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The Third Mathematics Education Revolution

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Introduction

The three mathematics education revolutions in my life were called the New Math, the reaction to it called "Back to Basics", and the current one, which is best illustrated by the NCTM Standards.

In the New Math period, many mathematicians were active in what would now be called education reform. However, there were also some mathematicians who expressed serious reservations about what was happening. Morris Kline was the most vocal, but far from the only one. A group of 64 mathematicians signed an article protesting the direction taken by the then current reforms. This article, "On the mathematics curriculum of the high school", was published in both the *American Mathematical Monthly* and in *The Mathematics Teacher* [1]. It was also reprinted in Kline's book *Why Johnny Can't Add* [16]. While the problems associated with the New Math were different from those the current reforms are causing, this article is still worth reading. Here is one paragraph from it:

> 6. 'Traditional' mathematics. The teaching of mathematics in the elementary and secondary schools lags far behind present day requirements and highly needs essential improvement: we emphatically subscribe to this almost universally accepted opinion. Yet the often heard assertion that the subject matter taught in the secondary schools is obsolete should be closely scrutinized and should not be taken simply at face value. Elementary algebra, plane and solid geometry, trigonometry, analytic geometry and the calculus are still fundamental, as they were 50 or 100 years ago: future users of mathematics must learn all these subjects whether they are preparing to become mathematicians, physical scientists, social scientists or engineers, and all these subjects can offer cultural values to the general students. The traditional high school curriculum comprises all these subjects, except calculus, to some extent; to drop any one of them would be disastrous.

Contrast this with what is being written now:

> California has taken the lead in upgrading mathematics education for the 21st century. But while reforms are gradually taking hold, the majority of classrooms still rely on a traditional mathematics curriculum, that, as one cynical observer remarked, is largely composed of eight years of 15th century arithmetic, two years of 17th century algebra and one year of 3rd century B.C. geometry. [21]

It is not clear what Calvin Moore meant when he quoted a cynical observer, but it should not mean that we should drop any of these topics. When the arithmetic developed in India passed through the Arab Near East to Italy, the result was the flowering of mathematics that not long afterwards led to the development of algebraic tools by Viète and others in the 16th century and then to the analytic geometry of Descartes and Fermat. This blending of algebra and geometry enriched both subjects, and led to the development of calculus. It is surprising that the NCTM Standards both pushes for the interconnection between different parts of mathematics and yet calls for less emphasis on conic sections [23, p. 127]. The algebraic treatment of conics is an ideal place to show how algebra and geometry can be blended, with both enriched as a result. The texts written before [23] was published in 1989 contained little about conics — just their definitions via a focus or foci and distance, such as the sum of the distance from the two foci being constant for an ellipse, the derivation of standard equations for them from the definitions, and the reduction to this standard form when the curve is translated. Translation of curves is now a major part of how graphing calculators are used in algebra. This leaves nothing to decrease without removing everything else. That is a pity, since what is dropped is a major part of the essential connection between geometry and algebra.

Similarly, Greek geometry played a vital role in the development of mathematics. While mathematics of some sort seems to have been developed in every culture, the idea that simpler parts can be used in a systematic way to develop more complicated parts was not wide-spread. For example, Chinese mathematics had an almost exclusively practical focus. Greek geometry reached China through the missionary Matteo Ricci in the first part of the 17th century. He and Xu Guanggi translated the first 6 books of Euclid's elements around 1607. Xu expressed great admiration for the ideas in this work. Here is a sample.

> [I]t looks complicated, in fact it is supremely simple, so we can use its simplicity to simplify the complexity of other thing; it looks difficult, in fact it is supremely easy, so we can use its ease to ease other difficulties. Ease arises from simplicity and simplicity from clarity; finally, its ingenuity lies in its clarity.

This is taken from [18]. For a view from India, consider the book review [24] of [14].

> I would like to end with a few comments on this thought-provoking and informative book. It is true that all the civilizations of the past have thought about questions in arithmetic. But it cannot be denied that modern mathematics, as it is understood today, does owe a great deal to the renaissance in Europe, which in turn was a miraculous revival of Greek thought.

These classical parts of mathematics are still of great use, and need to be taught and learned.

The "lead" in mathematics education which Moore wrote about California having taken comes from their 1992 Mathematics Framework [3]. In October, 1996, Wisconsin was in the process of trying to write State Standards. The Wisconsin Academy of Sciences, Arts and Letters appointed a committee to look at the Mathematics Standards and comment on them. At one meeting, I passed out a copy of the newly released California Mathematics Program Advisory [4] and said this seemed to be the first step in replacing this 1992 Framework. Tom Romberg, who chaired the committee which wrote the NCTM Curriculum and Evaluation Standards, replied: "About time". At the annual meeting of the National Council of Teachers of Mathematics in April, 1997, Romberg gave a talk in which he singled out these Frameworks as distorted, focusing almost exclusively on pedagogy instead of dealing with mathematical content. He said that he had written a warning letter to the people who were writing these Frameworks when it was in draft stage. This talk was based on a monograph Romberg is writing. The talk was titled "Sometimes it is frightening when people take you seriously: Reflections on the Standards".

Of the subjects listed in the paragraph quoted from [1], solid geometry has almost completely disappeared from the high school curriculum. Analytic geometry has disappeared with the exception of straight lines and circles. Due to the heavy reliance on graphing calculators, the only form of a straight line most students in calculus know from high school is $y = mx + b$. When asked to find the equation of a line through two points or the equation of a line with a given slope but a given point off the y-axis, most students have no idea what to do. The new standard text definition of a parabola is the equation

$$y = ax^2.$$

No geometry is mentioned.

If there were nothing more to the latest revolution in mathematics education than statements like the one made by Moore, we could probably relax and say this will pass. However, textbooks have been written to "conform" to the NCTM Standards and some of them are sorely lacking what is needed to improve mathematics education. There is a lot of rhetoric about teaching for understanding, yet here are some examples of what we get.

A Case Study

The Addison-Wesley book *Focus on Algebra* [5] was described but not named in an article which appeared in the Christian Science Monitor under the heading "'Rain Forest' Algebra Course Teaches Everything but Algebra" [13]. The author of the article, Marianne Jennings, is a professor at Arizona State University and had a daughter in a class which used this book as a text. The daughter was getting an A in beginning algebra, but had no idea how to solve an equation. We can look at the book and see that equations are not introduced until page 165, and are first considered as functions. The first solution of a linear equation comes on page 218, and is by guessing and checking. Next, graphs are used to solve them. Then come algebra tiles. If you have never seen algebra tiles, let me strongly suggest you find a relatively new first year algebra book and read about them. They will be described a bit later, but you really have to see them in full color to appreciate what is happening in school algebra. Finally, on page 255, a method is introduced which has some "power", something which is written about regularly in reform circles but is seldom encountered.

Ms. Jennings was not convinced that any algebra was being done until her daughter could solve

$$3x = x + 4$$

by subtracting x from both sides and then dividing both sides by 2 to get $x = 2$. One of the many problems in the New Math was that parents did not understand what was being done or why. Ms. Jennings knew her daughter could not solve the equation above by what she thought was a simple method, so went to school to ask what was happening. The daughter's teacher said: "We don't plug and chug anymore. We're teaching them to think." The teacher assured her that in five years her daughter and the other kids would be great in math.

I lived through the New Math as a parent, and was immediately concerned by the response of the teacher. I had heard all of this before, so called Ms. Jennings to find out which book her daughter's class was using. After finding a copy in a local high school, I looked at it. Some of my concerns follow.

The book is over 800 pages long. In Howson's book [12] on textbooks written for the TIMSS study, he points out that US textbooks are much longer than those in the other countries whose books he looked at. If more significant material were covered than in other countries, or if the explanations were better, the length might not be a problem. Unfortunately, neither is the case.

Lack of Reasoning

To see some of the failings of *Focus on Algebra*, consider the comment made by the teacher, "we teach our students to think". In the review section of the last chapter, there are some questions dealing with irrational numbers. One is to explain why $\sqrt{4}$ is rational and why $\sqrt{5}$ is irrational. In the teacher's edition, the answers are $\sqrt{4} = 2$ which is rational and $\sqrt{5}$, in its decimal form, does not terminate or repeat and therefore cannot be written as an integer over an integer. The teacher who thought this book was teaching students to think might think that this is an example of thinking, but there is no one in the world who knows how to prove that the decimal expansion of $\sqrt{5}$ does not repeat without first showing that it is not rational.

The substitution of rote comments that are misleading for substance should be contrasted with the treatment of irrationality of square roots in both a Japanese text [17] and in the book written for the Gelfand Outreach Program [9]. Each of these contains a proof of the irrationality of the square root of 2. With this knowledge, it would be appropriate to ask the students to explain why square root of 5 is irrational.

The last chapter is titled "Functions and the Structure of Algebra". It starts with a brief comment about the golden rectangle and the golden ratio, but without mentioning similarity. Students are asked why the golden rectangle might be more attractive than other rectangles. A possible response which is suggested is that the shapes are not too long and not too square. At this point, the golden ratio is said to be a number approximately equal to 1.618. In the first section, it is said to be the number

$$g = (1 + \sqrt{5})/2$$

which is approximately 1.618034. Then the authors move immediately to the Fibonacci sequence. The word "sequence" does not appear in the index, nor are any other sequences mentioned there, so one wonders why these are not given the traditional name of Fibonacci numbers. The first seven Fibonacci numbers are stated and the rule of formation is given. The first problem is to write out the first 15 numbers in the Fibonacci sequence and find the ratio of each number to the previous number. The students are to use a calculator to approximate each ratio to six decimal places. Then the students are asked to compare their ratios with the value of the golden ratio.

The authors continue by writing that the golden ratio and $\sqrt{5}$ are both irrational numbers, but you can combine them in two surprising expressions to find *any* number in the Fibonacci sequence. Then formulas are stated when n is odd and when n is even. While a^n has been used for general a, it is not used here with $a = -1$ to give a formula which holds for both n odd and n even, so separate formulas have to be given. The students are asked to find the thirtieth number in the Fibonacci sequence, but not to show that the defining property of Fibonacci numbers is satisfied by the numbers stated in the two formulas. This is just one of many places where teaching students to think is passed over in favor of having them do calculations which do not lead anywhere.

A little later in this section, students are asked to use a graphing utility to repeatedly multiply the matrix A

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The first part is to find A^2, A^3, A^4, A^5, and A^6, and then to describe the patterns which they see in the entries of these matrices. Leaving aside the fact that a "graphing utility" is not what was meant, the students should not use any kind of computing aid to do this. They need to do the calculations by hand to try to understand what occurs. There is more to this than just spotting the Fibonacci numbers as the entries. Students should understand how these numbers arise, how the defining relation for the Fibonacci numbers leads to their occurrence, and be able to explain this in words. It is too early to introduce formal induction arguments, but this example can be used as a way to lead up to induction in a couple of years. One of the reasons for having books appear in a series is to consider just such points.

If something like this were done often, the claim of teaching the students to think could be justified. It is not done at all.

Pseudo-Science and Bad History

The first thing one notices when looking at *Focus on Algebra* is the lavish use of color illustrations, frequently irrelevant to the mathematics being discussed. These illustrations can be very distracting, and some high school students have remarked on this to their teachers.

At the start of Chapter 3, Section 2, there is an illustration of a wooden carving from Mali. The text is about the Dogon and their interest in astronomy. Part of what is stated is relevant to the mathematics to be considered in this section, but part of it is totally inappropriate in a school text. "Anthropologists studying the Dogon in the 1940s reported that without the aid of telescopes or other instruments, the Dogon had discovered that Jupiter has satellites, and that Saturn has rings. Neither fact is apparent to the naked eye. Even more amazing, the Dogon claimed that an invisible star of enormous density orbits the star Sirius once every 50 years. Not until 1925 had astronomers discovered that a so-called 'white dwarf' — a dark, tiny, and incredibly dense star — circled Sirius." The reference in the margin of the teacher's edition is to an article in National Geographic. A fuller reference would be [10]. It is very unlikely either the students or the teacher will be able to explain what happened to Marcel Griaule, and how he was fooled by Ogotemmeli. I don't know if Ogotemmeli read French and had seen a French astronomy book, as was suggested to me by an anthropologist I asked about this claim, or if the possible reason given in the margin of the text about the Dogon learning of these facts from outsiders before contact with anthropologists is true. However, this type of pseudo-science has no place in a mathematics book. See [7] for other comments on this page.

There are some poor historical comments. The section on the golden ratio has the following: "It is said that Theano, wife of Pythagoras, did original work on the golden rectangle." I asked the mathematical historian Roger Cooke about this. Here is part of his reply:

> The 'it is said that' phrase ought to be punishable by death, likewise 'man sagt, dass' and 'on dit que'. All that is known of Pythagoras is due to later commentators, mostly Diogenes Laertius (3rd century AD, 700 years AFTER Pythagoras). It's only due to DL that we know of Pythagoras' wife and children. I don't recall offhand that he mentioned any work done by the wife, but he may have. I'm not sure he's a reliable source, since he includes two previous incarnations in his life of Pythagoras and gave two contradictory accounts of his death.

The list of reviewers of this book does not contain either mathematicians or mathematical historians. Both should be used to check the text for accuracy.

What about the Algebra?

What about some of the topics which have traditionally been a major part of a first year algebra course? Factoring is one, and a very good eighth grade teacher I have been corresponding with claims that it takes at least six weeks for students to learn this well. At the same time they are learning this, they are finally learning the arithmetic many of them have not learned well enough. Multiplication of binomials comes in Chapter 8, and it starts with algebra tiles. Here is a description, but it pales in comparison to seeing them in print. The number 1 is represented by a small yellow square. A red square of the same size represents -1. A yellow rectangle of the same width but longer represents x, and a red one of the same size represents $-x$. To represent x^2, a yellow square is used which has the same length as the long side of the rectangle which represents x. The set of tiles I bought was made carefully so that the length of the rectangle is not an integer multiple of the width, so that students will not think that x can be constructed from a fixed number of ones. These are first used to solve linear equations. There is a reference to work by Confrey and Lanier [6] that is said to show that many students at all secondary levels are still at the developmental level where learning must be facilitated by concrete and pictorial representations of concepts. I read the paper on which this claim is said to come and the summary is not accurate. They said that some students seem to benefit from this. From comments from teachers, it is clear that there are also students for whom use of algebra tiles is completely inappropriate. These objects have a very restricted use, for there are few equations which can be modeled with a moderate number of tiles. Their restricted usefulness is only one of their drawbacks. They can become a crutch which makes it hard to progress to a higher, more abstract level.

The reason given for using algebra tiles is to help students who seem not to be ready for the abstraction which comes with the use of letters to denote objects. A better solution to this problem would be to gently start using letters in elementary school, as is done in Russia. See [20] and [25]. While students who are having trouble with mathematics are being shortchanged, the recent TIMSS results suggest that our better students are equally disadvantaged in comparison to their peers in other countries. In the British TIMSS report, it is claimed that 25% of the US eighth grade students are in the bottom 25% of students internationally, while only 18% are in the top 25% [15, p. 22]. In [2, p. 31] it is claimed that only 5% of the US students in the eighth grade sample are in the top 10%, and 45% in the top half. This should be contrasted with the results from Singapore, where 45% are in the top 10%, 74% in the top 25%, 94% in the top half, and only 1% in the bottom 25%. Their texts are very interesting, and one does not see the irrelevant illustrations that clutter up our texts. Algebra tiles are not used.

When multiplication of binomials is introduced, the first example is $(x+2) \times (x+5)$. Algebra tiles are laid out on the outside of a rectangle with sides $x+5$ and $x+2$, and students are asked to fill in the rectangle using one x^2 tile and as many x-tiles and unit tiles as needed. It is here that the fact that the long and short lengths are not related by small integers is useful. One other drawback about algebra tiles is that students would tend to get the idea that x is larger than 1, and so is x^2. In fairness, the authors also use the distributive rule to do this multiplication, but only as a secondary method.

In the margin on page 598, FOIL makes its usual appearance. Almost all college students know what FOIL means, but many of their professors do not. It is an acronym used to illustrate $(a+b)(c+d) = ac + ad + bc + bd$, of First, Outer, Inner, Last. Here it is illustrated by $(2x+1)(x-3)$ rather than the usual $(a+b)(c+d)$. I have asked many people why FOIL is used, and have yet to hear a good explanation. Betty Phillips told me that she and Glenda Lappan have been trying to get rid of FOIL for over 20 years. Gail Burrill told me that FOIL was outlawed at Whitnall High School where she taught for many years. Yet, it is in a textbook she coauthored. It is in a series of video tapes on algebra which Sol Garfunkel had made. When I asked him about this, he said that he had not thought hard about it, and just used what others had used. He now thinks this was an error. At least in the present case, FOIL is not in the book the students use. However, it is presented as

> Tips from Teachers: You may want to teach the students the FOIL method ... as a mnemonic for multiplying two binomials....

Factoring is then done. First, this is done with algebra tiles. Then polynomials of degree two are factored by inspection, as well as by the two special cases; using the difference of two squares and using $(a+b)^2$. On one page, it is suggested that the teacher multiply $(x-1)(x^2+x+1)$ and $(x-1)(x^3+x^2+x+1)$, and then ask the students to guess what the result will be when the next case is considered. However, there is no follow-up to this work, so what has been started will be lost for almost all students. The need to come to closure with what is started is not one of the strengths of our current reforms. The same failure to follow up occurs with respect to Pascal's triangle on the same page.

The work on factoring is usually a prelude to quadratic equations, and is so here as well. Quadratic functions and equations are the subject of the next chapter. The first method of solving quadratic equations is with a graphing calculator or with a graph on a computer screen. The possibility of drawing a graph by hand is mentioned. The word "parabola" is used, but the geometric definition is not mentioned. A parabola is just the graph of the quadratic equation

$$y = ax^2 + bx + c.$$

There are problems which ask students to explain something. For example, problem 12 on page 642 is:

> The graph of $y_1 = rx^2 + 3$ is a parabola that is much wider than that of $y_2 = sx^2 + 3$. Which is larger, r or s? Explain.

The answer in the margin of the teacher's edition is "$|s| > |r|$; The larger the absolute value of the coefficient of x^2, the narrower the graph." Is it any wonder that our students in college have trouble explaining something when they are told that this statement is an explanation? The answer could have said something about why this is true, such as: "if $s > r > 0$, then it takes a smaller value of x to reach a given height for y_2 than for y_1."

Other ways of solving quadratic equations are introduced. Tables of values of the quadratic function are one way, factoring is another. Not all quadratic equations have rational roots, so a method using square roots is introduced, and used to solve equations easily reducible to $x^2 = a$. However, nothing is said about how square roots can be computed. The least that should have been done is to explain how to take square roots by bisection. A very nice treatment of square roots and cube roots from a historical perspective is contained in [11]. This is a marvelous book, written over a ten year period by a physician. It puts us to shame that a first rate book like this is written by an amateur rather than by a professional mathematician or mathematics educator.

Finally, there is a section on "The Quadratic Formula". Here is a quote from this section:

> Today most people solve quadratic equations using a formula that is based on a geometric model. The *Quadratic Formula* is one of the most important formulas in mathematics.

The quadratic formula is stated, then an example is given of its use. Students are then given problems to do using it. However, there is no derivation, nor is anything said about what the "geometric model" mentioned above is. This must mean the old way the Babylonians used to solve quadratic equations. I asked one of the authors why the quadratic formula was not derived and was told the following. Quadratic equations come near the end of the usual algebra course and many teachers do not give a derivation of the quadratic formula. Since it is important that students see a derivation of it, and most states now require three years of mathematics, the derivation was put off until the next algebra book in this series *Focus on Advanced Algebra*. In Wisconsin, two years of high school mathematics are required, not three. I was told by another person associated with this series that the real reason was that the quadratic formula has to be in a first year algebra book before the book can be considered for adoption in Texas. Nothing is said in the Texas guidelines about what has to be done with the quadratic formula, so a derivation does not have to be given. The authors have moved systems of linear equations from the second algebra book to the first, and they felt that the derivation of the quadratic formula could wait. My cynical view is that systems of linear equations allow matrices to be used to solve them, and graphing calculators can be used to solve systems of two by two linear equations, including finding inverses of matrices.

Focus on Algebra is far from the worst of the new books. However, it contains a representative sample of what is being called "a mile wide and an inch deep". Another similar book is [26]. This book has much more use of algebra tiles. A review of this book is posted on the World Wide Web; see [22]. Here is one quote from this review.

> Most of the problems of the book end with the word: "Explain." But the book or the teacher edition never offers any explanation.

Notice the similarity between these books. They ask for explanations, but do not teach the students how to explain things.

To put the treatment of the quadratic formula into perspective, consider the following.

Math Camp is a one month program for bright high school students. At Math Camp, in August, 1996, I mentioned that some of our new programs do not give a derivation of the quadratic formula in the first year algebra course, and some do not give this until 12th grade, when many students are no longer taking mathematics. A young woman from Turkey expressed surprise, since her class had been taught this in seventh grade.

Is Another Revolution Coming?

There are many other examples which could be given. Let me close with two examples of what I hope will be in the next mathematics education changes.

First, consider trigonometry. There are just a few central ideas behind trigonometry. The first is the idea of similarity. This allows one to define the trigonometric functions so that they apply to all right triangles as functions of only one variable. Right triangles are determined by the Pythagorean theorem, so combined with similarity it is possible to define sine and cosine as the coordinates of points on the unit circle. The second important fact is decomposition. Any triangle can be decomposed into a pair of right triangles. This allows the trigonometric functions to be used in the study of arbitrary triangles, and allows triangles to be used in the study of polygons. One can use the invariance of the circle under rotations to prove the addition formula for cosine, but there are very simple direct proofs using decomposition. See the forthcoming book [8] by I. M. Gelfand and Mark Saul for some examples. Everything in elementary trigonometry is a corollary of these few facts, but not such easy corollaries that one can stop here. The usual facts need to be learned, and technical skill needs to be developed and practiced. However, all of this is much easier if the fundamentals are always kept in mind.

Finally, there is an important book [19] coming out shortly. One chapter deals with the computational problem of $1\frac{3}{4}$ divided by $\frac{1}{2}$ and attempts to illustrate this by story problems. One of the Chinese teachers said she would not use division by $\frac{1}{2}$ to illustrate division by fractions, since it is easy to see the answers without dividing by fractions. She suggested using $1\frac{3}{4}$ divided by $\frac{4}{5}$, and gave a problem exemplifying this calculation.

Contrast this story with the following passage in the NCTM Curriculum Standards, page 96. "The mastery of a small number of basic facts with common fractions (e.g. $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$; $\frac{3}{4}+\frac{1}{2}=1\frac{1}{4}$; and $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$) and with decimals (e.g. $0.1+0.1=0.2$ and $0.1\times 0.1=0.01$) contributes to students' readiness to learn estimation and for concept development and problem solving. This proficiency in the addition, subtraction, and multiplication of fractions and mixed numbers should be limited to those with simple denominators that can be visualized concretely or pictorially and are apt to occur in a real-world setting; such computation promotes conceptual understanding of the operations. This is not to suggest, however, that valuable instruction time should be devoted to exercises like $\frac{17}{24}+\frac{5}{18}$ or $5\frac{3}{4}\times 4\frac{1}{4}$, which are much harder to visualize and unlikely to occur in real-life situations. Division of fractions should be approached conceptually. An understanding of what happens when one divides by a fractional number (less than or greater than 1) is essential."

I wrote Liping Ma to ask for comments on this quotation from the NCTM Standards. She replied as follows:

> I would like to claim some interesting and important relationship between basic facts with common fractions (e.g. $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$; $\frac{3}{4}+\frac{1}{2}=1\frac{1}{4}$; and $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$) and with decimals that can be visualized concretely and those much harder to visualize and unlikely to occur in real-life situations. In

> fact, without a conceptual understanding of the former, it will be unlikely for one to understand the latter. However, unless one's understanding of the former is deepened and solidified by the latter (which is not as hard as people imagine), the primary conceptual understanding is still very limited and superficial and therefore too fragile to make connections to other concepts of the subject. So, students' mathematical power will be generated from a connection of the "basic facts" and "abstract concepts", rather than emphasizing or ignoring either of them.

Ma's book may be the start toward a balanced view of mathematics education which we have long needed. Like a stool which needs three legs to be stable, mathematics education needs three components: good problems, with many of them being multistep ones, a lot of technical skill, and then a broader view which contains the abstract nature of mathematics and proofs. One does not get all of these at once, but a good mathematics program has them as goals and makes incremental steps toward them at all levels.

References

[1] Lars Ahlfors et al, "On the mathematics curriculum of the high school", American Mathematical Monthly, **69** (1962), 189–193; Mathematics Teacher, **55** (1962), 191–195.

[2] Albert E. Beaton et al., *Mathematics achievement in the middle school years: IEA's Third International Mathematics and Science Study (TIMSS)*, TIMSS International Study Center, Boston College, Chestnut Hill, MA, 1996.

[3] California State Board of Education, *Mathematics framework for California public schools*, California Dept. of Education, Sacramento, 1992.

[4] California State Board of Education, Mathematics Program Advisory, http://wwwgoldmine.cde.ca.gov/ci/branch/eltdiv/mathadv.htm.

[5] Randall I. Charles et al., *Addison-Wesley Secondary Math, Focus on Algebra*, Addison-Wesley, Menlo Park, CA, 1996.

[6] Jere Confrey and Perry Lanier, "Students' mathematical abilities: a focus for the improvement of teaching general mathematics", *School Science and Mathematics* **80**, 549–556.

[7] Martin Gardner, "The new New Math", *The New York Review of Books*, September 24, 1998, 9–12.

[8] I.M. Gelfand and Mark Saul, *Trigonometry*, Birkhäuser, Boston, to appear.

[9] I.M. Gelfand and A. Shen, *Algebra*, Birkhäuser, Boston, 1993.

[10] Marcel Griaule, *Conversations with Ogotemmeli*, Oxford Univ. Press, 1965.

[11] Jan Gullberg, *Mathematics: from the birth of numbers*, Norton, New York, 1997.

[12] Geoffrey Howson, *Mathematics textbooks: a comparative study of grade 8 texts*, Pacific Educational Press, Vancouver, BC, 1995.

[13] Marianne M. Jennings, "'Rain Forest' algebra course teaches everything but algebra", Christian Science Monitor, April 2, 1996.

[14] George Gheverghese Joseph, "The crest of the peacock", Penguin, London and New York, 1990.

[15] Wendy Keys, Sue Harris and Cres Fernandes, Third International Mathematics and Science Study, First National Report, Part 1, Achievement in Mathematics and Science at Age 13 in England, National Foundation for Educational Research, Upton Park, Slough, Berkshire SL1 2DQ, England, 1996.

[16] Morris Kline, *Why Johnny can't add: the failure of the new math*, St. Martin's Press, New York, 1973.

[17] Kunihiko Kodaira, *Japanese Grade 9 Mathematics*, 1984, translated by Hiromi Nagata, Univ. of Chicago School Mathematics Project, Chicago, 1992.

[18] Li Yen and Du Shiran, *Chinese mathematics: a concise history*, translated by John N. Crosley and Andrew W. C. Lun, Clarendon Press, Oxford, 1987.

[19] Liping Ma, *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*, Lawrence Erlbaum Associates, Hillsdale, NJ, to appear in 1999.

[20] M. I. Moro and M. A. Bantova, *Russian Grade 2 Mathematics*, translated by Robert Silverman, Univ. of Chicago School Mathematics Project, Chicago, 1992.

[21] Calvin Moore, "California's math reforms aim to inspire economy of thought", Sacramento Bee, October 18, 1995.

[22] Motohico Mulase, review of [26]. See http://ourworld.compuserve.com/homepages/mathman/pinkbook.htm.

[23] National Council of Teachers of Mathematics, Curriculum and Evaluation Standards, Reston, VA, 1989.

[24] R. Sridharan, Review of [14], *Current Science* **70** (1996), 753–754. Reprinted in *Resonance* **1**:6, June 1996, 90–93.

[25] Zalman Usiskin, "Doing algebra in grades K-4", *Teaching Children Mathematics*, 1997, 346–356.

[26] A. Wah and H. Picciotto, *Algebra: themes, concepts, tools*, Creative Publications, Mountain View, CA, 1994.

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