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Reflections on Teacher Education

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Prologue

Our panel's topic is one of the hotly discussed national issues: K–12 mathematics instruction. Mathematics is an important thinking tool, being

- a way of looking at and making sense of the world;
- a way of, for example, checking out the validity of claims made by advertisers or politicians;
- a way of reasoning.

Certain mental habits, observable in young children, should be cultivated in school rather than inhibited or allowed to atrophy.

Mathematics is particularly well suited for exercising mental muscles. Even those who do not achieve virtuosity can find that the psychological rewards of feeling mentally fit and gaining control of one's reasoning powers are just as great as the rewards of physical fitness and control of one's body.

Classroom teachers, always considered essential in the education of our children, are now expected to play even more crucial roles. Reformers eager to save public education see teachers as "agents of change". Advocates of children who see so many deprived of functioning families want teachers to act in *loco parentis*. Future employers have discovered that they no longer need human robots but flexible thinkers who can work in teams to solve problems with tools appropriate to the task. They are joining educators (who want to build communities of future adults able to cope with the demands of the next century) in viewing teachers as coaches in the acquisition of skills, and as guides in discussions of conjectures and verifications of results. Our demands on teachers are overwhelming indeed.

I want to talk about the role mathematicians might play vis-à-vis schools and teachers. If time permits, I should like to make this discussion less theoretical by telling anecdotes from my personal experience in some inner city schools of New York City, observations in an "alternative school", regular visits to planning sessions of mathematics and science teachers in a recently created "small" public school, and monthly "math suppers" with about a dozen dedicated teachers who work in various schools: private, suburban, alternative, a newly created secondary school modeled after Central Park East, "Coalition" schools, and a community college.[1]

Many teachers are confronted by the conflict of wanting to teach a curriculum that they consider to be right for their students, and on the other hand feeling that they must help students pass tests that they consider irrelevant, but that afford entry to certain careers and/or future studies. Mathematicians may help resolve this conflict.

How Can Mathematicians Help Mathematics Instruction in K–12?

University and industry research mathematicians can make a difference; look at Leon Henkin, Paul Sally, Arnold Ross, Henry Pollak, or, among the next generation, Phil Wagreich, Herb Clemens, Judy Roitman and others. Here are two ways they can help:

- By working, as almost silent partners, with teachers on planning classes and seeing how the plan works, discussing it afterwards with the teachers (perhaps also the students), and by improving the strategies for the next round of teaching that topic.

For most mathematicians, getting into contact with classroom teachers in their schools takes time, patience, and luck. We have to take down barriers. Educators say "What do these ivory tower mathematicians know about adolescents and about pressures from districts, parents, administrators?"; and university mathematicians say: "Look what these people are doing to our beautiful subject! And to some of our brightest students!" Teachers are often overburdened not only with too many classes, too many students per class, but also with non-instructional chores and bureaucratic paperwork. They cannot find the time to get together with mathematicians, and often are afraid of being found wanting mathematically. Mathematicians feel they are not trained to teach children and should do that which they are good at and which is rewarded: research. It took me more than a decade to be welcomed in schools.

Seeing teachers in action made me admire them, especially for those talents they have and I lack, such as engaging groups of kids in tasks they perform collaboratively. But I have found most teachers weak in mathematics. Many enjoy what I can offer, in planning sessions outside their classrooms, particularly

[1]Time did not permit. But I was inspired by Dick Askey's careful comments in his article (published in this volume) to use the same mathematical example to fashion a strategy richer in mathematical and pedagogical opportunities than a mere enumeration of weaknesses in the text. I called the paper *When you find a lemon, make lemonade!* It is included as an appendix to this paper; see page 90.

my connecting a student question or error to some part of mathematics or to an application or generalization or to a test item. One can go quite deep (but in small doses), and one must follow through. A great part of learning involves connecting a new piece of knowledge to something you already know or at least vaguely remember. But you need to have something in your experience to which you can connect the new event. Mathematicians have lots of such things in their heads, teachers have some, and students have only a few and need to build more. Nobody can say "this reminds me of a joke" if he has never heard that joke. Learning to make connections is one of the most important skills to emphasize in school.

- By looking critically, with teachers, at materials (textbooks, manipulatives, videotapes) adopted or recommended by their schools or by regional or national committees; by getting teachers to suggest what to emphasize, what to omit, and why, and how to correct or supplement weak sections of the text; by visiting a group of teachers who are creating a new curriculum unit. There are now some good texts that are making a difference.

Good teachers, at any level, rarely follow a textbook faithfully, even if they have authored it. In selecting a text, then modifying it, we use our mathematical knowledge as well as our classroom experience. By listening to students and reading their assignments, we become aware of what students find difficult and why; we note where misconceptions and anxieties are formed; our mathematical knowledge together with our curriculum agenda combine to lead students into meaningful mathematics.

I cannot think of better "staff development" than critiquing, correcting, supplementing and trying out a curricular unit. This should become a regular part of teacher education in collaborative set-ups where at least one non-obtrusive mathematician takes part. The success of such efforts depends on the personalities and interests of the participants. The arts of connection-making and of salvaging the useful parts of errors help, and the fear of exposing gaps in knowledge can get in the way.

Interested mathematicians are likely to find some obstacles, such as the barriers described above, and not being invited to give feedback on preliminary versions of a new curriculum, either because a publisher co-sponsors it and has a copyright, or because the creators fear their material may be misused by untrained teachers and cause more harm than good. It may even happen that research mathematicians are asked to serve on advisory committees to curriculum projects and find themselves "token mathematicians", neither consulted nor heeded. After they resign from the advisory committee, their names keep appearing on the list of members.

New and Old Styles of Teaching

Some years ago I saw a television program that featured two teachers: The first was shown lecturing to a class in a most engaging and stimulating way; the second was shown asking some challenging questions, getting groups of 4 or 5 to work together, walking around the class room stopping occasionally to clarify or ask a student to clarify a point that arose, and demanding a verbal description from a member of each group of what results or partial results had been arrived at. I was delighted by the performance of each and impressed by their knowledge of the topics studied in each class. It would never occur to me to persuade either one of these teachers to adopt the methods of the other, although I believe more young students benefit from the latter model because of their active engagement and intellectual struggles and the necessity of talking and listening to other group members who may bring a different perspective to the problem under consideration.

Let us encourage diversity in teaching styles even as we take into account diversity in learning styles. Different styles will reach different children; let all experience learning in more than one way. Let us not cramp a teacher's style! If the teacher is coerced into a mode felt to be unnatural, he or she will become less effective.

Reformers as well as traditionalists want students to reason mathematically, to express their thoughts and to examine one another's methods of attacking and solving a problem. Lecturers can reveal their own more informed thoughts in presenting a solution, or they can show students the stunningly clever thoughts of some great mathematicians by lecturing about their ideas. The pitfall to avoid is to have the class say "I could never have thought of any of these things; I might as well give up." It is the second teacher who can better demonstrate to students that, yes, they can indeed think of some pretty clever things. University mathematicians do just this with their doctoral candidates and should keep it in mind when they think of a young novice. The main difference is that the doctoral student must think of something completely new, while young students are discovering something well-known, yet new to them. Both teachers and students work and learn best when they feel that they are doing their own work and not just following somebody else's unexplained prescriptions.

The "basic skills" people should stop vilifying the "concepts" people by characterizing them as sentimental "hand holders" who lack appreciation of mathematical rigor; and the "concepts" people should stop thinking of the "skills" people as mindless drill masters without either aesthetic or deductive tendencies. Clearly, mathematical activities require both: understanding of concepts and facility with technique. But we should realize that some people first learn rules and algorithms and then become interested in why these work so well, while others are unable to accept recipes unless their origins and logic are explained. Most of us combine the two modes.

What Does all This Have to do with Teacher Education?

Both teachers shown in the TV show mentioned at the beginning of the previous section were well grounded in mathematics. I don't know how many courses or credits each had, but they knew that mathematics is not just a collection of "facts", and they recognized that learning mathematics involves doing mathematics. In contrast, the well worn caricatures of the traditional drill master (who concentrates on endless basics) and of the overzealous "facilitator" (who neglects proofs and questions of structure) would, with better mathematical background, become more like the models I saw on TV.

I want to cite the work of two people: Deborah Ball [Ba] asks "Not How Much, but What Kind?" of mathematics should teachers know? [Ba] And Eileen Fernandez, in a work in progress [Fe], asks "How does a teacher's knowledge affect her response to unanticipated student questions?"

The first of these focuses on the irrelevance of most quantitative certification requirements, e.g. 12 credits of mathematics courses above College Algebra; instead she examines what teachers make of a taped student discussion, how they interpret what is going on in students' heads as they argue about whether or not 6 is an even number, and what mathematical knowledge would help them guide the students in fruitful directions.

The second of these, also concerned with what students are thinking (when they are not thinking what we think they ought to be thinking) examines teacher responses to what students are saying, what errors the students are making, where they get stuck. She will probably conclude that the teachers' responses depend very much on the connections these teachers make to things they know, to mathematics they have internalized. Both researchers evidently consider teachers' mathematical strengths crucial because of their role in teacher-student interactions and in finding fruitful paths out of student errors and into important domains not necessarily next in the syllabus.

As college instructors of prospective teachers, we should give students an opportunity to become engaged in the kind of explorations and collaborations that many educational reformers advocate, yet do not provide for the practitioners. We, as well as the teachers we educate, should stop making assumptions about who learns in what ways. Some claim children are incapable of abstraction or generalization—they are wrong. Some look with disdain on the utilitarian "real world" views of mathematics. They too are wrong. What is needed is a balance, and if some of us have biases in one direction, let these be counterbalanced by colleagues of different opinions; but let us all try to take our students' talents and interests into account as we develop their mathematical teaching crafts.

In conclusion, I want to thank my panel colleagues for providing balance to the raging controversies on reform. I was cheered by George Andrews's article [An], and by David Mathews's response and Andrews's rejoinder in the same volume. There have been many thoughtful articles and reports on significant educational insights. Many of these preceded the reports we hear and read about. Yet, when you visit most class rooms, you see no change (except perhaps more anxiety on the part of teachers and administrators about meeting new standards or preparing students for changes in tests). Similarly, we see little change (with some notable exceptions) in the practices of most teacher education departments, and we see almost no change in the mathematics courses or seminars that Liberal Arts colleges recommend to prospective teachers. We need strong consortia of collaboratives to give prospective teachers the pedagogical and mathematical experiences that would prepare them best for what they would like and what we expect them to do. It is up to mathematicians to take such initiatives.

Appendix: When you Find a Lemon, Make Lemonade!

A recent text contains the problem:

Explain why $\sqrt{4}$ is rational while $\sqrt{5}$ is irrational.

The teachers' edition of this text suggests the answer:

> "$\sqrt{4} = 2$ which is rational; $\sqrt{5}$, in its decimal form, does not terminate or repeat and therefore cannot be written as an integer over an integer."

Now we don't know what the class has studied prior to being given this problem; presumably they know the definition of "rational" and, my guess is that they have seen a proof of the irrationality of $\sqrt{2}$, in which case they might try to cook up a similar proof for $\sqrt{5}$. But now suppose the teacher tells his class what the teachers' edition says. Two questions would immediately arise:

1. How do you know that the decimal form of $\sqrt{5}$ does not repeat or terminate?
2. Even if you knew that, how would the irrationality of $\sqrt{5}$ follow from the fact that its decimal representation is infinite and nonrepeating?

This would lead to two discussions which may be carried on with the entire class; or two groups might be formed, each pondering one of these questions. About Question 1, the following conversation might take place:

Student A: When I put 5 into my calculator and press the $\sqrt{\ }$ button, I get 2.2860679775, and this is only 10 places. This does not tell us if there will be a repeat, only that we don't see one so far.

Student B: My sister is in college and can get something called "double digit accuracy" on her computer.

Student C: So what! Suppose you get twenty or even 1024 places, which, they tell me, some computers can get, you still would not know if, may be after a million places, some string of digits starts repeating.

Student A: I remember that my grandmother told me she learned a way of calculating square roots in school long before there were computers or calculators, and that this was even harder than long division; but you could get as many decimals as you want.

Student C: And if you sat there all your life, you still wouldn't know if digits might repeat eventually.

At this point the class would conclude that the advice they got is not very helpful, because there is no way of verifying the claim that the decimal expansion neither repeats nor terminates. Here, we might hear a subdiscussion where A promises to ask his grandmother to show him the method she learned, and somebody else volunteering an uncle who knows how to extract the square root of a number N by first guessing a nearby integer, say x_0, and then refining the guess by averaging x_0 and N/x_0 and using this as the next approximation x_1, and then continuing this process, getting successive approximations $x_n = \frac{1}{2}\bigl(x_{n-1} + N/x_{n-1}\bigr)$. At a future class meeting, the uncle's and grandmother's methods may even be compared for efficiency and accuracy; for example, the students can determine after how many iterations in the uncle's method they get the 10 digits the computer gave them, etc.

Now let us turn to Question 2. Suppose we knew by some feat of omniscience that $\sqrt{5}$ has an infinite, nonrepeating decimal representation. Then we may hear

Student C: Well, if we knew that all rational numbers have terminating or repeating decimal expansions, then we could conclude that a number whose decimal expansion neither repeats nor terminates must be irrational simply because, if it were rational, this could not happen.

Student D: Oh yeah! We had examples like this in the school I went to last year, in a unit on Logic. Some fancy name was used, like 'contrapositive' I think. Well, how would you prove that rational numbers have repeating or terminating decimal expansions?

Student E: Well, how do you find the decimal representation of a number that can be written in the form $\frac{a}{b}$ with a and b whole numbers, $b \neq 0$? You would divide a by b.

And here comes a good reason for studying the division algorithm instead of using the calculator. In this discussion, students would probably take some examples, perhaps $2/3$ or $1/7$ or $3/5$ or some other fraction with a small denominator and would be led to some notable observations. For example, if a fraction in lowest terms has a denominator with no prime factors except 2 and 5, then its decimal representation must terminate. This will reinforce the notion that a terminating decimal is merely a fraction whose denominator is a power of 10. In the case $3/5$ above, we merely multiply top and bottom by 2 and get $3 \times 2/5 \times 2$ or $6/10 = 0.6$. In the other examples, they would see that in the long division, remainders keep popping up, and since the only nonzero ones are $1, 2, \ldots, b-1$ (where b is the denominator), we see by the pigeonhole principle that one of these remainders must, after at most $b-1$ steps, pop up again. And when this happens, the whole process repeats and we get a periodic decimal expansion.

The discussions of questions 1 and 2 need to be merged and the conclusions summarized. Yes, the decimal representation of rational numbers is periodic or terminating, so a number with infinite, nonperiodic decimal representation is indeed irrational; the trouble is we cannot tell that $\sqrt{5}$ is in this category, so we need to figure out another way of proving its irrationality.

We may now recall (or look for the first time at) the proof of the irrationality of $\sqrt{2}$ and try to construct its analogue for $\sqrt{5}$ — and more generally for $\sqrt{M}$, where the integer M is not a perfect square — and probe a bit into elementary number theory. We may also recall that an argument for the existence of $\sqrt{2}$ on the number line was the fact that it has geometric representations, for example as the length of the diagonal of a unit square, and we come up with the geometric analogue: $\sqrt{5}$ is the length of the diagonal of a 1×2 rectangle. Both will reinforce the theorem of Pythagoras. Students who like geometry may find other ways of constructing $\sqrt{5}$ and other square roots.

Another outcome may be a more precise application of the very elementary and extremely useful pigeonhole principle. It shows, for example, that the period of the decimal expansion of a rational number in lowest terms cannot be longer than the size of its denominator. So if the computer output of 2^{10} digits shows no periodicity, we could conclude that, if $\sqrt{5}$ were rational, its denominator would have to be at least 1023. Though the pigeonhole principle is understandable to everybody, the difficulty in applying it is often due to the difficulty we have in deciding which are the objects to be stashed, and which are the pigeonholes that must hold these objects.

Computer buffs in the class may enjoy writing a program for the iteration scheme that the hypothetical uncle has contributed, see above.

I am not suggesting that all these connections will be made; but I do want to call attention to the benefits of analyzing a suggested method for solving a problem. We can learn a lot even if the method cannot be carried out in finite time. We may also gain a better appreciation for an easily applicable, efficient, and correct method for solving our problem.

References

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