we will also be using.

of degree 18 in $\mathbb{P}^2(\mathbb{C})$ with 21 quadruple points, namely the *reflex of the Caylean* of Klein's quartic. Since the singularities are a union of orbits and the minimal G-orbit has 21 points, we obtain our 21 points which are at least quadruple. By the Plücker formula, we will know that the 21-point orbit is precisely quadruple and that there are no other singularities as soon as we show that the genus is 10. But by definition of the Caylean, there is a G equivariant rational mapping of the Hessian onto the Caylean which associates to a point x of the Hessian the line that joins x to the corresponding point of the Steinerian. This mapping must then be birational by Lemma 13.8.

The singular loci of invariant curves are unions of G orbits, but it is conceivable that there is some degeneration when specializing the general result to Klein's quartic. There can be no degeneration in the case of the reflex Caylean since there can be no orbit with less than 21 points. An argument is needed in the case of the reflex Steinerian, however, to verify that the singularities include the 42-point orbit and an 84-point orbit. We have verified this explicitly for the 42-point orbit but not for an 84-point orbit because of difficulties in computing the right 84-point orbit. The verification can however be completed by the following steps: (1) verify that the orbits O_{21}, O_{24}, O_{28}, O_{56} do not lie in the singular locus; (2) verify that the points of the 42-point orbit are nodes. For if (1) is verified, it will follow that all singularities of the reflex Steinerian lie on the 42-point orbit and on 84-point orbits. And if (2) is verified, it will follow that the 42-point orbit cannot be more singular than it is in the general case. In particular, O_{42} makes the same contribution to the Plücker formula as in the general case, so the same is true for the remaining orbits. Since the genus of the reflex Steinerian must be the same as the genus of $\mathcal{H}$, it follows that the remaining singularities must come from an 84-point orbit. The actual verification of (1) follows from Lemma 16.1. As for (2), the orbit O_{42} arises from i-eigenspaces in $\mathbb{C}^3$ of elements of order 4 in G. Since the eigenvalues of such an element are $1, \pm i$, the action of an element of order 4 on the tangent space to such a fixed point p has eigenvalues $-1, -i$ and acts, in suitable coordinates, by

$$(U, V) \mapsto (-U, -iV).$$

In the local ring at p, the defining equation must have the form

$$\phi(U, V) = 0,$$

where

$$\phi(U, V) = \sum_{n=0}^{\infty} \phi_n(U, V)$$

and where ϕ_n is the homogeneous part of degree n of ϕ. Since the point lies on the curve, we have $\phi_0 = 0$. Since the curve is defined by an invariant, we have

$$\phi_n(-U, -iV) = \phi_n(U, V)$$

for all n. This implies at once that $\phi_1 = 0$ and $\phi_2(U,V) = cU^2$. This proves that the points of the 42-point orbit are ordinary cusps of the curve, since we know that the curve can't have multiplicity greater than or equal to 3 on O_{42}. This completes the proof of Proposition 16.2. $\square$

Remark 16.3. Another natural idea for getting another invariant curve of degree 18 and genus 10 is to consider the image of the Hessian under the dual mapping of the Klein curve, i.e. the mapping which associates to each point x of the Hessian the polar line of x with respect to the Klein curve. This mapping is certainly G-equivariant and the coordinates of the mapping are cubics, so one gets a curve of degree 18. The genus is again 10 for the same reasons as before. By Corollary 14.7, it must be one of the curves we have already mentioned. In fact, according to [Berzolari 1903–15], this curve is the same as the dual of the Steinerian curve. Berzolari refers to [Cremona 1861; Clebsch 1876; 1891; Kötter 1887; Voss 1887] for this beautiful result: *the line joining a point x of the Hessian to the corresponding point y of the Steinerian is tangent to the Steinerian at y!* We also note that since the cubics form the adjoint system for $\mathcal{H}$, the dual Steinerian can therefore be regarded as the projection of the canonical curve of $\mathcal{H}$ in $\mathbb{P}^9(\mathbb{C})$ from the unique G-invariant $\mathbb{P}^6(\mathbb{C})$ onto the unique G-invariant $\mathbb{P}^2(\mathbb{C})$.

Corollary 16.4. *The curve F_4 is the reflex of the Caylean of the Klein curve.*

Proof. By Lemma 9.7, the curve F_4 does have quadruple points. The result now follows from Proposition 16.2. $\square$

Appendix: Matrices for Some Generators of G

We present here matrices according to which elements of each of the conjugacy classes of G act in the three-dimensional representation of G we are considering. The notation for representatives of conjugacy classes of G follows that of [Conway et al. 1985]. We do not claim that $4A^2 = 2A$, only that they be conjugate. One can also find a discussion of explicit matrices for this representation in [Weil 1974, § 115; Klein and Fricke 1890–92, § III.5, pp. 703–705].

$$7A = \begin{pmatrix} \zeta_7 & 0 & 0 \\ 0 & \zeta_7^4 & 0 \\ 0 & 0 & \zeta_7^2 \end{pmatrix} \qquad 7B = 7A^{-1} \qquad 3A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$2A = -\frac{1}{\sqrt{-7}} \begin{pmatrix} \zeta_7^2 - \zeta_7^5 & \zeta_7^4 - \zeta_7^3 & \zeta_7 - \zeta_7^6 \\ \zeta_7^4 - \zeta_7^3 & \zeta_7 - \zeta_7^6 & \zeta_7^2 - \zeta_7^5 \\ \zeta_7 - \zeta_7^6 & \zeta_7^2 - \zeta_7^5 & \zeta_7^4 - \zeta_7^3 \end{pmatrix} \qquad 4A = 7A^3 \cdot 2A$$

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