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Afterword

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> The opposite of talking is not listening. The opposite of talking is waiting.
>
> — Fran Lebowitz

Mathematicians are accustomed in their professional discourse to conditions which are alien to all other disciplines: On any given issue, there is a universally recognizable correct answer. If there is disagreement, it is because one side or the other does not correctly understand the situation. Therefore, the proper response to disagreement is to attack ruthlessly until the truth becomes clear. Once that happens, those in error will admit it gracefully and move on.

We sometimes make the mistake of expecting the same conditions to apply in arguments about mathematics education. Particularly damaging is the belief that there is no such thing as being half-right; there is nothing to be salvaged in the practices of one's opponents. Unfortunately, Fran Lebowitz's quip describes only too well much of the debate about mathematics education. One of the great pleasures of organizing this conference was to have witnessed some genuine listening. For example, the working groups on The First Two Years of University Mathematics and on Outreach to High Schools contained prominent representatives from opposite sides of the debate on mathematics education reform, yet forged remarkably unified position papers after two days of intense debate.

This is only a first step, however. In addition to listening to each other, we need to take the next step and learn to listen to voices from outside our profession.

The first group of people that we should listen to is our students. I recently had a very illuminating conversation with my daughter. Doing her homework one evening, she said:

"Dad, 14 sevenths is 2, right?" When I answered "yes", she said:

"Good, I just wanted to make sure that it wasn't division."

"But it *is* division; how else can you get the answer?"

"Oh, I was using fractions."

"How do you use fractions to show 14 sevenths is equal to 2?"

"Well... 14 is equal to 2 times 7, and the 7s cancel."

"Doesn't that mean that 14 divided by 7 is 2?" (Long pause.)

"Oh ... yeah."

My daughter is good at both division and fractions. Without this exchange, I would never have guessed that she wasn't clear on the connection between the two. Our mathematical training does not equip us to make such guesses, because it is in the nature of mathematical progress to erase the missteps in our journey towards insight. By listening carefully to our students we can eliminate the guesswork and detect the missing connections in their understanding.

We can also learn what makes sense to them and what doesn't. A business calculus class may not have much taste for mathematical abstractions, but can demonstrate great mathematical proficiency when presented with the same ideas in a concrete context that they know. The article by William Vélez and Joseph Watkins in this volume illustrates the power of presenting mathematics in contexts that mean something to the students.

The second group we should listen to is teachers and those who study teaching. (This includes, to some extent, ourselves. However, it must be admitted that university teaching has been pervaded by a dilettantish attitude which discourages serious discussion of teaching philosophy.) I include in this group anyone who takes teaching seriously as a profession; someone who can piece together with clever detective work the thinking of students, or who looks at a syllabus as more than just a list of textbook headings, or who delights in constructing homework problems that make students think about what they are doing. This group includes many school teachers and education researchers; Anneli Lax writes persuasively about what can be learned by listening to them. The group also includes mathematicians at the college level, on all sides of the educational debate, for whom, in Hyman Bass's words, "the practice of teaching has become a part of professional consciousness and collegial communication, not unlike their professional practice of mathematics itself." Jerry Uhl and William Davis in one article and Hung-Hsi Wu in another write about their college teaching experiences in a way which goes beyond personal anecdote to provide valuable professional insights.

Others worth listening to are those who use mathematics, and whose students we teach. We often invoke the opinions of engineers and scientists to defend our positions, but how often do we bother to go back to the source? I once spent a couple of hours talking to a colleague in our chemistry department. Although I obtained some useful examples for multivariable calculus, perhaps the most useful thing I discovered was the way he visualized functions of two variables. To my surprise, he did not regard the surface graph as the central geometric object, but rather the contour diagram. All his geometric reasoning about the behaviour of functions proceeded directly from the contours; it was only of marginal interest to him that the contours could be related to a surface in three dimensions. This insight fundamentally changed the way I taught multivariable calculus. The article by Dorothy Wallace gives more examples of what can be learned by talking to our colleagues in other disciplines.

Of course, we, as mathematicians, have a role as speakers as well as listeners. We must judge what are the important concepts, make sure that what is being taught is correct, and ensure a balance between technique, theory, and applications. Most of us, I think, have no trouble filling this role; it is, as Oliver Wendell Holmes said, "the province of knowledge to speak." He added that "it is the privilege of wisdom to listen." I would like to thank the many who exercised that privilege during the two days of this conference.