The Eightfold Way
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The Geometry of Klein's Riemann Surface

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ABSTRACT. Starting from the hyperbolic definition of Klein's surface we prove platonicity, derive the two classical equations $W^7 = Z(Z-1)^2$ between meromorphic functions and $x^3y+y^3z+z^3x = 0$ between holomorphic forms, describe a pair of pants decomposition in terms of which the automorphisms can be seen, find a basis for the holomorphic forms for which all periods (hence the Jacobian) can be computed, and from which the lattice of rhombic tori can be determined which are covered by Klein's surface.

1. Introduction

In autumn 1993, in front of the MSRI in Berkeley, a marble sculpture by Helaman Ferguson called *The Eightfold Way* was revealed. This sculpture shows a compact Riemann surface of genus 3 with tetrahedral symmetry and with a tessellation by 24 distorted heptagons. The base of the sculpture is a disc tessellated by hyperbolic 120°-heptagons, thus suggesting that one should imagine that the surface is "really" tessellated by these regular hyperbolic polygons. In the celebration speech Bill Thurston explained how to see the surface as a hyperbolic analogue of the Platonic solids: Its symmetry group is so large that any symmetry of each of the 24 regular heptagons extends to a symmetry of the whole surface — a fact that can be checked "by hand" in front of the model: Extend any symmetry to the neighboring heptagons, continue along arbitrary paths and find that the continuation is independent of the chosen path. The hyperbolic description was already given by Felix Klein after whom the surface is named. The large number of symmetries — we just mentioned a group of order $24 \cdot 7 = 168$ — later turned out to be maximal: Hurwitz [1893] showed that a compact Riemann surface of genus $g \geq 2$ has at most $84(g-1)$ automorphisms and the same number of antiautomorphisms. The next surface where Hurwitz's bound is sharp is treated in [1965]; see also [Lehner and Newman 1967; Kulkarni 1982], as well as Macbeath's article in this volume.

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The sculpture introduced Klein's surface to many non-experts. Of course the question came up how the hyperbolic definition of the surface (as illustrated by the sculpture) could be related to the rather different algebraic descriptions. For example, the equation

$$W^7 = Z(Z-1)^2$$

relates two meromorphic functions on the surface, and

$$x^3y + y^3z + z^3x = 0$$

relates three holomorphic 1-forms. The answer to this question is known in general: The uniformization theorem implies that every Riemann surface of genus $g \geq 2$ has a hyperbolic metric, that is, a metric of constant curvature -1, and the existence of sufficiently many meromorphic functions implies that every compact Riemann surface has an algebraic description. But it is very rare that one can pass *explicitly* from one description to the other.

There were other natural questions. In the hyperbolic picture one sees cyclic automorphism groups of order 2, 3, and 7 — what are the quotient surfaces? Topologically, this can easily be answered with the Euler number of a tessellated surface, $\chi = F - E + V$, if one takes a tessellation that passes to the quotient. Moreover, we will identify the quotient map under the order 7 subgroup with the meromorphic function Z in the first equation. By contrast, we do not know a group theoretic definition of the other function, W; it is constructed in the hyperbolic picture with the help of the Riemann mapping theorem. — The quotient surfaces by the other groups above, those of order 2 and 3, are always tori. This has another known consequence: Klein's surface does not doubly cover the sphere, it is not "hyperelliptic" — but it also leads to more questions: What tori appear as quotients? The differential of a holomorphic map to a torus is a holomorphic 1-form whose period integrals (along arbitrary closed curves on the surface) *are* the lattice of the torus. So again, the question is highly transcendental in general and explicit answers are rare.

Here the answer is possible, since we can identify Klein's surface in yet another representation of compact Riemann surfaces. Consider the Riemann sphere endowed with a flat metric with cone singularities. Riemann surfaces can be described as coverings over such a sphere that are suitably branched over the cone singularities. In this situation one has a developing map from the Riemann surface to the complex plane. Its differential is a holomorphic 1-form on the universal cover whose zeros are at the cone singularities. With a good choice of the flat metric this 1-form actually descends to the compact Riemann surface! (Already this step rarely succeeds.) In the special case of Klein's surface we find with the help of the 7-fold covering mentioned above *three different* such representations. This gives a basis of the holomorphic 1-forms — in fact the forms x, y, z of the second equation above — for which the *periods can be computed* via the Euclidean geometry of the flat metrics. At this point the Jacobian of the surface is determined. We proceed to find linear combinations of the basis 1-forms so that their periods are a lattice in $\mathbb{C}$. This shows that the Jacobian is the product of three times the same rhombic torus with diagonal ratio $\sqrt{7} : 1$. This torus has "complex multiplication", namely we can map its lattice to an index 2 sublattice by multiplication with $\bigl(1+\sqrt{-7}\,\bigr)/2$. This leads to recognizing the lattice as the ring of integers in the quadratic number field $\mathbb{Q}\bigl(\sqrt{-7}\,\bigr)$ and to see that this torus is defined over $\mathbb{Q}$.

We learnt from [Rodríguez and González-Aguilera 1994] that the hyperbolic description of the Fermat quartic,

$$x^4 + y^4 + z^4 = 0,$$

is surprisingly similar to Klein's surface. In fact, *each* Fermat surface is platonically tessellated by π/k-triangles; the area of these tiles is $\pi(k-3)/k$, which is always larger than the area $\pi/7$ of the $2\pi/7$-triangles, which are 56 platonic tiles for Klein's surface. Also, Jacobians and, for $k=4$, quotient tori can be computed with the methods outlined above. We included this only because we found a comparison instructive. The result is less exciting because the questions above can be answered for the Fermat case in each description separately.

Acknowledgment. This work started from discussions about the Ferguson sculpture while the authors enjoyed the hospitality of the MSRI. Conversations with W. Thurston and M. Wolf were then particularly helpful. Later a large number of colleagues helped us with comments, questions and advice.

This paper is organized as follows:

SECTION 2: Summarizes a few facts from the *group theoretic* treatment of platonic surfaces.

SECTION 3: Treats *two genus* 2 platonic surfaces. Together they show many phenomena that we will also encounter with Klein's surface, but they are much simpler. We hope this will help the reader to see more quickly where we are heading in the discussion of Klein's surface.

SECTION 4: Deduces *Klein's surface* from assumptions that require less than its full symmetry, derives the above equations and proves platonicity.

SECTION 5: Describes a *pairs of pants* decomposition that emphasizes the symmetries of one S_4 subgroup of the automorphism group. These pants also allow to list the conjugacy classes of all automorphisms.

SECTION 6: Discusses and compares the *Fermat surfaces*, in particular the quartic.

SECTION 7: Introduces *flat cone metrics*. In terms of these we construct holomorphic forms with computable periods, determine the Jacobians of the discussed examples and find explicit maps to tori. We prove that all quotient tori of Klein's surface are the same rhombic torus with diagonal ratio $\sqrt{7} : 1$.

2. Triangle Groups and Platonic Surfaces

Platonically tessellated Riemann surfaces and the structure of triangle groups are closely related. To give some background information we summarize the following known facts.

A *symmetry* of a Riemann surface is an isometry with respect to the hyperbolic metric on it. An *automorphism* is an orientation-preserving symmetry. This is the same as a conformal automorphism. Thus we do not mean that a symmetry has to be the symmetry of some embedding (like the sculpture) or immersion of the surface [Schulte and Wills 1985].

A tessellation of a Riemann surface is *platonic* if the symmetry group acts transitively on flags of faces, edges and vertices. Such a tessellation is also called a *regular map* [Coxeter and Moser 1957]. Finally, a Riemann surface is called platonic if it has some platonic tessellation.

Suppose now that we have a Riemann surface M^2 that is platonically tessellated by regular k-gons with angle $2\pi/l$. The stabilizer of one polygon in the symmetry group of the surface then contains at least the dihedral group of the polygon. Consequently there is a subgroup of the symmetry group that has as a fundamental domain a hyperbolic triangle with angles $\pi/2$, π/k, π/l. We will call such triangles from now on $(2, k, l)$-triangles. Observe that the order of this group is

$$\begin{aligned} \text{order} &= \text{hyperbolic area}(M^2)\big/\text{area}(2,\,k,\,l)\text{-triangle} \\ &= -2\pi\cdot\chi(M^2)\Big/\left(\frac{\pi}{2}-\frac{\pi}{k}-\frac{\pi}{l}\right), \end{aligned}$$

and half as many automorphisms are orientation-preserving. The smallest possible areas of such triangles are

$$\text{area}(2,3,7)=\pi/42\,,\quad \text{area}(2,3,8)=\pi/24\,,\quad \text{area}(2,4,5)=\pi/20\,.$$

Now consider the group generated by the reflections in the edges of a $(2, k, l)$-triangle in the hyperbolic plane; this group is called a triangle group. It acts simply transitively on the set of triangles. The covering map from $\mathbb{H}^2$ to M^2 maps triangles to triangles; the preimage of each triangle defines the classes of equivalent triangles in $\mathbb{H}^2$. The deck group of M^2 acts simply transitively on each equivalence class and, because we assumed M^2 to be platonically tessellated, it is also true that the (anti-)automorphism group of M^2 acts simply transitively on the set of equivalence classes. This shows that the deck group of the surface is a (fixed point free) normal subgroup of the triangle group.

Vice versa, given a fixed point free normal subgroup N of a $(2, k, l)$-triangle group G, then we define a Riemann surface M^2 as the quotient of $\mathbb{H}^2$ by N. This surface is tessellated by the $(2, k, l)$-triangles and the factor group G/N acts simply transitively on these triangles. In $\mathbb{H}^2$ the $(2, k, l)$-triangles of course fit together to a pair of dual platonic tessellations, one by k-gons with angle $2\pi/l$, the other by l-gons with angle $2\pi/k$. Both tessellations descend to tessellations of the quotient surface (namely: Consider the projection of the polygon centers in $\mathbb{H}^2$ to the surface, we recover a polygon tessellation of the surface as the Dirichlet cells around the projected set of centers). They are still platonic.

It is therefore in an obvious way equivalent to consider compact platonically tessellated Riemann surfaces or finite index normal subgroups of triangle groups. (We may even allow $(2, k, \infty)$-triangles, noncompact finite area triangles with one 0-angle.) See [Bujalance and Singerman 1985; Singerman 1995].

Meromorphic functions and forms are now accessible from this group theoretic approach as automorphic functions and forms on the hyperbolic plane with respect to the deck group of the surface. With the group-theoretical approach one does not always find the simplest equations [Streit 1996]. By contrast, in our discussion of Klein's surface we will construct on it simple functions and forms for which we do *not* know a group theoretic definition. We use the following two methods:

MEROMORPHIC FUNCTIONS: We map one tile of the tessellated Riemann surface to a suitable spherical domain with the Riemann mapping theorem; we extend this map by reflection across the boundary and finally check that the extension is compatible with the identifications.

HOLOMORPHIC 1-FORMS: We take exterior derivatives of developing maps of flat cone metrics and check by holonomy considerations whether they are well defined on the surface.

3. Two Platonic Surfaces of Genus Two

We explain with the simplest hyperbolic examples how symmetries can be used to derive algebraic equations.

3.1. The $\pi/5$-case. Let's try to construct a genus 2 surface M^2 that is platonically tessellated by F equilateral $\pi/5$-triangles. Such a triangulation must have $E = \frac{3}{2} \cdot F$ edges and $V = \frac{3}{10} \cdot F$ vertices, since 10 triangles meet at a vertex. Euler's formula then gives

$$\chi(M^2) = -2 = F \cdot \left(1 - \frac{3}{2} + \frac{3}{10}\right), \qquad F = 10, \qquad V = 3\,.$$

Equivalently, we could have used the Gauß-Bonnet formula

$$-2\pi \cdot \chi(M^2) = \text{area}(M^2) = F \cdot \text{area(triangle)} = F \cdot \frac{2\pi}{5}\,.$$

These ten triangles fit around one vertex to form a $2\pi/5$-decagon, which is already a fundamental domain for the surface we want to construct. What remains to be done is to give suitable identifications. We consider only identifications that satisfy necessary conditions for *platonic* tessellations. For example, we want the $2\pi/5$-rotations around the center of the decagon to extend to symmetries of the surface. This implies that the identification of one pair of edges determines all the others. Since the angles at five decagon vertices sum up to 2π the edge identifications have to identify every second vertex. This leaves only two possibilities, which will turn out to define the same surface: Identify edge 1 to edge 6 or to edge 4. Both cases are promising, because further necessary conditions for platonicity are satisfied.

Synthetic arguments in Euclidean and hyperbolic geometry are very much the same: One can compose two reflections in orthogonal lines to obtain a 180° rotation; one can join the centers of two 180° rotations by a geodesic and take it and a perpendicular geodesic through either center as such reflection lines; this shows that the composition of two involutions "translates" the geodesic through their centers. In the hyperbolic case this is the only invariant geodesic, it is also called *axis* of the translation.

Platonicity implies that the midpoints of edges are centers of 180°-rotations. On a compact platonic surface one can therefore extend any geodesic connection of midpoints of edges by applying involutions until one gets a closed geodesic. (Note that these extensions meet the edges, at the involution centers, always with the same angle and there are only finitely many edges.) This means that we always find translations in the deck group that are generated by involutions. Therefore, if we want to construct a platonic surface, then it is a good sign if already the identification translations are products of involutions. This is true for both identification candidates above: For the identification of the opposite edges (say) 1 and 6 take as centers the midpoint of the decagon and the midpoint of edge 6; the translation that identifies edges 1 and 4 is the product of the involutions around the marked midpoints of the radial triangle edges 1 and 3. See Figure 1 for the axes of these translations.

We are now going to construct meromorphic functions on M^2 since this leads to an algebraic definition of the surface. Namely, if two functions have either no common branch points or else at common branch points relatively prime branching orders then they provide near any point holomorphic coordinates — that is, an atlas for the surface. To turn this into a definition one needs to specify the change-of-coordinates and the classical procedure is to do this by giving an algebraic relation between the two functions. Therefore, to describe a specific hyperbolic surface algebraically means that one has to construct two meromorphic functions that one understands so well that one can deduce their algebraic relation. There is no general procedure to achieve this. In highly symmetric situations one can divide by a sufficiently large symmetry group and check whether the quotient Riemann surface is a sphere. Any identification of this quotient sphere with $\mathbb{C} \cup \{\infty\}$ turns the quotient map into a meromorphic function. This method is sufficient for the following genus 2 examples. Another way to construct meromorphic functions is to use the Riemann mapping theorem together with the reflection principle to produce first maps from a fundamental domain of an appropriate group action on the surface to some domain on the sphere and extend this by reflection to a map from the whole surface to the

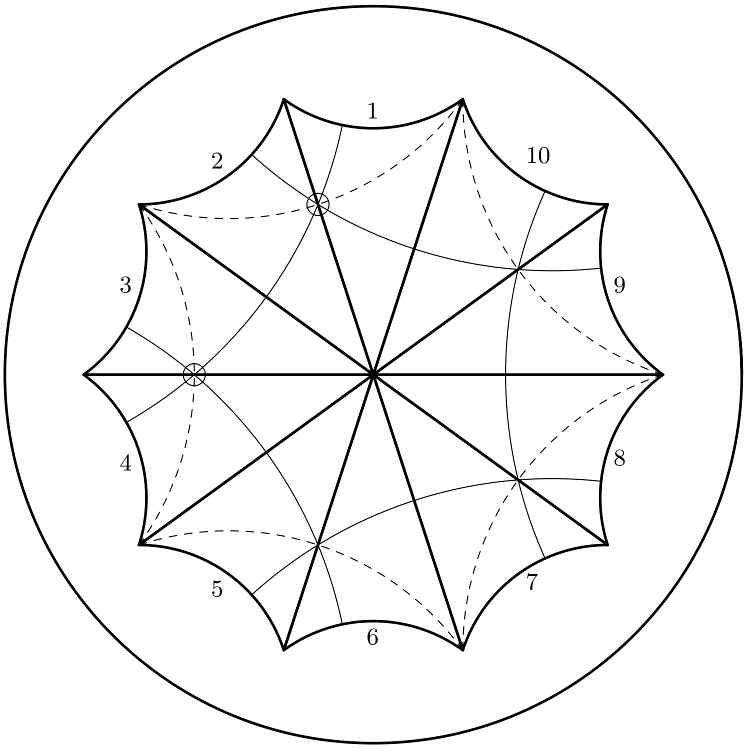

Figure 1. Decagon composed of ten equilateral hyperbolic $\frac{\pi}{5}$-triangles.

sphere. This method will be of importance for Klein's surface, and we will explain it with the simpler functions on the genus 2 surfaces.

To define the first function, look at the order 5 rotation group around the center of the decagon. This group respects the identification and therefore acts on the surface by isometries. It has 3 fixed points, namely the center of the decagon and the two identified sets of vertices. Using Euler's formula we see that the quotient surface is a sphere: Take any triangulation of M^2 that is invariant under the rotation group. Then the quotient surface is also triangulated and denoting by f the number of fixed points of the rotations on M^2, we compute its Euler number

$$\tfrac{1}{5}\big((V-f)-E+F\big)+f \;=\; \tfrac{1}{5}(-2-f)+f \;\in\; \{0,2\},$$

which reproves $f=3$ and shows that $\chi=2$ for the quotient.

This function can also be understood via a Riemann mapping problem: Imagine that the ten triangles are alternatingly colored black and white, "Riemann map" a white triangle to the upper half plane, "Möbius normalize" so that the vertices go to $0,\ 1,\ \infty$, and extend analytically by reflection in the radial boundaries to a map from the decagon to a fivefold covering of the sphere, branched

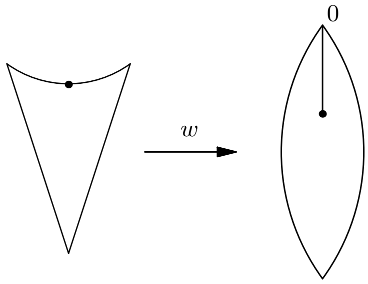

Figure 2. Mapping a hyperbolic $\frac{\pi}{5}$-triangle to a spherical $\frac{2\pi}{5}$-sector.

over ∞ and with slits from 0 to 1 on each sheet. Finally identify the edges of the slits in the same way as the preimage edges of the decagon. Therefore we can either see the quotient sphere as isometric to the double of a hyperbolic $2\pi/5$-triangle, which gives a singular hyperbolic metric on the sphere, or we can see M^2 as a fivefold covering of the Riemann sphere, branched over $0,\ 1,\ \infty$. In any case we have obtained — for both identification patterns — a meromorphic function z on M^2 that sends the three vertices of the triangulation as fivefold branch points to $0,\ 1,\ \infty$.

For a second function, we can consider the quotient of M^2 by any involution to obtain

$$\tfrac{1}{2}\left((V-f)-E+F\right)+f \;=\; \tfrac{1}{2}(-2-f)+f \;\in\; \{0,2\}\,;$$

hence $f=2$ or $f=6$. In both of our cases take as the involution one of those that were used to define the identifications and observe that we have $f=6$ (for the first identification we have as fixed points the midpoint of the decagon and the identified midpoints of opposite edges) so that the quotient by this involution again is a sphere. We normalize this meromorphic quotient function w on M^2 up to scaling by sending the midpoint of the decagon to ∞ and the two other vertices of the triangulation to 0 (and similarly for the other identification pattern).

Since reflection in the radial triangle edges passes to the quotient we can also understand the function w as mapping each triangle to a spherical $2\pi/5$-sector that is bounded by great circle arcs from 0 to ∞ and has a straight slit in the direction of the angle bisector, as in Figure 2. By scaling we may take the slit to have arbitrary length.

Simply by comparing the divisors of z and w we see that w^5 and $z(z-1)$ are proportional functions and (after scaling w) we obtain

$$w^5 = z(z-1),$$

which is a defining equation for M^2, the same for both identification patterns.

We will now be disappointed and find that the triangle tessellation is *not* platonic. One way to see this is to check that the involutions around midpoints of edges that were *not* used to define the identifications are not compatible with them. A more algebraic way is to produce too many holomorphic 1-forms by considering the following divisor table, where we define $y = z/(z-1)$:

vertices	V_1	V_2	V_3
z	0^5	1^5	∞^5
w	0	0	∞^2
y	0^5	∞^5	1^5
dy/y	∞	∞	0^4
$w \cdot (dy/y)$	$\star$	$\star$	0^2
$w^2 \cdot (dy/y)$	0	0	$\star$

Now suppose that M^2 were platonically tessellated. Then the 120°-rotation of one triangle would extend to a symmetry of the whole surface. This implies that we could cyclically permute the divisor of the holomorphic 1-form $w^2 \cdot dy/y$ to get divisors of other forms. The quotient of two of these would be a meromorphic function with only one simple pole, a contradiction.

Fortunately, we have not lost completely since we can platonically tessellate M^2 with two $\pi/5$-pentagons by joining even numbered neighboring vertices of the decagon, dashed in Figure 1. This is not quite as good as hoped for, but also on Klein's surface we will find platonic and other non-platonic tessellations by regular polygons.

3.2. The $\pi/4$-case. Next we will construct a more symmetric platonic genus 2 surface; its automorphism group has order 48, the maximum for genus 2 [Burnside 1911]. The quotient sphere is the double of the hyperbolic $(2,3,8)$-triangle—which is less than twice as big as the doubled $(2,3,7)$-triangle in Klein's case. We want the surface to be platonically tessellated by equilateral $\pi/4$-triangles. Since eight such triangles fit around one vertex we have

$$\chi(M^2) = -2 = F \cdot \left(1 - \tfrac{3}{2} + \tfrac{3}{8}\right), \qquad F = 16, \qquad V = 6\,.$$

The eight triangles around one center vertex form a small $\pi/2$-octagon. The remaining 8 triangles can be placed along the edges. No other pattern would be possible for a platonic surface, because the 45°-rotation around the center vertex extends to a symmetry of the surface. Hence we expect as a fundamental domain of our surface a big regular $\pi/4$-octagon (Figure 3).

Again we try the identification of opposite edges by hyperbolic translations, this time no other candidate is possible by platonicity. As before, these translations are compositions of two 180°-rotations (around the midpoint of a boundary edge and around the center of the octagon, both of which are triangle vertices). Moreover, all vertices (angle $2\pi/8$) of the big octagon are identified to one vertex to give a smooth hyperbolic genus 2 surface M^2. The 180°-rotation around the midpoint of the octagon is an involution of M^2 whose fixed points are the 6 vertices of the 16 $\pi/4$-triangles. The projection z to the quotient goes again to a sphere. One easily checks that this involution commutes with all reflections in the triangle edges so that these reflections and their fixed points pass to the

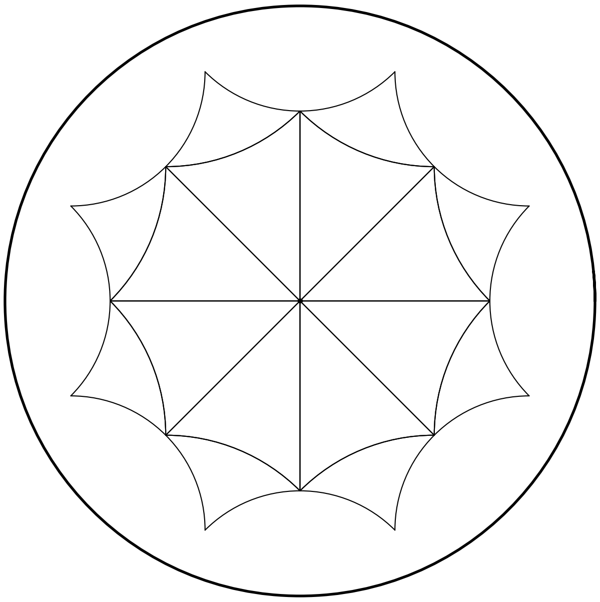

Figure 3. Hyperbolic $\frac{\pi}{4}$-octagon with 16 equilateral triangles.

quotient. Hence the hyperbolic quotient metric on the sphere is necessarily given by the octahedral tessellation by hyperbolic $\pi/4$-triangles. We can assume that the octahedron has its vertices in $0, \pm 1, \pm\mathbf{i}$ and ∞.

As before, this quotient map can also be defined independently: first Riemann-map a hyperbolic $\pi/4$-triangle to a spherical $\pi/2$-triangle, then extend analytically by reflection in the edges and check compatibility with the identifications.

Since M^2 is only a double covering over the sphere with known branch values, we have the following equation for this Riemann surface:

$$w^2 = z \cdot \frac{z-1}{z+1} \cdot \frac{z-\mathbf{i}}{z+\mathbf{i}}\,.$$

We still have to prove platonicity. Since all the reflections in symmetry lines of the octagon are clearly compatible with the identifications we only have to check that the involution around the midpoint of one radial triangle edge is also compatible. This can be seen by checking in the tessellated hyperbolic plane that *any* two vertices that are two triangle edges apart are equivalent under the identifications. It can also be seen on the doubly covered octahedral tessellation of the sphere by introducing three branch cuts and checking that $180°$-rotation around the midpoint of an octahedron edge on one sheet extends to a symmetry of the double cover. One observes that this involution has only two fixed points since at the antipodal point of the sphere the sheets are interchanged; the quotient map therefore only goes to a torus. Since this involution commutes with a reflection of M^2 the quotient torus has also such a symmetry, called a complex conjugation; the fixed point set of this torus reflection has *two* components — that is, the torus is rectangular. In the case of Klein's surface all the involutions will give quotient maps to rhombic tori.

With platonicity established we can interpret the function w above as the quotient map under the rotations of order 3 around the center of one triangle. These rotations can be seen on the octahedral sphere as follows: Consider a 120°-rotation of the octahedron around the centers of two opposite triangles. This map lifts to an isometry of M^2 with the desired property. It has four fixed points over the two fixed points of the rotation of the octahedron.

Clearly, a fundamental domain for the group of all automorphisms now is one third of one $\pi/4$-triangle, i.e., two (2, 3, 8)-triangles, each of area $\pi/24$. This gives for the order of the automorphism group $-2\pi\chi/(2\pi/24) = 48$. Why is this the maximal order for genus 2? A proof of Hurwitz' theorem begins by dividing a Riemann surface of genus ≥ 2, endowed with its hyperbolic metric, by the full group of automorphisms. These are also hyperbolic isometries. The quotient is a Riemann surface with larger Euler number and a hyperbolic metric with π/k_i cone singularities. The automorphism group is maximal (for the considered genus) if the hyperbolic area of the quotient surface is minimal. The two smallest quotients are the doubles of the hyperbolic (2, 3, 7)- and (2, 3, 8)-triangles. Therefore we have to show that (2, 3, 7) does not occur for genus 2. But already a cyclic group of prime order $p \geq 7$ is impossible for genus 2, since, from the Euler number of the quotient, the possibilities for the number f of fixed points of this group are given by

$$\frac{1}{p} \cdot \big((V-f) - E + F\big) + f \;=\; -\frac{1}{p}(2+f) + f \;\in\; \{0, 2\},$$

or

$$f \;\in\; \left\{\frac{2}{p-1},\; 2 + \frac{4}{p-1}\right\} \subset \mathbb{Z}\,, \quad \text{with } p \;\in\; \{2, 3, 5\}\,.$$

4. The Hyperbolic Description of Klein's Surface

Klein's surface is more complicated than our examples of genus 2, and the construction will take some time. Moreover, since we cannot construct some famous surface without using some knowledge about it, we do not even have a well defined problem yet. One could start with the 24 tiles of the platonic tessellation by 120°-heptagons mentioned in the introduction. We found it interesting that Klein's surface is already determined by much less than its full symmetry, and by asking less we are rather naturally led to a fourteengon as a fundamental domain together with the correct identifications. The heptagons then fit into this fundamental domain in a way that can be described easily and platonicity follows with short arguments.

In analogy to the first genus 2 example we will look for a genus 3 surface tessellated by (rather big) $\pi/7$-triangles such that reflections in the edges extend to antiautomorphisms of the surface. There are only two such Riemann surfaces and both have a cyclic group of order 7 as automorphisms. But only one of the two has the 120° rotations around triangle centers as automorphisms. We construct one function by exploiting the order 7 rotation group and we find a second function with the Riemann mapping theorem. For both surfaces we derive an algebraic equation. For Klein's identification pattern we prove platonicity and finally complete the picture by a pairs of pants decomposition in terms of which all the remaining symmetries, in particular the symmetry subgroups, have simple descriptions.

4.1. Consequences of Euler's formula and of platonicity. First, for a given tessellation by $2\pi/3$-heptagons we obtain from Euler's formula the numbers F of faces, $V = \frac{7}{3} \cdot F$ of vertices and $E = \frac{7}{2} \cdot F$ of edges:

$$\chi(M^2) = -4 = F \cdot (1 - \tfrac{7}{2} + \tfrac{7}{3}) \quad \Longrightarrow \quad F = 24,\ V = 56,\ E = 84\,.$$

In the dual tessellation by $\frac{2\pi}{7}$-triangles the numbers F and V are interchanged. These numbers are too large to easily talk about individual tiles. By contrast, a tessellation by big $\pi/7$-triangles (of area $4\pi/7$ each) needs $F = 14$ of them to have the required total area 8π for a hyperbolic genus 3 surface; such a tessellation has $E = 21$ edges and $V = 3$ vertices.

Next consider a cyclic rotation group of prime order p on a surface of genus 3 with f fixed points. The Euler number for the quotient surface is given by

$$\frac{1}{p}\big((V-f) - E + F\big) + f \;=\; \frac{1}{p}(-4-f) + f \;\in\; \{-2, 0, 2\},$$

or

$$f \in \left\{-2 + \frac{2}{p-1},\ \frac{4}{p-1},\ 2 + \frac{6}{p-1}\right\} \cap \mathbb{Z}.$$

Therefore $p = 7$ is the maximal prime order, $f = 3$ in that case and the quotient is a sphere. A genus 3 surface with an order 7 cyclic group of automorphisms therefore has a natural quotient map to the sphere. To view this map as a specific meromorphic function we identify the quotient sphere with $\mathbb{C} \cup \{\infty\}$ by sending the three fixed points to $0, 1, \infty$.

Furthermore, an involution ($p = 2$) must have $f = 0$, 4, or 8 fixed points. To discuss these possibilities further, note that an involution of a platonic tessellation by $2\pi/3$-heptagons cannot have its fixed points at vertices or centers of faces of the tessellation. Thus the fixed points are at edge midpoints. In such a case f must divide the number E of edges, therefore $f = 8$ cannot occur for an involution of our heptagon tessellation with E=84 edges — which shows in particular that the quotient is never a sphere, which is to say, Klein's surface is *not* hyperelliptic. We will see later that *all* involutions give quotient tori, $f = 0$ does not occur.

Platonicity further implies that we have a rotation group of order $p = 3$ around each of the heptagon vertices; we just computed its number f of fixed points:

$$f\Big|_{p=3} \in \{2, 5\}\,.$$

Here $f = 5$ is excluded because it does not divide $V = 56$. So we have $f = 2$ and the quotient is a torus.

4.2. A fundamental domain from big triangles. Because of the desired cyclic symmetry group of order 7 we arrange the 14 big triangles around one center vertex to form a $2\pi/7$-fourteengon (Figure 4) and we see that all the odd and all the even vertices have to be identified to give a smooth hyperbolic surface. This leaves three possibilities: identify edge 1 to edge 4, 6 or 8. The last case has the 180°-rotation around the center as an involution with $f = 8$ fixed points (namely the center and the pairwise identified midpoints of fourteengon edges); that is, the quotient is a sphere. So this example is a hyperelliptic surface. As in the $\pi/5$-case we have found two quotient functions and their divisors give the equation

$$w^7 = z(z-1)\,.$$

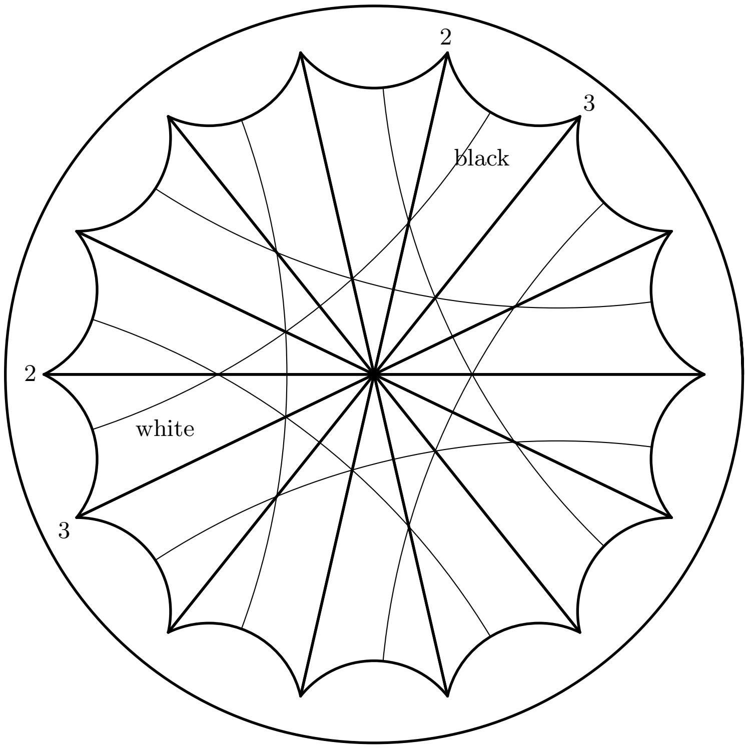

Figure 4. Hyperbolic fourteengon made from equilateral $\frac{\pi}{7}$-triangles, with translation axes.

Identification of edge 1 to 4 leads to the same hyperelliptic surface and therefore leaves the identification of edge 1 to edge 6 as the only candidate for some platonic surface, which we will prove to be Klein's surface. Note that this identification of the fourteengon edges is the hyperbolic description of the surface given in Klein's work; see the lithographic plate in [Klein 1879]. We concentrate on this case now and reveal further symmetries.

If one wants to check whether some expected symmetry is compatible with the identifications then the just given rule has the disadvantage that, for using it, one needs a rather large piece of the tessellation in the hyperbolic plane. We begin with a more convenient reformulation. Color the fourteen big triangles alternatingly black and white. Each black edge (of the fourteengon) is identified with the white edge that is counterclockwise 5 steps ahead (or the white edges with the black ones 9 steps ahead). We call the fourteengon center vertex 1, the left endpoint of a black edge vertex 2 and its right endpoint vertex 3. The identification rule can be restated as follows: Under the identification translation of a black edge to a white one, vertex 2 (as seen from vertex 1) is rotated by $2 \cdot 2\pi/7$ around the center and the triangle adjacent to this black edge is, at vertex 2, rotated by $1 \cdot 2\pi/7$; similarly, vertex 3 is rotated by $3 \cdot 2\pi/7$ around the center and the same triangle adjacent to this black edge is rotated around vertex 3 by $-1 \cdot 2\pi/7$. This can be expressed in a simpler way if one observes

$$2 \cdot \{1,2,4\} = \{2,4,1\} \pmod 7, \qquad 4 \cdot \{1,2,4\} = \{4,1,2\} \pmod 7.$$

The identification rule now is: Rotation around vertex 1 by $1 \cdot 2\pi/7$ is rotation at vertex 2 by $4 \cdot 2\pi/7$ and at vertex 3 by $2 \cdot 2\pi/7$. The high symmetry of Klein's surface is apparent in the fact that this rule remains the same (mod 7) if we cyclically permute the vertices. — We remark that our description of Klein's surface in terms of flat cone metrics on a thrice punctured sphere will start from here.

To apply the new rule we consider a tessellation of the hyperbolic plane by the black and white $\pi/7$-triangles. Mark the equivalence classes of triangles from 1 to 14 and the vertices from 1 to 3, and observe that the identification rule allows us to pick an arbitrary triangle from each equivalence class and still know how to identify. The 120° rotation around any triangle center cyclically permutes the (equivalence classes of) vertices, but we saw that the identification rule is not affected by this change. Similarly, reflection in a triangle edge interchanges the black and white triangles and thereby the cyclic orientation of their vertices, but again, this does not change the identification rule. These reflections generate the order 7 rotational symmetry and therefore pass to the quotient sphere. This means that we can again understand the quotient map (under this symmetry group) via a Riemann mapping problem: Map a black triangle to the upper half plane, normalize so that the vertices 3, 2, 1 go to 0, 1, ∞ and extend by reflection.

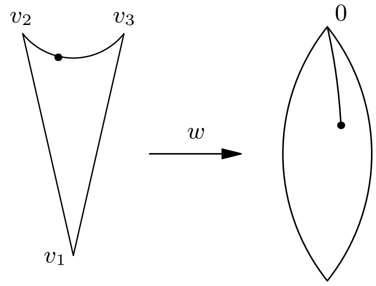

Figure 5. Equilateral hyperbolic $\frac{\pi}{7}$-triangle mapped to a spherical slit domain.

4.3. A second function and equations. We define a second function with the Riemann mapping theorem. Map one of the black triangles to a spherical domain that is bounded by two great circles from 0 to ∞ with angle $3 \cdot \pi/7$ at ∞ and has a great circle slit from 0 dividing the angle at 0 as $2 : 1$, counterclockwise the bigger angle first. (The length of the slit can be changed by scaling this map.)

This map can be extended analytically by reflection in the edges (around ∞) to cover the sphere three times. The slits in these three sheets are such that always in two sheets there are slits above each other, and these are not above a slit in the third sheet. This forced identification of the slits is compatible with the identifications of the edges of the fourteengon since the rotation angles $\{4, 1, 2\} \cdot 2\pi/7$ counterclockwise at the vertices of a black triangle are the same as the rotation angles $\{-3, 1, 2\} \cdot 2\pi/7$ at the vertices of the spherical domain. We compare the divisors of this function w and the above quotient function z and find that the functions w^7 and $z(z-1)^2$ are proportional. We can scale w to give us one of the known equations,

$$w^7 = z(z-1)^2 \,.$$

We do not know a group theoretic definition of the function w. Also, observe that the derivation did not use that this surface is platonic. Next we derive from this equation the even more famous quartic equation. It exhibits not only the order 7 symmetry (which was built in by construction) but gives another proof of the order 3 symmetry (independent of the above one). Consider this divisor table:

vertices	V_3	V_2	V_1
z	0^7	1^7	∞^7
w	0	0^2	∞^3
$v = w^2/(z-1)$	0^2	∞^3	0
$u = (z-1)/w^3$	∞^3	0	0^2
$\xi = v\,dz/z$	0	0^3	$\star$
$\omega = u\,dz/(1-z)$	0^3	$\star$	0
$\eta = u\,dz/(z(z-1))$	$\star$	0	0^3
$u\,z$	0^4	0	$\infty^5 r$

First, if we define

$$x := (1-z)/w^2 = -v^{-1}, \qquad y := -(1-z)/w^3 = +u$$

then the first equation implies the quartic equation

$$x^3\,y + y^3 + x = 0\,.$$

Of course, the substitution can be inverted: $w = -x/y, \quad z = 1 - x^3/y^2$. Secondly we see from the divisor table that the functions x, y are quotients of *holomorphic* 1-forms, namely

$$x = \xi/\omega, \quad y = \eta/\omega.$$

This observation gives an additional interpretation to the equation in its homogeneous form

$$\xi^3\eta + \eta^3\omega + \omega^3\xi \;=\; 0$$

as an equation between explicitly known holomorphic 1-forms. The projective embedding defined by this equation is called the canonical curve.

We see an order 3 symmetry as the permutation of the coordinates and an order 7 symmetry by multiplying ξ, η, ω with powers of a seventh root of unity, namely ζ^1, ζ^4, ζ^2.

The existence of the function u, v, w with single poles of order $3 < g+1$, and $u \cdot z$ of single pole of order 5, prove that V_1, V_2, V_3 are Weierstraß points with non-gap sequence $(3, 5, 6)$ and hence of weight 1. After platonicity is proved, we know that all the heptagon centers are such Weierstraß points. These are all since $g^3 - g = 24$ is the total weight.

4.4. The heptagon tessellation. We now add the heptagon tessellation to the previous picture. This will allow to prove platonicity with rather little effort. Notice that from now on the emphasis is on the involutions of the surface, they were not visible so far.

One $2\pi/3$-heptagon can be tessellated by fourteen $(2,3,7)$-triangles that fit together around its center. The big $\pi/7$-triangle has 24 times the area of one $(2,3,7)$-triangle. We now explain how to tessellate one (called “the first”) of the black big triangles by 24 of the small $(2,3,7)$-triangles; compare Figure 6. Take half a heptagon (tessellated by seven of the small triangles) to the left of its diameter, with the vertex 1 at the upper end and half an edge from vertex 4 to the lower end of the diameter; now reflect the lowest $(2,3,7)$-triangle in the half edge to give us eight small triangles that already tessellate one third of the big triangle. (The lowest vertex will be the center of the fourteengon.) 120°-rotations around heptagon vertex 2 complete the desired tessellation of the big triangle. Now extend by reflections to Klein’s tessellation of the hyperbolic plane by $(2,3,7)$-triangles and notice that these can be grouped either to a tessellation by heptagons or by big triangles, the vertices of the latter being centers of certain heptagons.

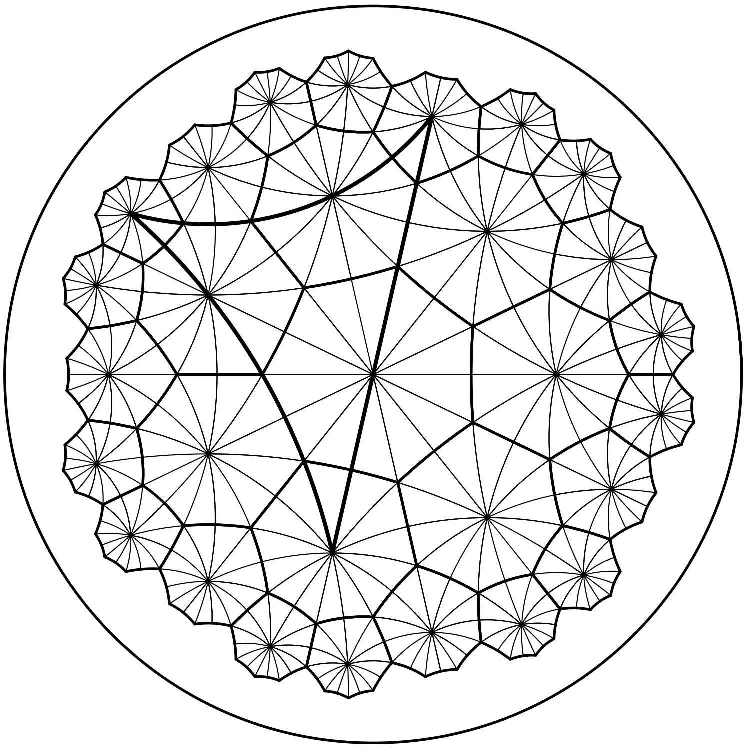

Figure 6. The tessellations by $\frac{\pi}{7}$-triangles and $\frac{\pi}{3}$-heptagons fit together.

Next we recover the above identification of the fourteengon edges as seen with the heptagons. The outer edge of the just tessellated big triangle joins counterclockwise fourteengon vertices, which we number 1 and 2. Connect midpoints of adjacent edges of the heptagon around the center of the fourteengon and extend these geodesics until they hit the fourteengon boundary. Notice that they are precisely eight such segments long. In other words, the eight segment translations along these geodesics give Klein's identification of the fourteengon edges! Notice also that the edges of the black triangles are indeed identified 5 steps forward (9 steps for the white ones). It is justified to quote Klein's lithographic plate again [Klein 1879].

We have now tessellated the above Riemann surface by 24 regular heptagons. Each vertex of the big triangle is the center of one heptagon and around each of these is a ring of seven heptagons. The identification translations are compositions of involutions (in the hyperbolic plane) around midpoints of heptagon edges that are four segments along a zigzag (called the Petrie polygon) apart — another indication that we have a platonic surface (Figure 7).

What remains to be checked? We know that the identification translations generate the deck group of a Riemann surface that also has the described order 3

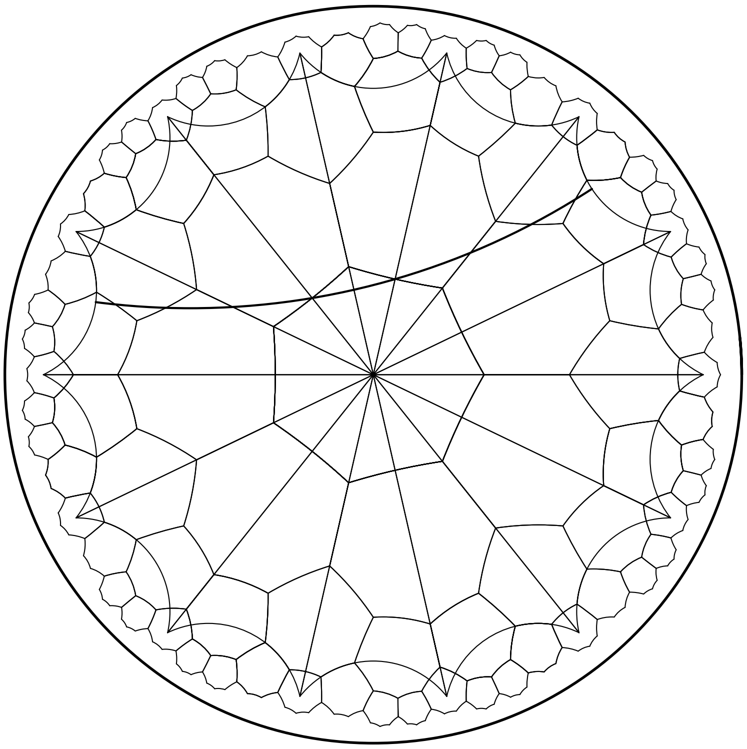

Figure 7. An eight-step geodesic crosses heptagon edge midpoints.

and order 7 rotational symmetries and the reflections in big triangle edges. Our Riemann surface will be platonic if *all* the eight segment geodesics that connect midpoints of heptagon edges in the hyperbolic plane connect equivalent points under the deck group, a new condition only for those that were not used to *define* the identifications. It is sufficient to check this for all the eight segment geodesics that meet the fourteengon fundamental domain. (If one discusses only candidates for identification generators and does not know a fundamental domain then at this point much more work is required.) Modulo reflections in symmetry lines through the center there are only four different eight segment geodesics that meet the fundamental domain. With the 120°-rotations these can be rotated into ones that were used to define the identifications! Now the hyperbolic description is complete enough to see platonicity, because the 180°-rotations around midpoints of heptagon edges in the hyperbolic plane always send equivalent points to equivalent ones.

How about other closed geodesics? If one connects the midpoints of second nearest heptagon edges and extends by applying 180°-rotations around the endpoints then these geodesics close after *six* such steps. Similarly, if one connects the midpoints of third nearest edges then these close again after eight steps.

Finally, also the symmetry lines close up: Let M be the midpoint of an edge, extend the edge across the neighboring heptagon, cross another heptagon, pass along another edge and cross a third heptagon to the midpoint M' of the opposite edge; M and M' are antipodal points of a closed geodesic (which is fixed under a reflection symmetry of the surface). All these geodesics are longer than the eight step ones used above and we did not see a fundamental domain that shows that one can take translations along them as generators of the deck group.

Finally, the heptagons also provide the connection with number theory: Puncture the surface in *all* the heptagon centers and choose a new complete hyperbolic metric by tessellating each punctured heptagon by seven $(3, 3, \infty)$-triangles. In the upper half plane model one such triangle is the well known fundamental domain for $SL_2(\mathbb{Z})$ and seven of them around the cusp at ∞ give the translation by 7 as one of the identification elements. This already connects Klein's surface with the congruence subgroup mod 7. In fact, $\Gamma(7)$ is the normal subgroup of the triangle group $SL_2(\mathbb{Z})$ (see Section 2) that is the deck group of this representation of Klein's surface. It is in this form that the surface first appears in [Klein 1879]; see [Rauch and Lewittes 1970; Gray 1982].

5. Oblique Pants and Isometry Subgroups

Pairs of pants decompositions are frequently used tools in the hyperbolic geometry of Riemann surfaces. One pair of pants is a Riemann surface of genus 0 bounded by three simply closed geodesics; it is further cut by shortest connections between the closed geodesics into two congruent right-angled hexagons. One builds Riemann surfaces by identifying pants along geodesics of the same length; the Fenchel–Nielsen coordinates are the lengths of these closed geodesics plus twist parameters since one can rotate the two pants against each other before the identification. If the hexagon vertices of neighboring pants coincide, the twist is $0°$ (or $180°$). Riemann surfaces have so many different pairs of pants decompositions that we need to say what we want to achieve for Klein's surface. The main motivation is to quickly understand symmetry subgroups that contain (many) involutions. The big triangle tessellation is not preserved under any involution and the heptagon tessellation has too many pieces. We find pairs of pants that are bounded by eight segment geodesics (the ones used in the previous section) in such a way that the twelve common vertices of the eight pant hexagons are fixed points of involutions. We will describe all types of symmetry subgroups with orders prime to seven in terms of this one pant decomposition. Our pant hexagons are not right angled but they have zero twist parameters. We have only found right angled pants with nonzero twists, therefore the oblique hexagons seem rather natural — to give a simpler example: one can subdivide parallelogram tori into rectangles, but only with a "twist"; that is, certain vertices lie on edges of other rectangles.

Our first goal is to develop a feeling for the shape of the surface — “versinnlichen” in Klein’s words. Therefore we begin by giving a 1-parameter family of genus 3 surfaces, embedded in $\mathbb{R}^3$ with tetrahedral symmetry and with the full permutation group S_4 contained in the automorphism group. In this case one can visualize the rectangular quotient tori; this may help to appreciate the obliqueness of Klein’s surface, which has no rectangular quotient tori.

Take a tetrahedral tessellation of the unit sphere and take a tube around its edge graph such that the tube not only respects the tetrahedral symmetry in $\mathbb{R}^3$ but also the conformal inversion in the unit sphere. Cut the legs between vertices by symmetry planes. This gives congruent pants, each with a 120° symmetry. The sphere cuts the pants into right-angled hexagons. One can interchange any two pants while mapping the others to themselves with a conformal map of $\mathbb{R}^3$ as follows: Invert in the unit sphere and then reflect in the plane of any of the great circle arcs of the tetrahedral graph with which we started.

An order 3 rotation subgroup commutes with the reflections in symmetry planes that contain the rotation axis, modulo this rotation group. The reflections pass to one reflection of the quotient torus, which clearly has two fixed point components. This makes the quotient torus rectangular. The 180°-rotations also commute with certain reflectional symmetries; they also descend with two fixed point components to the quotient torus.

In a last step we can ignore the embedding in $\mathbb{R}^3$ and identify the pants, using the *same twist* along all six closed geodesics. This will keep S_4 as a subgroup of automorphisms, but the complex conjugation on the quotient tori is generally lost. One of these surfaces is Klein’s, another one is the Fermat quartic, see Section 6. For these special surfaces the quotient tori have again reflectional symmetries, but these are more difficult to imagine.

5.1. Pants for Klein’s surface. Now we describe pants for Klein’s surface in terms of the heptagon tessellation; see Figure 8. Because of the previous description we look for pants with a 120° symmetry. Select P, Q as fixed points of an order 3 rotation group. P, Q are opposite vertices of any pair of heptagons with a common edge e. We call e a symmetry line “between” P and Q; the edges “through” P or Q are not symmetry lines of the pants. Apply the rotations around P, Q to our first pair of heptagons. We obtain the three heptagons adjacent to P and the three adjacent to Q. Together they have the correct area for one pair of pants, and they are identified to a pair of pants along the three symmetry edges “between” P and Q, but the three pant boundaries are not yet closed geodesics, they are zig-zag boundaries made of eight heptagon edges. Next, extend the three edges from P slightly beyond the neighboring vertices until they orthogonally meet three of the closed eight segment geodesics. Observe that these three geodesics are also met orthogonally by the extension of the three symmetry edges between the heptagons around P and the heptagons around Q. This means that these three eight step closed geodesics cut a pair of pants out of

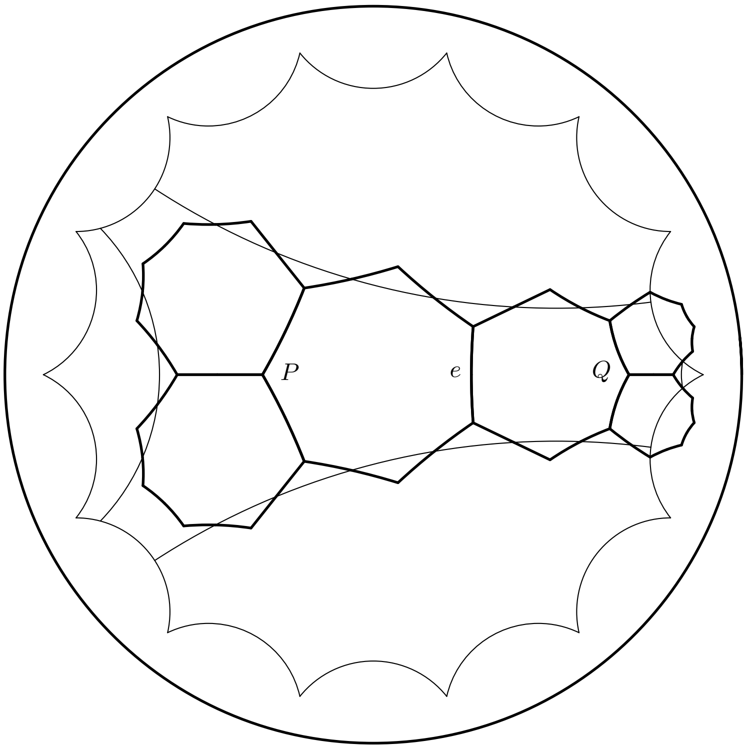

Figure 8. One pair of pants in the fourteengon fundamental domain.

the surface, which can be viewed as a smoothed version of the six heptagons. And the symmetry lines between P, Q cut these pants into two right-angled hexagons. (Precisely these pants have to be used in the initial description of an embedded surface. Since the hexagons of neighboring pants do not have common vertices we need a twist by one eighth of the total length of the boundary.)

Observe that reflection in the edges through P passes to a reflection of the quotient torus, quotient under the rotation group around P. This torus is made out of one pair of pants with two holes identified, the third hole is closed by one third of a pant, which is cut and identified along edges through P, Q. One can check that the fixed point set in the quotient has only one component so that the torus is rhombic. We find in Section 7 that its diagonals have a length ratio $\sqrt{7} : 1$.

Finally, we have to get the neighboring three pants, preferably with the help of an involution. Therefore we locate the *fixed points of one involution*: If one rotates around the midpoint M of any heptagon edge, then the two eight step geodesics through M are reversed so that their antipodal points $N, \overline{N}$ have also to be fixed points. Through both points there are again two eight step closed geodesics that get reversed; since the total number of fixed points is already

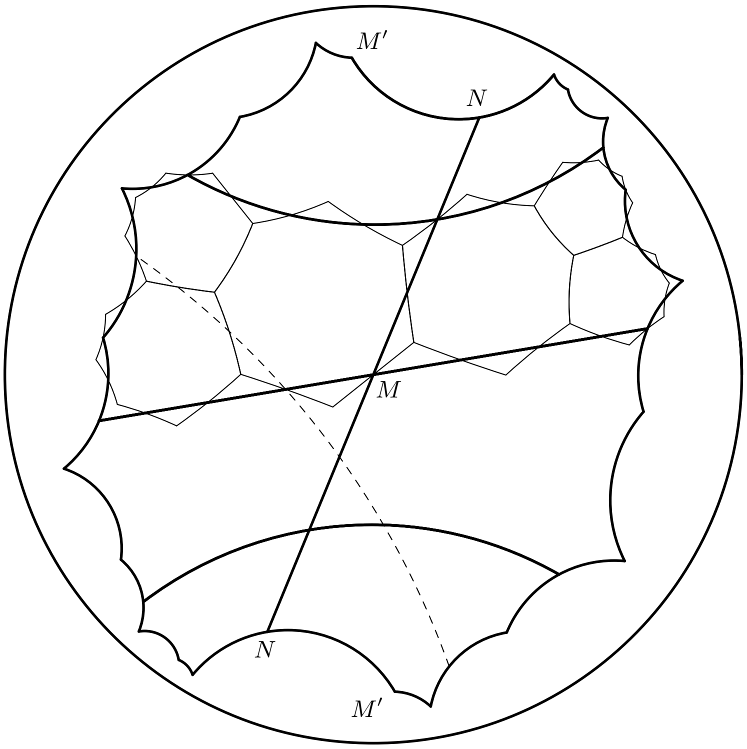

Figure 9. Eight big hexagons give four oblique pants and another fundamental domain.

known to be 4, these last two closed geodesics must meet in their common antipodal point M'.

We can use this intersection pattern of quadruples of eight step geodesics to get a pairs of pants decomposition without twist. First change the right angled hexagons to *oblique* ones: Instead of cutting the first pants into hexagons by the three symmetry edges between P and Q we use eight step geodesics through the midpoints of these edges, see Figure 9; at the first edge we have two choices, the other two are determined by the 120° symmetry. Observe that the sum of adjacent hexagon angles remains $= \pi$ and that the hexagon vertices are moved to involution centers! The edge lengths of the hexagons are now one quarter and one half, respectively, of our closed eight step geodesics. (Interpret the present description on the compact surface, but use a drawing in $\mathbb{H}^2$, Figure 9, to represent it.) The six vertices of the two hexagons of one pair of pants are three pairs of antipodal points on the three boundary geodesics — each of which consists of two long hexagon edges. Therefore each boundary geodesic can be rotated by 180° around the hexagon vertices on it and this moves the first pair of pants to three other pants, on the other side of each boundary geodesic. Note that the short edges that cut the first pair of pants into hexagons extend and also cut the neighboring pants into hexagons, i.e., these oblique pants fit together without twist and their hexagon vertices are involution centers. — We remark that the two conformal parameters of the family initially described in this section are in the present hyperbolic picture the ratio of adjacent hexagon angles and the ratio of adjacent edge lengths (recall the $120°$ symmetry of each hexagon).

So far the description emphasized an order 3 symmetry. We modify the description to emphasize a less obvious cyclic symmetry group of order 4; see Figure 9 above. Start with one of the closed geodesics, divided by involution centers M_i into four short hexagon edges (there are two possibilities for this subdivision, choose one). Through each of the four M_i choose the other closed geodesic, each divided by its antipodal point $\overline{M}_i$ into two long hexagon edges. (In the drawing in $\mathbb{H}^2$ each $\overline{M}_i$ is seen twice, each pair is connected by an eight step identification geodesic.) Through each of these four antipodes $\overline{M}_i$ we have again a unique other closed geodesic, but these are now pairwise the same ones — because we described above how four of them join the four fixed points of an involution. These last two geodesics therefore consist of the remaining short edges (sixteen in the $\mathbb{H}^2$-drawing) of our eight pant hexagons, so that we now have reached all the vertices. It remains to close the hexagons with long edges that fill up two more closed geodesics. We think of the hexagons as black and white, in a checkerboard fashion. Since there are $84 \cdot 2/8 = 21$ such closed geodesics we have $21 \cdot 2/3 = 14$ of these pant decompositions. Platonicity implies that the automorphism group is transitive on the set of 21 closed geodesics so that the isotropy group of each has order eight. We want to show that one such isotropy group leaves only one geodesic invariant. (Recall that the order 7 isotropy group of one heptagon has *three* invariant heptagons.) We can only propose proofs where the reader has to check how the eight step geodesics pass through a tessellation, by either pant hexagons or the earlier big triangles. Consider a tessellation by the big $\pi/7$-triangles that is kept invariant by a group of order 21, the order 7 rotations around the three common vertices and the fourteen order 3 rotations around the centers of the white big triangles (with the other fixed point of each rotation in the "opposite" black triangle). We claim that this group acts simply transitively on the 21 eight step closed geodesics. One can see this by following the geodesics that meet one of these triangles into the neighboring ones. Modulo its $\pm 120°$-rotations one white triangle is only met by three different kinds of eight step geodesics; already in one of the neighboring white triangles can one see that they are in fact all equivalent under this group.

5.2. Conjugacy classes and isometries. As a reward for this effort we can now describe all the isometries and also the subgroups of the automorphism group.

List of the conjugacy classes of the 168 orientation-preserving isometries. We have already characterized the isometries by sets that are left invariant; we only have to count that all 168 isometries have been found. Platonicity shows that all the isometries with the same characterization are in one conjugacy class.

ORDER 1: The class of the identity contains 1 element.

ORDER 2: The class of the involutions consists of 21 elements since each involution has four of the 84 edge midpoints as fixed points.

ORDER 3: The class of order three rotations has 56 elements since each of these rotations has two vertices as fixed points, and by interchanging the two fixed points with an involution one can conjugate one rotation and its inverse.

ORDER 7: There are two classes of order 7 rotations, each with 24 elements. Namely, with order 3 rotations can we cyclicly permute the three fixed points of one order 7 rotation and this conjugates the order 7 rotation with its second and fourth power; this gives two classes of three elements for each triple of fixed points, but each of the 24 heptagon centers can be mapped to every other one because of platonicity. The two classes of 24 elements are distinct, since only antiautomorphisms interchange black and white triangles.

ORDER 4: The class of order four translations of one eight step geodesic has 42 elements since each of the 21 closed geodesics has two such fixed point free translations and each translation is in the isotropy group of only one closed geodesic.

Altogether we have listed 168 isometries. So there are no isometries that we have not characterized, in particular no fixed point free involutions, and thus no genus 2 quotients of Klein's surface. The list also shows that the automorphism group is simple: Any normal subgroup has to consist of a union of full conjugacy classes, always including $\{id\}$; but its order has to divide 168, which is clearly impossible with the numbers from our list.

List of subgroups, assuming one fixed pairs of pants decomposition.

ORDER 2: Rotation around the midpoint of a short edge interchanges the adjacent black and white hexagons; every white hexagon has a black image. There are 21 of these subgroups.

ORDER 3: Cyclic rotation of one pair of pants into itself; cyclic permutation of the other pants. There are 28 of these subgroups.

ORDER 6: The two symmetries just given combine to the full isotropy group of *one* pair of pants. The decomposition into hexagons by short edges is not determined. There are 28 of these subgroups.

ORDER 4: From the construction of the pants we know the cyclic group generated by two step translations of a closed geodesic made of short edges. The uncolored tiling is preserved. There are 21 of these subgroups.

ORDER 4: The 180°-rotations around the twelve vertices of our pant hexagons form a Klein Four-group that preserves the colored tiling. There are 14 of these subgroups.

Order 8: Extend the cyclic translation subgroup of order 4 by the 180° rotation around the midpoint of one of the translated short edges. This is the isotropy group of an eight step closed geodesic. There are 21 of these subgroups.

Order 12: The full invariance group of the colored tessellation contains in addition to the above Klein group the order 3 rotations of each of the pants. There are 14 of these subgroups.

Order 24: All the above combine to the full invariance group of the uncolored tiling, abstractly this is the permutation group S_4. There are 14 of these subgroups.

Order 7: We know this as the invariance group of one heptagon. There are 8 of these subgroups.

Order 21: The invariance group of the tiling by 14 big triangles; no black and white triangles are interchanged; the isotropy of one triangle has order 3. There are 8 of these subgroups.

Order 14: Would contain an order 7 subgroup and an involution, hence at least 7 involutions and more order 7 rotations — too many.

Order 84: Would be a normal subgroup, which we excluded already.

For the remaining divisors of 168, namely 28, 42, 56, we have not found such a simple connection to the geometry. It is known that such subgroups do not occur, because an order 7 rotation and an involution generate the whole group.

6. Fermat Surfaces $x^k + y^k + z^k = 0$ Are Platonic

We add this section because, from the hyperbolic point of view, the Fermat quartic $x^4 + y^4 + z^4 = 0$ turns out to be surprisingly similar to Klein's surface. It has a platonic tessellation by twelve $2\pi/3$-octagons — one obtains the identification translations in the hyperbolic plane (which is tessellated by these octagons) if one joins two neighboring midpoints of edges and extends this geodesic to *six* such segments (see Figure 10 on page 35). Finally there is also a decomposition into congruent and 120°-symmetric pairs of pants that can be cut into oblique hexagons whose twelve common vertices are centers of involutions; this makes the Fermat quartic also a member of the 2-parameter family with at least S_4-symmetry, which we described in Section 5. Actually, *all* Fermat curves $x^k + y^k + z^k = 0$ can be described uniformly with their platonic tessellations. The hyperbolic picture is closer to this equation than in Klein's case, because the equation shows all the automorphisms immediately: One can independently multiply x and y by k-th roots of unity to obtain order k cyclic subgroups; cyclic permutation of the variables gives an order 3 rotation. In fact, any permutation of the variables gives an automorphism — including involutions, which were so hidden for Klein's surface. Also one can either derive from the equation the hyperbolic description or vice versa.

We start with a tessellated hyperbolic surface, point out obvious functions that have no common branch points and satisfy the Fermat equation: The rotations of order 3 and of order k point to a tessellation by π/k-triangles. To get their number we compute the Euler characteristic: The meromorphic function $f := x/z$ has k-fold zeros where the values of the function $g := y/z$ are k-th roots of -1; the k simple poles of both functions agree. The differential df has therefore $k \cdot (k-1)$ zeros and $2k$ poles, which gives $\chi = -k^2 + 3k$. Our expected triangle tessellation therefore has $F = 2k^2$ faces, $E = 3k^2$ edges and $V = 3k$ vertices and the dual tessellation consists of $3k$ regular $2k$-gons with angle $2\pi/3$. First we consider all tessellations in the hyperbolic plane. $2k$ of the triangles fit together around one vertex to form a regular $2k$-gon with angle $2\pi/k$. Into this we inscribe two regular k-gons with angle π/k by joining neighboring even-numbered and odd-numbered vertices, respectively. (Note that edges of these two polygons are symmetry lines of the triangle tessellation, the intersection of the two k-gons therefore is a regular $2k$-gon with angle $2\pi/3$.) We extend one of these inscribed k-gons (called red) to a tessellation of the hyperbolic plane and color its tiles in checkerboard fashion red and green; the other inscribed k-gon, called blue, we extend to a blue/yellow checkerboard tessellation. Note that the midpoints of the red and the blue k-gons agree; the vertices of the red/green ones are the midpoints of the yellow ones and vice versa, the vertices of the blue/yellow ones are the midpoints of the green ones. Now we define with the Riemann mapping theorem two functions on the hyperbolic plane, which we will recognize as the functions f, g above. Any checkerboard tessellation of the hyperbolic plane having an even number of regular polygons meeting at each vertex can be used in the same way: Map one tile to one hemisphere; we can keep its symmetry by first mapping a fundamental triangle (of the tiles symmetry group) to the corresponding sector of the hemisphere and then extend by reflection around the midpoint of the tile; finally extend by reflection in the edges of the tiles to the hyperbolic plane. We apply this by mapping a yellow and a green tile to the unit disc, normalizing so that the vertices go to k-th roots of -1. The functions, which we now call f and g, then have simple poles at the common centers of the red and blue polygons, respectively, and each has simple zeros at the other midpoints of its tiles, i.e., at the k-fold branch points of the other function whose branch values are k-th roots of -1. This gives the inhomogeneous Fermat equation

$$f^k + g^k + 1 = 0.$$

If we now identify points in the hyperbolic plane that are not separated by this pair of functions, then we are given a surface together with two tessellations by $2k$ regular k-gons; the vertices of both of them define a tessellation by $2k^2$ equilateral π/k-triangles. As a platonic tessellation the automorphism group would have to have order $6k^2$, but we already exhibited that many automorphisms of the Fermat equation — so this proves platonicity of the triangle tessellation and its dual, and then also of the tessellations by the k-gons. In particular this includes platonic tessellations with $\pi/5$- and $\pi/7$-triangles that we failed to obtain in the earlier attempts. Note also that $k = 3$ gives the triangle tessellation of the hexagonal torus.

We add some more details to the quartic, $k = 4$ (Figure 10). The k-gon tessellation consists of eight $\pi/4$-squares; they fit together around one vertex and give as fundamental domain a sixteengon with vertex angles alternately $\pi/4$ and $2\pi/4$ (and the vertices are identified to three points). There is only one possible edge identification pattern: If one wants the platonic symmetries around the center and notes the different angles at the vertices then the edge from a $\pi/4$-vertex clockwise to a $2\pi/4$-vertex can only be identified with the edges numbered 6 (translation axes in Figure 10) or 14 (clockwise). But the identifying translations are too short in the second case: The axis from edge 1 to edge 14 is two $\pi/4$-triangle edges long, which is only one half of a (vertex-)diameter of the sixteengon, a contradiction to platonicity. Now look at the dual of the triangle tessellation, by twelve $2\pi/3$-octagons (of which the figure shows nine) and note that we obtain the determined edge identification as composition of two involutions: Join two neighboring midpoints of edges of the central octagon and extend this (dithered) geodesic until it meets the boundary of the sixteengon fundamental domain. It is then six segments long; that is, the identification translation along this geodesic translates by a distance of six segments. Hence

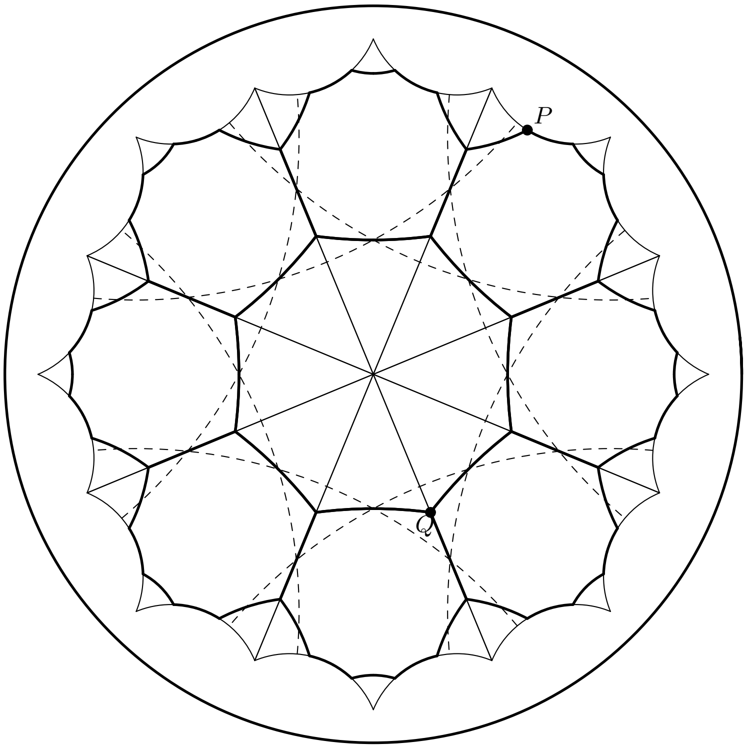

Figure 10. Fundamental domain for the Fermat quartic, with translation axes.

this translation can be written as a composition of two involutions whose centers are three segments apart, as claimed. This completes the hyperbolic description of the Fermat quartic with tiles and identifications.

We remark that the quotient by the 180°-rotation around the center is easy to see in both pictures: The fixed points are the four vertices of one tessellation by eight $\pi/4$-squares, the quotient therefore is the square torus tessellated by four squares; algebraically we have to identify points $(f,\, g) \sim (-f,\, g)$, i.e., we get the torus $x^2 + y^4 + 1 = 0$. Because of the order three symmetry we have three such quotient maps. An 180°-rotation around the midpoint of an octagon edge has also four fixed points so that the quotient is also a torus, a rectangular torus, because one reflection descends to the quotient with two fixed point components; for more information one has to compute. One may count that there are no other involutions, which proves that the Fermat quartic has no degree two projection to the sphere and therefore is not hyperelliptic; it also says that there are no fixed point free involutions; that is, our second genus 2 surface in Section 3.2, the one tessellated by the same $\pi/4$-triangles, is not a quotient of the Fermat quartic.

Finally we describe a pairs of pants decomposition such that the pant hexagons have involution centers as vertices. Number the vertices of the central $2\pi/3$-octagon of the fundamental domain. Choose vertex 6 as center of an order 3 rotation; the midpoint of the last edge (between vertex 8 and 1) is also midpoint between the *two* fixed points of this rotation; extend the octagon diameter from vertex 1 to 5 to a closed geodesic (a diameter of the fundamental domain) and rotate it by $\pm 2\pi/3$ around the chosen vertex 6; these three closed geodesics cut out of the surface a pair of pants tessellated by six half-octagons. Now we cut it into two hexagons: Join the midpoints of those two octagons, which have the mentioned edge from vertex 8 to 1 in common, across this edge; then also rotate this connection by $\pm 2\pi/3$ to obtain the desired hexagons. Finally obtain the neighboring pairs of pants by the involutions in the hexagon vertices — one does have to check that they do not overlap, but this is easy since the closed geodesics that we used to cut the pants into hexagons again traverse all four pairs of pants along short hexagon edges. The description of the symmetry subgroups is now very similar to the case of Klein's surface and will be omitted.

7. Cone Metrics and Maps to Tori

As we have seen above, certain quotients of Klein's surface are rhombic tori and we would like to know more about them. While we don't have any arguments using hyperbolic geometry to obtain this information, there is a surprisingly simple way using flat geometry. The idea is as follows: Suppose we have a holomorphic map from M^2 to some torus. Its exterior derivative will be a well defined holomorphic 1-form on M^2 with the special property that all its periods lie in a lattice in $\mathbb{C}$. Vice versa, the integral of such a 1-form will define a map to a torus whose lattice is spanned by the periods of the 1-form. There are two problems with this method:

It is rarely the case that one can write down holomorphic 1-forms for a Riemann surface. An exception are the hyperelliptic surfaces in their normal form $y^2 = P(x)$ where one can multiply the meromorphic form dx by rational functions in x and y to obtain a basis of holomorphic forms. But even if one can find holomorphic 1-forms then it is most unlikely that one can integrate them to compute their periods.

Flat geometry helps to overcome both problems simultaneously: Any holomorphic 1-form ω determines a flat metric on the surface that is singular in the zeroes of ω and that has trivial linear holonomy (parallel translation around any closed curve is the identity). This flat metric can be given by taking $|\omega|$ as its line element. Another way to describe it is as follows: Integrate ω to obtain a locally defined map from the surface to $\mathbb{C}$. Use this map to pull back the metric from $\mathbb{C}$ to the surface. A neighborhood around a zero of order k of ω is isometric to a euclidean cone with cone angle $2\pi(k+1)$, as can be seen in a local coordinate. And vice versa, specifying a flat *cone metric* without linear holonomy will always define a surface together with a holomorphic 1-form. The periods of this 1-form are just the translational part of the affine holonomy of the flat metric, which can be read off by developing the flat metric. Hence we have a method to construct Riemann surfaces with one holomorphic 1-form and full control over the periods. Usually one does not know whether two surfaces constructed by two different flat metrics coincide. The reason why we succeed with Klein's surface is that, surprisingly, we can apply the construction in three different ways so that we can produce three different 1-forms. This means that we have to show that three different cone metrics define the same conformal structure, which is difficult in general. But Klein's surface can be nicely described as a branched covering over the sphere with only three branch points, see Section 4.2, which allows to reduce this problem to the fact that there is only one conformal structure on the 3-punctured sphere.

For convenience, we introduce $[a,\, b,\, c]$-triangles, which are by definition euclidean triangles with angles

$$\frac{a\pi}{a+b+c}, \quad \frac{b\pi}{a+b+c}, \quad \frac{c\pi}{a+b+c}.$$

7.1. Again a definition of Klein's surface. Let $S = \mathbb{C}P^1 - \{P_1, P_2, P_3\}$ be a three punctured sphere. We construct a branched 7-fold covering over S that has branching order 7 at each P_i as follows: Choose another point P_0 in S and non-intersecting slits from P_0 to the punctures. Cut S along these slits, call the slit sphere S' and the edges at the slit from P_0 to P_i denote by a_i and a'_i. Now take 7 copies of S' and glue edge a_j in copy number i to edge a'_j in copy number $i + d_j \pmod 7$, where

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 4.$$

This defines a connected compact Riemann surface M^2 with a holomorphic branched covering map $\pi : M^2 \to S$. We call the branch points on M^2 also P_1, P_2, P_3. Viewing the sphere as the union of two triangles with vertices P_i, M^2 becomes then the union of fourteen triangles and Euler's formula gives $\chi(M^2) = -4$, $g = 3$. Equivalently, we could have used the Riemann–Hurwitz formula.

This description coincides with the one given in 4.2 — we have only switched from the identification of edges to slits for convenience. The reader can check again that the order 3 automorphism ϕ of the sphere that permutes the P_i lifts to M^2.

Observe that this description comes with a deck transformation of order 7.

7.2. Construction of holomorphic 1-forms. Now we want to construct holomorphic 1-forms on M^2. Consider a euclidean triangle with angles $\alpha_i \cdot \pi/7$ at the vertices P_i, for $i = 1, 2, 3$. Take the double to get a flat metric on the 3-punctured sphere, which also defines a conformal structure on the whole sphere. Because there is only one such structure, we can identify *any* doubled triangle with S and pull back the flat metric to M^2. In this metric a neighborhood of the branch points P_i on M^2 is isometric to a euclidean cone with cone angle $\alpha_i \cdot 2\pi$.

Remark that if we would take instead of the euclidean triangle a hyperbolic $2\pi/7$-triangle, this would give a hyperbolic metric without singular points — the same one that we know already.

After selecting a base point and a base direction on the universal cover $\hat{M}^2$ of $M^2 - \{P_1, P_2, P_3\}$, consider the developing map

$$\mathrm{dev} : \hat{M}^2 \to \mathbb{C}$$

of this flat metric. Let γ be a deck transformation of $\hat{M}^2$. Then $\mathrm{dev}(z)$ and $\mathrm{dev}(\gamma z)$ differ by an isometry of $\mathbb{C}$ and $\alpha(z)$ and $\alpha(\gamma z)$ with $\alpha = d\,\mathrm{dev}$ differ by a rotation. We want the holomorphic 1-form α to descend to M^2 and we therefore want all these rotations to be the identity. This is equivalent to having trivial linear holonomy of the flat cone metric on M^2. We call triangles such that the cone metric on M^2 above has this property *admissible*.

Because a triangle has a simpler geometry than a cone metric on M, we will do the holonomy computation on S and therefore need to be able to recognize closed curves on S that lift to closed curves on M^2:

Let c be a closed curve on S and $A_j = \#(c, a_j) =$ algebraic intersection number of c with the slit a_j. Let $\tilde{c}$ be any lift of c to M^2. Then $\tilde{c}$ is closed in M^2 if and only if $A_1 d_1 + A_2 d_2 + A_3 d_3 \equiv 0 \pmod 7$, because by crossing the slit a_j we change from copy i to copy $i + d_j$, the contributions from all crossed slits add up and we want to arrive in the same copy as we started.

To compute the linear holonomy of the curve c we modify it at every intersection with a slit as follows: Instead of crossing the slit a_j, we prefer to walk around the point P_j. The new curve will never cross a slit and therefore be ho-

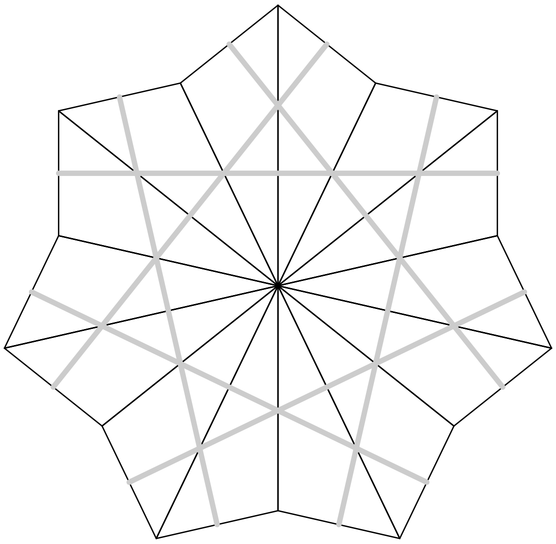

Figure 11. The flat fourteengon fundamental domain represents the surface together with a holomorphic 1-form.

motopically trivial, hence without linear holonomy. But each time we modified the curve c at the slit a_j, we changed the linear holonomy by a rotation by the cone angle $\alpha_j \cdot 2\pi/7$. This sums up to

$$\mathrm{hol}(c) = \text{rotation by } (A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3) \cdot \frac{2\pi}{7}\,.$$

Therefore, the linear holonomy of each closed curve in M^2 is trivial if and only if whenever $A_1 d_1 + A_2 d_2 + A_3 d_3 \equiv 0 \pmod 7$, then $A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 \equiv 0 \pmod 7$. This is here the case for

$$(\alpha_i) = (1, 2, 4),\ (2, 4, 1),\ (4, 1, 2).$$

These are all the same triangles with differently labeled vertices. Corresponding to these three possibilities of choosing cone metrics we obtain *three different* holomorphic 1-forms ω_i on M^2.

All this is illustrated in Figure 11, which shows the fourteengon fundamental domain of Figure 4 where each $\pi/7$-triangle is replaced by an admissible euclidean triangle. That this metric gives rise to a holomorphic 1-form is instantly visible because the identifications of edges are achieved by euclidean parallel translations.

Observe that the scaling of an ω_i is well defined as soon as we have chosen a fixed triangle, but up to now there is no natural way and no necessity to do this.

As mentioned above, a cone angle $2\pi k$ of a cone metric causes a zero of order $k-1$ of the 1-form defined by the derivative of the developing map. So we obtain for the divisors of the ω_i on M^2 the expressions

$$\begin{aligned}(\omega_1) &= P_2 + 3P_3\,,\\ (\omega_2) &= P_1 + 3P_2\,,\\ (\omega_3) &= P_3 + 3P_1\,.\end{aligned}$$

This allows us to derive the equation for Klein's surface in a different way than in the hyperbolic discussion, because there the holomorphic 1-forms were only obtained after we had the first equation:

Set $f = \omega_1/\omega_3$ and $g = \omega_2/\omega_3$. We have

$$\begin{aligned}(f) &= -3P_1 + P_2 + 2P_3\,,\\ (g) &= -2P_1 + 3P_2 - P_3\,,\\ (g^3 f) &= -9P_1 + 10P_2 - P_3\,,\\ (f^3) &= -9P_1 + 3P_2 + 6P_3\,,\\ (fg^2) &= -7P_1 + 7P_2\,.\end{aligned}$$

Assuming that P_1 is mapped to ∞, P_2 to 0 and P_3 to -1 by π (see Section 7.1), so that $(\pi) = -7P_1 + 7P_2$ we see from the above table that (after scaling fg^2) fg^2 and π coincide. Therefore $fg^2 + 1$ has a zero of order 7 and $g^3 f + g$ a zero of order 6 in P_3. From the divisor table we see that there is only one pole of order 9 at P_1 which is completely compensated by the zeros at P_2 and P_3; hence $(g^3 f + g) = -9P_1 + 3P_2 + 6P_3$, and after a second normalization we have

$$f^3 + g^3 f + g \equiv 0 \quad \Longrightarrow \quad \left(\frac{\omega_1}{\omega_3}\right)^3 + \left(\frac{\omega_2}{\omega_3}\right)^3 \frac{\omega_1}{\omega_3} + \frac{\omega_2}{\omega_3} = 0,$$

so that the 1-forms themselves — suitably scaled — satisfy one equation for the Klein surface:

$$\omega_1\omega_2^3 + \omega_2\omega_3^3 + \omega_3\omega_1^3 = 0\ .$$

Note that the other equation can be written as

$$\pi(\pi - 1)^2 = f^7$$

by comparing divisors and scaling f.

7.3. Finding maps to tori. Now we want to find maps from M^2 to tori. First we determine the Jacobian. As explained above, we have to look for holomorphic 1-forms whose periods span a lattice in $\mathbb{C}$. Because M^2 is of genus 3, any holomorphic 1-form is a linear combination of the three forms ω_i above. We start by computing their periods. Because we have everything reduced to triangles, this is an exercise in euclidean geometry.

Consider an admissible triangle with angles $\alpha_i \cdot \pi/7$ in P_i and take the double. Choose the base point P_0 very close to P_1 and consider loops c_1, c_2 at P_0: c_1 is

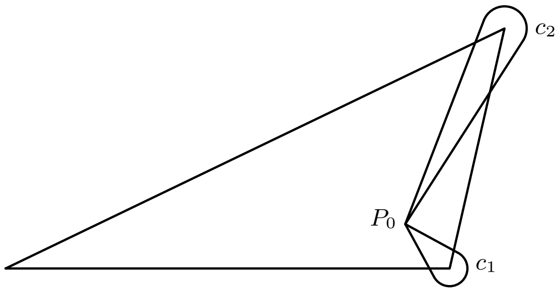

Figure 12. The integration paths.

a loop around P_1 and c_2 a loop around P_2, both with winding number 1. Every closed loop in S will be homotopic to a product of these two loops. Consider

$$\eta_k = c_1^k c_2 c_1^{5-k}, \quad \text{for } k = 1, \ldots, 6.$$

These curves will have closed lifts to M^2 and it is easy to see that they furnish us with a homology base of M^2, for example as follows: Take the cone metric with $\alpha_1 = 1$ and repeatedly reflect the [1, 2, 4]-triangle to arrange all the fourteen copies around P_1. Since a lift of c_1 is a $2\pi/7$-arc around the center one can see that the $\eta_0, \ldots, \eta_6$, are homotopic to the eight step closed geodesics that we used to identify edges of our fourteengon.

To compute their affine holonomy with respect to a cone metric on M^2, we can as well work on S. Recall how the development is constructed in this simple case: Follow the path starting at P_0 until it meets the boundary of the triangle (which is thought of as the upper hemisphere). Continue in the reflected triangle the portion of the path on the other hemisphere until it hits a triangle boundary again. Keep continuing until the endpoint of the path in S is reached and we have constructed the developped path in $\mathbb{C}$. This shows that $\operatorname{dev}(c_1)$ consists of a rotation by $\alpha_1 \cdot 2\pi/7$ and a translational part that can be made arbitrarily small since the holonomy is independent of how close we chose P_1 to P_0 . On the other hand, $\operatorname{dev}(c_2)$ consists (again up to an arbitrarily small error) of a translation by twice the height of the triangle with vertex P_1 followed by a rotation of angle $\alpha_2 \cdot 2\pi/7$. Since the last rotations do not change the endpoint of the developped path we obtain

$$\operatorname{dev}(\eta_k) = \zeta^{k\cdot\alpha_1} \cdot h_1 \quad \text{with} \quad \zeta = e^{2\pi \mathbf{i}/7}$$

and h_1 denotes the length of the height. Because we are still free to scale the triangles independently, we do this in a way that the periods look as simple as possible, namely we scale the height h_1 to length 1. So the triangles under consideration will have different size, but we obtain the periods as

$$\int_{\eta_k} \omega_1 = \zeta^k\,, \quad \int_{\eta_k} \omega_2 = \zeta^{2k}\,, \quad \int_{\eta_k} \omega_3 = \zeta^{4k}\,.$$

This gives a base for the lattice of the Jacobian of M^2.

As explained above, we now want to find linear combinations

$$\omega = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$$

such that the periods of ω span a lattice in $\mathbb{C}$. Remark that if this is the case for some a_i, it will also be true for $\zeta^{d_i}a_i$. This has its geometric reason in the fact that the covering transformation of order 7 acts on the ω_i by multiplication with ζ^{d_i}. The corresponding maps to tori differ therefore only by an automorphism. We guess the first example of such a map to a torus:

The first map to a torus. Take $a_1 = a_2 = a_3 = 1$. Here $\omega = \omega_1 + \omega_2 + \omega_3$ and the periods $e_k := \int_{\eta_k} \omega$ lie in the lattice spanned by

$$v_1 = e_1 = e_2 = e_4 = \zeta^1 + \zeta^2 + \zeta^4 \quad \text{and} \quad v_2 = e_3 = e_5 = e_6 = \zeta^3 + \zeta^5 + \zeta^6\,.$$

Observe that

$$\left|\zeta^1 + \zeta^2 + \zeta^4\right|^2 = \left|\zeta^3 + \zeta^5 + \zeta^6\right|^2 = (\zeta^1 + \zeta^2 + \zeta^4)\cdot(\zeta^6 + \zeta^5 + \zeta^3) = 2\,;$$

hence

$$\left|\zeta^1 + \zeta^2 + \zeta^4 - \zeta^3 - \zeta^5 - \zeta^6\right|^2 = 7, \quad \zeta^1 + \zeta^2 + \zeta^4 = \frac{-1+\sqrt{-7}}{2}\,,$$

so that we obtain a map $\psi := \int \omega$ onto a rhombic torus $\mathbb{T}$ with edge length $\sqrt{2}$ and diagonal lengths $\sqrt{7}$ and 1. The lattice points are the ring of integers in the quadratic number field $\mathbb{Q}(\sqrt{-7})$. This implies that the torus has complex multiplication: Multiplication by any integer in $\mathbb{Q}(\sqrt{-7})$ maps the lattice into itself and therefore induces a covering of the torus over itself, in particular coverings of degree 2 and 7.

The standard basis for this lattice Γ is $\{1, \tau\}$, where τ is defined as $\frac{1}{2}(-1+\sqrt{-7})$. The Weierstraß $\wp$-function for Γ is a degree 4 function for the index 2 sublattice $\tau\cdot\Gamma$ and $\wp(z/\tau)/\tau^2$ is the Weierstraß $\wp$-function for $\tau\cdot\Gamma$. Starting from these two functions one can derive the following equation for the torus, which is defined over $\mathbb{Q}$:

$$q'^2 = 7q^3 - 5q - 2\,.$$

Remember that we are hunting for the quotient tori of Klein's surface by the automorphism groups of order 2 and 3. Because we have scaled the triangles that define the ω_i to different size, it is unlikely that their sum will give us a 1-form invariant under the order 3 rotation. But we might have with ψ a degree 2 quotient map. To decide this, we compute the degree of ψ. With respect to the basis η_k of $H^1(M^2,\mathbb{Z})$ and the basis v_1, v_2 of $H^1(\mathbb{T},\mathbb{Z})$, we have the matrix representation

$$\psi_* = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

To switch to cohomology, we need the intersection matrix I for our homology basis and its inverse, which represents the cup product with respect to the dual basis. The first can be read off from Figure 4:

$$I = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & +1 & 1 & 0 & -1 & -1 \\ 0 & 0 & +1 & +1 & 0 & -1 \\ -1 & 0 & 0 & +1 & 1 & 0 \end{pmatrix}, \qquad I^{-1} = \begin{pmatrix} 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

Denoting the dual basis of η_k by β_k and that of v_i by γ_i, we compute

$$\begin{aligned} \deg\psi = \deg\psi \cdot \int_{\mathbb{T}} \gamma_1 \wedge \gamma_2 &= \int_{M^2} \psi^*\gamma_1 \wedge \psi^*\gamma_2 \\ &= \int_{M^2} (\beta_1+\beta_2+\beta_4) \wedge (\beta_3+\beta_5+\beta_6) \\ &= 1+1+0+1+0+1+1+1+1 = 7\,. \end{aligned}$$

This is certainly a surprise, because we couldn't find any degree seven map to a torus in the hyperbolic setting. This means especially that ψ is *not* a quotient map.

The second map to a torus. As already mentioned, the above ω is not invariant under the triangle rotation automorphism ϕ of order 3, because we have normalized the ω_i using triangles of different size. By taking one fixed triangle size for all 1-forms, that is, by only permuting the labels of the vertices, we will obtain differently scaled 1-forms $\tilde{\omega}_i$, which now do have the invariance property

$$\phi^*\tilde{\omega}_i = \tilde{\omega}_{i+1}\,.$$

This means that

$$\int_{\eta_k} \tilde{\omega}_i = \zeta^{kd_i} \int_{\eta_0} \tilde{\omega}_i$$

with

$$\int_{\eta_0} \tilde{\omega}_i = \vec{h}_i\,,$$

where $\vec{h}_i \in \mathbb{C}$ is the height based at P_i in one fixed triangle $P_1P_2P_3$ with the angles $\beta_i := \alpha_i\pi/7$ at P_i, for $\alpha_i \in \{1, 2, 4\}$. Denote by h_i the norm of $\vec{h}_i$. Then compute

$$\vec{h}_2 = e^{-\mathbf{i}\beta_3}\frac{h_2}{h_1}\vec{h}_1, \qquad \frac{h_2}{h_1} = \frac{\sin\beta_1}{\sin\beta_2},$$

so that

$$\vec{h}_2 = e^{\mathbf{i}(\pi-\beta_3)}\frac{\sin\beta_1}{\sin\beta_2}\vec{h}_1\,, \quad \vec{h}_3 = e^{\mathbf{i}(\pi-\beta_1)}\frac{\sin\beta_2}{\sin\beta_3}\vec{h}_2\,.$$

Now introduce temporarily $\xi = e^{2\pi \mathbf{i}/14} = -\zeta^4$ and $\beta = \beta_1$. Write $\approx$ for equality up to a non-zero factor independent of k. Then

$$\begin{aligned}
\int_{\eta_k} \tilde\omega_1 + \tilde\omega_2 + \tilde\omega_3 &= \zeta^k \vec h_1 + \zeta^{2k}\vec h_2 + \zeta^{4k}\vec h_3 \\
&= \left(\zeta^k - \zeta^{2k} e^{-\mathbf{i}\beta_3}\frac{\sin\beta_1}{\sin\beta_2} + \zeta^{4k} e^{-\mathbf{i}(\beta_1+\beta_3)}\frac{\sin\beta_1}{\sin\beta_3}\right)\vec h_1 \\
&= \left(\zeta^k - \zeta^{2k}\xi^{-4}\frac{\sin\beta}{\sin 2\beta} + \zeta^{4k}\xi^{-5}\frac{\sin\beta}{\sin 4\beta}\right)\vec h_1 \\
&\approx \frac{1}{\xi-\xi^{-1}}\xi^{2k} - \frac{1}{\xi^2-\xi^{-2}}\xi^{4k-4} + \frac{1}{\xi^4-\xi^{-4}}\xi^{8k-5} \\
&\approx \frac{\zeta^k}{\zeta-1} + \frac{\zeta^{2k+2}}{\zeta^2-1} + \frac{\zeta^{4k-1}}{\zeta^4-1} \\
&\approx \frac{\zeta^k}{\zeta-1} + \frac{\zeta^{2k}}{1-\zeta^5} + \frac{\zeta^{4k}}{\zeta^5-\zeta}\,.
\end{aligned}$$

Denote this last expression for the period over η_k by e_k. One easily computes

$$\begin{aligned}
e_2 &= 0\,,\\
e_0 &= -e_4 = e_1 - e_3\,,\\
e_5 &= -e_0 - e_1\,,\\
e_6 &= e_5 + 3e_0\,,
\end{aligned}$$

so that

$$\begin{aligned}
\tilde v_1 &:= -e_3 = 1 + \zeta^2 - \zeta^3 - \zeta^4\,,\\
\tilde v_2 &:= e_1 = -1 + \zeta^3 + \zeta^4 - \zeta^5
\end{aligned}$$

constitute a basis for the lattice spanned by all periods e_k. So this time we obtain a map $\tilde\psi$ to a torus as the quotient map $X \to X/(\phi)$. Using the above mapping degree argument, one finds indeed that $\deg\tilde\psi = 3$.

Remarkably, the quotients of the period vectors of the two tori agree:

$$v_1 \cdot \tilde v_2 = 2(\zeta^5 - \zeta^2) = v_2 \cdot \tilde v_1,$$

so that $\tilde\psi$ is a different map to the same torus $\mathbb{T}$.

The thrid map to a torus. Finally, we know two ways to find a degree 2 map to a torus. The first is to guess. This works as follows: Any holomorphic map $\psi : M^2 \to T^2$ will induce a complex linear map $\operatorname{Jac} M^2 \to \operatorname{Jac} T^2 = T^2$ and therefore a direct factor of $\operatorname{Jac} M^2$. After having found two such factors, there has to be a third so that the Jacobian of M^2 is up to a covering the complex product of three 1-dimensional tori. To find the third factor one just has to compute the kernel of the two linear maps that are the projections of $\operatorname{Jac} M^2$ to the tori already found and write down a projection onto this kernel. So the recipe is: Take the cross product of the 1-forms that define the maps to the two tori with respect to the basis ω_i.

For instance, we can take for the first torus the linear combination $\omega_1+\omega_2+\omega_3$ and for the second $\zeta\omega_1+\zeta^2\omega_2+\zeta^4\omega_3$, which is obtained from the first by applying an order 7 rotation. Hence also $(\zeta^4-\zeta^2)\omega_1+(\zeta-\zeta^4)\omega_2+(\zeta^2-\zeta)\omega_3$ integrates to a map to a torus, we compute the period integrals to

$$\begin{aligned} e_0 &= e_1 = 0\,, \\ -e_2 &= e_4 = \sqrt{-7}\,, \\ e_3 &= 3-\zeta^3-\zeta^5-\zeta^6 = \frac{7+\sqrt{-7}}{2}\,, \\ e_5 &= -3+\zeta+\zeta^2+\zeta^4 = -e_3+e_4\,. \end{aligned}$$

Taking $v_1 = e_4$ and $v_2 = e_3$ we get a basis for our familiar torus $\mathbb{T}$, and the computation of the mapping degree yields 2. Being a twofold covering, this map must be the quotient map of an involution.

The other way to find such a torus is analogous to the approach for the second torus: We just have to find a 1-form invariant under an involution. But while the order 3 rotations were apparent from the construction of the surface, this is not the case for the involutions, and we don't know a geometric method to derive the operation of an involution on the ω_i by euclidean or hyperbolic means. On the other hand, this operation already occurs in [Klein 1879] who used an algebraic-geometric description of his surface to obtain this map as

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} \longmapsto \frac{1}{\sqrt{-7}} \begin{pmatrix} -\zeta^2+\zeta^5 & \zeta^3-\zeta^4 & -\zeta+\zeta^6 \\ \zeta^3-\zeta^4 & -\zeta+\zeta^6 & -\zeta^2+\zeta^5 \\ -\zeta+\zeta^6 & -\zeta^2+\zeta^5 & \zeta^3-\zeta^4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix},$$

where $A^3B+B^3C+C^3A=0$. This information can be used to obtain the above and other invariant 1-forms — one has only to be aware of the fact that one has to take the scaled 1-forms $\tilde{\omega}_i$, which satisfy $\tilde{\omega}_1\tilde{\omega}_2^3+\tilde{\omega}_2\tilde{\omega}_3^3+\tilde{\omega}_3\tilde{\omega}_1^3=0$.

7.4. Further computations for our examples. It is possible to compute the Jacobians of all the hyperbolic examples we have given in the preceding sections using cone metrics. This is quite straightforward. For instance, for the genus 2 surface constructed from hyperbolic $\pi/5$-triangles one uses as the conformal definition a 5-fold covering over the 3-punctured sphere analogous to step 1 above. Here one has to take $d_1 = 1$, $d_2 = 1$ and $d_3 = 3$. Using the same reasoning as in Section 7.2, one finds that admissible triangles for *this* covering are [1, 1, 3]- or [2, 2, 1]-triangles. This gives holomorphic 1-forms with divisors $2P_3$ and P_1+P_2, which can be used to derive an equation for the surface:

Introduce $w := \omega_2/\omega_1$ and denote the covering projection by z, normalized so that P_1, P_2, P_3 are mapped to 0, 1, ∞. After scaling w we obtain the equation from Section 3.1, $w^5 = z(z-1)$. The same computation as in 7.3 gives for the Jacobian of the surface the following basis matrix of the lattice

$$\begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^3 & \zeta & \zeta^4 \end{pmatrix} \quad \text{where } \zeta = e^{2\pi \mathbf{i}/5}.$$

The other genus 2-example is slightly more complicated. It is described as a double covering M^2 over a sphere punctured at octahedron vertices. So, instead of starting with the 3-punctured sphere as above we now have to start with a 6-punctured sphere. Here we can clearly see the limitations of our method: It is rarely the case that two different cone metrics on a 6-punctured sphere define the same conformal structure. However the octahedron itself is symmetric enough so that we can achieve this: Represent the conformal structure of the octahedron by the Riemann sphere with punctures at the images of the octahedron vertices under stereographic projection, that is at the fourth roots of unity, at 0 and at ∞. Then the map $z \mapsto z^4$ defines a branched covering over the 3-punctured sphere S, which we can handle. This means that any doubled triangle metric on S when lifted to the octahedron defines the same conformal structure. The metrics we obtain in this way can be described geometrically as follows: Instead of constructing the octahedron from equilateral triangles, it is allowed to construct it from isosceles triangles (bases along the equator). It is even allowed to take two different heights over the same base for the upper and the lower hemisphere. For instance, we can choose the triangles in such a way that the cone angles on the octahedron are π in ∞, 3π in 0 and also π at the roots of unity by taking four [2, 3, 3]-triangles and four [6, 1, 1]-triangles.

This cone metric on the octahedron is now admissible in the sense of Section 5.1: its lift to the double cover M^2 has no linear holonomy! This is an immediate consequence of these three facts:

- each branch point has order 2,
- each cone angle is an *odd* multiple of π,
- a closed curve on S has a closed lift to M^2 if and only if it crosses an even number of slits.

If we denote the branch points over 0 and ∞ by P_+ and P_-, respectively, we have found a holomorphic 1-form ω_1 with divisor $2P_+$. By interchanging the angles given to 0 and ∞, we obtain a 1-form ω_2 with divisor $2P_-$.

These two 1-forms are not sufficient to produce an equation for the surface. But this will be possible by using another *meromorphic* 1-form, also constructed using cone metrics: Represent the conformal structure of the octahedron just by the flat euclidean plane, where all cone points save ∞ have cone angle 2π and ∞ has -2π. The lift of this metric to M^2 (recall that all vertices are simple branch points) defines a meromorphic 1-form ω_3 with divisor $-3P_- + P_+ + P_1 + P_2 + P_3 + P_4$ where P_i denote the preimages of the roots of unity. Introduce the function $v = \omega_3/\omega_2$ and denote the covering projection from M^2 to the octahedron by z. Comparing divisors and scaling v now gives the equation

$$v^2 = z(z^4 - 1)$$

which is equivalent to the one in Section 3.2, put $w = v/((z+1)(z+\mathbf{i}))$.

Now we compute a basis for the lattice of the Jacobian of M^2. As a homology basis on M^2 we can take lifts of the 4 loops on the octahedron that start near 0 and go once around one root of unity. Because we have cone angle π at the roots of unity for both holomorphic 1-forms, these curves develop to straight segments of equal length, which we scale to 1. The directions can be easily obtained from the different cone angles at 0 and we get for the period matrix of ω_1, ω_2

$$\begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^3 & \zeta^6 & \zeta \end{pmatrix} = \begin{pmatrix} 1 & \frac{\sqrt{2}}{2}(1+\mathbf{i}) & \mathbf{i} & \frac{\sqrt{2}}{2}(-1+\mathbf{i}) \\ 1 & \frac{\sqrt{2}}{2}(-1+\mathbf{i}) & -\mathbf{i} & \frac{\sqrt{2}}{2}(1+\mathbf{i}) \end{pmatrix},$$

where $\zeta = e^{2\pi\mathbf{i}/8}$. From this it follows that $\omega_1+\omega_2$ and $\omega_1-\omega_2$ can be integrated to give maps to the isomorphic tori with lattices spanned by $\big(1,\ \sqrt{-2}/2\big)$ and $\big(1,\ 2\sqrt{-2}\big)$. They also have complex multiplication, as can be seen by folding a sheet of A4 paper.[1]

Similarly one can compute the Jacobians of all the Fermat surfaces. We carry this out for the quartic:

Here X will always be the four punctured sphere with punctures at the points $1, \mathbf{i}, -1, -\mathbf{i}$ or $0, 1, -1, \infty$. This is the only four punctured sphere for which we can sometimes describe *different* admissible cone metrics explicitly.

We will construct a branched covering M^2 of genus 3 over X very similar to the construction of Klein's surface, but this time using 4 slits instead of 3 and taking only a fourfold covering. We choose all the four numbers d_i that we need to specify the identifications to be 1. Using the Riemann–Hurwitz formula one can check that the so-defined surface has genus 3. Now one has to be careful to choose cone metrics on X, because we have to guarantee that different cone metrics live on the *same* 4-punctured sphere, namely X. This is done economically by representing X as a double cover over S such that $\mathbf{i}$, $-\mathbf{i}$ are mapped to ∞ and 1, -1 are mapped to 0, 1 without branching. Then *admissible* triangles on S in the sense that their lift to M^2 via X has no linear holonomy are given as [1, 5, 2]-, [5, 1, 2]- and $[2, 2, 4]$-triangles. These lift to three cone metrics on X with the following angles:

1	$\mathbf{i}$	-1	$-\mathbf{i}$
$\pi/2$	$\pi/2$	$5\pi/2$	$\pi/2$
$5\pi/2$	$\pi/2$	$\pi/2$	$\pi/2$
π	π	π	π

Counting the branching orders, this gives holomorphic 1-forms on M^2 with divisors $4P_2$, $4P_1$, and $P_1 + P_2 + P_3 + P_4$. And they can be used to derive the equation $x^4 + y^4 + z^4 = 0$.

[1]The ISO series of paper sizes, A1, A2, ..., has the property that cutting in half an An sheet yields two A$(n+1)$ sheets similar to the original one.

For the computation of the Jacobian we want to use the curves c_1, c_2 defined in Section 7.3 and use the lifts of the curves

$$\eta_k = c_1^k c_2 c_1^{5-k} \quad \text{for } k = 0, \dots, 5$$

to M^2 via X as a homology basis. Remark that these curves have closed lifts on M^2 because we have decided to start at P_1, which is an eightfold branch point of M^2 over 0 — starting at P_2 or P_3 would not produce closed curves. But having been careful gives after checking that the η_k form indeed a homology basis the following period matrix of the ω_i:

$$\begin{pmatrix} 1 & \zeta & \mathbf{i} & \zeta^3 & -1 & \zeta^5 \\ 1 & \zeta^5 & \mathbf{i} & \zeta^7 & -1 & \zeta \\ 1 & \mathbf{i} & -1 & -\mathbf{i} & 1 & \mathbf{i} \end{pmatrix}, \quad \zeta = e^{2\pi \mathbf{i}/8}.$$

So we see that ω_3, $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$ integrate to maps to the square torus — which gives another proof that this M^2 does not cover the second genus 2-example: If this were the case, there would be a nontrivial map between their respective Jacobians by the universal property of Jacobians, inducing a nontrivial map from the square torus to the A4-torus (page 47, footnote). Such a map does not exist.

References

[Bujalance and Singerman 1985] E. Bujalance and D. Singerman, “The symmetry type of a Riemann surface”, *Proc. London Math. Soc.* (3) **51**:3 (1985), 501–519.

[Burnside 1911] W. Burnside, *Theory of groups of finite order*, Cambridge U. Press, 1911.

[Coxeter and Moser 1957] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Springer, Berlin, 1957. Fourth edition, 1980.

[Gray 1982] J. Gray, “From the history of a simple group”, *Math. Intelligencer* **4**:2 (1982), 59–67. Reprinted in this collection.

[Hurwitz 1893] A. Hurwitz, “Über algebraische Gebilde mit eindeutigen Transformationen in sich”, *Math. Annalen* **41** (1893), 403–442.

[Klein 1879] F. Klein, “Ueber die Transformationen siebenter Ordnung der elliptischen Funktionen”, *Math. Annalen* **14** (1879), 428–471. Reprinted as [Klein 1923, LXXXIV, pp. 90–136]. Translated in this collection.

[Klein 1923] F. Klein, *Gesammelte Mathematische Abhandlungen, 3: Elliptische Funktionen* etc., edited by R. Fricke et al., Springer, Berlin, 1923. Reprinted by Springer, 1973.

[Kulkarni 1982] R. S. Kulkarni, “Pseudofree actions and Hurwitz's $84(g-1)$ theorem”, *Math. Annalen* **261**:2 (1982), 209–226.

[Lehner and Newman 1967] J. Lehner and M. Newman, “On Riemann surfaces with maximal automorphism groups”, *Glasgow Math. J.* **8** (1967), 102–112.

[Macbeath 1965] A. M. Macbeath, “On a curve of genus 7”, *Proc. London Math. Soc.* (3) **15** (1965), 527–542.

[Rauch and Lewittes 1970] H. E. Rauch and J. Lewittes, "The Riemann surface of Klein with 168 automorphisms", pp. 297–308 in *Problems in analysis* (Princeton, 1969), edited by R. C. Gunning, Princeton Univ. Press, Princeton, N.J., 1970.

[Rodríguez and González-Aguilera 1994] R. E. Rodríguez and V. González-Aguilera, "Fermat's quartic curve, Klein's curve and the tetrahedron", pp. 43–62 in *Extremal Riemann surfaces* (San Francisco, 1995), edited by J. R. Quine and P. Sarnak, Contemp. Math. **201**, Amer. Math. Soc., Providence, RI, 1997. Preprint, 1994.

[Schulte and Wills 1985] E. Schulte and J. M. Wills, "A polyhedral realization of Felix Klein's map $\{3,7\}_8$ on a Riemann surface of genus 3", *J. London Math. Soc.* (2) **32**:3 (1985), 539–547.

[Singerman 1995] D. Singerman, "Mirrors on Riemann surfaces", pp. 411–417 in *Second International Conference on Algebra* (Barnaul, 1991), edited by L. A. Bokut' et al., Contemp. Math. **184**, Amer. Math. Soc., Providence, RI, 1995.

[Streit 1996] M. Streit, "Homology, Belyĭ functions and canonical curves", *Manuscripta Math.* **90**:4 (1996), 489–509.

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