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On the Role of Proof in Calculus Courses

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I would like to consider two questions:

- Should students see proofs in a standard calculus course?
- Should students do proofs in a standard calculus course?

I use the word "should" with all its moral overtones because I think that to many this is as much a moral issue as a pedagogical one. On the other hand, I have a hard time distinguishing moral considerations from considerations of taste, where the dictum *de gustibus non est disputandum* applies. Therefore I will for the most part think of "should" as "need" or "must" or even "does it help". I do believe these questions are at the root of some of the debate about calculus reform, but I have heard very little thoughtful discussion of the issue. I must, however, apologize that what I say is woefully uninformed by research in mathematical education, and all I intend is to begin a dialogue.

Before I proceed I would like to differentiate between two pedagogical uses of proof. The first, which I'll call Proof I, is part of a process of formalization and organization. In this setting, the student is presumed to have an effective, reliable understanding of concepts and results; the goal is to develop a formal language with which those results can be proved true. I would consider the proof that the limit of a sum is the sum of the limits as Proof I. The second sort of proof, which I'll call Proof II, is less formal, and is used to answer a question that is in doubt. Students don't often encounter such questions in a calculus course. Here is a very simple example: if the functions f and g are twice differentiable and their graphs are concave up, must the graph of $f+g$ be concave up? The answer is not obvious if one thinks only pictorially and one could well start looking at examples; but of course, once the question is interpreted in terms of the sign of the second derivative, the question easily yields its answer and a proof.

Should Calculus Students See Proofs?

I will list some commonly given reasons why they should, and comment on those reasons. (I am sure there are other reasons, and I am sure my comments are not without bias. Again, I intend only to begin a dialogue.)

1. Proofs help students understand concepts and believe results. This certainly hasn't been my experience, at least with Proof I. Does it help students to see a proof that the limit of the sum is the sum of the limits? To put this in a historical perspective, would it have helped Newton or Euler? Of course, the context I have established for Proof I assumes effective understanding and belief already. However, I don't think that merely seeing Proof I adds much to understanding or belief. On the other hand, I feel Proof II does help, but I have no solid evidence.

2. It is useful in later mathematics courses for students to see proofs in a calculus course. The obvious question here is: What later course? The half-life of a calculus student is one semester, and well under 10% of the students in a standard calculus class will ever major in mathematics. Whether it is useful or not for potential mathematics majors, we better have a good reason for the other 90% of the class. As for the majors themselves, I am sure there is some value in seeing proofs, but I'm not sure how much. Having seen my best students forget in one semester the derivative of the arctangent, I don't have a lot of faith in the vertical recall of students in vertically structured courses. Will they remember the Mean Value Theorem when they get to a junior analysis course? Will the dim memory of having seen the proof help? Was it worth confusing the rest of the class? Now I'm talking about Proof I. Seeing Proof II may help more in later courses, but not nearly so much as doing Proof II.

3. Proofs are part of students' cultural heritage, which they should appreciate the same way they appreciate the theory of relativity or Huckleberry Finn — even if they don't understand it. I am sympathetic to this view, and I have to admit that ten years ago I used to force my class to memorize the ε-δ definition of limit as a form of poetry. I guess that, as I've gotten older, I've become less impressed with the grandeur of this definition. There is better poetry to be remembered. Or there are Kepler's Laws. Maybe I've just found what I feel are better things to do with the little time I have with my students.

4. Proofs are what we mathematicians do, and students should see what we do. I am not so sure we do spend that much time doing Proof I, except in writing textbooks or teaching. Our research is mostly Proof II. Anyway, although I believe what I do is interesting, I'm pretty sure that most of my students don't. They might be expecting proofs from me because that is what a mathematician does, but they are not looking forward to it.

5. Proofs are beautiful. Some of Proof I is beautiful and some isn't. I don't find the limit of the sum proof especially appealing, but I know a breathtaking proof that if $f' > 0$ on an interval then f is increasing (it does not involve the Mean Value Theorem). I am very sympathetic to this motive, just as I am to the cultural one; they may be close to the same thing. The trouble is that although most people are struck instantly by the beauty of a Chardin still-life, there seems to be much less innate appreciation for beauty in thought. An untrained or inattentive eye misses much but it can still see a lot; an untrained or inattentive mind misses everything.

6. Proofs build character. It is hard to argue with this. I used to tell my students the same thing about techniques of integration. Lots of things build character — pain and suffering for example. I always remember the story I heard at West Point that Patton (I think it was) said he had no fear of war after having to do boardwork in front of a calculus class as a cadet. I'm not so sure that says anything I want to hear about the public's attitude towards mathematics, when even war is to be feared less. There are more constructive ways to build character. Moreover, this is akin to viewing calculus as a filter, and ten years later it should not be necessary to repeat the reasons why calculus should be a "pump and not a filter".

Should Calculus Students Do Proofs?

It may seem odd that I have phrased this as a separate question. Scientists and mathematicians generally believe their disciplines can only be learned by doing. Yet mathematicians have backed so far away from asking proofs of students that it is rare to find the verb "prove" in a textbook or exam problem. Rather, the word is "show", "justify", or "explain why". Students know there is a difference. They associate "proof" with something very formal, like writing the answer in latin, and they are always afraid we might mean "prove" when we say "show". We allay their fears; we talk about proof, but we don't put it on the exam. Is material learned if it does not appear on the exam? Does a tree falling in the forest make a sound if there is no one to hear it?

I find it a sad state of affairs that much of the debate about calculus reform has devolved to arguments about whether students in a calculus course should see formal definitions and proofs, with the tacit admission that it is out of the question for students to write proofs of their own. You know you're poor when you fight over crumbs. How did we get to this point, where we dare not use the word "prove" in calculus problems? Many I am sure will cite the quality and attitude of their students, but blaming students only provides solace to the instructor. I think one reason proof by students has been abandoned is that too often it was Proof I we were asking for. Since we knew students have trouble with proofs, we chose proofs that were more mechanical, such as *epsilon-delta* proofs which mostly involve manipulation of inequalities. One might even call this type of proof remedial. The trouble is that students don't react any better to this type of remediation than to other types. They see proofs as pointless exercises in saying things the instructor wants to hear. One of the important lessons of the Treisman programs for students at risk is that students respond better to challenging problems involving thought than to simple problems involving drill and mechanical manipulation. Maybe if we had been asking more Proof II problems, we would not have given up asking students to do proofs.

Reasons for students doing proofs are no different than the reasons given before for seeing proofs, but this time the reasons are more cogent for Proof II. I really do believe that students doing Proof II are gaining understanding. I am sure that doing Proof II helps mathematics majors for later courses; I even feel that non-majors are not wasting their time in doing Proof II, especially science majors. Students doing Proof II are working much more the way mathematicians do. The beauty of proof is much more likely to be appreciated when the question is in doubt and it is up to the student to grapple with the problem, even if unsuccessfully. Doing Proof II, although perhaps less of a discipline than doing Proof I, is a more constructive way to build character; in any case, the only character built by *seeing* proofs is the ability to sit still while someone is talking to you.

The only one of the reasons for teaching proofs that I'm not so sure about as a reason for doing them is the appeal to cultural heritage. It is possible to appreciate music or literature without composing or writing. Perhaps an attentive students can learn to appreciate mathematical culture by seeing proofs, rather than doing them, and Proof I is as good as Proof II. After all, that is exactly what good popularization of mathematics achieves.

So where are we going to find Proof II problems for calculus students to do? Some areas are replete with good Proof II problems that students can do: geometry, number theory, graph theory. Other areas are not so good. I've always thought linear algebra much more oriented toward Proof I, and for that reason it always seemed odd to me to use it as a vehicle to introduce proof to students, especially when the practical applications of linear algebra are so useful to so many. I believe that introductory calculus has plenty of Proof II problems that students can do. Although such problems don't appear on college exams and in texts, the Advanced Placement exam for years has had a final problem which was usually Proof II, frequently involving a functional equation, and those problems have provided food for thought in countless AP classrooms. Many of the longer projects given in calculus reform texts or supplementary materials are Proof II or could be put in that form. Proof II is often more verbal than algebraic, and the calculus reform emphasis on writing and non-algebraic approaches opens up more possibilities for Proof II. In fact, much of the culture shock felt by students in upper level, theoretical courses may be due more to how little they have been asked to write in words a coherent mathematical argument or explanation, than to whether they have seen formal definitions and proofs.

In any case, at this point I think I owe some more examples of Proof II problems than the one I gave earlier. Here are a few to get started.

(i) If the graph of f is increasing and concave down for all x, then the graph of f has a horizontal asymptote. Prove or give a counterexample.
(ii) The student is given the graph of a differentiable function f on the interval $[0, 1]$ such that $f(0) = f(1)$. Clearly at some points the slope is positive and at other points the slope is negative. Prove that the average of the slopes is exactly zero.
(iii) Prove that the function $f(x) = x^2 + \cos(kx)$ has either infinitely many points of inflection or none at all, depending on the value of k (this was on the 1995 AP exam).
(iv) One could define the derivative of f at $x = a$ as the limit of difference quotients of the form $(f(a+h)-f(a-h))/(2h)$, instead of $(f(a+h)-f(a))/h$. For example, graphing calculators use the first quotient rather than the second to estimate the derivative. Does it make a difference in the definition? If the first limit exists, must the second? If the second limit exists, must the first? Prove or give counterexamples.
(v) Prove that the equation $\sin(x) + x = c$ has exactly one solution, no matter what the value of c.
(vi) (from multivariable calculus) Suppose that $f(x, y) = g(x) + h(y)$ and that $g' < 0$ for $x < a$, $g'(a) = 0$, and $g' > 0$ for $x > a$, and that $h' < 0$ for $y < b$, $h'(b) = 0$, and $h' > 0$ for $y > b$. Prove that f has a local minimum at $x = a$, $y = b$.

Some of these problems are pretty hard, but they are not out of the question for a good calculus student. The point is that problems like these should be in a calculus course, and at present they are not.

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