Reports from the Working Groups

How the Working Groups Worked

The participants of the conference were divided into small working groups in the afternoons, each charged with a different topic. The task of each working group was to come up with a coherent, concise report on its deliberations, with concrete recommendations on how to improve mathematics education with respect to its topic.

Before meeting each day, participants in each working group were asked to fill out a questionnaire on their topic for the day; the answers provided the basis for that day's discussion. The group's reporter collected the questionnaires at the end of the conference.

Each group had a discussion leader and a reporter. The discussion leader's job was to keep the group working and on track. The reporter's job was to take notes, to collect the questionnaires, and to write up the final report.

The final report divides roughly into two parts, one for each day: (1) a summary of previous activities of the working group members, with an assessment of successes and failures; and (2) recommended goals and strategies to achieve those goals.

It was neither necessary nor possible in all cases that the group come to a consensus. Nor was it possible to avoid speculation and uncertainty. However, each group was advised to make its best effort at finding areas of common purpose, and recommending specific strategies.

The Renewal of Teaching in Research Departments

Members: Harvey Keynes (Discussion Leader), Al Taylor, Richard Falk, Leon Henkin, Lars-Ake Lindahl, Richard Montgomery, Dan Shapiro, Donald St. Mary, Donald Martin (Reporter), Susan Montgomery

Questions for Day 1

- What is the current state of teaching in your department? What is good or bad about it?
- What steps have you or your department taken to improve teaching?
- How does being in a research department affect your teaching?

Questions for Day 2

- What goals would you set for your department's teaching effort? (Consider curriculum, teaching practices, and infrastructure.)
- What strategies would you recommend to attain those goals? What obstacles, such as workload or reward structure, stand in the way?
- As a research department, what specific advantages or disadvantages do you have to offer your students?

Introduction. Perhaps the single greatest challenge to research departments is the renewal of teaching. Many departments are under great external pressure to do a better job teaching and to pay more attention to the needs of students heading into technical, rather than research, careers; at the same time, internal incentives remain much as they have been: slanted towards research.

The group listed the following points as basic elements in the mission of a research department:

- Research universities have a dual role of basic research and teaching.
- Every faculty member should be a good teacher.
- The measure of good teaching is that the students learn and become engaged in the learning process.
- There should be an atmosphere in the department which is conducive to students learning.
- Research departments should, in addition to the traditional work of preparing students for graduate school, provide mathematics education for the technical workforce and leadership in training of K-12 teachers.

The strategies recommended to achieve a renewal of teaching fell into three broad areas: (1) changes in faculty attitudes; (2) programmatic and curricular changes; and (3) pedagogical changes.

Faculty attitudes. Most group members felt that faculty at their institutions took teaching fairly seriously, although one mentioned a small group (most, but not all, quite senior) who 'felt that researchers need not be concerned with good teaching.' However, as one group member put it:

> The feeling of professors seems to be that most reform efforts or other ways to improve learning involve money: smaller class size, and larger teaching staff. The administration may supply some support, but not much. Some professors blame the students for inability to learn.

It was generally agreed that the reward structure needed to be changed. This was reflected in responses to the question, How does being in a research department affect your teaching?

> While being a good teacher is encouraged, it is still understood that I will basically be judged on my research output. This creates a conflict concerning time. . . .
>
> Not as much time available for students as one would like, and not much energy for restructuring the curriculum. Research record is the major criterion for promotions, raises, and university recognition, so there is not much incentive for really good teaching except to maintain one's self-respect, and a feeling of responsibility toward students.
>
> Of course the traditional structure of rewards is geared entirely toward research, and any extra time spent on teaching is discouraged. . . .

Apart from fundamental changes in the reward structure, the following measures were recommended:

- There should be more opportunities for faculty to meet students in small classes or groups.
- Faculty should be teach entire sequences with the same student group.
- There should be more regular contact between senior and junior faculty members.
- There should be improved and more extensive professional development opportunities for faculty (junior and senior).

Programmatic and curricular changes. Some departments were in the process of implementing new curricula in lower division courses, others had traditional curricula. One group member summarized the advantages and disadvantages of a traditional curriculum:

ADVANTAGES: Well-organized curricula, most people quite responsible, works OK in a traditional way.

DISADVANTAGES: Very hard to change anything. . . . Very little outreach to students; a minimal number of math majors . . . .

The group made the following programmatic recommendations:

- Programs for undergraduates not going to graduate school should be strengthened.
- More effective professional master's programs should be developed.
- Departments should make greater recruitment efforts for math majors, by offering different tracks (applied, actuarial, ...), stressing career opportunities, offering case-based courses, finding internships and undergraduate research opportunities.

Pedagogical changes. There was a consensus that experimentation with different teaching styles and with the use of technology should be encouraged, although caution was advised:

- More experimentation with teaching and pedagogy should be encouraged for some faculty.
- Effective use of technology should be encouraged at all levels of mathematics.
- There should be increased efforts to develop and use effective measures to determine if teaching approaches, alternate pedagogies, and different content are improving student learning.

The Use of Technology in the Teaching of Mathematics

Members: Peter Alfeld, Kirby Baker, Angela Cheer, Estela Gavosto (Discussion Leader), Ben Halperin (Reporter), Tom Judson, Abel Klein, Gerardo Lafferriere, Charles Lamb, John Orr, Bob Welland

Questions for Day 1

- Give a specific example of how you have used technology in your teaching (Which courses, how much use, who used it, what was it used for: motivation, illustration, heuristic arguments, numerical computations, symbolic computations, graphical work, more advanced problems).
- How successful was it? How did it help or hinder student learning, compared to teaching the same topic without technology?

Questions for Day 2

- In general, what are the problems to think about when considering the use of technology? (Effect on later courses where technology is used, training for the workplace, ...) Distinguish between problems that are inherent in the technology and those coming from the implementation.
- What do you think the future role of technology in teaching mathematics should be?

Introduction. The discussions in our group were very lively during the two days that the group met. The dominating concern was that technology is already a part of everyday life. In the near future, whether we like it or not, it will be an essential part in the teaching of mathematics. The choice for research mathematicians is what role we will play in this change. In our group, we covered a broad range of issues. Not surprisingly, the computational capabilities of the technology were not discussed much. Main issues were the potential of the technology as a communication tool and the demand for it in the workplace.

Recommendations. We want to urge all mathematicians to take an active role in the use of technology in the teaching of mathematics. We are the only ones that have the mathematical knowledge and the teaching experience to make it worthwhile.

We should take a realistic (optimistic) approach about students' abilities and computer availability. There are two main ways that we think that the technology can be productive in the teaching of mathematics:

- as a computational tool and
- as a communication tool.

The big questions about the computer as a computational tool (numerical and symbolical computations and graphics) are related to the students' learning. In particular, the following questions should be addressed:

- How can technology be used to further students thinking beyond the traditional contents of a course?
- How can technology be used to teach "remedial" mathematics like deficiencies in algebra?
- How can technology be used to help student develop independent visualization skills?

We are witnessing an explosion in the increase of the use of technology as a communication tool. Electronic mail is a very powerful way of facilitating communication among students and instructors. The use of the web is in exploratory stages. Some of its possible uses are the following.

- Online Courses
- Traditional courses:
  - web course notes
  - discussion groups
  - on-line office hours
  - pre-testing and testing
  - on-line demonstrations

Physical considerations should also be a very important consideration from the students' perspective. Actually, there are very different environments where technology can be used:

- the computer laboratory;
- the traditional classroom with equipment for computer demonstrations;
- the traditional classroom with portable technology like a graphing calculator or calculator based laboratory; and
- web–based activities.

All these different environments seem to converge. A primary obstacle for the implementation of any new technology is the training of the instructors. Serious consideration and effort should be given to this matter.

We want to conclude by saying again that it is imperative that research mathematicians participate in defining the technology that will be used to teach mathematics in the near future.

Different Teaching Methods

Members: Greg Baker, David Epstein, Ted Gamelin, Sid Graham (Reporter), Ole Hald (Discussion Leader), Delphine Hwang, Suzanne Lewis, Randy McCarthy, Brad Shelton, John Sims, Robert Underwood

Questions for Day 1

- What is your favorite teaching method? (Standard lecture method, collaborative learning, labs, etc.)
- What experiments have you tried in order to improve your teaching methodology? (Getting students actively involved in the lecture, group work, lab projects, using technology, etc.)
- What did you want to achieve with your experiment? Did you achieve it? Were there any unexpected results?

Questions for Day 2

- What goals would you set for alternative teaching methods? (For example, getting students to read, write, think for themselves, take responsibility for their work.)
- What strategies do you recommend to achieve these goals? What obstacles stand in the way?

The participants in this working group exchanged ideas on teaching techniques that they either use regularly or with which they have experimented. Over time, enthusiastic mathematics instructors have attempted various new and old teaching techniques. The wheel is constantly being reinvented. Nevertheless, the characteristics of the body of students taking calculus are constantly changing, teaching resources are changing, and teaching techniques require constant experimentation and adaptation to new circumstances. It is a worthwhile exercise to compile an assortment of teaching techniques, and to describe their assets and pitfalls. We begin here to create such a collection.

We describe here several of the techniques that various committee members have used in recent years. The techniques we list are all designed to work in large lecture sections; many of the techniques are adaptable to the small class format as well.

Summary of previous activities

Advisory Committee of Students, or Ombudsmen (*Randy McCarthy*). This technique has been used by Randy McCarthy to deal with the problem of communication between students and instructor in large lecture classes. He forms an advisory committee of students, which meets regularly with the instructor to communicate the students' concerns. The advisory committee students — or *ombudsmen* — have regular communication with individuals or groups of students, and make it their business to be in synch with the needs and feelings of the class. Students open up to their peers on topics which they would be most uncomfortable discussing directly with the instructor. The ombudsmen develop a good working relationship with the instructor, and the resulting good feeling permeates the rest of the class.

Use of Worksheets and Breaks in Large Lecture Classes (Ole Hald). Ole Hald has used a version of this technique at the University of California at Berkeley. For a calculus class of about 400 students, of which 300 might be present at any given lecture, about 100 worksheets are distributed very quickly at the beginning of the class by a large number of assistants. Hald might lecture for ten minutes and then direct the students to the first problem on the worksheet. For instance, he might show them how to integrate $\cos(x)$, and the worksheet problem might be to integrate $x + \cos(3x)$. The students work in groups of three or four. Though the sheets are not handed in, the students are asked to put their names on the top of the sheet, and this way they learn about each other. If the classroom is large enough, students are asked to leave every third row empty, and then Hald can circulate around the room and reach all students. After ten minutes on the worksheet, there might be another short lecture segment, then back to the worksheet.

Use of TA-administered 15-minute oral exams to cover theory (Ole Hald). Ole Hald is experimenting with this technique at U.C. Berkeley. The main idea is to dedicate a week (or four days) of the schedule to oral exams on theory. The students are given ten exam topics to prepare. They sign up for exams at 15-minute intervals. The exam is administered by a TA, and another TA is present as a scribe. At the beginning of the exam the student is offered two topics and selects one. Then the student is given eight minutes (no more!) to present the topic (such as an easy theorem about convergence of series and its proof from the text). In the next few minutes the student responds to questions from the proctors; these questions are designed to test understanding. The student leaves, and the two TAs assign a grade. The grading is on a scale of ten points. For a perfect presentation, the student is allocated eight points, and the student's score fluctuates up or down from this base level by at most two points according to the responses to the questions. (Thus, to guarantee a score of six points, the student must simply to commit to memory the presentations of nine — out of ten — theorems and proofs.) At the beginning of the exam the student is permitted to reject the two topics offered and ask for two others, at a cost of two points deducted from the final score. How much total TA time this requires depends on the size of the TA sections (not the class size). It does require extra TA time and intensive effort during the oral exam week. No classes are given during that period. The instructor gives some training to the TA's before the oral exams, and the instructor visits exams randomly. The student is allowed to bring a friend as witness to the exam, and some students have brought teddy bears to their oral exams.

The two students participating in our workshop session, Delphine Hwang and Suzanne Lewis, have taken these oral exams. They feel that the method is effective. They practiced their presentations with each other before the exams and they learned a lot in the process.

Hald has also used variants of this technique in smaller, more advanced undergraduate courses.

Use of Summary Statements and Paragraphs (John Sims). The idea is to have the students write summary paragraphs of individual sections and chapters of the text. This forces students to think through the material on their own and to organize their own synopses. It helps students to prepare for exams. One way to implement the idea is to give the hour exams in two parts. One part is given in class; the other part is a take-home exam that includes a request for the written summary. The summary might be three sentences long, and the concatenated summaries give a good overview of the course.

Use of Crib Sheets. Crib sheets are very brief sets of notes taken into an exam. This is a widely-used technique; it has many variations. It also encourages the students to come up with a short overview of the material in their own words. Some instructors prefer to distribute the cards for the crib notes, and even to use colored cards so that they are easily recognizable in exams. Others allow the students to use their own materials. There are widely differing views on whether students should be allowed to use calculators in exams.

Selective Grading of Homework. In an era of diminishing resources, it is often impossible to read all the homework of all the students. Yet it is very important that students do the homework, and on a regular basis. And it is important that they get regular feedback on their work.

One compromise is to require that the students turn in substantial homework, and have the reader grade only a portion of the homework problems. If there are not even resources for this option, then it is better to require the students to hand in homework and simply check it as done than to give the students the responsibility for doing the homework and checking their answers against some posted answer guide.

One method of selecting homework for grading, used by Ray Redheffer at UCLA, is to select which of several assignments will be handed in for grading by a random choice, as follows. Ray gives two homework assignments in every class meeting. He has the students bring the two completed assignments to class in the following lecture. At the beginning of the class Redheffer asks three students to spell their last names, which he writes on the blackboard. He counts the total number of letters in the names, and the parity determines which of the two assignments is to be turned in. A byproduct of this method is that Redheffer learns the names of a number of students in his large lecture.

Another method of solving the reading problem has been used in large calculus courses at UCLA by Bob Brown. While no homework is handed in, there is a weekly quiz in recitation section consisting of exactly one homework problem from a designated list. The problem is chosen at random, and it might (or might not) be different for each recitation section.

Use of Mastery Exams (David Epstein). These are exams given during the term that a student must pass at a level of 100% in order to pass the course. The course might have two midterms, a final, eight quizzes, and six mastery exams. If the students passes all the mastery exams then he/she is guaranteed a grade of C in the course. Grades on midterms and other parts of the course then serve to raise the student's aggregate grade from the entry level of C.

The mastery exams are designed to be straightforward and simple, and the student may repeat each mastery exam as often as necessary. Of course the professor will require considerable help (from TAs or other assistants) to administer such a system.

Recommendations. The working group recommends that an appropriate education committee of one of the professional societies maintain a Web site that would serve as a repository for descriptions of teaching techniques, including details of implementation, commentary on hidden difficulties, what works, and what does not work. The Web site should also maintain a bibliographic listing of books or articles on teaching techniques.

The First Two Years of University Mathematics

Members: Joseph Ball, Christopher Grant, Peter Lax, Robert Megginson (Reporter), Kenneth Millett, Wayne Raskind, Thomas Tucker, Joseph Watkins, Hung-Hsi Wu (Discussion Leader)

Questions for Day 1

- Describe the composition of the freshman and sophomore mathematics class of your institution. (Math majors, science and engineering students, business students, premeds, general education students.)
- Have you undertaken any projects involving freshman and sophomore courses? Consider both curriculum and teaching practice.
- What would you change if you undertook the same project today?

Questions for Day 2

- Consider the factors that affect your department's ability to do a good job teaching these courses (workload, homework policy, grading practices, availability of technology, student preparation, faculty commitment, curriculum). Which ones can your department affect, and how can it do so in a systematic way?
- What strategies do you recommend to achieve a mathematics program that your department teaches well, and that satisfies the standards of both your department and the home department of your students?

This working group endeavored to discover the distribution of students by major in the first two years of mathematics at the institutions represented by the working group members. It also discussed what projects are being undertaken in freshman/sophomore level courses at these institutions and what has been learned from them, and what strategies would help the departments achieve a well-taught mathematics program for these students.

The distribution of students by major in these courses varies greatly from institution to institution. None of the working group members reported more than about 20% mathematics majors in these courses, and one institution reported only 2%. Though not all members were able to report figures, an estimate for the median percentage of mathematics majors in these courses at these institutions would be about 10%. The distribution of non-mathematics-majors in these courses varies greatly depending on the mission of the institution. For example, in one institution about 50% of the Calculus I students are premedical, while another has no such students because it has no premedical program. It is clear that, looking over all these institutions, there is no "typical" distribution of students — according to major — in the first two years of mathematics.

Almost all of the institutions represented by this working group are doing some sort of experimentation with or modification of their mathematics courses for the first two years. Some of the projects are significant across-the-board calculus reform efforts, while others involve smaller modifications to individual sections of a course by the instructor teaching the section; there are yet other efforts that are a mixture of the two.

Summary of previous activities. Virginia Tech has three paths through calculus, with each path appropriate for a different group of disciplines. Some sections of differential equations courses have been taught for individual disciplines. *Mathematica* is used regularly in engineering calculus. There have been experiments with self-paced, interactive, computer-based instruction in precalculus. There is also an Emerging Scholars Program involving mandatory extra problem sessions — to give students with marginal backgrounds a better chance of success.

In the past several years, BYU has had three strands of calculus: one taught from the Harvard Consortium text, one from the `Mathematica`-based text by Stroyan, and one from Ellis & Gulick. The teacher of each section is free to choose one of the texts according to personal preference, and no differentiation between the courses taught from these various texts is made in the timetable. Starting in the second term of the 1996–1997 academic year, all students in calculus will take a common competency exam. "Teaching trios" are being introduced, each consisting of a senior faculty member, a newer faculty member, and a graduate student; these three team members visit each others' classes. A course has been implemented that is designed specifically to teach students about proofs.

NYU has integrated applications extensively into its calculus courses. Currently, full use of graphing calculators is being made in these courses. There has been some experimentation with the Harvard calculus materials. While the instructors liked them, this feeling was definitely not shared by the students.

The University of Michigan has large-scale calculus and precalculus reform efforts underway, and uses the Harvard Consortium's precalculus and calculus texts in all regular sections of their Precalculus, Calculus I, and Calculus II courses. Group methods and graphing calculator technology are used extensively, and a week-long training program at the end of August prepares the instructors to teach these courses.

UCSB has two calculus tracks, one for the mathematicians, scientists, and engineers that is taught from the Harvard text, and one for the life scientists and economics majors taught from *Calculus in Context*. There has been quite a bit of experimentation with pedagogy in these courses. Currently, the following courses are undergoing renovation and development: Mathematics for Elementary Teachers; a course for majors to aid the transition to upper division courses; and problem solving and history courses for future teachers. There is also a substantial workshop program, associated with the traditional "barrier" courses, directed toward increasing the participation and success of women and underrepresented students. With the passage of California Proposition 209, there is increased pressure threatening such programs at UCSB, especially in mathematics. UCSB is also currently in the midst of a counter-reaction to efforts to increase the effectiveness of the program for the first two years; resulting negotiations have extended the scope of the discussion to bring in faculty members who have seldom, if ever, taught many of these courses.

USC has experimented with reform in the past, and the working group member from USC, Wayne Raskind, is a member of the Harvard Calculus Consortium. The current main effort at USC is along traditional lines, using the Stewart text. There are plans for an enhanced calculus course for students who come in with AP credit, with some linear algebra included in the first semester of this course. Some efforts are being made in the direction of group learning.

While Colgate has no large-scale projects to change courses in the first two years of mathematics, various instructors are experimenting with changes in individual sections. A few are using weekly computer assignments, and a few have used the Harvard Consortium text. (Two years ago, the department agreed on a core syllabus for Calculus I and II, but instructors are free to choose their own texts. There are common final exams.) All sections use graphing calculators, but there is no formally prescribed way in which they are to be used in the courses, so much of their use is similar to that of traditional calculators.

Arizona has had a major presence in the Harvard Calculus Consortium; several of it faculty are members of that consortium. The Harvard texts are used for the first three semesters of calculus. There is a training program, lasting several days, for those new to the teaching of these courses. On occasion, some sections have used other texts and teaching styles. The Mathematics Department has recently designated calculus and precalculus "czars" whose job is to provide institutional memory and consistency for the calculus and precalculus courses. The role of the precalculus czar is particularly important at the moment, since the precalculus program is changing in response to increased entry requirements for the Arizona university system.

The finite mathematics and calculus courses for business students at Arizona are moving toward the use of spreadsheets as part of the pedagogy of those courses. Two consortium members have written a text for differential equations, and technological ideas are being added to the linear algebra course. One general difficulty currently affecting teaching assignments is a mandate from the Board of Regents that 2/3 of all freshman and sophomore teaching be done by faculty with Ph.D.s.

About two years ago, the chair of this working group, along with a colleague, reorganized the first two years of calculus at Berkeley. The principal changes were to eliminate linear algebra from multivariable calculus, to use the same book (Stewart) for three semesters, and to make the two sophomore courses (multivariable calculus and ODE–linear algebra) independent of each other. There is another effort underway, again spearheaded by the chair of this group with another colleague, to have the "soft" calculus sequence at Berkeley use the Harvard Consortium's *Applied Calculus* text.

Recommendations. Much has been learned from the above projects. There were comments from several members that assessment is a critical issue, and one that would perhaps be better thought out if some of the projects were to be undertaken anew. It was observed that all too often departments make significant changes in programs or texts without adequate procedures in place to assess the results. Agreement should be reached ahead of time on such issues as what needs to be improved, what the goals of any proposed reform are, what the strategies are to reach those goals, and other related issues. (In several of the universities represented by this group, assessment of reform efforts has shown no discernible difference between test scores of students coming from traditional and reform sections, but it was observed that this can have different interpretations depending on the design of the tests. The scope of assessment can be fairly broad, and should probably take into account aspects of student self-esteem and empowerment.)

It was also observed that the workload needed to support curricular reform projects can be great, and that in retrospect this could be taken into account to a greater extent. Another observation was that more communication with client departments about the nature and extent of the reform efforts could in certain cases have been useful.

The working group's discussion of possible changes to mathematics programs in the first two years focused on three issues: **(i)** the role and form of linear algebra in the first two years, **(ii)** the role of abstract mathematical reasoning in introductory courses (by whatever name one wishes to attach to it, whether *proof,* or *rigor,* or *theory*), and **(iii)** the need to keep lines of communication open to client departments. No agreement was reached on the role and form of linear algebra, with some preferring that it be strongly motivated by applications and computation, and others wishing to see more of the structure and theory of linear algebra introduced.

There was general agreement that, in the first two years, students should be shown the need for rigorous mathematical argument in appropriate situations, rather than being exposed only to heuristic arguments. There was also general agreement that the instructor should always seek the best way to convey *understanding* of an idea, result, or technique; this goal may at times require proof and at other times call for less rigorous arguments based, for example, on graphical intuition.

Some members of the working group felt that mathematically correct proofs of several main results should be given in all introductory mathematics courses, to the extent of perhaps four or five such proofs per semester; also that the students should be shown how to construct rigorous mathematical arguments themselves, lest they suddenly discover that they are not properly equipped to continue in mathematics when they reach the more proof-intensive advanced courses. Others felt that this is not so important in the first courses, but that instead appropriate opportunities could be sought to develop in the students' minds an understanding of the need for proof and generalization in arguments. One example that was given involves first showing students that $x/e^x \to 0$ as $x \to \infty$, then doing the same for x^2/e^x, x^3/e^x, and x^4/e^x. This investigation does provide heuristic evidence that $x^n/e^x \to 0$ as $x \to \infty$ whenever n is a positive integer. However, the students might feel uncomfortable with the generalization since, for example, x^{1000} grows so much faster than x^4. In addition, after seeing the four special cases, the students will probably conclude that it would be good just to do this once rigorously for x^n/e^x and be done with it.

The working group also discussed the need to design courses to meet the needs of client departments, and possible mechanisms for keeping lines of communication open with those departments. It was recognized that one common difficulty involving lines of communications is that modifications can be made in consultation with certain persons in client departments who may have gone on sabbatical or may not be in the appropriate roles to influence the process at the time the changes are actually implemented. It was suggested that stability can be achieved by having some persons in long-term roles to make decisions of this sort and to interface with their counterparts in the other departments. Some members also felt that efforts to assure long-term stability in courses — in particular to standardize course content and classroom pedagogy over a long time period and over multiple sections of a course — bump up against serious issues of academic freedom; and this tends to be a force against such stabilization.

The Mathematics Major

Members: Jorgen Andersen, John Brothers, Ralph Cohen, Stephen Fisher, Andrew Gleason, James Lin, Lea Murphy, Richard Montgomery, Y. S. Poon, Ken Ross (Reporter), Anthony Tromba (Discussion Leader)

Questions for Day 1

- Are you satisfied with the experience for mathematics majors at your institution? In what ways have you tried to improve it? (Tracks for different professional interests, undergraduate research projects, math clubs, mentoring, capstone courses, summer internships.)
- What worked and what didn't?
- Do you have many students from other departments taking your upper division courses? How do these students compare with your majors?

Questions for Day 2

- What should we be trying to provide most of our math majors: training for graduate school, training for professional careers, a general education in mathematics?
- What strategies do you recommend for improving the education of math majors? At what stage should majors be introduced to proofs?

Summary of Previous Activities. It has been the case historically that mathematics majors have been produced by schools of every type, from liberal arts college to technical institute. Any discussion of the mathematics major should take into account the various types of mathematics majors that there are, and where they are trained.

Here are some general trends concerning the major; these are distilled from questionnaires that we distributed among the panel members:

- Tracking in the mathematics major, also known as a choice of "options", is a common device for helping the student to have a focus for his/her studies. A typical example of a track is an "applied mathematics track". It consists of a specific curriculum of courses and activities that will help an undergraduate student to train as a mathematics major with applied skills. Such a curriculum will contain some specific mathematics courses, such as numerical matrix theory and partial differential equations; it will also include courses outside the department, such as computer science, physics, and engineering courses. Other standard tracks include "computational math track" or a "biomathematics track" or a "statistics track" or a "mathematical physics track".
- Student math clubs, such as $\pi\mu\varepsilon$, are often unsuccessful; this is so because they are dependent on the involvement of a dedicated faculty member and on a few key students to act as catalysts. The math club can be more successful if some of the meetings are career-oriented and if alumni are involved. In

particular, alumni can provide the students with valuable career information, and also give living credibility to the notion that mathematics is something that real people use in the real world.

- The trend in other departments such as engineering, biology, and chemistry has been to increase the number of course requirements needed to complete their majors. The net result is that students from other departments have less time than perhaps they once did to take additional mathematics courses.
- Surprises included: Some of the best students in upper division courses at colleges represented on our panel are in fact high school students; at two of the schools represented on the panel, the math majors comprised 0.5 percent of the student body. It has been a nationwide trend that the number of science majors in general, and the number of mathematics majors in particular, has fallen steadily over the last decade or more. Reasons for this decline include

(i) Student interests have moved in new directions, so it is less natural for the brightest students to go into math than it may have been twenty years ago.

(ii) Many students today graduate high school not having the study skills and the intellectual values that students were expected to have twenty or more years ago. As a result, even if they come to college thinking that they want to major in mathematics, they find that they cannot cope with the workload and the intellectual depth and they consequently switch majors.

(iii) Students find mathematics to be dry, uninteresting, and not engaging. They do not have a clear picture of what they will do with a mathematics degree once they graduate (this despite the fact that more and more fields are becoming mathematized).

(iv) Concomitant with (iii), students are easily enticed by the ready success and easy entree offered by majors like business. Traditional subject areas on the whole, and mathematics in particular, have suffered a loss of student interest and therefore a loss of majors.

The use of technology was discussed. Maple, Mathematica, MatLab and other computer algebra/graphing systems have been used successfully in calculus and upper-level courses without detracting from the main goals of the courses. Students often experience some initial discomfort in learning the syntax of one of these languages and in becoming inured to formulating mathematical questions in the computer environment. Once they internalize the language, then this problem begins to fade away. The curriculum is especially effective if students see the same computer language in several courses.

A basic question we should ask ourselves is, "What do we want technology to do?" Do we use it to illustrate concepts, sometimes using vivid graphics? Do we want it to take over tedious calculations so that students can concentrate on higher order ideas? Do we want the computer to perform massive calculations that would be infeasible to do by hand? Do we want to use the computer as a modeling tool? Or is it to be a primary learning tool? Is learning to use software part of students' training to make them more marketable? We theoretical mathematicians often find ourselves unfamiliar with this territory; but in today's climate it is essential that we become conversant with the relevant issues.

Recommendations. Mathematics majors should be shown both the intrinsic beauty and the applicability of mathematics. They should also understand that mathematics is a living subject with a fascinating history that has made substantial contributions to our culture. Such considerations deserve to be a part of all mathematics courses at every level.

With the introduction of technology and also more applications in our courses, some other topics will have to be eliminated. We cannot cover all of the old topics and cram in the new and expect to teach coherent courses. Many of our current sophomore-level courses used to be taught in the junior year, so our expectations have been raised at the same time that the students' skills seem to be dropping.

It is also the case that many of the new teaching techniques—group learning, self-discovery, and computer labs are just three instances—use time in ways that are unfamiliar to many of us. The more traditional lecture teaching method is a highly structured didactic device which gives the instructor complete control of the use of time. An instructor inured to that classical methodology will, in effect, have to be retrained so that he/she can use the new techniques effectively and well.

Mathematics majors must know the difference between a precise statement and a vague assertion; some in the working group felt that every undergraduate mathematics major should take a course with serious focus on proofs and should be able to write a proof of some sort themselves. We noted that there is a difference between (i) understanding the ideas and (ii) studying the proofs in detail, or even in understanding the details. Getting lost in the epsilons and deltas and missing "the big picture" is all too common and very undesirable. Equally unpalatable is the prospect of a student getting the big picture but not understanding the inner workings of the subject. A good mathematics course is a balancing act among abstraction, careful rigor, precision, and problem-solving.

It is also critical for graduating mathematics majors to have been presented with the history of the development of ideas in mathematics. There are important ideas and important theorems that do not develop in a vacuum and which must be seen as part of the natural development of the subject.

We recommend that more attention be given, both in courses and in textbooks, to the broader context into which any particular idea fits. To achieve this goal, instructors may have to decide to prioritize what must be proved and consequently which proofs must be sketched or in fact omitted entirely. Furthermore, it should be made clear that commonplace mathematical ideas were (historically) often achieved only after a long and difficult period of development. For example, progress in understanding basic concepts like the real numbers, limits, and continuity was very slow. Students should come to understand the reason for the development of abstract and axiomatic approaches.

This highly interactive melange of ideas should be kept in mind when departments are revising syllabi and creating new courses. A new course is not simply a list of topics. It is a symbiotic collection of ideas, the body of which fits into an historical and a scientific context. The course should also fit rather naturally into the mathematics curriculum; and that role of the course should be apparent to instructor and student alike.

The Education of Non-Mathematics Majors

Members: Adeniran Adeboye, Stephen Greenfield, Jean Larson, Ashley Reiter (Reporter), Dorothy Wallace (Discussion Leader)

Questions for Day 1

- What projects have you been involved with regarding the education of non-majors? (Consider individual courses, such as calculus for scientists, engineers, and general education courses; and consider larger projects, such as interdisciplinary projects with other departments.)
- What worked and what didn't?

Questions for Day 2

- What mathematics should the average student know? (Algebra? Basic Probability? Geometry?)
- What is the appropriate balance of drill and conceptual understanding for the average student? (Algebraic skills, ability to analyze data in a graphical or numerical form, applications of mathematics, etc.)
- What strategies do you recommend to teach students the knowledge and skills they need?

Much of the discussion of undergraduate mathematics focuses quite naturally on the needs of mathematics majors. However, non-math majors make up the overwhelming majority of the undergraduate students taught by departments of mathematics at U.S. colleges and universities. We should think of these students when we analyze our curriculum and our course offerings. The purpose of this report is to initiate a discussion centered on those students whose need for mathematics is directly related to their study in some other subject or in order to satisfy a degree requirement.

Summary of discussion. We think that students interested in physics, chemistry, engineering, computer science, and biochemistry need skill in mathematical modeling and the ability to use mathematical techniques outside of mathematics classes. A more recent development is that psychology, economics, earth sciences, nursing, business, and other majors who are not from a core science are also required to learn some mathematics. At many schools there is a quantification requirement, thus in effect mandating that every student at the school take some mathematics. The mathematics department should be prepared to rise to this challenge, and to fill the need. Apart from abstract philosophical reasons that can be mounted to justify such an action, it is also crucial for the survival and well being of the mathematics department that it be prepared to meet any and all curricular math needs on campus.

The students who come to us for service courses should be able to recognize when a problem in their field calls for a mathematical approach and then to follow the problem from the original considerations to a mathematical description (including some consideration of the simplifications and errors that modeling may introduce). They should then understand the predictions of the model, and use those predictions to check the validity of the model. They should also understand the precision and power of mathematics and its logical structure and cohesiveness. All of our service courses should be designed with these needs in mind. Our sensitivity to student and to curricular needs may get us more "converts", and it also should make the students more reliable and appreciative consumers of mathematics.

The needs of students of biology, nursing, psychology, economics, earth sciences and business may be a bit different from those in the hard sciences. The qualitative needs described in the last paragraph will still apply; but a business student will want to learn some mathematical modeling, qualitative calculus ideas (rates of change and numerical approaches, phase portraits, etc.), linear algebra, statistics, and some areas of discrete mathematics. This mix of basic analytical techniques will serve these clients better than some more traditional subjects.

Future teachers also have particular needs. Prospective teachers should learn mathematics that goes well beyond what they will teach, but that is salient to what they will teach. We also need to help teachers develop mathematical ideas and provide them with mathematics that they can use in their classrooms, while modeling pedagogy appropriate for their use.

We noted that liberal arts students often must fulfill a quantitative skills requirement. This population is large and growing, and may not be well-served by the type of precalculus course now being offered. A history or literature student may want to know some mathematical history and some mathematical culture; that student's need for analytical problem solving skills, or for integral calculus, or for vector analysis will be much less.

Pre-professionals need certain skills for exam preparation (e.g., the MCAT, LSAT, and GMAT). The mathematics department would be well advised to become acquainted with those exams. Consultation with relevant departments is also recommended.

Recommendations. Clearly the mathematical needs of non-math-majors will vary significantly with the university and academic major. Each department must therefore do its own analysis. Communication with faculty members in other departments and other programs can be extremely valuable in finding out what students in other majors actually need. We strongly urge that math departments engage in a dialogue with representatives of other faculty groups to try to discover what is needed and how to fulfill these needs most efficiently. Important questions to consider are:

- What is the mission of your institution?
- Who are the students in your classes (with respect to major, career track, graduation requirements)?
- For each major, what math is specifically used and where?

While the particular ways in which non-math majors can be best served will vary from institution to institution, the following general recommendations may be appropriate at most schools:

- It would be ideal if individual senior faculty were responsible for specific non-major courses for a period of several years. Each course stewardship should have a duration of (about) five years. It would also be ideal if the "ownership" of a course changed on a regular, scheduled basis. The purpose of this system is to facilitate communication with other departments and interest groups, as well as to develop among faculty a sense of responsibility for the education of non-math majors.
- Too often in the past have we funneled non-majors into some form of calculus course. In many instances this choice fit poorly the needs of our clients. As noted above, many non-majors—business majors and psychology majors come first to mind—could make good use of some statistics, some finite math, some problem solving skills, and *certain parts of calculus.* We should endeavor to design service courses that actually fit the needs and values of our clients. Service courses should be regularly re-examined to insure that they are kept timely and continue to meet the needs of the departments and the students they are intended to serve. This review should be conducted in consultation with client departments.
- All courses, regardless of content, should be presented with a view towards communicating something of the mathematical endeavor. This can be done in many ways. For example, we could give a historical perspective, showing what sorts of questions mathematicians have asked, and what efforts have been made to answer these questions.
- When a service course is designed and implemented, the language and the intellectual values of the client disciplines should be kept in mind. An effort should be made to guarantee that the course is both valuable and credible to the students who will be taking the course.

- Pedagogy in recent years has responded to changes in generally available technology and to changes in educational style. Graduate students, as prospective college teachers, should be helped to respond effectively to these opportunities. They should be made aware of the many new teaching techniques that are being developed, and also of the many technological tools that now exist. Graduate programs in mathematics have a responsibility to their students to enable them to teach effectively.

Departments may have difficult debates when dealing with some of these suggestions. Change is rarely easy and painless. We therefore further suggest:

- Maintain civility and respect for colleagues across disciplines and within the math department.
- Build on existing friendships and camaraderie when possible.
- Allow each teacher more flexibility to teach each section in a way which is consistent with his or her own philosophy, at least until consensus is reached.
- Be aware that it is quite difficult for an individual faculty member, regardless of age and level of experience, to analyze his or her own teaching and to master new teaching techniques. A departmental system of mentoring and faculty development should be formed.
- Resolution of outside requirements and departmental needs is worthy of detailed discussion.
- Be aware of political pressure points at your school (again, ask client departments) which can be used to help in developing the service curriculum.
- Some external support, such as a supportive dean, funding to purchase hardware, or released time for faculty to plan or restructure courses may be very helpful in this transition.

The teaching of service courses has long been the "silent partner" in the mathematics enterprise. The stunning growth of mathematics departments in the 1960s was due in large part to the need for more faculty because of substantially increased enrollments in service courses. We need to take a more active role in developing service courses that accurately address the needs and curricular requirements of the departments that we profess to serve. If we do not do so, and in a credible way, then our clients will begin to develop their own mathematics courses; such a development would be to our detriment. In the long run, well-thought-out service courses will be important to the health of mathematics.

Outreach to Other Departments

Members: Chris Anderson, Barbara Bath, Marjorie Enneking, Terry Herdman (Discussion Leader), Paul M. Weichsel (Reporter)

Questions for Day 1

- What students and what departments do you serve in your mathematics courses?
- To what extent have you worked with other departments in the design or delivery of mathematics courses? (Engineering, Sciences, Business, Social Sciences)
- Which projects were successful and which weren't?

Questions for Day 2

- What strategies do you recommend for finding out the needs of other departments?
- For each group of students that you serve, what mathematics do you think they need?
- What strategies do you recommend for setting up successful, sustained joint activities with other departments, and which activities do you think are likely to be most fruitful? (Joint committees, regular consultation, finding out how they use mathematics, team teaching, interdisciplinary courses.)

Outreach has been a fact of mathematics department life for the past forty years. But changes in the university structure, in societal values, and in the needs of the student body demand that the mathematics department take a more active role in developing and nurturing its outreach activities. This work group report explores some of these new needs.

Basic principles.

I. *Outreach must be based on a notion of* ***partnership*** *rather than a client-provider relationship.* There should be regular meetings between mathematics department representatives and client department (physics, engineering, psychology, etc.) representatives in order to determine the shape and content of any given service course. An effort should be made to have some continuity in the membership of these interface committees. Since different departments often have entirely different vocabularies and value systems, real effort must be made to open lines of communication. Committee members should examine texts and discuss syllabi. Department chairmen should have a role on these committees.

II. *Lines of communication with other units need to be kept active on a continuing basis whether or not both units are engaged in a specific joint project.* There should probably be twice-yearly meetings just so that the departments can touch base with each other and assess the status of courses that are in place. This is also a time when new needs and ideas can be broached.

III. *Mathematicians need to be prepared to invest the time and effort necessary to learn enough about other disciplines and help others to learn about ours in order to make interaction fruitful.* While the responsibility for communication lies on both sides of the fence, we must bear in mind that *we are the serving department*. The mathematics department's health and well-being depends on how well we are *perceived to be fulfilling our service role*. Therefore we must be prepared to go the extra distance to communicate with our clients.

Recommendations and examples.

(i) *Engage in interdisciplinary research which may lead to interdisciplinary educational activities.* Such collaboration will not only be mutually beneficial on a scholarly level, but will help faculty to develop the common vocabulary, and also the trust and rapport, that is essential to any kind of cooperative venture.

(ii) *Work with other departments to develop joint majors/minors and double majors.* In today's climate, the pure mathematics major is playing an ever smaller role in the educational milieu. Collaboration with other departments will help the mathematics department to develop viable programs in statistics, biomathematics, mathematical physics, and other applied disciplines.

(iii) *Examples of course development*:

- Team-taught courses. Ideally, the "team" should consist both of math department members and of faculty from other departments. The subject matter should be of interest to students and faculty from both fields, and also to faculty in departments across campus. Team-taught courses are an effective device for increasing campus appreciation of the mathematics department and the contribution that it makes to the educational effort.
- Courses jointly developed by several departments and taught by one. Examples might include a course on applied partial differential equations, one on general relativity, and one on mathematical genetics.
- Custom courses designed for special audiences. This could include courses in acoustics or in wavelet algorithms or in applications of C^* algebras to physics.
- General education courses. These would be courses for non-majors, often students majoring in the social sciences or humanities who need a "quantification" course for a breadth requirement. It will also include K–12 teachers.
- Work with other departments to ensure the mathematical integrity of their courses. This activity has the usual pitfall that the participating mathematician(s) might be seen as "talking down" to their clients. Care should be taken by those who are trying to help other departments to make their courses mathematically honest.

- Work with other departments to make other courses appropriate for math majors. The same caution as in the last item applies. This can be a fertile field of interaction with other departments, as obviously those departments want to attract more students (particularly math students) to their courses.

(iv) *Establish a unit consisting of faculty on temporary assignment from their home departments to jointly develop and teach interdisciplinary courses.* This is a sound idea that will obviously require administrative support — in the form of release time, or flexible scheduling, or support staff. It may also require pecuniary resources — for equipment or staff or guest speakers.

(v) *Campus-wide committees to engage in course revision over an extended period when relevant.* Mathematics plays a central role in any College of Arts and Sciences. As we approach the millennium, mathematics impinges on all sciences and on many other subjects as well. Campus-wide committees help to institutionalize the pivotal role of mathematics and also help to open up lines of communication both with client departments and with the administration.

(vi) *Work cooperatively with other departments on software and hardware decisions.* Most science departments outside of mathematics do not use TEX. Only some use `Mathematica`; many instead use `MatLab`. Physics departments still use `Macsyma` in certain applications. It would be well for us to become aware of how other departments use technology; such knowledge is certainly essential in designing new courses and new curricula.

Traditionally, little formal effort has been expended by mathematics departments in trying to develop relationships with other departments. In those happy circumstances where a productive relationship did develop, it was usually through the serendipity of a few individuals. It would be well to institutionalize the lines of communications between math and its allied physics, engineering, premed programs, and so forth. Clearly the long-term health and prosperity of the mathematics enterprise is increasingly dependent on such communication. Institutional support for these outreach activities is essential — including adequate reward and appropriate recognition.

Outreach to High Schools

Members: Gunnar Carlsson, Phil Curtis, Dan Fendel, Neal Koblitz, Anneli Lax (Discussion Leader), Judith Roitman, Tom Sallee (Reporter), Martin Scharlemann, Alina Stancu, Abigail Thompson, David Wright, William Yslas Vélez

Questions for Day 1

- What work with high schools have you been involved in? (Visiting classes, in-service, co-op programs, programs for high school students, summer courses, faculty volunteering in high schools.)
- Which were fruitful and which weren't?
- Do you have any programs for teacher training at your institution?

Questions for Day 2

- What do you think is the appropriate role for university faculty in the development of high school curricula and teaching methods?
- What strategies do you recommend to improve the communication between local high schools and your institution?
- How do you think your institution can help high school students learn mathematics?

It is too easy for university mathematics faculty to bemoan the preparation and the attitudes toward learning that our entering freshmen exhibit. In fact university faculty can play an active role in helping high schools to prepare their students for college and university learning. It should be noted explicitly that the "high school method" of learning and the "college/university" method of learning are quite distinct. And students must be shown what the difference is and then taught how to pass from one mode of learning to the next. Certainly cooperation between high schools and colleges can play a major role in making such a program effective.

Recommendations

1. Mathematics departments, in partnership with other faculty responsible for teacher education, should establish strong links with local pre-college institutions. This partnership should involve activities that entail exchange of faculty and administrators.
   - The university math department should invite in-service teachers to visit the university (or college). Visiting teachers may meet mathematics faculty and participate informally and formally in mathematical activities ranging from chats about curricular priorities to seminars about selected topics in mathematics and/or innovative instructional strategies. Visiting teachers may also be involved in formal courses geared to teachers' needs and interests, and to allow teachers to teach some elementary college courses.

The street does not run only one way. University faculty may also visit colleges, junior colleges, and high schools and may participate in their activities. Of course there should be constant communication among the different faculties while these activities are taking place.

- University faculty can offer advice, in a collegial atmosphere, on how to connect mandated syllabi and tests to some serious, sound mathematics. Conversely, university faculty can learn more about what is going on in high schools and junior colleges, and may thereby become better prepared to teach the students who enter their universities.
- The cooperative efforts of university, college, high school, and other faculty can help to establish a mechanism for college undergraduates to become involved with secondary and middle schools and with classroom mathematics teachers. Such a program would pave the way for mathematics majors into pre-college teaching as a career.
- The university faculty member can convey to students, teachers, and educators that understanding mathematics involves sustained hard work and has its own rewards for those who responsibly undertake its study. Also, faculty at all levels and at every institution of learning should understand that students should be held to the high academic standards we believe they are capable of meeting.

2. The activities mentioned in item 1 above should be built-in features of universities and colleges made possible by university administrators and department chairs without the constant distractions of finding grant money. Logistics and solvency issues should be jointly addressed by schools, by their university partners, and cooperatively by their administrators.
3. Mathematicians should actively cooperate with teachers and educators in formulating and implementing federal, state and district policies. These should include policies related to
   - mathematics teacher certification requirements;
   - the continuing education of teachers (staff development), e.g. devising course and seminar offerings appropriate for pre- and in-service teachers;
   - the writing of state frameworks;
   - the writing of new curricula and related materials;
   - constructively criticizing computer software and textbooks recommended for adoption.

In any state of the union, mathematics professors are the ultimate authorities on the subject area of mathematics. But they also have considerable expertise in areas of pedagogy. Exercised diplomatically, professors can use that expertise to help shape curricula, text choices, and even values and attitudes throughout the state. In particular, professors should take up the long-ignored gauntlet of developing a fruitful interaction with high school teachers and administrators.

Such an interaction can only help both styles of institution to serve each other more effectively, and will also result in meaningful exchanges both of information and of personnel.

Research Mathematicians and Research in Mathematics Education

Members: Hyman Bass (Discussion Leader), Kenneth Bogart, Michael Fried, Cathy Kessell, Alfred Manaster, Steve Monk (Reporter), Blake Peterson

Questions for Day 1

- What interests you particularly about education research?
- What education research have you undertaken or made use of?
- What interactions between education researchers and research mathematicians have you taken part in or have you found particularly worthwhile?

Questions for Day 2

- What specific types of mathematics education research do you recommend as likely to be most useful and accessible to mathematics teachers in the classroom?
- What specific types of mathematics education research do you recommend as likely to be most useful and accessible to those designing mathematics curricula?

Mathematics education research is a field of inquiry into the nature of mathematical learning, as well as into the practice of mathematics teaching. It provides a foundation and methods for designing diverse teaching strategies and for studying their effects. The study of mathematical learning investigates the process by which students give meaning to and learn to employ mathematical ideas and practices, by making connections with and updating their prior knowledge and experience. Such investigations not only provide basic knowledge essential to the development of curricula and materials, but can significantly inform teaching practice as well.

It is vitally important for the mathematics research community to become better acquainted with the field of mathematics education research, with the many insights and perspectives on student learning it affords, and with the applications to practice it suggests. Actions should be taken in this direction by individuals as well as professional societies and institutions. As teachers and professionals, mathematics researchers should become acquainted and/or engaged with research in mathematics education in a number of ways. Among these are:

User of Information. Among the products of mathematics education research are "first order" informational studies about students, teaching, and learning that should be helpful to college teachers. These include studies of: (a) the effects on student attitudes and on student learning of work in collaborative groups; (b) transfer and non-transfer of knowledge between apparently related domains; (c) patterns of retention (or loss) of students' knowledge of mathematical ideas and procedures during their study in a particular course; and (d) relationships between student preparation for entry-level courses and course performance. Although such studies rarely give decisive answers to pedagogical questions, they provide a critical basis for the planning that takes place before teaching a course, as well as a framework for reflections on the effects of this teaching.

Teacher-Researcher. To improve or enrich one's teaching requires careful reflection on one's teaching experience. This process is made more disciplined and effective by the adoption of various techniques and strategies for gathering and objectively analyzing data. These include: keeping a teaching journal and collecting anonymous student writings about their mathematical ideas; carefully following the work and progress of particular students; systematically examining data with other teachers. Mathematics researchers should also consider the use of such techniques as means of pursuing deeper questions about their own teaching and their students' learning, as well as becoming more thoughtful and critical users of research in mathematics education.

Collaboration with Researchers in Mathematics Education. Mathematics education is fundamentally an interdisciplinary field of inquiry that draws on expertise in mathematics, psychology, and sociology. Many of its key ideas and methods originate in fields such as anthropology and philosophy. Since few individuals have strong backgrounds in such diverse fields, mathematics education depends on inter-disciplinary collaboration for its development. Mathematicians can make their most important contributions to the field of mathematics education through collaborations with scholars in this field who show a strong orientation toward mathematics. As is the case in other inter-disciplinary work involving mathematics, such collaborations are likely to be productive only when there is openness, honesty, and respect on all sides. This takes both care and thoughtfulness as to how mathematics might look to a professional in another field, who is interested in mathematics but not an expert in it, and how one who is steeped in mathematics can come to share in the point of view and expertise of a field quite different from mathematics.

Professional mathematics societies and institutions can play a number of roles with respect to helping individual mathematicians become involved with research in mathematics education in these ways. Among these are: publication of expository papers and annotated lists of references to studies that will be most immediately helpful to college mathematics teachers; invitations for invited addresses on research in mathematics education at society meetings and colloquia; inclusion in the programs of professional meetings of workshops and short courses on ways of carrying out teacher research; and active encouragement of collaboration between professional mathematicians and researchers in mathematics education. Above all, they should find ways to foster increased communication and collaboration between the communities of researchers in mathematics and researchers in mathematics education.