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Hurwitz Groups and Surfaces

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Abstract. Hurwitz not only gave an upper bound for the number of automorphisms of a compact Riemann surface of genus greater than 2, but also gave a characterization of which finite groups could be groups of automorphisms achieving this bound. In practice, however, the identification of such groups and of the surfaces they act on is difficult except in special cases. We survey what is known.

1. How I Got Started on Hurwitz Groups

One day in the late 1950's, rereading Siegel's article [1945] entitled "Some remarks on discontinuous groups", I was struck by his proof that the smallest area of fundamental region for a Fuchsian group is $\pi/21$.

Siegel notes the remarkable similarity between the arithmetic used in his proof and the arithmetic in Hurwitz's proof that a curve of genus $g \geq 2$ has no more than $84(g-1)$ birational self-transformations. That, he said, is not surprising because of the theory of uniformization. That was all — no indication where to find Hurwitz's paper, at that time unknown to me. (Siegel is one of my heroes, but, it must be confessed, he was not very good at citing references.)

I did know about uniformization, and I made that connection at once. However, I had some trouble tracking down Hurwitz's theorem. Finally, thanks to the late Professor W. L. Edge, I read Hurwitz's paper [1893], which invoked Klein's surface as an example to show that his bound was attained. So at last, by a very tortuous path, I unearthed this chapter of mathematics, which has fascinated me ever since.

Hurwitz left open the question whether there was any other surface with the maximum number $84(g-1)$ of automorphisms, as we now call them. Only one other such surface was found, by Fricke, in the sixty years to 1961. My own first contribution [Macbeath 1961] was a proof that there are infinitely many of them.

My research changed direction when I became aware of Klein's curve and Hurwitz's theorem. I was driven to think more and more about Riemann surfaces with many automorphisms. It was natural to progress to Riemann surfaces in general and to Teichmüller spaces. Friends and colleagues, whether their first interest might be geometry, algebra, analysis or number theory, found points of contact with this work. It is a truly central piece of mathematics.

The Klein surface is the Riemann surface of the algebraic curve with equation, in homogeneous coordinates $x : y : z$,

$$x^3y + y^3z + z^3x = 0. \tag{1}$$

Klein [1879] showed that it is mapped on itself by 168 analytic transformations. Since the equation is real, the surface is also mapped on itself by complex conjuation, which can be composed with the analytic maps to give a further 168 antianalytic mappings, yielding a group of order 336. Klein concentrated his attention on the subgroup of index 2 and order 168.

2. Klein

That group is the second smallest simple noncommutative group. (From now on we will write "simple group" for "simple noncommutative group".) It belongs to two infinite families, $\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$. For Klein it would certainly have been $\mathrm{PSL}(2,7)$ (if the notation had been invented), because he approached the situation — group *and* Riemann surface — by studying the modular group $\mathbf{\Gamma}(1)$ of all functions

$$z \mapsto \frac{pz+q}{rz+s}, \tag{2}$$

where $p, q, r, s \in \mathbb{Z}$, and $ps - qr = 1$. These are permutations of the upper half-plane $\mathbb{U} := \{z \in \mathbb{C} \mid i(\bar{z} - z) > 0\}$.

Since the integers are a discrete subset of the reals, $\mathbf{\Gamma}(1)$ is, in any reasonable sense, a discontinuous group of mappings. The upper half-plane is a Riemann surface, so its quotient surface $\mathbb{U}/\mathbf{\Gamma}(1)$ is also a Riemann surface — a sphere with one missing point, or *puncture*. This is a slight disappointment if we are looking for interesting Riemann surfaces. Subgroups of $\mathbf{\Gamma}(1)$ might do better.

The *congruence subgroups* $\mathbf{\Gamma}(n)$, which consist of mappings (2) such that

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \equiv \pm\,\mathrm{Id} \pmod{n},$$

are the first to jump up and hit us. Being the kernel of a homomorphism, $\mathbf{\Gamma}(n)$ is a normal subgroup of $\mathbf{\Gamma}(1)$, and the factor group acts on the quotient surface as a group of automorphisms.

The quotient surfaces for $\mathbf{\Gamma}(2)$, $\mathbf{\Gamma}(3)$, $\mathbf{\Gamma}(4)$ and $\mathbf{\Gamma}(5)$ are spheres with 3, 4, 6 and 12 punctures. The factor groups include the symmetry groups of the platonic solids (tetrahedron, octahedron and icosahedron). The quotient surface of $\mathbf{\Gamma}(6)$, a torus with twelve punctures, is slightly more interesting, but the factor group $\mathbf{\Gamma}(1)/\mathbf{\Gamma}(6)$ is rather dull. Klein had studied all these groups in detail from the viewpoints of complex analysis and projective geometry.

At $\mathbf{\Gamma}(7)$, he found buried treasure. This surface has genus 3 with 24 punctures. The punctures are "removable singularities" and pose no problem, so he had a Riemann surface of genus 3 with 168 automorphisms. The quotient group is what we get when we replace the integers in (2) by their residue classes modulo 7, a group now denoted by $\mathrm{PSL}(2,7)$.

Now when we know that a surface S exists, then by general theory there is a pair of meromorphic functions x, y on S that can distinguish any one point of S from any other. The functions satisfy a polynomial identity $F(x,y) = 0$, defining an algebraic curve. The curve has a vector space of abelian differentials, it has Weierstrass points and all the other good things that an algebraic curve possesses. Having found S, some of us might be content to rest on our laurels in the mere knowledge that these things exist, but Klein was made of the right stuff. He had to know what they were.

Not only did he find the equation (1) — no mean achievement from such meagre data — he also found explicitly all the biholomorphic mappings as 3×3 matrices. These define projective transformations mapping the curve on itself. By doing this he closed one of two gaps in Jordan's list [1878] of finite ternary linear groups, as Fricke points out [1926, footnote on p. 182].

It sems that Klein *began* with the differentials, and then worked out the linear mappings induced on them by the 168 automorphisms. He studied the invariants of this linear group, finding three basic invariants that are connected by equation (1).

The Riemann surface of the curve (1) is a 168-sheeted covering of the sphere, branched over three points of the sphere.

Above one of these points the 168 sheets join together in sevens to give 24 points of the surface. These are the *points of inflection.* They are also the Weierstrass points.

Above another branch point, there are 84 points of the surface, where the sheets join in twos. These are the *sextactic* points, through which pass a conic section that has six-fold contact with the curve.

Above the third branch point the sheets join in threes to give 56 points of the surface. These 56 points are the points of contact of (1) with the 28 *bitangents*, or lines that are tangent to the curve at two points.

All these facts were discovered by Klein.

The numbers 2, 3, 7 reflect the fact that the universal cover of the whole picture is the triangle group $(2,3,7)$ acting on $\mathbb{U}$. The modular group $\mathbf{\Gamma}(1)$ is the triangle group $(2,3,\infty)$. Replacing ∞ by 7 amounts to removing the removable singularities.

For more detail see [Fricke 1926, p. 182–235] or the translation of Klein's article in this volume.

3. Hurwitz

Hurwitz's paper [1893] is a bold piece of work. His aim, as the title indicates, was to study the general situation of an algebraic curve and a group $\mathbf{\Gamma}$ of automorphisms. It had been proved by Schwarz that the automorphism group of a curve of genus $g \geq 2$ is finite, so he assumed that $g \geq 2$.

His approach was topological, considering the Riemann surface of the curve as a branched covering of the quotient surface of $\mathbf{\Gamma}$-orbits. He worked out the relation between the genus of the surface, the genus of the quotient surface and the branching numbers. We now call this the *Riemann–Hurwitz relation*. From this he worked out the upper bound $84(g-1)$ mentioned by Siegel.

Among other results, Hurwitz also proved that the action of the automorphism group on the abelian differentials is faithful, and that the order of any single automorphism cannot exceed $10(g-1)$. He also showed that a finite group can be realized as a group of $84(g-1)$ automorphisms of a surface of genus $g \geq 2$ *if and only if it is generated by two elements t, u such that*

$$t^2 = u^3 = (tu)^7 = 1.$$

Such a group is now called a *Hurwitz group*. Even if we did not know anything about Fuchsian groups, we would nowadays feel forced to invent the abstract *triangle group*

$$\langle t, u \mid t^2 = u^3 = (tu)^7 = 1 \rangle \tag{3}$$

and to rephrase the result:

The Hurwitz groups are precisely the finite homomorphic images of (3).

The problem of finding surfaces with $84(g-1)$ automorphisms is now reduced to a purely group-theoretic question. Without making the connection to Riemann surfaces, G. A. Miller [1902] proved that there are infinitely many Hurwitz groups.

4. Poincaré

The introduction of Fuchsian groups by Poincaré [1882] had a strong influence on our way of thinking about Riemann surfaces. Though his work was a decade before Hurwitz's, it is quite clear that Hurwitz was writing without reference to it, and perhaps he did not know of it. Some very effective work on automorphisms, for example [Accola 1968], has been done quite recently without any mention of Fuchsian groups, using covering space theory.

For me, though, the intuitive picture gained from Fuchsian groups is all-important. I see the automorphism group as a tiling of the surface, the quotient surface being what we get when we identify matched edges of any one tile. By rolling the whole surface out on to the simply connected universal covering surface $\mathbb{U}$ we get a coarser tiling (of $\mathbb{U}$) whose matching gives the target surface of the automorphism group. Each of the coarse tiles is a mosaic of finer tiles, whose edge-matching gives the quotient surface.

All the fine tiles have the same hyperbolic area, say a, and all the coarse tiles have the same area A. The order n of the automorphism group is the number of fine tiles that fit into one coarse tile. Therefore

$$A = na.$$

This is the *Riemann–Hurwitz relation*, which, in this form, seems blindingly obvious. To use the relation effectively, we must rewrite the areas A, a in terms of algebraic invariants of the Fuchsian groups. Still, the use of Fuchsian groups makes everything more transparent.

Poincaré recognised $\mathbb{U}$ as the hyperbolic plane, which does not admit arbitrarily fine congruent tilings. Siegel's theorem is an exact quantitative expression of this. The upper half-plane $\mathbb{U}$ had been known as the target space for the modular group before Poincaré was even born, and hyperbolic (or, as it was then called, noneuclidean) geometry had been studied for its own sake. With the metric

$$\frac{|dz|}{y},$$

$\mathbb{U}$ had been used as a model for hyperbolic geometry by Beltrami (see, e.g., [Stillwell 1996]). Regarded by many mathematicians as a gigantic counterexample designed to show that Euclid's geometry could not be deduced without the parallel postulate, hyperbolic geometry had to wait until Poincaré to be synthesized with the modular figure and admitted to mathematical respectability.

Klein's approach to his curve, involving the modular group, is much closer to Poincaré than to Hurwitz. Even though the groundwork was done without specific mention of general Fuchsian groups, the ideas were in the air, just waiting for someone like Poincaré to crystallize them.

For historical and mathematical insight on the emergence of hyperbolic geometry from the shadows see the collection [Stillwell 1996]. It seems ironic that Klein had written much earlier about both modular functions and "noneuclidean geometry". He had all the expertise to make the connection, but somehow he did not. Perhaps it is understandable that he was at times less than generous to the youthful Poincaré, who had burst like a supernova on the mathematical scene.

As we have seen, Klein had plenty of reason to feel good about himself, and it would have cost nothing to be more cordial.

5. From 1893 to 1960

Between Hurwitz's paper and about 1960, there was a certain amount of routine work on automorphisms of Riemann surfaces, but very little of real significance. Fricke discovered the Hurwitz group $\mathrm{PSL}(2, 2^3)$ of order 504 and genus 7, and Wiman [1895b; 1895a] improved the bound for the order of an automorphism from $10(g-1)$ to $2(2g+1)$, which is best possible. Wiman also worked out all interesting automorphism groups for surfaces of genus 2, 3, 4, 5 and 6 by using methods of classical algebraic geometry. A lot of labour was involved.

Siegel's "remark", taking only two lines of text, was also a major contribution. One normally thinks of Siegel as an analyst and a number theorist, but here we see geometric inspiration as well.

6. Hurwitz Groups

In 1900 the most up-to-date list of finite simple groups was to be found at the end of Dickson's book [1900, Chapter XV, particularly § 290]. By 1954, there was still no advance. Then Chevalley's paper [1955] appeared, starting off the avalanche that culminated in the classification of all finite simple groups in the 1980's.

Now the search for Hurwitz groups requires a knowledge of finite simple groups. It is easy to show that the factor group of a Hurwitz group modulo any maximal normal subgroup is a simple group and also a Hurwitz group. So our obvious strategy for finding Hurwitz groups is first to comb the simple groups for Hurwitz groups and then to find extensions building on these as factor groups. See [Macbeath 1990].

Not knowing Miller's work [1902], I started from scratch. I did not need more than Dickson's book and a little basic topology of surfaces to find the following two results [Macbeath 1961; 1969].

- $\mathrm{PSL}(2,q)$, where $q = p^m$, p prime, is a Hurwitz group if and only if *either* $q = 7$, *or* $q = p \equiv \pm 1 \pmod 7$, *or* $q = p^3$, $p \equiv \pm 2, \pm 3 \pmod 7$.
- If G is a Hurwitz group of order $84(g-1)$, for $0 < n \in \mathbb{Z}$, then there is a group $G(n)$ of order $84(g-1)n^{2g}$ that is also a Hurwitz group. The group $G(n)$ is an extension of a product of $2g$ copies of the finite cyclic group $\mathbb{Z}/n$ by G.

The first of these results is proved by manipulating 2×2 matrix equations in the finite field $\mathrm{GF}(q)$. The second is proved by applying Fuchsian group theory to the groups of the coarse and fine tilings just mentioned. The group of the coarse tiling is the fundamental group of the surface of genus g, which abelianizes to give a product of $2g$ copies of $\mathbb{Z}$, the infinite cyclic group. Hence the exponent $2g$ in the expression for the order of $G(n)$.

It struck me forcibly at the time, and still seems remarkable, that this is all so heavily group-theoretic. The methods indicated allow us to construct a great variety of Hurwitz groups. The second theorem allows us to derive "towers" $G(p)$, $G(p)(q)$, $G(p)(q)(r)$, $\ldots$.

Indeed there is no need to look for abelian kernels only in this process. It has been observed by J. M. Cohen (oral communication) that a similar method proves that, given any simple group H, one can construct a Hurwitz group with H in its composition series. When we build towers, the order of the group, and therefore the genus of the surface, increases dramatically as a function of the number of building blocks in the tower.

In the opposite direction, Cohen [1981] proved that $\mathrm{PSL}(3, q)$ is *not* a Hurwitz group unless $q = 2$.

One can also look for permutation solutions of (3). Experimentation with permutations of given degree n can be done graphically. A pair of permutations t, u satisfying (3) can be drawn as a graph with triangles for the 3-cycles of u and edges of a different colour joining points to their t-images. It is not difficult to manipulate things so that $(tu)^7 = 1$. The generated group is transitive if the graph is connected.

One finds by trial that the only permutation solutions for degrees 7 and 8 give us back $\mathrm{PSL}(2, 7)$ — the original Klein surface! At degree 9 we find $\mathrm{PSL}(2, 2^3)$, already found by Fricke. There is no transitive group for $10 \leq n \leq 13$. For $n = 14$, we have $\mathrm{PSL}(2, 13)$, and for $n = 15$ we have the alternating group A_{15}.

By systematically combining graphs — and a lot of ingenuity — Conder [1980] proved that A_n is a Hurwitz group for $n \geq 168$. He also determined specifically which A_n are not Hurwitz groups for $16 \leq n \leq 167$.

Sah [1969] produced a lot of information about Hurwitz groups and also about other groups acting on Riemann surfaces. He showed that the Ree groups ${}^2G_2(3^p)$ are all Hurwitz groups.

As far as I know, $\mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, A_n, and the Ree groups ${}^2G_2(3^p)$ are the only infinite series of finite simple groups where we know precisely which ones are Hurwitz groups.

During the search for finite simple groups, eleven "sporadic" simple groups were found to be Hurwitz groups. These are listed in Conder's excellent survey article [1990]. It contains some more techniques for producing Hurwitz groups, and is the best place to get further information about them.

7. The Wider Picture

The Hurwitz groups, then, proved to be surprisingly interesting. Apart from Miller's paper, presentations including (3) are found scattered through the literature. In [Coxeter and Moser 1957, p. 96] we find a presentation displaying $\mathrm{PSL}(2, 7)$ as a Hurwitz group and $\mathrm{PSL}(2, 13)$ as a Hurwitz group in two different ways. We can deduce that $\mathrm{PSL}(2, 13)$ acts on *two* Riemann surfaces of genus 14. (We know now that there is a third one.)

Now, the first few Hurwitz groups, in order, act on surfaces of genus 3, 7, 14 and 17, and the admissible genera seem to become more sparse as they get larger. For every $g \geq 2$, though, there is a maximum order $\mu(g)$ for an automorphism group, and it is not difficult to show that, for any $g \geq 2$ there is a group of order

$8g+8$ acting on some surface of genus g. We therefore have

$$8g+8 \leq \mu(g) \leq 84(g-1). \tag{4}$$

Klein's and other surfaces show that the upper bound is sharp. Independently, and about the same time, Bob Accola in Providence and Colin Maclachlan in Birmingham, England, found the lower bound and proved that it too is sharp. Each of them produced an infinite family of g with $\mu(g) = 8g+8$, and the two families are not only distinct but disjoint! Maclachlan used the language of Fuchsian groups, but Accola, like Hurwitz, worked without them. See [Accola 1968; Maclachlan 1969].

Folk also looked at special kinds of groups. Wiman had dealt with *cyclic* groups, but there was more to say. Harvey [1966] found, for each n, the smallest genus of a surface with an automorphism of order n. This gave Wiman's bound as an easy corollary. Maclachlan [1965] found the upper bound for *abelian* automorphism groups. Accola was the first to observe that the order of a *soluble* group of automorphisms cannot exceed $48(g-1)$. Zomorrodian [1985] found the upper bound for *nilpotent* groups. See also [Macbeath 1984]. Maclachlan and Gromadzki [1989] found the bound for *supersoluble* groups.

The aim of these workers was to find the largest or smallest group in some particular category, but there is a good reason for looking at *all* the automorphism groups acting on surfaces of a given genus, whether or not they have any extreme value. Here is why.

For every g we have a *Teichmüller space* $\mathbb{T}_g$ of "marked Riemann surfaces" analogous to the modular figure in genus 1. Topologically $\mathbb{T}_g$ is a euclidean space of $6g-6$ real dimensions. The *mapping class group* M_g is a discontinuous group acting on $\mathbb{T}_g$. The quotient space $\mathbb{T}_g/M_g$ is the space R_g of all closed Riemann surfaces of genus g. The quotient mapping $\mathbb{T}_g \to R_g$ is a branched covering and the points where ramification occurs are the Riemann surfaces with nontrivial automorphisms.

To understand this situation it is necessary to get some understanding of the whole set of groups involved as well as the dimensions of the subspaces of $\mathbb{T}_g$ consisting of the fixed point sets for each group.

Though we have a good understanding of Teichmüller space, there is a lot we don't know about the configuration of interlocking fixed point sets, or *branch loci*, as they are called. Even for fairly small values of g, the number of possible groups, including cyclic and dihedral groups, is quite large and the same group may act in several topologically different ways, as we saw with $\mathrm{PSL}(2,13)$.

Some people have tried to outflank the problem, by taking a given group and finding all the genera of surfaces on which it acts. Harvey [1966] did this for cyclic groups, and his work was extended by Lloyd [1972]. More recently, Kulkarni [1987] has shown that, for *any* group, the admissible values of g settle ultimately into a periodic pattern modulo the prime factors of its order. For more information see [Kulkarni 1987; Harvey 1971; Macbeath and Singerman 1975]. Much remains to be done in this direction.

The title of Hurwitz's paper, freely translated, is *Algebraic structures with one-to-one self-mappings*. Most of the paper deals with the general picture of an algebraic curve with automorphisms, looking closely at the branched covering of the quotient surface by the target surface. The bound $84(g-1)$ falls out as a by-product.

The spirit of Hurwitz's work is consistent with studying the general picture and not beoming obsessed with one particular Fuchsian group, which, by an arithmetical accident, happens to have the smallest quotient area. That is in the spirit of Klein and Fricke too. Their books on modular and automorphic functions [Klein and Fricke 1890–92; Fricke and Klein 1912] give many examples of curves with fairly large non-Hurwitz automorphism groups.

8. Conclusion

It is appropriate to reflect how much Klein knew about his curve, and how little we know about all the Hurwitz surfaces we have constructed. Apart from Klein's curve and the curve of genus 7, we know equations for no other curve with $84(g-1)$ automorphisms. Each one of them is an isolated point of $\mathbb{T}_g$, so the problem makes good sense. The only really useful tool that has emerged in looking for equations seems to be the Lefschetz fixed point formula [Macbeath 1965; 1973]. For $\mathrm{PSL}(2,2^3)$, it worked, but for $\mathrm{PSL}(2,13)$ it was not quite enough. On the other hand, there are limits to what we can expect to do. The genus of the curve on which the Hurwitz group A_{15} acts is 7783776001. Even with modern computers, the calculation of an equation might be difficult even if we had a program to do it.

Let us pay tribute to Klein: he may not have known as much group theory as we do, but he knew a whole lot more about other things.

References

[Accola 1968] R. D. M. Accola, "On the number of automorphisms of a closed Riemann surface", *Trans. Amer. Math. Soc.* **131** (1968), 398–408.

[Chevalley 1955] C. Chevalley, "Sur certains groupes simples", *Tōhoku Math. J.* (2) **7** (1955), 14–66.

[Cohen 1981] J. Cohen, "On non-Hurwitz groups and noncongruence subgroups of the modular group", *Glasgow Math. J.* **22**:1 (1981), 1–7.

[Conder 1980] M. D. E. Conder, "Generators for alternating and symmetric groups", *J. London Math. Soc.* (2) **22**:1 (1980), 75–86.

[Conder 1990] M. Conder, "Hurwitz groups: a brief survey", *Bull. Amer. Math. Soc.* (*N.S.*) **23**:2 (1990), 359–370.

[Coxeter and Moser 1957] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Springer, Berlin, 1957. Fourth edition, 1980.

[Dickson 1900] L. E. Dickson, *Linear Groups, with an exposition of the Galois Field Theory*, Chicago, Leipzig, 1900. Reprinted by Dover, New York, 1958.

[Fricke 1926] R. Fricke, *Lehrbuch der Algebra*, Vieweg, Braunschweig, 1926.

[Fricke and Klein 1912] R. Fricke and F. Klein, *Vorlesungen über die theorie der automorphen Functionen, 2: die functionentheoretischen Ausführungen und die Anwendungen*, Teubner, Leipzig, 1912. Reprinted by Johnson Reprint, New York, and Teubner, Stuttgart, 1965.

[Gromadzki and Maclachlan 1989] G. Gromadzki and C. Maclachlan, "Supersoluble groups of automorphisms of compact Riemann surfaces", *Glasgow Math. J.* **31**:3 (1989), 321–327.

[Harvey 1966] W. J. Harvey, "Cyclic groups of automorphisms of a compact Riemann surface", *Quart. J. Math. Oxford* (2) **17** (1966), 86–97.

[Harvey 1971] W. J. Harvey, "On branch loci in Teichmüller space", *Trans. Amer. Math. Soc.* **153** (1971), 387–399.

[Hurwitz 1893] A. Hurwitz, "Über algebraische Gebilde mit eindeutigen Transformationen in sich", *Math. Annalen* **41** (1893), 403–442.

[Jordan 1878] M. C. Jordan, "Mémoire sur les équations différentielles linéaires à intégrale algébrique", *J. für reine angew. Math.* **84** (1878), 89–215. Reprinted as pp. 13–140 of his *Œuvres*, vol. 2, Gauthier-Villars, Paris, 1962.

[Klein 1879] F. Klein, "Ueber die Transformationen siebenter Ordnung der elliptischen Funktionen", *Math. Annalen* **14** (1879), 428–471. Reprinted as [Klein 1923, LXXXIV, pp. 90–136]. Translated in this collection.

[Klein 1923] F. Klein, *Gesammelte Mathematische Abhandlungen, 3: Elliptische Funktionen* etc., edited by R. Fricke et al., Springer, Berlin, 1923. Reprinted by Springer, 1973.

[Klein and Fricke 1890–92] F. Klein, *Vorlesungen über die Theorie der elliptischen Modulfunctionen,* ausgearbeitet und vervollständigt von Robert Fricke (2 vol.), Teubner, Leipzig, 1890–92. Reprinted by Johnson Reprint, New York, 1966.

[Kulkarni 1987] R. S. Kulkarni, "Symmetries of surfaces", *Topology* **26**:2 (1987), 195–203.

[Lloyd 1972] E. K. Lloyd, "Riemann surface transformation groups", *J. Combinatorial Theory Ser. A* **13** (1972), 17–27.

[Macbeath 1961] A. M. Macbeath, "On a theorem of Hurwitz", *Proc. Glasgow Math. Assoc.* **5** (1961), 90–96.

[Macbeath 1965] A. M. Macbeath, "On a curve of genus 7", *Proc. London Math. Soc.* (3) **15** (1965), 527–542.

[Macbeath 1969] A. M. Macbeath, "Generators of the linear fractional groups", pp. 14–32 in *Number Theory* (Houston, 1967), edited by W. J. Leveque and E. G. Straus, Proc. Sympos. Pure Math. **12**, Amer. Math. Soc., Providence, 1969.

[Macbeath 1973] A. M. Macbeath, "Action of automorphisms of a compact Riemann surface on the first homology group", *Bull. London Math. Soc.* **5** (1973), 103–108.

[Macbeath 1984] A. M. Macbeath, "Residual nilpotency of Fuchsian groups", *Illinois J. Math.* **28**:2 (1984), 299–311.

[Macbeath 1990] A. M. Macbeath, "Automorphisms of Riemann surfaces", pp. 107–112 in *Combinatorial group theory* (College Park, MD, 1988), edited by B. Fine et al., Contemporary Mathematics **109**, Amer. Math. Soc., Providence, RI, 1990.

[Macbeath and Singerman 1975] A. M. Macbeath and D. Singerman, "Spaces of subgroups and Teichmüller space", *Proc. London Math. Soc.* (3) **31**:2 (1975), 211–256.

[Maclachlan 1965] C. Maclachlan, "Abelian groups of automorphisms of compact Riemann surfaces", *Proc. London Math. Soc.* (3) **15** (1965), 699–712.

[Maclachlan 1969] C. Maclachlan, "A bound for the number of automorphisms of a compact Riemann surface", *J. London Math. Soc.* **44** (1969), 265–272.

[Miller 1902] G. A. Miller, "Groups defined by the order of two generators and the order of their product", *Amer. J. Math.* **24** (1902), 96–100.

[Poincaré 1882] H. Poincaré, "Théorie des groupes fuchsiens", *Acta Mathematica* **1** (1882), 1–62.

[Sah 1969] C.-h. Sah, "Groups related to compact Riemann surfaces", *Acta Math.* **123** (1969), 13–42.

[Siegel 1945] C. L. Siegel, "Some remarks on discontinuous groups", *Ann. of Math.* (2) **46** (1945), 708–718.

[Stillwell 1996] J. Stillwell, *Sources of hyperbolic geometry*, History of Mathematics, Amer. Math. Soc. and London Math. Soc., 1996.

[Wiman 1895a] A. Wiman, "Über die hyperelliptischen Curven und diejenigen von Geschlechte $p = 3$, welche eindeutigen Transformationen in sich zulassen", *Bihang till Kongl. Svenska vetenskaps-akademiens handlingar* **21**:I-1 (1895), 1–23.

[Wiman 1895b] A. Wiman, "Ueber die algebraischen Curven von den Geschlechten $p = 4$, 5 und 6, welche eindeutigen Transformationen in sich besitzen", *Bihang till Kongl. Svenska vetenskaps-akademiens handlingar* **21**:I-3 (1895), 1–41.

[Zomorrodian 1985] R. Zomorrodian, "Nilpotent automorphism groups of Riemann surfaces", *Trans. Amer. Math. Soc.* **288**:1 (1985), 241–255.

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