The Eightfold Way
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MSRI and the Klein Quartic

On November 14, 1993, a marble and serpentine sculpture was unveiled at the Mathematical Sciences Research Institute in Berkeley, an event that marked one of the ways in which MSRI has been reaching out beyond its traditional role. The work had been commissioned from the famous mathematical sculptor Helaman Ferguson, thanks to a generous donation from Mitsubishi Electric Research Laboratories (MERL) made for the purpose. This sculpture, and the mathematical object that lies behind it, are the subject of this book.

Felix Klein discovered in 1878 that a certain surface, whose equation (in complex projective coordinates) he gave very simply as $x^3y + y^3z + z^3x = 0$, has a number of remarkable properties, including an incredible 336-fold symmetry. He arrived at it as a quotient of the upper complex half-plane by a modular group — the group of fractional linear transformations whose coefficients are integers and that reduce to the identity modulo 7. Since then, the same structure has come up in different guises in many areas of mathematics.

Ferguson's sculpture, *The Eightfold Way*, is a distillation of the beauty and remarkable properties of the Klein quartic. (See Plate 1 following page 150.) At the base is a two-color stone mosaic, representing the uniformization of the surface: a regular hyperbolic tesselation shown in the Poincaré model. Rising out of the central tile, a seven-sided black column cups the artist's Carrara marble rendition of the surface, which highlights its tetrahedral symmetry. The name *The Eightfold Way* is explained by the ridges and grooves that crisscross the otherwise smooth hand-polished surface: they represent the same tesselation, after the surface has folded over itself. If you run your finger along these curves, alternating left and right turns at each corner, you always come back to the beginning after eight turns. In the words of Claire Ferguson, the overall effect is that of "...a symphony of elegant counterpoint — as if Gothic tracery and Alhambra tilings were united in one work."

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