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Is the Mathematics We Do the Mathematics We Teach?

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In a recent article [7], William Thurston called attention to the sad state of the mathematics classroom:

> We go through the motions of saying for the record what the students "ought" to learn while students grapple with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material "covered" in the course, and then grading the homework and tests on a scale that requires little understanding. We assume the problem is with students rather than communication: that the students either don't have what it takes, or else just don't care. Outsiders are amazed at this phenomenon, but within the mathematical community, we dismiss it with shrugs.

This brings up the question: Does what is taught in the typical mathematics course even qualify as mathematics?

In another article in the same issue of the *Bulletin* [4], Saunders Mac Lane offered:

intuition – trial – error – speculation – conjecture – proof

as a sequence for understanding of mathematics. In contrast, the sequence in place in most modern mathematics courses is:

lecture – memorization – test.

Most working mathematicians agree with MacLane's description, thus leaving the inescapable conclusion that the mathematics we do is not the same as what is commonly offered in the classroom. The questions for the next century are:

- Can the mathematics we offer in the classroom be more like the mathematics we do?

- Can we ignite students' mathematical interest?
- What role can computers play in dealing with these questions?

Phillip J. Davis [2] indicates how the answer to the computer question sets up answers to the others:

> The capabilities of all mathematicians are elevated by their association with computation. The transformation by the computer of triangle geometry and of many other areas has, paradoxically, reconfirmed and strengthened the the vital role of humans in the wonderful activity known as mathematics. Put it even more strongly: mathematics develops in such a way that the role of the mathematician is always manifest...
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> In connection with visual output, I have even argued for the recognition of "visual theorems"... where what the eye "sees" need not even be verbalized let alone formalized in traditional formal mathematical language... subtle feeling that that language cannot even name, let alone set forth...
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> As regards mathematical education, I think the message is clear. Classical proof must move over and share the educational stage and time with other means of arriving at mathematical evidence and knowledge. Mathematical textbooks must modify the often deadening rigidity of the Euclidean model of exposition.

Calculus&Mathematica as a Prototype Reaction to the Issues and Questions

Calculus&Mathematica [1] is a new computer laboratory calculus course developed at University of Illinois at Urbana-Champaign and the Ohio State University expressly to deal with the questions and the issues raised above. The course is freshly built from the ground up. The purpose of the course, the ways of implanting mathematical ideas into students minds, the ways of motivating students to learn, and the ways of making students retain the important ideas have all been rethought.

As a result, Calculus&Mathematica is the most thoroughly new calculus course available today, and it presents a new model for successful learning of calculationally heavy sciences. Not screened from the essence of calculus by labor-intensive calculations and plots, students in Calculus&Mathematica get right to the good stuff. From the very beginning, they see calculus emerge as the first course in scientific measurement, calculation, and modeling. Students also see calculus as a highly visual and often experimental scientific endeavor just as research mathematics is. The medium is a live electronic interactive text composed of lessons written in Mathematica Notebooks. Each interactive Calculus&Mathematica lesson consists of the following set of Mathematica Notebooks:

- Basics Notebook, for the fundamental ideas,
- Tutorials Notebook, for sample uses of the basic ideas,

- Give It a Try Notebook, for actual student work, and a
- Literacy Sheet: for what a student should be able to handle away from the machine.

For an annotated example, see http://www-cm.math.uiuc.edu/work/examples.

The National Research Council report *Moving Beyond Myths* [5] describes Calculus&Mathematica as follows:

> An innovative calculus course... [which] uses the full symbolic, numeric, graphic, and text capabilities of a powerful computer algebra system. Significantly, there is no textbook for this course — only a sequence of electronic notebooks.
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> Each notebook begins with basic problems introducing the new ideas, followed by tutorial problems on techniques and applications. Both problem sets have "electronically active" solutions to support student learning. The notebook closes with a section called "Give-it-a-try," where no solutions are given. Students use both the built-in word processor and the graphic and calculating software to build their own notebooks to solve these problems, which are submitted electronically for comments and grading.
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> Notebooks have the versatility to allow re-working of examples with different numbers and functions, to provide for the insertion of commentary to explain concepts, to incorporate graphs, and plots as desired by students, and to launch routines that extend the complexity of the problem. The instructional focus is on the computer laboratory and the electronic notebook, with less than one hour per week spent in the classroom. Students spend more time than in a traditional course and arrive at a better understanding, since they have the freedom to investigate, rethink, redo and adapt. Moreover, creating course notebooks strengthens students' sense of accomplishment.

Unlike point-and-click multimedia and programs that merely turn pages, Calculus&Mathematica presents examples that can be modified by the student and rerun; so that each example in Calculus&Mathematica is as many active interactive examples as the student wants.

The premise behind Calculus&Mathematica is that if students have the opportunity to go about their work in a manner similar to the manner that working research mathematicians go about their work, they have a good chance for success. Here are some of the principles on which Calculus&Mathematica is based:

Communicate new ideas visually and experimentally; get an idea across before putting language on. Unique to Calculus&Mathematica is the attempt to get mathematical ideas into the students' minds visually before words are put on. To paraphrase Stephen Jay Gould: Scholars are trained to analyze words, but students are visual animals. Well-conceived visualizations are not frills, they are foci for modes of thought. The course is driven by well-chosen, re-executable, interactive computer graphics and student-produced graphics inviting the students to experiment, to construct for themselves, to describe, and to explain what's happening in their own words.

Through interactive visualizations, Calculus&Mathematica tries to stick the basic calculus ideas into the students' unconscious minds before it transfers the ideas into English. For instance, students experiment with simultaneous plots of $f[x]$ and $f'[x]$ to acquire an understanding of the meaning of the derivative. Students experiment with plots of the exponential function and are imprinted with its awesome growth. Students who have never heard of convergence experiment with plots of functions and their Taylor series expansions, soon discovering that the convergence is what advanced mathematicians call "uniform on certain compact intervals."' And they invent the word "cohabitation" to describe what they see. Students experiment with running trajectories through vector fields and become comfortable with vector fields. They know that gradient fields drain at relative maximums. As a result of their experience, most of them can tell you why a solution of Laplace's equation cannot have an interior maximum. Reason: The gradient field of a solution of Laplace's equation has no sinks or sources.

Always give the students the opportunity for a creative response; give the students an active role in their own learning. Don't try to think for the students. Calculus&Mathematica students take an active role in their own learning by selecting material from the electronically alive Basics and Tutorials to learn (and possibly rework) as needed, and at their own pace. If a point doesn't get through, then they are free to modify and rerun as they see fit. At all times, they have the opportunity to pursue their learning actively and creatively. This lone aspect of C&M puts C&M at a great distance from lecture-based calculus courses and the new passive point-and-click multimedia courses coming onto the market. In the final analysis, this aspect of C&M is totally natural because this is the way research scientists do their work.

Approach mathematics as a science, not as a language or as a liturgy. Often mathematics is taught as a ritual or a liturgy in which the professor functions as curator of the dogma and arbiter of truth. Sometimes mathematics is taught as a language, a language which, as Blaise Pascal pointed out, "must be fixed in [the student's] memory because it means nothing to [the student's] intelligence." All too rarely is mathematics taught as the science that it is. The Calculus&Mathematica course attempts to teach mathematics as a science in which the student is the active investigator. With wise use of the computer to help introduce the ideas through the eyes, Calculus&Mathematica replaces the usual sequence,

lecture – memorization – tests,

with this variant of Mac Lane's sequence:

visualization – trial – error – speculation – explanation.

In this format, calculus becomes the same as the mathematical activity in which active mathematicians engage.

Ask students for explanations, not proofs. The words "prove" and "show" are the most terrifying words inexperienced math students ever encounter. The word "explain" is not so terrifying because explanations are usually assumed to be not so formal as a proof. On the other hand, a good explanation usually contains the main ideas of a formal proof; so that concentrating on explanations instead of formal proofs does not degrade mathematical understanding. In fact, rigor and understanding are often separate: Rigor is in one part of the brain, but understanding permeates the brain, the heart and the soul.

Rigor without understanding and understanding without rigor are both possible. In any case, the ability to recite a memorized proof of a theorem is not the same as understanding the theorem. The real goal is to understand. And that's what Phillip Davis is talking about when he says, "Classical proof must move over and share the educational stage and time with other means of arriving at mathematical evidence and knowledge."

Use a computer-based, genuinely interactive text. Conventional printed texts have a paralyzing effect on learning because they force the student into a passive, subservient role. Thomas S. Kuhn explains it best [3]: "Science students accept theories on the authority of teacher and text, not because of evidence. What alternatives have they, or what competence?"

The Calculus&Mathematica electronically alive interactive text, in which every example is as many examples as the student needs, is an environment in which the student can accumulate as much evidence as the student requires. The result: The student actively learns, in part, on the basis of the student's own authority and not just on the authority of the teacher or of the text.

Eliminate introductory lectures. Thurston [7] wrote: "Mathematicians have developed habits of communication that are often dysfunctional... most of the audience at an average colloquium talk gets little of value from it." Just as mathematics colloquium talks are usually failures, so introductory lectures in mathematics classes are usually failures. Reasons:

- Introductory lectures are full of answers to questions that have not been asked.
- By necessity, introductory lectures are full of precise terms not yet understood by the students.
- Introductory lectures provide the strong temptation for the teacher to try to do the thinking for the students.
- Introductory lectures tend to center the course on the lecturer instead of the students.

To paraphrase Schopenhauer: Attending introductory lectures is equivalent to thinking with someone else's head instead of with one's own. Instead of introductory lectures in Calculus&Mathematica, regular discussions are held, but not until the visual ideas have congealed in the students' minds as a result of their lab experience. These discussions emphasize answers to questions the students ask.

Motivate students to want to learn by serving up problems whose importance is recognized by the students. "What's this stuff good for?" is a question often heard from students in ordinary calculus courses but seldom heard from Calculus&Mathematica students. The reason is that the mix of student problems in C&M puts students in a position to try calculus out to see what calculus can do for them in terms of their own lives and in terms of their own planned professional futures in measurement, calculation and science. Students think carefully about how to apportion their efforts as part of their planned futures. Possessing an uncanny ability to recognize frivolous or artificial classroom problems, students usually tune out of ordinary calculus courses, but they rarely lose interest in Calculus&Mathematica.

Keep the language in the vernacular. Students fail in writing about mathematics because their textbooks are written in language that they cannot understand. As a result, they resort to rote memorization because much of what they read and hear means little to their intellects. Paul Halmos even went so far as to say that the job of the mathematics teacher is to translate the textbook into the vernacular. It does not have to be this way. Calculus&Mathematica is written in the vernacular in words, phrases and sentences that the students can understand and adapt in their own writing.

Give the students a chance to organize their thoughts by explaining themselves in writing. Calculus&Mathematica students visually absorb ideas uncorrupted by strange words, and they address the problem of communicating what they have learned only after they have a visual understanding of the idea under discussion. The first step is to visually determine what the truth is; the second step is to explain it. Students in ordinary calculus courses are deprived of the excitement of discovery and explanation. C&M students write a lot of mathematics and they are unexpectedly good at it. There are two reasons for this talent:

- The Mathematica Notebook front end gives the students a unified environment for graphics, calculations and write-ups.
- The language used in Calculus&Mathematica is informal enough for the students to adapt it to their own writing.

Give the students the opportunity to learn the mathematics and the programming in context. Ordinary attempts to bring applications into calculus tend to separate the mathematics from the applications. Similarly, ordinary attempts to bring technology into calculus tend to separate the mathematics from the technology. Calculus&Mathematica always puts the mathematics in the context of measurement and puts the programming in the context of mathematics. Most importantly, C&M exploits the technology in an effort to introduce new ideas.

As a result, the applications, the programming, and the mathematics all feed off each other. A C&M student put it best: "I have started to notice aspects of one class carrying over to another. Similarities in fields I thought unrelated before. An interconnection between math and language and programming and everything just kind of fits together a little better now."

Give the students professional tools. Students preparing for careers in a calculational science see computers or workstations running Mathematica as professional tools. Believing that the ability to use professional tools is part of their overall education, C&M students typically throw themselves into using Mathematica-equipped computers. They understand, perhaps better than their teachers, what vistas these professional tools open up.

Does it Work?

The study by Kyunmee Park and Kenneth Travers [6], which compares standard calculus and Calculus&Mathematica, states: "Generally the findings from an achievement test, concept maps, and interviews were all favorable to C&M students. The C&M group obtained a higher level of conceptual understanding than did the standard group without much loss of [hand] computational proficiency… [Some believe] that a laboratory course in calculus is very time consuming, and that students can become overly dependent on Mathematica. But this research found that the C&M course allowed the students to spend less time on computations and better direct themselves to conceptual understanding. Accordingly there was an increase in the students' conceptual achievement without a serious decrease in computational achievement… Furthermore, the C&M group's disposition toward mathematics and the computer was far more positive than that of the standard group… Generally, the C&M group seemed to more clearly understand the nature of the derivative and the integral than did the standard group… A positive side effect of the [computer] lab was the rapport that was established among the students. When students gathered around the computer, worked together, and shared and developed ideas, a great deal of mathematics was learned. … [Computer] capabilities helped students discover and test mathematical results in much the same way that a physics or chemistry student uses the laboratory to discover and test scientific laws. Those capabilities provided the opportunities for the students to consider more open-ended questions and to encounter more realistic problems than often found in traditional calculus texts."

Students who enter calculus with high expectations and motivations resulting from their own professional plans in a calculational science are likely to blossom in C&M. This includes high percentages of engineering students and math students. It also includes motivated rural high school students in the C&M Distance Education Program at Illinois. Life science students at Illinois have done so well in C&M sections designed for life science students that the School of Life Sciences at Illinois has financed C&M labs for all of their freshman students.

We have been personally overwhelmed by the way students have thrown themselves into Calculus&Mathematica. We hope that Calculus&Mathematica and better courses to follow will help to pave the way to a time at which mathematics becomes just as alive for its students as it is for its practitioners.

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