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# On the Education of Mathematics Majors

HUNG-HSI WU

## Re-examination of Standard Upper-Division Courses

Upper division courses in college are where math majors learn real mathematics. For the first time they get to examine the foundations of algebra, geometry and analysis, come face-to-face with the deductive nature of mathematics on a consistent basis and, most importantly, learn to do serious theorem-proving. For reasons not unlike these, most mathematicians enjoy teaching these courses more than others. While teaching graduate courses may be professionally more satisfying, it also involves more work, and the teaching of lower division courses — calculus and elementary discrete mathematics — is a strenuous exercise in the suppression of one's basic mathematical impulses. By contrast, the teaching of upper division courses involves no more than doing elementary mathematics the *usual* way: abstract definitions can be offered without apology and theorems are proved as a matter of principle. This is something we can all do on automatic pilot.

But have we been on automatic pilot for too long?

Mathematicians approach these courses as a training ground for future mathematicians. Even a casual perusal of the existing textbooks would readily confirm this fact. We look at upper division courses as the first steps of a journey of ten thousand miles: in order to give students a firm foundation for future research, we feed them technicality after technicality. If they do not fully grasp some of the things they are taught, they will when they get to graduate school or, if necessary, a few years after they start their research. Then they will put everything together. In short, we build the undergraduate education program for our majors on the principle of *delayed gratification*. Whatever their misgivings for the time being, students will benefit in the long haul.

This is the abbreviated version of a longer paper [W4] which discusses the same issues from a slightly different perspective, that of training prospective school teachers. I am indebted to Richard Askey, Paul Clopton, Ole Hald, Ken Ross, and Andre Toom for valuable comments.

However, even the most conservative estimate nation-wide would put no more than 20% of these majors as potential graduate students in mathematics. The remaining 80% — the overwhelming majority — look at the last two years of college as the grand finale of their mathematical experience. To them, their mathematical future is *now*. Given this fact, how should we teach them if we have their welfare in mind? We would want them to understand better the many things they were imperfectly taught in school. We also want them to know what mathematics is about and how mathematics is done. In addition, we owe it to them to give them a sample of the best that mathematics has to offer: some of the major ideas and great theorems in the history of mathematics. But an unflinching assessment would show that, at least for this 80%, we have failed at every step. Take for instance the standard one semester course on complex functions. At the end of such a course, it is perfectly feasible to explain the meaning and significance of the Riemann Mapping theorem, the Dirichlet problem, and the Riemann Hypothesis. Embedded in these three topics are ideas that have helped shape the course of mathematics in the past century and a half, and are therefore ideas which would interest these majors. But how many students in complex functions know about this piece of history even if on rare occasions the Riemann Mapping Theorem is stated and proved? Instead, such courses spend the time on the proofs of the general form of the Cauchy theorem and other technical facts. Another conspicuous example is the recent proof of Fermat's Last Theorem. How many of our majors have the vaguest idea that this proof is an achievement of "enormous humanistic importance" (words of Elliott Lieb)? If we are unhappy about the answers to these questions, we have only ourselves to blame. After all, we are the ones who design this guided tour of the edifice we call mathematics, a tour that allows these majors to see only the nuts and bolts in its foundation but never its splendor or even the *raison d'être* of some of its interior designs. We have let them down. In our effort to nurture future mathematicians, which is undoubtedly an essential goal of mathematics education, we have neglected the education of the remaining 80% of our majors who put themselves under our care. We forget that they too are part of our charge.

The usual defense of this philosophy of education would argue that, far from a case of neglect, this system came about by design. For, by giving students a firm technical grounding with ample exposure to abstract theorems and rigorous proofs, we also give them the tools to explore on their own. Eventually, they will acquire the necessary perspective and overall understanding of the details so that they will look back on all they have learned and enlightenment ensues. Or so the theory goes.

This is what I call the Intellectual Trickle-down Theory of Learning: aim the teaching at the best students, and somehow the rest will take care of themselves. In practice, however, most of the students who do not go on to graduate school in mathematics are not among those with a strong enough interest or firm enough mastery of the fundamentals to dig deeper for further understanding.

Consequently, the college education of these students is long on technical details that they cannot digest but short on the minimal essential information that would enable them to understand even the elementary facts from "an advanced standpoint". They go out into the world impoverished in both technique and information for the simple reason that we never had them in mind when we designed our curriculum.

What lends a sense of urgency to this unhappy situation is the presence of future school mathematics teachers among the 80% in question. When they go into the classroom so mathematically ill-equipped, they cannot help but victimize the next generation of students. Some of the latter come back to the university and the vicious circle continues. We pay a high price indeed for our neglect.

This particular aspect of our collective failure to educate the majority of majors bears on the current mathematics education reform in K–14 (i.e., from kindergarten to the sophomore year in college). This reform (cf. [W1, W2, W3]) has by and large ignored the critical issue of the technical inadequacy of school mathematics teachers.[1] Not coincidentally, there is a corresponding de-emphasis of content knowledge in favor of pedagogy in schools of education across the nation [K, NAR]. For example, it was already explicitly pointed out back in 1983 that "Half of the newly appointed mathematics, science, and English teachers are not qualified to teach these subjects..." [NAR]. To this vitally important area of school education, the most direct contribution the mathematical community can make would seem to be to design a better education for prospective school teachers. Our failure to do this was what gave the original impetus to the writing of this paper.[2]

Let me illustrate the various threads of the preceding discussion with a concrete example. In the spring semester of 1996, I taught a course in introductory algebra which was attended by math majors who did not intend to continue with graduate study in mathematics, including a few prospective school teachers. Before I discussed the field of rational numbers, I asked how many of them knew *why* $1/(1/5) = 5$. I waited a long time, but not a single hand was raised while a few shook their heads.[3] One cannot understand the significance of this fact unless one realizes that the subject of fractions is one of the sore spots in school mathematics education. Herb Clemens [C] relates the following story:

> Last August, I began a week of fractions classes at a workshop for elementary teachers with a graph paper explanation of why $\frac{2/7}{1/9} = 2\frac{4}{7}$. The reaction of my audience astounded me. Several of the teachers present were

[1] I am referring to the generic case, of course. There are many excellent teachers.

[2] In this context, I am obliged to point out that this paper does not do justice to the complex issue of how best to educate prospective teachers; what it does is merely to suggest ways to give these prospective teachers a better mathematical training *given the existing requirements of a math major.*

[3] My colleague Ole Hald has suggested to me that one explanation of the lack of response could be the lack of a proper context for the students to understand such a question.

> simply terrified. None of my protestations about this being a preview, none of my "Don't worry" statements had any effect.

Or, take another statement from the NCTM Standards [NC, p. 96]:

> This is not to suggest, however, that valuable instruction time should be devoted to exercises like $\frac{17}{24} + \frac{5}{18}$ or $5\frac{3}{4} \times 4\frac{1}{4}$, which are much harder to visualize and unlikely to occur in real-life situations.

This suggestion concerning the teaching of fractions occurs in the Standard on Estimation and Computation in Grades 5–8 of [NC]. It is difficult for a mathematician to imagine that students going into high school (9th grade) would have trouble *computing* simple products and sums such as above, but this difficulty evaporates as soon as we look into how fractions are taught in grades 4–8. Take the standard Addison-Wesley Mathematics for Grades 4–8, for example [EI]. There the definition of the addition of fractions unnecessarily brings in the LCM of denominators, and the division of fractions is defined using circular reasoning (for the latter, see p. 232 of Grade 6, p. 182 of Grade 7 and p. 210 of Grade 8), to name just two problems off-hand. Unfortunately, students do not get to learn substantially more about the rational number system in high school because, once there, they take algebra which assumes they know how to compute with rational numbers. Thus by the time students come to the university, very often their understanding of the rationals remains in the primitive state reflected in the two quotations above.

In the light of this glaring weakness in our average math major's mathematical arsenal, let us take a look at what he or she is taught about the rationals in a typical course in introductory algebra. We first introduce the notion of an integral domain $D$, and construct out of $D$ its quotient field by introducing equivalence classes of ordered pairs $\{(p, q)\}$, show that addition and multiplication among these equivalence classes are well defined and that they form a field $F$. Then we define the canonical injective homomorphism from $D$ to $F$, identify $D$ with its image, and explain to the students that henceforth all nonzero elements of $D$ will have multiplicative inverses in $F$. After all that, we will mention that if we replace $D$ by the integers, then the $F$ above would be the field of rational numbers. Whether or not one would explicitly point out the relationship between the common assertion $1/(1/b) = b$ and the general fact that $(b^{-1})^{-1} = b$ would depend very much on the instructor, and in any case, even if this is done, it would be a passing remark and no more. In the meantime, the average math major is overwhelmed by this onslaught of newly acquired concepts: integral domain, field, equivalence class, injection, and homomorphism. To most beginners, these are things at best half understood. Thus expecting them to come to grips with the rational numbers by way of the concept of a quotient field is clearly no more than a forlorn hope. It therefore comes as no surprise that, in the midst of such uneasiness and uncertainty, the average student would fail to gain any new insight into such fundamentals as $1/(1/b) = b$ or $(-p)(-q) = pq$. Some of them soon go into school classrooms as teachers and, things being what they are, their students too can be counted on to fear such simple tasks as $\frac{17}{24} + \frac{5}{8}$.

## Some Proposed Changes

One way to resolve the difficulty concerning the education of our math majors who do not go on to graduate work in mathematics is to teach them in a separate track. There are obstacles that stand in the way of such a simple recommendation, to be sure, but none seems to be more formidable than the suspicion that it is really *infra dig* for a "good" department to offer "watered down" courses to its own majors. Thus even if money is available to teach two tracks, fighting this prejudice will not be easy. The education of over 80% of our majors is however too serious an issue to be glossed over by institutional or professional prejudices. It is time that we meet this problem head on by discussing it in public.

There is perhaps no better explanation of why "different" is not the same as "watered down" than to list a few of what I believe to be the desirable characteristics of courses designed for students who do not pursue graduate work in mathematics.

(1) *Only proofs of truly basic theorems are given, but whatever proofs are given should be complete and rigorous.* On the one hand, we are doing battle with time: given that there are many topics we want the students to be informed about, the proofs of some of our favorite theorems (e.g., the Jordan canonical form of a linear transformation or the implicit function theorem) would have to go in favor of other issues of compelling interest (e.g., historical background or motivation). On the other hand, we also want them to understand that proofs are the underpinning of mathematics. Thus any time we present a proof, we must make sure that it counts.

(2) *In contrast with the normal courses which are relentlessly "forward-looking" (i.e., the far-better-things-to-come in graduate courses), considerable time should be devoted to "looking back".* In other words, there should be an emphasis on shedding light on elementary mathematics from an advanced viewpoint. One example is the cleaning up of the confusion about rational numbers (cf. the discussion in the preceding section). Another is the elucidation of Euclidean geometry and axiomatic systems.

(3) *Keep the course on as concrete a level as possible, and introduce abstractions only when absolutely necessary.* The potency of abstract considerations should not be minimized, but we have to be aware of the point of minimal return, when any more abstraction would decrease rather than promote students' understanding. The example of the construction of a quotient field from an integral domain in the context of rational numbers has already been given above.

(4) *Ample historical background should be provided.* This idea is by now so widely accepted that no argument need be given save to point out that a knowl-

edge of the evolution of a concept or theorem helps to break down students' resistance to abstractions.

(5) *Provide students with some perspective on each subject, including the presentation of surveys of advanced topics.* For example, in discussing diagonalization of matrices, students would benefit from a discussion of the various canonical forms — *without proofs* — because they need to understand that diagonalization is not an isolated trick but a small part of the general attempt to simplify and classify all mathematical objects. Along this line, I cannot help but be amazed by the general tendency in most texts to refrain from discussing any topic that is "out of logical order". The desire to develop the subject *ab initio* is well taken, but what is there to fear about exposing students to interesting advanced ideas without proofs, so long as the advanced nature of the material is made clear to them? The way we learn is hardly "logically-linear"; otherwise there would be no incentive for any of us to go to colloquium lectures. So why deny the students the same opportunity to learn when we can so easily provide it?

(6) *Give motivation at every opportunity.* The usual complaint about school mathematics degenerating into rote-learning is fundamentally a reflection of the fact that most teachers were themselves never exposed to any motivation for every concept, lemma, and theorem that they learned. It is incumbent on the college instructors to break this vicious circle.

As an example of how to implement some of these changes, consider the teaching of *Dedekind cuts* to students in introductory analysis. As is well-known, there are two kinds of Dedekind cuts. If we try to construct $\mathbb{R}$ out of $\mathbb{Q}$, then we would define:

(1) A **real number** is an ordered pair $(A, B)$ of nonempty subsets of $\mathbb{Q}$, so that $A \cup B = \mathbb{Q}$, and $a < b$, $\forall\, a \in A$, $\forall\, b \in B$.

On the other hand, if we try to define $\mathbb{R}$ as a complete ordered field, then we postulate (in the original words of Dedekind, 1872):

(2) "If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions" [D, p. 11].

Beginners are usually befuddled by either of these quaint statements. Now suppose in the classroom we have to use (1) to construct $\mathbb{R}$. We would first point out the relationship between (1) and (2), describe the state of the calculus in Dedekind's time, and make students understand that there was a real need for a non-mystical approach to the real numbers. Clearly something had to be done, and Dedekind's was the first successful contribution. Did he get this idea out of the blue? Hardly. Again, in his own words (1887): "... if one regards the irrational number as the ratio of two measurable quantities, then is this manner of determining it already set forth in the clearest possible way in the celebrated definition which Euclid gives of the equality of two ratios (*Elements*, V, 5). This same most ancient conviction has been the source of my theory ... to lay the foundations for the introduction of irrational numbers into arithmetic" [D, pp. 39–40]. Students at this point should be curious about this famous definition in *Elements*, V, 5. Historians agree that this was in fact the creation of Eudoxus (ca. 408–355 B.C.), and it states [E, p. 114]:

"Magnitudes are said to be **in the same ratio**, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and the third, and any equimultiples whatever of the second and the fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of the latter equimultiples respectively taken in corresponding order."

Naturally, few can penetrate this kind of prose. This then gives the instructor an excellent opportunity to extol the virtues of the symbolic notation, something that most students take for granted without any appreciation. In symbols, the preceding paragraph merely states, word for word: "$a_1 : a_2 = a_3 : a_4$ if and only if given any positive integers $m$ and $n$, $ma_1 > na_2$ implies $ma_3 > na_4$, $ma_1 = na_2$ implies $ma_3 = na_4$, and $ma_1 < na_2$ implies $ma_3 < na_4$." One can make this even more accessible by using modern mathematical language: "$a_1/a_2 = a_3/a_4$ if and only if for any rational number $n/m$, the following conditions hold: $n/m < a_1/a_2$ implies $n/m < a_3/a_4$, $n/m = a_1/a_2$ implies $n/m = a_3/a_4$, and $n/m > a_1/a_2$ implies $n/m > a_3/a_4$. Or, more simply:

> "$a_1/a_2 = a_3/a_4$ if and only if the cuts in $\mathbb{Q}$ induced by both $a_1/a_2$ and $a_3/a_4$ are equal."

This then brings us to the previous statement (2) of Dedekind.

It remains to explain to students why Eudoxus would even dream of such a tortuous definition of **equal ratio**. Couldn't he just divide the two pairs of real numbers? Now is the time to discuss the Greeks' veneration of rational numbers in the 5th century B.C., the absence of any concept of "real numbers" back then, the subsequent crisis precipitated by the discovery of incommensurable quantities, and Eudoxus' brilliant achievement in defining incommensurable quantities using only the rationals. Dedekind's insight was to realize that Eudoxus' *ad hoc* definition in fact suffices to pin down the real number field.

Such a detour in a beginning course in analysis takes time. If one's aim is to usher students to the frontier of research in the most efficient manner possible, this detour would be ill-advised. But if one instead tries to instill a little understanding together with some mathematical culture in these students before they leave mathematics, the detour would be well worth it. So it becomes important to know whether we are teaching the 20% or the other 80%.

## An Experiment

What I will take up next is the description of a small experiment I am currently conducting at Berkeley in an attempt to implement the preceding philosophy of teaching. Starting in the spring semester of 1996, I have been offering an upper division course each semester specifically for "math majors who do not go on to graduate school in mathematics". At the time of writing (April 1997), I have taught introductory algebra and linear algebra, and am currently teaching differential geometry. The remaining courses I hope to have taught by June of 1999 (in some order) are history of mathematics, introductory analysis, complex functions and classical geometries.[4] Having taught two such courses and being in the middle of a third one, I would like to recount my experience in some detail. Since there is as yet no tradition of teaching upper divisional math courses with the philosophical orientation propounded here, others might find this account to be of some value.

Two general observations emerged from this experience. The first one is that it is very difficult, if not impossible, to find an appropriate text for such a course. Almost all the standard texts are written to prepare students for graduate work in mathematics. At the other extreme are a few texts that try to be "user-friendly" by trivializing the content. Neither would serve my purpose. The other observation is that in this approach to upper divisional instruction, the trade-offs are quite pronounced even for someone who is prepared for them. Let me be more specific by discussing separately the two courses I have taught thus far.

For the introductory algebra course, the announced goals of the course were the solutions of the three classical construction problems and the explanation of why the roots of certain polynomials of degree $\geq 5$ cannot be extracted from the coefficients by use of radicals. In more detail, the first two thirds of the course were devoted to the following topics:

> Part 1: The quadratic closure of $\mathbb{Q}$, constructible numbers, $\mathbb{Z}$ and $\mathbb{Q}$ revisited, Euclidean algorithm and prime factorization, congruences modulo $n$, fields, polynomials over a field, irreducibility and unique factorization of polynomials, Eisenstein, complex numbers and the fundamental theorem of algebra, roots of unity and cyclotomic polynomials, field extensions and their degrees, solutions of the construction problems, constructibility of regular polygons.

The last third of the course was more descriptive in nature. It consisted of the following:

> Part 2: Isomorphism of fields, automorphisms relative to a ground field, root fields, computations of automorphisms, groups and basic properties, solvable groups, Galois group of an extension, radical extensions, the theorems of Abel and Galois.

[4]The absence of number theory in this list may come as a surprise to some, but I did not create a separate number theory course for this track because I cannot imagine there is anything in a beginning number theory course that should not be known to one and all regardless of students' needs.

The main reason for the decision to direct the whole course towards these classical problems was the hope that students' prior familiarity with these problems would entice them to learn the abstract algebraic concepts needed for their solutions. My hope was not in vain: a few of my 19 students told me after the course was over that they had worked harder for this course than for other math courses because they could relate to what they were learning. Sadly, with the gutting of Euclidean Geometry and the de-emphasis of purely mathematical questions (without real-world relevance) in the present school mathematics education reform, soon students will go to college not knowing any of these famous problems. This is a potential educational crisis that mathematicians must do all they can to avert.

Another reason for this choice of materials was the historical connection. Since these problems gave impetus to almost all the important advances in algebra up to the nineteenth century, the course provided a natural setting to delve into the historical roots of the subject. I was able to discuss the slow emergence of the symbolic algebraic notation, Cardan, Tartaglia, Fermat, Gauss, Galois and Abel.

Compared with more standard approaches to introductory algebra, some losses and gains are worth noting. Because I consider the topics in Part 1 to be basic to a teacher's understanding of algebra, every theorem there was proved, with the exception (of course) of the fundamental theorem of algebra, the transcendence of $\pi$, and the theorem of Gauss on the constructibility of regular polygons. Students received, and welcomed, a careful and rigorous treatment of $\mathbb{Q}$ and $\mathbb{C}$. In addition, because Part 1 gives a detailed treatment of the polynomial ring over $\mathbb{R}$, these students got to understand this important object — important especially for the future teachers — much better than otherwise possible. For example, I made a point of proving for them the technique of partial fractions which is usually imperfectly stated and used for integration in calculus. On the other hand, this choice of topics entails certain glaring omissions: no PID's and no UFD's, no general construction of the quotient field of a domain, and no serious discussion of ideals, ring homomorphisms and quotient rings.

The omissions in Part 2 are much more serious. With only five weeks to treat these sophisticated topics, there was hardly time to prove any theorem other than the simplest. Even at the most basic level, there was no discussion of homomorphism between groups, and hence also no discussion of the relationship between normal subgroups and the kernel of a homomorphism, and theorems about the existence of subgroups of an appropriate order were hardly mentioned. In exchange, students were given exposure to the fantastic ideas of Galois theory — without proofs, to be sure — and the hope is that perhaps one day some of them would revisit the whole terrain on their own.

Implicit in the preceding syllabus is the fact that no noncommutative mathematics appears until two thirds of the way into the course when groups enter the discussion. This is by design. It seems to me that students taking such a course are confronted with proofs in a serious way for the first time, and that is enough of a hurdle without their being simultaneously overloaded by noncommutativity as well. Because it took mankind more than two thousand years after Euclid to face up to noncommutativity, it seems unfair that beginning students are not given a few weeks of reprieve before being saddled with it. The presentation in almost all the standard texts in algebra, beginning with groups and followed by rings and fields, mimics the order adopted by van der Waerden in his pioneering text of 1931 [WA]. However, van der Waerden was writing a research monograph, and there seems to be very little reason why undergraduate texts should follow suit without due regard for pedagogy. Along this line, let it be said that the first American undergraduate textbook on (modern) algebra, *A Survey of Modern Algebra* of Birkhoff and MacLane [BM], does begin with integers, commutative rings, and more than a hundred pages of commutative mathematics before launching into groups. There is a reason why Birkhoff–MacLane is still in print after 55 years. Having said that, I want to reiterate a serious concern in the teaching of such a course, which is the pressing need of an appropriate textbook.

Next, let us turn to linear algebra. Among upper division math courses, this course may be the only one which is as rich as calculus in nontrivial scientific applications. Furthermore, this is also a subject almost tailor-made for computers and therefore the consideration of computational simplicity plays an important role. For the benefit of those students who do not go on to higher mathematics, a course on linear algebra that emphasizes both of these facts in place of topics like the Cayley-Hamilton theorem and the rational canonical form would seem to be more educational.[5] While it is true that standard courses usually pay some attention to computational simplicity at the beginning when discussing Gaussian elimination, it is also true that they quickly drop this consideration the rest of the way. I would prefer that this consideration be the underlying theme of the whole course because, in scientific applications, savings in cost and time are important.[6] The syllabus for the course I gave in the fall of 1996 is then:

> Elementary row operations, Gaussian elimination, existence and uniqueness of LU decomposition of nonsingular square matrices, approximate solutions of ODE's by linear systems, review of vector spaces and associated concepts, LU decomposition in general, the row space, column space and solution space of a general matrix, the rank-nullity theorem, application to electrical networks, inner product spaces, orthogonal projection on a subspace, least square approximations, Gram-Schmidt, the QR decomposition of a nonsingular matrix, signal processing and the fast Fourier transform,

[5] Of course students should learn the Cayley-Hamilton theorem and the rational form too *if* there is time.

[6] The issue of roundoff errors was mentioned in my course but not pursued. It seems to me that this would be better handled in a course on numerical analysis.

determinants, eigenvalues and eigenvectors, diagonalization, applications to Fibonacci sequence, symmetric and Hermitian matrices, quadratic forms.

This is clearly a course on matrices rather than abstract vector spaces. Even so, some standard topics on matrices are conspicuous by their absence: the minimal polynomial, Cayley-Hamilton, Jordan canonical form and rational canonical form. But perhaps the most distressing aspect was my inability, due to lack of time, to impress on the students the advantage of knowing the coordinate-free point of view. It goes without saying that there was no mention in the course of invariant subspaces, dual spaces, and induced dual linear transformations. The difficulty with some of these deficiencies could have been alleviated by making reading assignments in an appropriate textbook, one in which the applications are presented with integrity and the abstract point of view is treated with the clarity and precision that befit a mathematics text. However, the scarcity of textbooks suitable for the kind of teaching under discussion continues to call attention to itself.

What remains to be said are the gains that go with such losses, namely, the interesting applications that throw a completely different light on abstract linear algebra. Personally, I must admit to having been enthralled by the applications of the fast Fourier transform to signal processing, and this sense of enchantment would be shared by the students too if the latter is properly explained. No less interesting is the way the QR decomposition helps save time and cost in formulating precise experimental laws using the least squares method. These rather pronounced trade-offs in such an approach to teaching will undoubtedly continue to invite strong reactions from each of us.

Finally, an ongoing concern of my colleagues is that by not proving *every* theorem, such courses run the risk of giving students a distorted perception of the fundamental nature of proofs in mathematics. Whether or not such a danger would be realized in the classroom seems to me to depend very much on the way a course is actually taught. There are no foolproof pedagogical strategies. For the cases at hand, I have appended the final exams of the preceding two courses for the readers' inspection. One can at least get a sense of what was emphasized in these courses even if not of what was actually achieved.

## The Summing Up

Upper division courses should enlarge a student's basis of mathematical knowledge. With this in mind, I find it necessary to lecture throughout these courses to ensure that a minimum number of topics be covered. At the same time, I also attach a great deal of importance to homework assignments. To help students with problem solving, which includes not only getting the solutions but also learning how to write intelligible proofs, I either pass out solution sets or schedule extra problem sessions each week. While one hesitates to make an unconditional recommendation of such time-consuming efforts as part of our teaching duty, one should also ask if most students can learn much in these courses by simply attending lectures.

It should be emphasized that what has been proposed above is an *alternative* to the standard courses, not a replacement. If we are still committed to the concept of the university as a repository of knowledge, then we must insure the continuity of this knowledge by producing future mathematicians. The standard upper division courses therefore play a vital role in honoring this commitment. On the other hand, a university is also an educational institution and the education of the majority of the math majors must also be taken seriously. The aim of the present proposal is therefore to suggest a way for the university to better fulfill this dual obligation, and *not* to suggest a radical change in undergraduate education. In practical terms, the suggested alternative can be implemented more easily in large institutions than in small colleges. In the former, scheduling parallel sections of the same course catering to different student clienteles does not present much difficulty. One can only hope that this suggestion, in one form or another, will be discussed in these institutions. In the smaller colleges, any change would seems to come only with some special effort or ingenuity.

Finally, the present proposal is very germane to the current education reform in K–14. There is a tendency in this reform to make sweeping changes: wholesale replacements of existing curricula or pedagogies are routinely recommended. The idea that one can offer alternatives to *some* but not all of what are already in place appears to be foreign to the reformers, as is the need to clearly delineate the liabilities as well as virtues of every one of the proposed changes. Education is a multi-faceted enterprise which, for better or for worse, involve both politicians and the public. We accept the fact that the latter two thrive in the world of hyperbole. However, most of the reform measures have been proposed in the academic community. I hope I will be forgiven for my temerity if I offer the reminder that, insofar as education is still an academic subject, the academics who propose education reform should make an effort to conform to the minimal standards of intellectual integrity and candor. Had such an effort been made, much of the rancor of the present reform would have disappeared and a more rational course of action would have resulted. For the sake of the next generation — and the reform is nothing if not about the welfare of the next generation — let us restore such integrity and candor to our discussions.

## Appendix: Sample Final Exams

**Introduction to Abstract Algebra (Math 113)**
**FINAL EXAM**
**May 13, 1996 8–11 am Prof. Wu**

**1.** (5%) Prove that for an integer $n$, $3 \mid n \Longleftrightarrow 3 \mid$ (sum of digits of $n$).

**2.** (5%) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ be a polynomial with integer coefficients, and let $r$ be a rational number such that $f(r) = 0$. Show that $r$ has to be an integer and $r \mid a_0$.

**3.** (5%) Find a minimal polynomial of $\sqrt[3]{1+\sqrt{3}}$ over $\mathbb{Q}$. (Be sure to prove that it is minimal.)

**4.** (5%) Let $n$ be a positive integer $\geq 2$ such that $n \mid (b^{n-1} - 1)$ for all integers $b$ which are not a multiple of $n$. What can you say about $n$?

**5.** (5%) Do the nonzero elements of $\mathbb{Z}_{13}$ form a cyclic group under multiplication? Give reasons.

**6.** (10%) Let $p$ be a prime.

(a) Prove: $p \Big| \binom{p}{k}$ for $k = 1, \ldots p-1$, where $\binom{p}{k} \equiv \dfrac{p!}{k!(p-k)!}$.

(b) Prove: the mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ defined by $f(k) = k^p$ for all $k \in \mathbb{Z}_p$ is a field isomorphism.

**7.** (10%) Is $x^4+2x+3$ irreducible over $\mathbb{R}$? Is it irreducible over $\mathbb{Q}$? Give reasons.

**8.** (10%) Let $F \equiv \{a + ib : a,\ b \in \%Q\}$ and let $K \equiv \mathbb{Q}[x]/(x^2+1)\mathbb{Q}[x]$. Show that $F$ is isomorphic to $K$ as fields by defining a map $\varphi : F \to K$ and show that $\varphi$ has all the requisite properties.

**9.** (10%) If $\beta$ is a root of $x^3-x+1$, find some $p(x) \in \mathbb{Q}[x]$ so that $(\beta^2-2)\, p(\beta) = 1$.

**10.** (10%) Let $\zeta = e^{i2\pi/3}$. Compute $(\mathbb{Q}(\zeta, \sqrt[3]{5}) : \mathbb{Q}(\zeta))$.

**11.** (25%) (In (a)–(d) below, each part could be done independently.)

(a) Assume that if $p$ is a prime, then $x^{p-1} + x^{p-2} + \cdots + 1$ is irreducible over $\mathbb{Q}$. Compute $(\mathbb{Q}(\cos(2\pi/7) + i\sin(2\pi/7)) : \mathbb{Q})$. (Each step should be clearly explained.)

(b) Suppose the regular 7-gon can be constructed with straightedge and compass. Explain why $(\mathbb{Q}(\cos(2\pi/7) : \mathbb{Q}) = 2^k$ for some $k \in \mathbb{Z}^+$.

(c) If $F \equiv \mathbb{Q}(\cos(2\pi/7))$, show that $(F(i\sin(2\pi/7)) : F) = 1$ or $2$.

(d) Use (b) and (c) to conclude that if the regular 7-gon can be constructed with straightedge and compass, then $(\mathbb{Q}(\cos(2\pi/7) + i\sin(2\pi/7)) : \mathbb{Q}) = 2^m$ for some $m \in \mathbb{Z}^+$.

(e) What can you conclude from (a) and (d)? What is your guess concerning the construction of the regular 11-gon, the regular 13-gon, the regular 23-gon, etc.?

**Linear Algebra (Math 110)**
**FINAL EXAM**
**Dec 11, 1996 12:30–3:30 pm Prof. Wu**

**1.** (5%) Find the determinant of $\begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}$ and show all your steps.

**2.** (15%) Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$. Find its eigenvalues and the corresponding eigenvectors. Also find a $3 \times 3$ matrix $S$ and a diagonal matrix $D$ so that $S^{-1}AS = D$.

**3.** (5%) If $Q$ is an $n \times n$ orthogonal matrix, what is $\det Q$? What are the eigenvalues of $Q$? Give reasons.

**4.** (15%) Let $F_k$ denote the Fourier matrix of dimension $k$. Define for each $n$:

$$Y_{2n} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{2n-1} \end{bmatrix}, \qquad Y_{\text{odd}} = \begin{bmatrix} y_1 \\ y_3 \\ y_5 \\ \vdots \\ y_{2n-1} \end{bmatrix}, \qquad Y_{\text{even}} = \begin{bmatrix} y_0 \\ y_2 \\ y_4 \\ \vdots \\ y_{2n-2} \end{bmatrix}$$

Also let $w = e^{i\frac{2\pi}{2n}}$ and $W = \begin{bmatrix} 1 & & & & \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{n-1} \end{bmatrix}$. Then the fast Fourier transform can be described as follows:

$$F_{2n}Y_{2n} = \begin{bmatrix} F_nY_{\text{even}} + WF_nY_{\text{odd}} \\ F_nY_{\text{even}} - WF_nY_{\text{odd}} \end{bmatrix}$$

Now let $\rho(2n)$ denote the minimum number of operations needed to compute $F_{2n}Y_{2n}$. Prove: $\rho(2^k) \leq k2^k$ for every integer $k \geq 1$. (Recall: an "operation" means either a multiplication-and-an-addition or a division.)

**5.** (10%) We want a plane $y = C + Dt + Ez$ in $y - t - z$ space that "best fits" (in the sense of least squares) the following data: $y = 3$ when $t = 1$ and $z = 1$; $y = 5$ when $t = 2$ and $z = 1$; $y = 6$ when $t = 0$ and $z = 3$; and $y = 0$ when $t = 0$ and $z = 0$. Set up, but do not solve the $3 \times 3$ linear system of equations that the least squares solution $C$, $D$, $E$ must satisfy.

**6.** (10%) Let $v_1, \ldots, v_k$ be eigenvectors of an $n \times n$ matrix $A$ corresponding to *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, respectively, where $k \leq n$. Prove that $v_1, \ldots, v_k$ are linearly independent.

**7.** (10%) Suppose a real $n \times n$ matrix $A$ has $n$ distinct real eigenvalues. Is there necessarily a *real* $n \times n$ matrix $S$ so that $S^{-1}AS$ is diagonal? Explain.

**8.** (10%) If the eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct), what are the eigenvalues of $A^k$ where $k$ is an integer $\geq 1$? If $A$ is nonsingular, what are the eigenvalues of $A^{-k}$ for $k \geq 1$? Give reason.

**9.** (15%) Let $P$ be the projection matrix which projects $\mathbb{R}^n$ onto a $k$-dimensional subspace $W \subset \mathbb{R}^n$, where $0 < k < n$. Enumerate all the eigenvalues of $P$ and for each eigenvalue, describe all its eigenvectors.

**10.** (5%) Let $A$ be an $n \times n$ matrix and let $A'$ be obtained from $A$ be Gaussian elimination. Do $A$ and $A'$ necessarily have the same eigenvalues? Give reasons.

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Hung-Hsi Wu
University of California Berkeley
Department of Mathematics
Berkeley, CA 94720-3840 United States
wu@math.berkeley.edu