The Eightfold Way
MSRI Publications
Volume **35**, 1998

The Eightfold Way: A Mathematical Sculpture by Helaman Ferguson

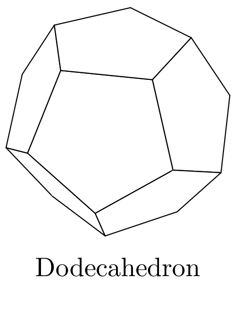
WILLIAM P. THURSTON

This introduction to *The Eightfold Way* and the Klein quartic was written for the sculpture's inauguration. On that occasion it was distributed, together with the illustration on Plate 2, to a public that included not only mathematicians but many friends of MSRI and other people with an interest in mathematics. Thurston was the Director of MSRI from 1992 to 1997.

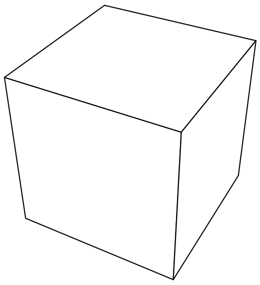
Mathematics is full of amazing beauty, yet the beauty of mathematics is far removed from most people's everyday experience. The Mathematical Sciences Research Institute is committed to the search for ways to convey the beauty and spirit of mathematics beyond the circles of professional mathematicians.

As a step in this effort, MSRI (pronounced "Emissary") has installed a first mathematical sculpture, *The Eightfold Way*, by Helaman Ferguson. The sculpture represents a beautiful mathematical construction that has been studied by mathematicians for more than a century, from many points of view: geometry, symmetry, group theory, algebraic geometry, topology, number theory, complex analysis. The surface depicted by the sculpture was discovered, along with many of its amazing properties, by the German mathematician Felix Klein in 1879, and is often referred to as the Klein quartic or the Klein curve in his honor.

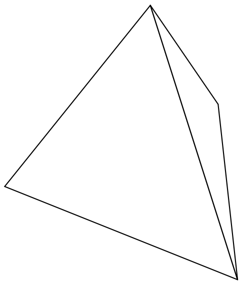
The abstract surface is impossible to render exactly in three-dimensional space, so the sculpture should be thought of as a kind of topological sketch. Ridges and valleys carved into the white marble surface divide it into 24 regions. Each region has 7 sides, and represents the ideal of a regular heptagon (7-gon). The 24 heptagons fit together in triples at 56 vertices. It is the pattern of the division of the surface into heptagons that carries the essence of the mathematics. The Klein quartic thus is an extension of the concept of a regular polyhedron, of which the dodecahedron, the cube and the tetrahedron are examples:



Dodecahedron



Cube



Tetrahedron

Even though the heptagons on the physical surface are not regular, the *pattern* of heptagons on the surface is completely symmetric — in fact, the pattern is just as symmetric as the pattern of pentagons on a dodecahedron. One way to get a sense of the symmetry is to place a finger on any edge. Trace out along the edge to the next intersection, and turn left. Now proceed to the next intersection and turn right. Continue in this way, making a total of 8 turns, LRLRLRLR. If you do this carefully, with concentration and contortion, you arrive back where you started. It doesn't matter where you start or in which direction you go: in 8 alternating turns, you always arrive back at the beginning. (Question: what happens when you do this on a tetrahedron, cube, or dodecahedron?)

In the pattern of heptagons on the surface, and of the 24 heptagons is equivalent to any other heptagon. Furthermore, if any heptagon is rotated by $\frac{1}{7}$th of a revolution, it still fits into the pattern in an identical way. This makes $24 \times 7 = 168$ ways that the pattern of heptagons on the surface can be mapped to itself. Mathematically, the pattern has order 168. When a heptagon is reflected along any of its altitudes, it still fits into the pattern in an identical way, making a total of 336-fold symmetry when the mirror-image transformations are allowed.

The circular base area of the sculpture is also tiled by heptagonal tiles, in a regular geometric pattern that resembles a honeycomb. The sides of the heptagons are arcs of circles; when these arcs are continued, they meet the boundary at a $90°$ angle. This circular base area is a map of the hyperbolic or non-Euclidean plane. In hyperbolic geometry, it is possible to construct regular heptagons whose angles are exactly $120°$; these heptagons fit together to tile the hyperbolic plane. The physical map of the hyperbolic plane is distorted, but in hyperbolic geometry itself all the heptagons have an identical size and shape.

The heptagonal tiling of the base and the heptagonal tiling of the surface are closely related. The 7-sided column that supports the sculpture starts off this relationship: it sweeps up from the central heptagon in the hyperbolic plane to one of the 24 heptagons on the surface. Imagine continuing this relationship. The 7 heptagons adjacent to the foot of the column sweep up and stretch to cover the 7 heptagons that border the top of the column, and so forth. The base area stretches out and wraps around the surface to completely encompass it; it continues stretching and wrapping around and around, infinitely often.

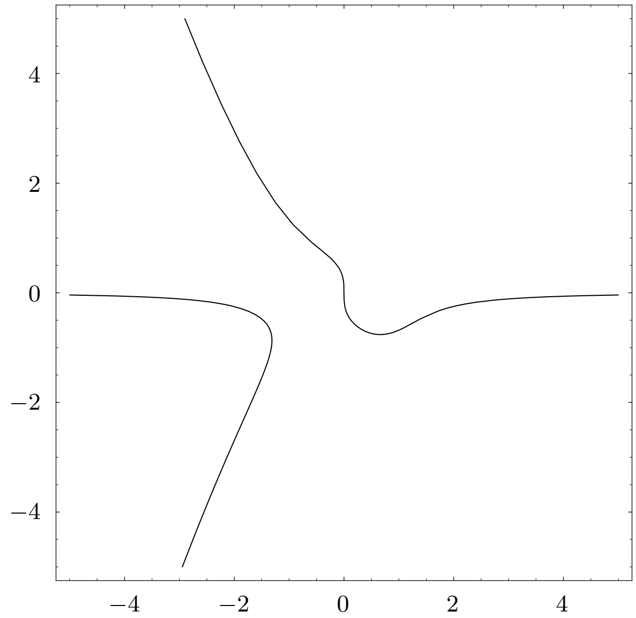
Plate 2 following page 150 shows the heptagonal hyperbolic honeycomb with a pattern superimposed to indicate what happens when it wraps around the surface. The infinite hyperbolic honeycomb is divided into 3 kinds of groups of 8 cells each, where each group is composed of a heptagon together with its 7 neighbors. There are red rings surrounding one person, green groups surrounding another person, and white groups with letters.

When the honeycomb is wrapped around the surface, equivalent groups wrap up to the same place on the surface. In other words, the pattern superimposed on the surface would have only one green group, one red group, and one white group, making 24 heptagons in all. You can check this out by testing the LRLRLRLR rule on the hyperbolic honeycomb. For instance, if you start on an edge that points in toward the central white area and has a red group on its left and a green group on its right, and proceed LRLRLRLR, you will arrive at another edge with red on its right and green on its left. If the initial edge pointed toward and "a" (say), the final edge also points toward an "a".

It is interesting to watch what happens when you rotate the pattern by a $\frac{1}{7}$ revolution about the central tile: red groups go to red groups, green groups go to green groups and white groups go to white groups. The person in the center of a green group rotates by $\frac{2}{7}$ revolution, and the person in the center of a red group rotates by $\frac{4}{7}$ revolution. The interpretation on the surface is that the 24 cells are grouped into 8 affinity groups of 3 each. The symmetries of the surface always take affinity groups to affinity groups. This is analogous to the dodecahedron, whose twelve pentagonal faces are divided into 6 affinity groups of 2 each, consisting of pairs of opposite faces.

The name "Klein quartic" or "Klein curve" refers to an algebraic description of the ideal surface that the sculpture represents, determined by the equation $x^3y + y^3 + x = 0$. (This equation is called a quartic or 4th-degree equation because the highest term x^3y has 3 x's and 1 y, making degree 4 in all.) The solutions to this equation in the (x, y)-plane form the curve shown at the top of the next page. [A more symmetric view is presented in Figure 10 on page 326. –Ed.]

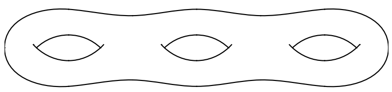
But when x and y are allowed to be complex numbers, there are many more solutions; in fact, the set of solutions forms a 2-dimensional surface in 4-dimensional space. The symmetry of the surface is reflected algebraically by the phenomenon that there are many possible substitutions that keep the equation the same. For instance, if you replace x by Y/X and y by $1/X$, the equation becomes $Y^3/X^4 + 1/X^3 + Y/X = 0$; if you multiply both sides by X^4 to clear denominators, you get the original equation. There are 168 essentially different algebraic "substitutions" that preserve the equation, one for each of the orientation-preserving symmetries of the surface. (Coordinates can be chosen so that the



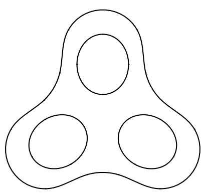
$x^3y + y^3 + x = 0$

center of the central tile of the hyperbolic honeycomb maps to the solution $(0, 0)$ to the equation, and a vertical line through that point maps to the curve graphed above. The substitution just described takes the integral sign to the person in a red ring, the person in a red ring to the person in a green ring, and the person in the green ring back to the integral sign.) Most of the substitutions are more complicated, involving complex algebraic numbers, and we won't describe them.

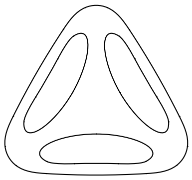
Topologically, the Klein curve is called a surface of genus 3 or a 3-holed torus. Why 3? Here is a standard picture of a 3-holed torus:



In topology, two figures are equivalent if one can be deformed into the other. So we can rearrange the holes (stretching but not tearing or gluing) however we like without changing the topology, for instance into



(to reveal a kind of 3-fold symmetry that was not evident before), and further into



which looks like the frame of a tetrahedron as seen from above. This is the approximate form of the sculpture, and it displays the maximal amount of the symmetry of the ideal surface that can be made directly visible in space.

The heptagonal hyperbolic honeycomb has an interesting relationship to the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

in which each number, starting with the third, is the sum of the preceding two. Imagine growing the hyperbolic honeycomb like a crystal, starting with the central white group of 8 heptagons as a seed. At each unit of time, adjoin a heptagon wherever there is a concave angle — that is, adjoin all the heptagons that touch at least two of the heptagons already present. At the first step, you will add 7 heptagons. Second, you will add the 14 green heptagons that fill out the next complete ring. On the third step, you add 21 heptagons, consisting of 14 red and 7 white heptagons (the centers of green groups). The sequence, if we include the ring of 7 white heptagons in the initial seed, goes

$$7, 7, 14, 21, 35, 56, 91, 147, 238, 385, 623, \ldots.$$

Each term is the sum of the preceding two: this sequence is 7 times the Fibonacci sequence!

The number of tiles grows very rapidly as you add additional layers. That is why the tiles around the edges must get quite small in the map of the hyperbolic plane: there are so many of them that otherwise they wouldn't fit. The base of the sculpture includes tiles corresponding to the first 7 terms of the sequence, making 231 tiles in all (or 232 if you include the spot where the column fits). The cover diagram shows the tiles for two additional terms plus a few scattered heptagons, making over 617.

Instructions for how to glue the 24 heptagons of the surface together can be constructed as follows. Label the 24 heptagons with different labels, say the letters `a` through `x`. Arrange these letters, together with a sharp sign (`#`), in a grid pattern in the plane that repeats very 7 units across, every 7 units up, and is symmetric about each `#`. All the information is contained in a 7×7 table. Notice that the 7×7 table is filled out by the `#` together with 2 occurrences of each of the 24 letters:

#	a	b	c	c	b	a	#	a	b	c	c	b	a	#
d	e	f	g	h	i	j	d	e	f	g	h	i	j	d
k	l	m	n	o	p	q	k	l	m	n	o	p	q	k
r	s	t	u	v	w	x	r	s	t	u	v	w	x	r
r	x	w	v	u	t	s	r	x	w	v	u	t	s	r
k	q	p	o	n	m	l	k	q	p	o	n	m	l	k
d	j	i	h	g	f	e	d	j	i	h	g	f	e	d
#	a	b	c	c	b	a	#	a	b	c	c	b	a	#
d	e	f	g	h	i	j	d	e	f	g	h	i	j	d
k	l	m	n	o	p	q	k	l	m	n	o	p	q	k
r	s	t	u	v	w	x	r	s	t	u	v	w	x	r
r	x	w	v	u	t	s	r	x	w	v	u	t	s	r
k	q	p	o	n	m	l	k	q	p	o	n	m	l	k
d	j	i	h	g	f	e	d	j	i	h	g	f	e	d
#	a	b	c	c	b	a	#	a	b	c	c	b	a	#
d	e	f	g	h	i	j	d	e	f	g	h	i	j	d
k	l	m	n	o	p	q	k	l	m	n	o	p	q	k
r	s	t	u	v	w	x	r	s	t	u	v	w	x	r
r	x	w	v	u	t	s	r	x	w	v	u	t	s	r
k	q	p	o	n	m	l	k	q	p	o	n	m	l	k
d	j	i	h	g	f	e	d	j	i	h	g	f	e	d
#	a	b	c	c	b	a	#	a	b	c	c	b	a	#

To determine what heptagons to glue to a given heptagon (call it `z`), find the letter of the heptagon in the table. It's always possible to construct a line segment that connects some `#` to `z` without going through any intermediate letters. Draw a line parallel to `#z` that is as close as possible to the right while still going through letters in the table. The letters along this line are the heptagons adjacent to `z`, in counterclockwise order. For example, the `a` heptagon is glued to the 7 heptagons in the second row of the table: `defghij`. The `e` heptagon is glued to `adltvnf`. The neighbors of `t` are slightly harder to determine: `t` is 2 units to the left and 3 units up from a `#`, so starting with an `l`, which is 2 units to the left and 2 units up, one can repeatedly go 2 over and 3 up, reading off `loirbve`.

Try labeling the blank heptagonal honeycomb on the next page, using this rule.

The spirit of mathematics and the essence of its beauty is remarkably fragile, because mathematics is about ideas and about thought. Mathematics takes place in the mind, and no two minds are the same. After many years of study and work, a mathematician may stumble on a vast and beautiful vista that unifies and simplifies many things that once appeared disparate and complicated. Mathematicians can share a beautiful mathematical vista with one another, but there is no camera that can easily capture an image of such a vista to convey it in full to people who have not trudged along many of the same trails.

We have only touched on a small part of the mathematical vista associated with this sculpture, but we hope that you can get form it some glimpse of the unity, the beauty, and the spirit of mathematics.

William P. Thurston
University of California Davis
Department of Mathematics
565 Kerr Hall
Davis, CA 95616
United States
wpt@math.ucdavis.edu