Several Complex Variables
MSRI Publications
Volume **37**, 1999

Local Holomorphic Equivalence of Real Analytic Submanifolds in $\mathbb{C}^N$

M. SALAH BAOUENDI AND LINDA PREISS ROTHSCHILD

Abstract. This paper presents some recent results of the authors jointly with Peter Ebenfelt concerning local biholomorphisms which map one real-analytic or real-algebraic submanifold of $\mathbb{C}^N$ into another. It is shown that under some optimal conditions such mappings are determined by their jets of a predetermined finite order at a given point. Under these conditions, if the manifolds are algebraic, it is also shown that the components of the holomorphic mappings must be algebraic functions. The stability group of self mappings is shown to be a finite dimensional Lie group for most points in the case of real-analytic holomorphically nondegenerate real hypersurfaces in $\mathbb{C}^N$. The notion of Segre sets associated to a point of a real-analytic CR submanifold of $\mathbb{C}^N$ is one of the main ingredients in this work. Properties of these sets and their relationship to minimality of these manifolds are discussed.

Introduction

We consider here some recent results concerning local biholomorphisms which map one real analytic (or real algebraic) subset of $\mathbb{C}^N$ into another such subset of the same dimension. One of the general questions studied is the following. Given $M, M' \subset \mathbb{C}^N$, germs of real analytic subsets at p and p' respectively with $\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} M'$, describe the (possibly empty) set of germs of biholomorphisms $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ with $H(M) \subset M'$.

Most of the new results stated here have been recently obtained in joint work with Peter Ebenfelt. We shall give precise definitions and specific references in the text. One of the results (Theorem 2) states that if $M \subset \mathbb{C}^N$ is a connected real analytic holomorphically nondegenerate CR manifold which is minimal at some point, then at most points $p \in M$, a germ H of a biholomorphism at p mapping M into M', another submanifold of $\mathbb{C}^N$, is determined by its jet at p of a finite order, depending only on M. This result is used to prove (Theorem 3)

Both authors were partially supported by National Science Foundation Grant DMS 95-01516.

that the real vector space of infinitesimal CR automorphisms of M is finite dimensional at every point.

Denote by $\mathrm{Aut}(M,p)$ the group of germs of biholomorphisms of $\mathbb{C}^N$ at p, fixing p and mapping M into itself. Theorem 4 and its corollaries show that if M is a holomorphically nondegenerate hypersurface, then for most points $p \in M$, $\mathrm{Aut}(M,p)$, equipped with its natural topology, is a finite dimensional Lie group parametrized by a subgroup of the jet group of $\mathbb{C}^N$ at 0 of a certain finite order. The proof of Theorem 4 gives an algorithm to determine all germs of biholomorphisms at p mapping the hypersurface M into another hypersurface M' and taking p to p'. The set of all such biholomorphisms (possibly empty) is parametrized by a real analytic, totally real submanifold of a finite order jet group of $\mathbb{C}^N$ at 0.

Section 6 deals with the special case where the real submanifolds M and M' are real algebraic, that is, defined by the vanishing of real valued polynomials. In particular, Theorem 8 implies that if M and M' are holomorphically nondegenerate generic algebraic manifolds of the same dimension, and if M is minimal at p, then any germ of a biholomorphism at p mapping M into M' is algebraic. Theorem 9 shows that holomorphic nondegeneracy and minimality are essentially necessary for the algebraicity of all such mappings.

A main ingredient in the proofs of the results stated in this paper is the use of the *Segre sets* associated to every point of a real analytic CR submanifold in $\mathbb{C}^N$. The description of these sets and their main properties is given in Section 1 and Theorem 1. One of these properties is that the complexification of the CR orbit of a point $p \in M$ coincides with the maximal Segre set at p. In particular, a real analytic generic submanifold M is minimal at p if and only if the maximal Segre set is of complex dimension N.

Bibliographical references relevant to the results given in this paper can be found at the end of each section of the text.

We shall give now some basic definitions. Most of the results described here can be reduced to the case where M and M' are real analytic generic submanifolds. Recall that a real analytic submanifold $M \subset \mathbb{C}^N$ is *generic* if near every $p \in M$, we may write

$$M = \{Z \in \mathbb{C}^N : \rho_j(Z, \overline{Z}) = 0 \text{ for } j = 1, \ldots, d\}, \tag{0.1}$$

where $\rho_1, \ldots, \rho_d$ are germs at p of real-valued real analytic functions satisfying $\partial\rho_1(p) \wedge \cdots \wedge \partial\rho_d(p) \neq 0$. Here

$$\partial\rho = \sum_{j=1}^{N} \frac{\partial\rho}{\partial Z_j} dZ_j.$$

More generally, we say that M is *CR* if $\dim_{\mathbb{R}}(T_pM \cap JT_pM)$ is constant for $p \in M$, where T_pM is the real tangent space of M at p, and J the anti-involution of the standard complex structure of $\mathbb{C}^N$. If M is CR, then $\dim_{\mathbb{R}} T_pM \cap JT_pM = 2n$

is even and n is called the *CR dimension* of M. In particular, if M is generic of codimension d, then $n = N - d$.

We say that a real submanifold of $\mathbb{C}^N$ is *holomorphically nondegenerate* if there is no germ of a nontrivial vector field

$$\sum_{j=1}^{N} c_j(Z)\frac{\partial}{\partial Z_j},$$

with $c_j(Z)$ holomorphic, tangent to an open subset of M. Another criterion of holomorphic nondegeneracy, which can be checked by a simple calculation, is the following. Let $L = (L_1, \ldots, L_n)$ be a basis for the CR vector fields of a generic manifold M near p. For any multi-index α put $L^\alpha = L_1^{\alpha_1} \ldots L_n^{\alpha_n}$. Introduce, for $j = 1, \ldots, d$ and any multi-index α, the $\mathbb{C}^N$-valued functions

$$V_{j\alpha}(Z, \bar{Z}) = L^\alpha \rho_{jZ}(Z, \bar{Z}), \tag{0.2}$$

where ρ_{jZ} denotes the gradient of ρ_j with respect to Z, with ρ_j as in (0.1). We say that M is *finitely nondegenerate* at $p \in M$ if there exists a positive integer k such that the span of the vectors $V_{j\alpha}(p, \bar{p})$, for $j = 1, \ldots, d$ and $|\alpha| \le k$, equals $\mathbb{C}^N$. If k is the smallest such integer we say that M is *k-nondegenerate* at p. These definitions are independent of the coordinate system used, the defining functions of M, and the choice of basis L. One can then check that if a generic manifold M is connected, then M is holomorphically nondegenerate if and only if it is finitely nondegenerate at some point $p \in M$. Another equivalent definition is that M is holomorphically nondegenerate if and only if it is essentially finite at some point $p \in M$.

It can also be shown that a connected, generic manifold M is holomorphically nondegenerate if and only if there exists a positive integer $l(M)$, $1 \le l(M) \le N - 1$, such that M is $l(M)$-nondegenerate at every point outside a proper real analytic subset of M. We shall call $l(M)$ the *Levi number* of M. Hence to determine holomorphic nondegeneracy, one need compute (0.2) for only finitely many multi-indices α. In particular, a connected real analytic hypersurface is Levi nondegenerate at some point if and only it its Levi number is 1. For a connected hypersurface in $\mathbb{C}^2$, Levi nondegeneracy at some point is equivalent to holomorphic nondegeneracy. However, in $\mathbb{C}^N$, for $N > 2$, there exist connected, real analytic holomorphically nondegenerate hypersurfaces which are nowhere Levi nondegenerate.

1. Segre Sets of a Germ of a CR Manifold

In this section, we introduce the Segre sets of a generic real analytic submanifold in $\mathbb{C}^N$ and recall some of their properties. We refer the reader to [Baouendi et al. 1996a] for a more detailed account of these sets; see also [Ebenfelt 1998]. Let M denote a generic real analytic submanifold in some neighborhood $U \subset \mathbb{C}^N$

of $p_0 \in M$. Let $\rho = (\rho_1, \dots, \rho_d)$ be defining functions of M near p_0 as in (0.1), and choose holomorphic coordinates $Z = (Z_1, \dots, Z_N)$ vanishing at p_0. Embed $\mathbb{C}^N$ in $\mathbb{C}^{2N} = \mathbb{C}^N_Z \times \mathbb{C}^N_\zeta$ as the real plane $\{(Z,\zeta) \in \mathbb{C}^{2N} : \zeta = \overline{Z}\}$. Denote by pr_Z and pr_ζ the projections of $\mathbb{C}^{2N}$ onto $\mathbb{C}^N_Z$ and $\mathbb{C}^N_\zeta$, respectively. The natural anti-holomorphic involution $\sharp$ in $\mathbb{C}^{2N}$ defined by

$$ {}^\sharp(Z,\zeta) = (\bar\zeta, \overline{Z}) \tag{1.1} $$

leaves the plane $\{(Z,\zeta) : \zeta = \overline{Z}\}$ invariant. This involution induces the usual anti-holomorphic involution in $\mathbb{C}^N$ by

$$ \mathbb{C}^N \ni Z \mapsto \mathrm{pr}_\zeta({}^\sharp\mathrm{pr}_Z^{-1}(Z)) = \bar Z \in \mathbb{C}^N. \tag{1.2} $$

Given a set S in $\mathbb{C}^N_Z$ we denote by ${}^*S$ the set in $\mathbb{C}^N_\zeta$ defined by

$$ {}^*S = \mathrm{pr}_\zeta({}^\sharp\mathrm{pr}_Z^{-1}(S)) = \{\zeta : \bar\zeta \in S\}. \tag{1.3} $$

We use the same notation for the corresponding transformation taking sets in $\mathbb{C}^N_\zeta$ to sets in $\mathbb{C}^N_Z$. Note that if X is a complex analytic set defined near Z^0 in some domain $\Omega \subset \mathbb{C}^N_Z$ by $h_1(Z) = \cdots = h_k(Z) = 0$, then ${}^*X$ is the complex analytic set in ${}^*\Omega \subset \mathbb{C}^N_\zeta$ defined near $\zeta^0 = \overline{Z}^0$ by $\bar h_1(\zeta) = \cdots = \bar h_k(\zeta) = 0$. Here, given a holomorphic function $h(Z)$ we use the notation $\bar h(Z) = \overline{h(\overline{Z})}$.

Let $\mathcal{M} \subset \mathbb{C}^{2N}$ be the complexification of M given by

$$ \mathcal{M} = \{(Z,\zeta) \in \mathbb{C}^{2N} : \rho(Z,\zeta) = 0\}. \tag{1.4} $$

This is a complex submanifold of codimension d in some neighborhood of 0 in $\mathbb{C}^{2N}$. We choose our neighborhood U in $\mathbb{C}^N$ so small that $U \times {}^*U \subset \mathbb{C}^{2N}$ is contained in the neighborhood where $\mathcal{M}$ is a manifold. Note that $\mathcal{M}$ is invariant under the involution $\sharp$ defined in (1.1).

We associate to M at p_0 a sequence of germs of sets $N_0, N_1, \dots, N_{j_0}$ at p_0 in $\mathbb{C}^N$ — *the Segre sets* of M at p_0 — defined as follows. Put $N_0 = \{p_0\}$ and define the consecutive sets inductively (the number j_0 will be defined later) by

$$ N_{j+1} = \mathrm{pr}_Z\big(\mathcal{M} \cap \mathrm{pr}_\zeta^{-1}({}^*N_j)\big) = \mathrm{pr}_Z\left(\mathcal{M} \cap {}^\sharp\mathrm{pr}_Z^{-1}(N_j)\right). \tag{1.5} $$

We shall assume that the open set U is fixed sufficiently small and make no further mention of it. These sets are, by definition, invariantly defined and arise naturally in the study of mappings between submanifolds, as will be seen in Section 2.

The sets N_j can also be described in terms of the defining equations $\rho(Z,\overline{Z}) = 0$. For instance,

$$ N_1 = \{Z : \rho(Z,0) = 0\} \tag{1.6} $$

and

$$ N_2 = \{Z : \exists\zeta^1 : \rho(Z,\zeta^1) = 0\,,\ \rho(0,\zeta^1) = 0\}. \tag{1.7} $$

We have the inclusions

$$N_0 \subset N_1 \subset \cdots \subset N_j \subset \cdots \tag{1.8}$$

and j_0 is the largest number j such that the generic dimension of N_j is the same as that of N_{j-1}. (The generic dimensions of the Segre sets stabilize for $j \geq j_0$.)

To show that the Segre sets are images of holomorphic mappings, it is useful to make use of appropriate holomorphic coordinates. Recall that we can find holomorphic coordinates $Z = (z, w)$, with $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^d$, vanishing at p_0 and such that M near p_0 is given by

$$w = Q(z, \bar{z}, \overline{w}) \quad \text{or} \quad \overline{w} = \overline{Q}(\bar{z}, z, w),$$

where $Q(z, \chi, \tau)$ is holomorphic in a neighborhood of 0 in $\mathbb{C}^{2n+d}$, valued in $\mathbb{C}^d$ and satisfies $Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \tau$. The coordinates $Z = (z, w)$ satisfying the above properties are called *normal coordinates* of M at p_0. In normal coordinates (z, w) one may use the definition above to express the Segre sets N_j for $j = 1, \ldots, j_0$ as images of germs at the origin of certain holomorphic mappings

$$\mathbb{C}^n \times \mathbb{C}^{(j-1)n} \ni (z, \Lambda) \mapsto (z, v^j(z, \Lambda)) \in \mathbb{C}^N. \tag{1.9}$$

We have

$$\begin{aligned} N_1 =& \{(z, 0) : z \in \mathbb{C}^n\}, \\ N_2 =& \{(z, Q(z, \chi, 0)) : z, \chi \in \mathbb{C}^n\}, \end{aligned}$$

and so forth. Thus, we can define the generic dimension d_j of N_j as the generic rank of the mapping (1.9).

So far we have considered only generic submanifolds. We may reduce to this case, since any real analytic CR manifold M is a generic manifold in a complex holomorphic submanifold $\mathcal{V}$ of $\mathbb{C}^N$, called the *intrinsic complexification* of M. The Segre sets of M at a point $p_0 \in M$ can be defined as subsets of $\mathbb{C}^N$ by the process described above just as for generic submanifolds, or they can be defined as subsets of $\mathcal{V}$ by identifying $\mathcal{V}$ near p_0 with $\mathbb{C}^K$ ($K = \dim \mathcal{V}$) and considering M as a generic submanifold of $\mathbb{C}^K$. It can be shown that these definitions are equivalent.

If M is a real analytic CR submanifold of $\mathbb{C}^N$ and $p_0 \in M$, then by Nagano's theorem [1966] there exists a real analytic CR submanifold of M through p_0 of minimum possible dimension and the same CR dimension as M. Such a manifold is called the *CR orbit* of p_0.

The main properties concerning the Segre sets that we shall use are summarized in the following theorem of the authors jointly with Ebenfelt.

THEOREM 1 [Baouendi et al. 1996a; 1998]. *Let M be a real analytic CR submanifold in $\mathbb{C}^N$, and let $p_0 \in M$. Denote by W the CR orbit of p_0 and by X the intrinsic complexification of W.*

(a) *The maximal Segre set N_{j_0} of M at p_0 is contained in X and contains an open subset of X arbitrarily close to p_0. In particular*, $d_{j_0} = \dim_{\mathbb{C}} X$.

(b) *There are holomorphic mappings defined near the origin* $Z_0(t_0), Z_1(t_1), \ldots,$ $Z_{j_0}(t_{j_0})$ *and* $s_0(t_1), \ldots, s_{j_0-1}(t_{j_0})$ *with*

$$\mathbb{C}^{d_j} \ni t_j \mapsto Z_j(t_j) \in \mathbb{C}^N, \quad \mathbb{C}^{d_j} \ni t_j \mapsto s_{j-1}(t_j) \in \mathbb{C}^{d_{j-1}}, \tag{1.10}$$

such that $Z_j(t_j)$ *is an immersion at the origin,* $Z_j(t_j) \in N_j$, *and such that*

$$\big(Z_j(t_j), \bar{Z}_{j-1}(s_{j-1}(t_j))\big) \in \mathcal{M}, \tag{1.11}$$

for $j = 1, \ldots, j_0$. *In addition* $Z_j(0),\ j = 1, \ldots, j_0$, *can be chosen arbitrarily close to* p_0.

Proof. Part (a) is contained in [Baouendi et al. 1996a, Theorem 2.2.1], and the mappings in part (b) are constructed in the paragraph following Assertion 3.3.2 of the same reference. □

Remark 1.12. For each j with $j = 0, 1, \ldots, j_0$, the holomorphic immersion $Z_j(t_j)$, in part (b) above provides a parametrization of an open piece of N_j. However, this piece of N_j need not contain the point p_0. Indeed, N_j need not even be a manifold at p_0.

Recall that a CR submanifold M is said to be *minimal* at a point $p_0 \in M$ if there is no proper CR submanifold of M through p_0 with the same CR dimension as M. Equivalently, M is minimal at p_0 if the CR orbit of p_0 is all of M. For a real analytic submanifold, this notion coincides with the notion of finite type in the sense of [Bloom and Graham 1977]; that is, M is of finite type at p_0 if the Lie algebra generated by the CR vector fields and their complex conjugates span the complex tangent space to M at p_0. It is easy to determine whether a hypersurface M is of finite type at p_0 by using a defining function for M near p_0. Furthermore, if a connected hypersurface M is holomorphically nondegenerate, it is of finite type at most points. (The converse is not true, however.) One of the main difficulties in higher codimension is that it is cumbersome to describe finite type in local coordinates. Furthermore, unlike in the hypersurface case, in general, holomorphic nondegeneracy does not imply the existence of a point of finite type.

One can check that if M is connected then M is minimal almost everywhere if and only if M is minimal at some point. The following is an immediate consequence of the theorem.

Corollary 1.13. *Let* M *be a real analytic generic submanifold in* $\mathbb{C}^N$ *and* $p_0 \in M$. *Then* M *is minimal at* p_0 *if and only if* $d_{j_0} = N$ *or, equivalently, if and only if the maximal Segre set at* p_0 *contains an open subset of* $\mathbb{C}^N$.

We note if M is a hypersurface, then $j_0 = 1$ if M is not minimal at p_0, and $j_0 = 2$ otherwise. We now describe the Segre sets at 0 for two generic manifolds in $\mathbb{C}^3$ of codimension 2, one minimal and one not minimal.

EXAMPLE 1.14. Consider $M \subset \mathbb{C}^3$ defined by

$$\operatorname{Im} w_1 = |z|^2, \quad \operatorname{Im} w_2 = \operatorname{Re} w_2 |z|^4.$$

In this example M is not minimal at 0. We have $j_0 = 2$ and the maximal Segre set of M at 0 is given by

$$N_2 = \{(z, w_1, w_2) : z \neq 0, w_2 = 0\} \cup \{0, 0, 0\}.$$

Here $d_2 = 2$, and N_2 is not a manifold at 0. However, the intersection of (the closure of) N_2 with M equals the CR orbit of 0.

EXAMPLE 1.15. Let $M \subset \mathbb{C}^3$ be the generic submanifold defined by

$$\operatorname{Im} w_1 = |z|^2, \quad \operatorname{Im} w_2 = |z|^4.$$

Then M is of finite type at 0. The Segre set N_2 at 0 is the manifold given by

$$N_2 = \{(z, w_1, w_2) : w_2 = -i w_1^2/2\}.$$

We have here $j_0 = 3$, and N_3 is given by

$$N_3 = \{(z, w_1, w_2) : w_2 = i w_1 (w_1/2 - 2z\chi)\,,\ \chi \in \mathbb{C}\}$$

and hence N_3 contains $\mathbb{C}^3$ minus the planes $\{z = 0\}$ and $\{w_1 = 0\}$.

Before concluding this section we point out that the Segre set N_1, introduced above, coincides with the so-called Segre variety, introduced in [Segre 1931] and used in [Webster 1977a; Diederich and Webster 1980; Diederich and Fornæss 1988; Chern and Ji 1995] and elsewhere. The subsequent Segre sets N_j are all unions of Segre varieties. We believe that the results described above are the first to explore Segre sets for manifolds of higher codimension and to use them characterize minimality. The notion of minimality as described in this section, was first introduced by Tumanov [1988a].

2. Holomorphic Mappings and Segre Sets

In this section we describe how the Segre sets constructed in Section 1 can be used to prove that mappings between CR manifolds are determined by their jets of a fixed order, under appropriate conditions on the manifolds. The main result of this section is the following.

THEOREM 2 [Baouendi et al. 1998]. *Let $M \subset \mathbb{C}^N$ be a connected real analytic, holomorphically nondegenerate CR submanifold with Levi number $l(M)$, and let d be the* (*real*) *codimension of M in its intrinsic complexificiation. Suppose that there is a point $p \in M$ at which M is minimal. Then for any $p_0 \in M$ there exists a finite set of points $p_1, \ldots, p_k \in M$, arbitrarily close to p_0, with the following property. If $M' \subset \mathbb{C}^N$ is another real analytic CR submanifold with*

$\dim_{\mathbb{R}} M' = \dim_{\mathbb{R}} M$, *and* F, G *are smooth CR diffeomorphisms of* M *into* M' *satisfying in some local coordinates* x *on* M

$$\frac{\partial^{|\alpha|} F}{\partial x^{\alpha}}(p_l) = \frac{\partial^{|\alpha|} G}{\partial x^{\alpha}}(p_l) \quad \text{for } l = 1, \ldots, k \text{ and } |\alpha| \le (d+1)l(M), \tag{2.1}$$

then $F \equiv G$ *in a neighborhood of* p_0 *in* M. *If* M *is minimal at* p_0, *then one can take* $k = 1$. *If*, *in addition*, M *is* $l(M)$*-nondegenerate at* p_0, *then one may take* $p_1 = p_0$.

Remarks. (i) Condition (2.1) can be expressed by saying that the $(d+1)l(M)$-jets of the mappings F and G coincide at all the points $p_1, \ldots, p_k$.

(ii) The choice of points $p_1, \ldots, p_k$ can be described as follows. Let $U_1, \ldots, U_k$ be the components of the set of minimal points of M in U, an arbitrarily small neighborhood of p_0 in M, which have p_0 in their closure. For each $l = 1, \ldots, k$, we may choose any p_l from the dense open subset of U_l consisting of those points which are $l(M)$-nondegenerate.

We shall give an indication of the proof of Theorem 2 only for the case where M is generic and is $l(M)$-nondegenerate and minimal at p_0. We start with the following proposition.

Proposition 2.2. *Let* $M, M' \subset \mathbb{C}^N$ *be real analytic generic submanifolds*, *and* $p_0 \in M$. *Assume that* M *is holomorphically nondegenerate and* $l(M)$*-nondegenerate at* p_0. *Let* H *be a germ of a biholomorphism of* $\mathbb{C}^N$ *at* p_0 *such that* $H(M) \subset M'$. *Then there are* $\mathbb{C}^N$ *valued functions* Ψ^{γ}, *holomorphic in all of their arguments*, *such that*

$$\frac{\partial^{|\gamma|} H}{\partial Z^{\gamma}}(Z) = \Psi^{\gamma}\left(Z,\ \zeta,\ \left(\frac{\partial^{|\alpha|} \overline{H}}{\partial \zeta^{\alpha}}(\zeta)\right)_{|\alpha| \le l(M)+|\gamma|}\right), \tag{2.3}$$

for all multi-indices γ *and all points* $(Z, \zeta) \in \mathcal{M}$ *near* $(p_0, \bar{p}_0)$. *Moreover*, *the functions* Ψ^{γ} *depend only on* M, M' *and*

$$\frac{\partial^{|\beta|} H}{\partial Z^{\beta}}(p_0), \quad \text{with } |\beta| \le l(M). \tag{2.4}$$

Proof. This follows from the definition of $l(M)$-nondegeneracy at p_0 and the use of the implicit function theorem. For details, see the proof of [Baouendi et al. 1996a, Assertion 3.3.1] and also [Baouendi and Rothschild 1995, Lemma 2.3]. □

We shall use Proposition 2.2 to give an outline of the proof of Theorem 2 under the more restrictive assumptions indicated above. Let N_j, for $j = 0, 1, \ldots, j_0$, be the Segre sets of M at p_0, and let $Z_0(t_0), \ldots, Z_{j_0}(t_{j_0})$ be the canonical parametrizations of the N_j's and $s_0(t_1), \ldots, s_{j_0-1}\ (t_{j_0})$ the associated maps as in Theorem 1. Since M is minimal at p_0, it follows from [Tumanov 1988a] that F and G extend holomorphically to a wedge with edge M near p_0. Hence, by a theorem of the first author jointly with Jacobowitz and Treves [Baouendi et al. 1985], F and G extend holomorphically to a full neighborhood of p_0 in $\mathbb{C}^N$, since finite nondegeneracy at p_0 implies essential finiteness at p_0. We again denote by F and G their holomorphic extensions to a neighborhood. Assumption (2.1) with $p_l = p_0$ then implies that

$$\frac{\partial^{|\alpha|}F}{\partial Z^\alpha}(p_0) = \frac{\partial^{|\alpha|}G}{\partial Z^\alpha}(p_0), \quad \text{for } |\alpha| \le (d+1)l(M). \tag{2.5}$$

By Proposition 2.2, there are functions Ψ^γ such that both F and G satisfy the identity (2.3) for $(Z,\zeta) \in \mathcal{M}$. Substituting (Z,ζ) in (2.3) by the left hand side of (1.11) and recalling that $Z_0(t_0) \equiv p_0$ (that is, it is the constant map), we deduce that F and G, as well as all their derivatives of all orders less than or equal to $dl(M)$ are identical on the first Segre set N_1. Note that since each N_j is the holomorphic image of a connected set, if two holomorphic functions agree on an open piece, they agree on all of N_j. Inductively we deduce that the restrictions of the mappings F and G to the Segre set N_j, as well as their derivatives of orders at least $((d+1)-j)l(M)$, are identical. The conclusion of Theorem 2 now follows from Theorem 1, since M minimal at p_0 implies that N_{j_0} contains an open piece of $\mathbb{C}^N$.

Theorem 2 is optimal in the sense that holomorphic nondegeneracy is necessary for its conclusion and that the condition that M is minimal almost everywhere is necessary in model cases. We have the following result.

PROPOSITION 2.6 [Baouendi et al. 1998]. *Let $M \subset \mathbb{C}^N$ be a connected real analytic CR submanifold.*

(i) *If M is holomorphically degenerate, then for any $p \in M$ and any integer $K > 0$ there exist local biholomorphisms F and G near p mapping M into itself and fixing p such that*

$$\frac{\partial^{|\alpha|}F}{\partial Z^\alpha}(p) = \frac{\partial^{|\alpha|}G}{\partial Z^\alpha}(p), \quad \text{for } |\alpha| \le K, \tag{2.7}$$

but $F \not\equiv G$ on M.

(ii) *If M is defined by the vanishing of weighted homogeneous polynomials, and nowhere minimal then for any $p \in M$ and any integer $K > 0$ there exist local biholomorphisms F and G near p mapping M into itself and fixing p such that* (2.7) *holds for all $|\alpha| \le K$, but $F \not\equiv G$ on M.*

In case M is a Levi-nondegenerate hypersurface (that is, $d = 1$ and $l(M) = 1$), Theorem 2 reduces to the result of Chern and Moser [1974] that a germ of a CR diffeomorphism is uniquely determined by its derivatives of order ≤ 2 at a point. Generalizations of this result for Levi nondegenerate manifolds of higher codimension were found later [Tumanov and Khenkin 1983; Tumanov 1988b]. More precise results for Levi nondegenerate hypersurfaces have been given by Beloshapka [1979] and Loboda [1981]. The notion of holomorphic nondegeneracy for hypersurfaces is due to Stanton [1995], who showed that it is a necessary and sufficient condition for the finite dimensionality of the space of "infinitesimal holomorphisms", as will be mentioned in Section 3. The notion of finite nondegeneracy was first introduced in the case of a hypersurface in our joint work with X. Huang [Baouendi et al. 1996b].

3. Infinitesimal CR Automorphisms

A smooth real vector field X defined in a neighborhood of p in M is an *infinitesimal holomorphism* if the local 1-parameter group of diffeomorphisms $\exp tX$ for t small extends to a local 1-parameter group of biholomorphisms of $\mathbb{C}^N$. More generally, X is called an *infinitesimal CR automorphism* if the mappings $\exp tX$ are CR diffeomorphisms. We denote by $\mathrm{hol}(M,p)$ the Lie algebra generated by the germs at p of the infinitesimal holomorphisms, and by $\mathrm{aut}(M,p)$ the one generated by the germs of infinitesimal CR automorphisms. Since every local biholomorphism preserving M restricts to a CR diffeomorphism of M into itself, it follows that $\mathrm{hol}(M,p) \subset \mathrm{aut}(M,p)$.

The following result gives the finite dimensionality of the larger space $\mathrm{aut}(M,p)$ not only for hypersurfaces, but also for CR manifolds of higher codimension.

THEOREM 3 [Baouendi et al. 1998]. *Let $M \subset \mathbb{C}^N$ be a real analytic, connected CR submanifold. If M is holomorphically nondegenerate, and minimal at some point, then*

$$\dim_{\mathbb{R}} \mathrm{aut}(M,p) < \infty \tag{3.1}$$

for all $p \in M$.

PROOF. Let $p_0 \in M$ and let $X^1, \ldots, X^m \in \mathrm{aut}(M,p_0)$ be linearly independent over $\mathbb{R}$. Let $x = (x_1, \ldots, x_r)$ be a local coordinate system on M near p_0 and vanishing at p_0. In this coordinate system, we may write

$$X^j = \sum_{l=1}^{r} \tilde{X}^j_l(x) \frac{\partial}{\partial x_l} = \tilde{X}^j(x) \cdot \frac{\partial}{\partial x}. \tag{3.2}$$

For $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we denote by $\Phi(t,x,y)$ the flow of the vector field $y_1 X_1 + \cdots + y_m X_m$, that is, the solution of

$$\begin{aligned} \frac{\partial \Phi}{\partial t}(t,x,y) &= \sum_{i=1}^{m} y_i \tilde{X}^i(\Phi(t,x,y)), \\ \Phi(0,x,y) &= x. \end{aligned} \tag{3.3}$$

Using elementary ODE arguments, one can show that by choosing $\delta > 0$ sufficiently small, there exists $c > 0$ such that the flows $\Phi(t,x,y)$ are smooth (C^∞) in $\{(t,x,y) \in \mathbb{R}^{1+r+m} : |t| \le 2,\ |x| \le c,\ |y| \le \delta\}$. Denote by $F(x,y)$ the corresponding time-one maps, that is,

$$F(x,y) = \Phi(1,x,y). \tag{3.4}$$

LEMMA 3.5. *There is δ', $0<\delta'<\delta$, such that for any fixed y^1, y^2 with $|y^1|, |y^2| \le \delta'$, if $F(x,y^1) \equiv F(x,y^2)$ for all x, $|x|\le c$, then necessarily $y^1 = y^2$.*

PROOF. It follows from (3.3) and (3.4) that

$$\frac{\partial F}{\partial y_i}(x,0) = \tilde{X}^i(x). \tag{3.6}$$

Thus, denoting by $\tilde{X}(x)$ the $r\times m$-matrix with column vectors $\tilde{X}^i(x)$, we have

$$\frac{\partial F}{\partial y}(x,0) = \tilde{X}(x). \tag{3.7}$$

By Taylor expansion we obtain

$$\|F(x,y^2) - F(x,y^1)\| \ge \left\| \frac{\partial F}{\partial y}(x,y^1)\cdot (y^2-y^1)\right\| - C\big|y^2-y^1\big|^2, \tag{3.8}$$

where $C>0$ is some uniform constant for $|y^1|, |y^2|\le\delta$. The linear independence of the vector fields $X^1,\dots,X^m$ over $\mathbb{R}$ implies that there is a constant C' such that

$$\|\tilde{X}(x)\cdot y\| \ge C'|y|. \tag{3.9}$$

The lemma follows by using (3.6), (3.9) and a standard compactness argument. $\square$

We proceed with the proof of Theorem 3. Denote by U the open neighborhood of p on M given by $|x|<c$. We make use of Theorem 2 with M replaced by U. Let $p_1,\dots,p_k$ be the points in U given by the theorem. By choosing the number $\delta'>0$ in Lemma 3.5 even smaller if necessary, we may assume that the maps $x\mapsto F(x,y)$, for $|y|<\delta'$, are CR diffeomorphisms of M. Consider the smooth mapping from $|y|<\delta'$ into $\mathbb{R}^\mu$ defined by

$$y \mapsto \left(\frac{\partial^{|\alpha|}F(p_l,y)}{\partial x^\alpha}\right)_{\substack{|\alpha|\le (d+1)l(M)\\ 1\le l\le k}} \in \mathbb{R}^\mu, \tag{3.10}$$

where μ equals $k\cdot r$ times the number of monomials in r variables of degree less than or equal to $(d+1)l(M)$. This mapping is injective for $|y|<\delta'$ in view of Theorem 2 and Lemma 3.5. Consequently, we have a smooth injective mapping from a neighborhood of the origin in $\mathbb{R}^m$ into $\mathbb{R}^\mu$. This implies that $m\le\mu$ and hence the desired finite dimensionality of the conclusion of Theorem 3. $\square$

As in the case of Theorem 2, here again the condition of holomorphic nondegeneracy is necessary for the conclusion of Theorem 3 to hold. Also, if M is not minimal at any point, but is defined by weighted homogeneous polynomials, then $\dim_{\mathbb{R}} \mathrm{hol}(M,p)$ is either 0 or ∞. This can be viewed as an analogue of Proposition 2.6.

We conclude this section by some bibliographical notes. Tanaka [1962] proved that $\mathrm{hol}(M,p)$ is a finite dimensional vector space if M is a real analytic Levi nondegenerate hypersurface. More recently Stanton [1995; 1996] proved that if M is a real analytic hypersurface, $\mathrm{hol}(M, p)$ is finite dimensional for any $p \in M$ if and only if M is holomorphically nondegenerate. Theorem 3 above generalizes Stanton's result. It should be also mentioned that the methods outlined here are quite different from those of [Stanton 1996].

4. Parametrization of Local Biholomorphisms Between Hypersurfaces

In this section and the next, we shall restrict ourselves to the case of hypersurfaces. Let $M \subset \mathbb{C}^N$ be a real analytic hypersurface and $p_0 \in M$. Denote by $\mathrm{hol}_0(M, p_0)$ the elements of $\mathrm{hol}(M, p_0)$ that vanish at p_0. Also denote by $\mathrm{Aut}(M, p_0)$ the set of all germs of biholomorphisms at p_0, fixing p_0 and mapping M into itself. Under the assumption that M is holomorphically nondegenerate the finite dimensionality of $\mathrm{hol}(M, p_0)$, which follows from Theorem 3 (and, as just mentioned, is in fact proved in [Stanton 1996]), implies that there is a unique topology on $\mathrm{Aut}(M, p_0)$, considered as an abstract group, such that the latter is a Lie transformation group with $\mathrm{hol}_0(M, p_0)$ as its Lie algebra (see [Kobayashi 1972, p. 13], for example). On the other hand $\mathrm{Aut}(M, p_0)$ has a natural inductive limit topology corresponding to uniform convergence on compact neighborhoods of p_0 in $\mathbb{C}^N$. One of the main results of this section (Corollary 4.2) implies that for almost all $p_0 \in M$ the two topologies on $\mathrm{Aut}(M, p_0)$ must coincide.

We shall first introduce some notation. Let k be a positive integer and $J^k_p = J^k(\mathbb{C}^N)_p$ the set of k-jets at p of holomorphic mappings from $\mathbb{C}^N$ to $\mathbb{C}^N$ fixing p. J^k_0 can be identified with the space of holomorphic polynomial mappings of degree $\le k$, mapping 0 to 0. Let $G^k = G^k(\mathbb{C}^N)$ be the complex Lie group consisting of those holomorphic mappings in J^k_0 with nonvanishing Jacobian determinant at 0. We take the coefficients $\Lambda = (\Lambda_\alpha)$ of the polynomials corresponding to the jets to be global coordinates of G^k. The group multiplication in G^k consists of composing the polynomial mappings and dropping the monomial terms of degree higher than k.

For $p, p' \in \mathbb{C}^N$, denote by $\mathcal{E}_{p,p'}$ the space of germs of holomorphic mappings $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$, (that is, $H(p) = p'$) with Jacobian determinant of H nonvanishing at p equipped with the natural inductive limit topology corresponding to uniform convergence on compact neighborhoods of p. We define a mapping $\eta_{p,p'} : \mathcal{E}_{p,p'} \to G^k$ as follows. For $H \in \mathcal{E}_{p,p'}$, let $F \in \mathcal{E}_{0,0}$ be defined by $F(Z) = H(Z + p) - p'$. Then $j_k(F)$, the k-jet of F at 0, is an element of G^k. We put $\eta_{p,p'}(H) = j_k(F)$. In local holomorphic coordinates Z near p we have $\eta_{p,p'}(H) = (\partial^\alpha_Z H(p))_{1\le|\alpha|\le k}$. The mapping $\eta_{p,p'}$ is continuous; composition of mappings is related to group multiplication in G^k by the identity

$$\eta_{p,p''}(H_2 \circ H_1) = \eta_{p',p''}(H_2) \cdot \eta_{p,p'}(H_1) \tag{4.1}$$

for any $H_1 \in \mathcal{E}_{p,p'}$ and $H_2 \in \mathcal{E}_{p',p''}$, where $\cdot$ denotes the group multiplication in G^k. We write η for $\eta_{p,p'}$ when there is no ambiguity.

If M and M' are two real analytic hypersurfaces in $\mathbb{C}^N$ with $p \in M$ and $p' \in M'$, denote by $\mathcal{F} = \mathcal{F}(M, p; M', p')$ the subset of $\mathcal{E}_{p,p'}$ consisting of those germs of mappings which send M into M', and equip $\mathcal{F}$ with the induced topology.

THEOREM 4 [Baouendi et al. 1997]. *Let M and M' be two real analytic hypersurfaces in $\mathbb{C}^N$ which are k_0-nondegenerate at p and p' respectively and let $\mathcal{F} = \mathcal{F}(M, p; M', p')$ as above. Then the restriction of the map $\eta : \mathcal{E}_{p,p'} \to G^{2k_0}$ to $\mathcal{F}$ is one-to-one; in addition, $\eta(\mathcal{F})$ is a totally real, closed, real analytic submanifold of G^{2k_0} (possibly empty) and η is a homeomorphism of $\mathcal{F}$ onto $\eta(\mathcal{F})$. Furthermore, global defining equations for the submanifold $\eta(\mathcal{F})$ can be explicitly constructed from local defining equations for M and M' near p and p'.*

With the notation above, we put $\mathrm{Aut}(M, p) = \mathcal{F}(M, p; M, p)$ and refer to it as the *stability group* of M at p. When $\mathrm{Aut}(M, p)$ is a Lie group with its natural topology, it is easy to show that $\mathrm{hol}_0(M, p)$, as defined above, is its Lie algebra. We have the following corollary of Theorem 4.

COROLLARY 4.2. *If, in addition to the assumptions of Theorem 4, $M = M'$ and $p = p'$, then $\eta(\mathcal{F})$ is a closed, totally real Lie subgroup $G(M, p)$ of G^{2k_0}. Hence the stability group $\mathrm{Aut}(M, p)$ of M at p has a natural Lie group structure. In general, for different (M, p) and (M', p'), $\eta(\mathcal{F})$ is either empty or is a coset of the subgroup $G(M, p)$.*

In the next section we shall give an outline of a proof of Theorem 4 which gives an algorithm to calculate $G(M, p)$ and, in particular, to determine whether two hypersurfaces are locally biholomorphically equivalent.

Since a connected, real analytic, holomorphically nondegenerate hypersurface M is $l(M)$-nondegenerate at every point outside a proper real analytic subset $V \subset M$, the following is also a consequence of Theorem 4.

COROLLARY 4.3. *Let M be a real analytic connected real hypersurface in $\mathbb{C}^N$ which is holomorphically nondegenerate. Let l be the Levi number of M. Then there is a proper real analytic subvariety $V \subset M$ such that for any $p \in M \backslash V$, η is a homeomorphism between $\mathrm{Aut}(M, p)$ and a closed, totally real Lie subgroup of G^{2l}.*

One may also generalize Theorem 4 to the case where p and p' are varying points in M and M' respectively. We first introduce some notation. If X and Y are two complex manifolds and k a positive integer, we denote by $J^k(X, Y)$ the complex manifold of k-jets of germs of holomorphic mappings from X to Y, that is,

$$J^k(X, Y) = \bigcup_{x \in X, y \in Y} J^k(X, Y)_{(x,y)}$$

where $J^k(X, Y)_{(x,y)}$ denotes the k-jets of germs at x of holomorphic mappings from X to Y and taking x to y. (See [Malgrange 1967; Golubitsky and Guillemin

1973], for example.) With this notation, $J^k(X,X)_{(x,x)}$ is the same as $J^k(X)_x$ introduced above with $X = \mathbb{C}^N$.

Denote by $E(X,Y)$ the set of germs of holomorphic mappings from X to Y equipped with its natural topology defined as follows. If $H_x \in E(X,Y)$ is a germ at x of a holomorphic mapping from X to Y which extends to a holomorphic mapping $H : U \to Y$, where $U \subset X$ is an open neighborhood of x, then a basis of open neighborhoods of H_x is given by

$$N_{U',V'} = \{F_p \in E(X,Y) : p \in U',\ F : U' \to V'\},$$

where U' is a relatively compact open neighborhood of x in U and V' is an open neighborhood of $H(x)$ in Y. In particular, a sequence $(H_j)_{x_j}$ converges to H_x if x_j converges to x and there exists a neighborhood U of x in X to which all the (H_j) and H extend, for sufficiently large j, and the H_j converge uniformly to H on compact subsets of U. This topology restricted to $E(X,Y)_{(x,y)}$ (the germs at x mapping x to y) coincides with the natural inductive topology mentioned above.

For every k there is a canonical mapping $\sigma_k : E(X,Y) \to J^k(X,Y)$. Note that $\sigma_k|_{E(X,Y)_{(p,p')}}$ is the same as the mapping $\eta_{p,p'}$ with $X = Y = \mathbb{C}^N$. It is easy to check that σ_k is continuous. If $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y$ then we denote by $G^k(X,Y)$ the open complex submanifold of $J^k(X,Y)$ given by those jets which are locally invertible. Similarly, we denote by $\mathcal{E}(X,Y)$ the open subset of $E(X,Y)$ consisting of the invertible germs. It is clear that the restriction of σ_k maps $\mathcal{E}(X,Y)$ to $G^k(X,Y)$.

If $M \subset X$ and $M' \subset Y$ are real analytic submanifolds, we let $E_{(M,M')}(X,Y)$ be the set of germs $H_p \in E(X,Y)$ with $p \in M$ which map a neighborhood of p in M into M'. Similarly, we denote by $\mathcal{E}_{(M,M')}(X,Y)$ those germs in $E_{(M,M')}(X,Y)$ which are invertible. Note that with $X = Y = \mathbb{C}^N$ we have

$$\mathcal{E}_{(M,M')}(X,Y)_{(p,p')} = \mathcal{F}(M,p;M',p').$$

We may now state a generalization of Theorem 4 with varying points p, p'.

Theorem 5 [Baouendi et al. 1997]. *Let X and Y be two complex manifolds of the same dimension, $M \subset X$ and $M' \subset Y$ two real analytic hypersurfaces, and k_0 a positive integer. Suppose that M and M' are both at most k_0-nondegenerate at every point. Then the mapping*

$$\sigma_{2k_0} : \mathcal{E}_{(M,M')}(X,Y) \to G^{2k_0}(X,Y)$$

is a homeomorphism onto its image Σ. Furthermore, Σ is a real analytic subset of $G^{2k_0}(X,Y)$, possibly empty, and each fiber $\Sigma \cap G^{2k_0}(X,Y)_{(p,p')}$, with $p \in M$, $p' \in M'$ is a real analytic submanifold.

From Theorem 5, together with some properties of subgroups of Lie groups, one may obtain the following result on the discreteness of $\mathrm{Aut}(M,p)$ in a neighborhood of p_0.

THEOREM 6 [Baouendi et al. 1997]. *Let M be a real analytic hypersurface in $\mathbb{C}^N$ finitely nondegenerate at p_0. If* $\mathrm{Aut}(M,p_0)$ *is a discrete group, then* $\mathrm{Aut}(M,p)$ *is also discrete for all p in a neighborhood of p_0 in M. Equivalently, if* $\mathrm{hol}_0(M,p_0)=\{0\}$ *then* $\mathrm{hol}_0(M,p)=\{0\}$ *for all p in a neighborhood of p_0 in M.*

EXAMPLE 4.4. Let M be the hypersurface given by

$$\operatorname{Im} w = |z|^2 + (\operatorname{Re} z^2)|z|^2.$$

Then by using the algorithm described in Section 5 below, one can show that $\mathrm{Aut}(M,0)$ consists of exactly two elements, namely the identity and the map $(z,w)\mapsto(-z,w)$. In particular, $\mathrm{hol}_0(M,0)=\{0\}$. Hence, by Theorem 6, $\mathrm{hol}_0(M,p)=\{0\}$ for all $p\in M$ near 0.

We mention here that there is a long history of results on transformation groups of Levi nondegenerate hypersurfaces, beginning with the seminal paper [Chern and Moser 1974]. (See also [Burns and Shnider 1977; Webster 1977b].) In particular, the fact that $\mathrm{Aut}(M,p)$ is a Lie group follows from [Chern and Moser 1974] when M is Levi nondegenerate at p. Further contributions were made by the Russian school (see for example the survey papers [Vitushkin 1985; Kruzhilin 1987], as well as the references therein). Results for higher-codimensional quadratic manifolds were obtained by Tumanov [1988b]. We point out that even for Levi nondegenerate hypersurfaces the approach given here is not based on [Chern and Moser 1974].

5. An Algorithm for Constructing the Set of All Mappings Between Two Real Analytic Hypersurfaces

Even in the case of a hypersurface, the parametrization of the Segre sets given by Theorem 1, is in general not an immersion onto a neighborhood of the base point p_0. Hence in the proof of Theorem 2, one goes to a nearby point to verify the uniqueness of the holomorphic mapping (which is already assumed to exist). By contrast, in the proof of Theorem 4, this method can no longer be used, because one has to know when a particular value of a parameter actually corresponds to a holomorphic mapping between the hypersurfaces in question.

In this section we outline the proof of Theorem 4, which actually gives an algorithm to construct the defining equations of the manifold

$$\Sigma_{p,p'} = \eta(\mathcal{F}(M,p;M',p'))$$

from defining equations of M and M' near p and p'. Moreover, for each $\Lambda\in G_0^{2k_0}$ the algorithm constructs a mapping which is the unique biholomorphic mapping H sending (M,p) into (M',p') with $\eta(H)=\Lambda$ for $\Lambda\in\Sigma_{p,p'}$. We give here the main steps of this algorithm.

STEP 1. We choose normal coordinates (z,w) and (z',w') for M and M' vanishing respectively at p and p'. We may write any $H \in \mathcal{F}(M,p;M',p')$ in the form $H=(f,g)$, such that the map is defined by $z'=f(z,w)$ and $w'=g(z,w)$. Note that it follows from the normality of the coordinates that $g(z,0)\equiv 0$. For each fixed k we choose coordinates Λ in G^k with $\Lambda=(\lambda_{z^\alpha w^j},\mu_{z^\alpha w^j})$, where $0<|\alpha|+j\le k$, such that if $H=(f,g)\in\mathcal{F}$, then the coordinates of $\eta(H)$ are defined by $\lambda_{z^\alpha w^j}=\partial_{z^\alpha w^j}f(0)$ and $\mu_{z^\alpha w^j}=\partial_{z^\alpha w^j}g(0)$. We identify an element in G^k with its coordinates Λ. We shall denote by G_0^k the submanifold of G^k consisting of those $\Lambda=(\lambda,\mu)$ for which $\mu_{z^\alpha}=0$ for all $0<|\alpha|\le k$. It is easily checked that G_0^k is actually a subgroup of G^k and hence a Lie group.

We apply (2.3) with $Z=(z,0)$ and $\zeta=0$. We obtain the following. There exist $\mathbb{C}^N$-valued functions $\Psi_j(z,\Lambda)$, with $j=0,1,2,\ldots$, each holomorphic in a neighborhood of $0\times G_0^{k_0+j}$ in $\mathbb{C}^n\times G_0^{k_0+j}$, such that if $H(z,w)\in\mathcal{F}(M,p;M',p')$ with $(\partial^\alpha H(0))_{|\alpha|\le k_0+j}=\Lambda_0\in G_0^{k_0+j}$, then

$$\partial_w^j H(z,0)=\Psi_j(z,\overline{\Lambda_0}),\quad \text{for } j=0,1,2,\ldots. \tag{5.1}$$

Furthermore, we have $\Psi_{0,N}(z,\Lambda)\equiv 0$, where $\Psi_{0,N}$ is the last component of the mapping Ψ_0. The fact that the Ψ_j do not depend on H follows from a close analysis of (2.3).

STEP 2. By taking $\gamma=0$, $Z=(z,Q(z,\chi,0))$ and $\zeta=(\chi,0)$ in (2.3), we find a $\mathbb{C}^N$-valued function $\Phi(z,\chi,\Lambda)$, holomorphic in a neighborhood of $0\times 0\times G_0^{2k_0}$ in $\mathbb{C}^n\times\mathbb{C}^n\times G_0^{2k_0}$, such that for $H\in\mathcal{F}(M,p;M',p')$ with $(\partial^\alpha H(0))_{|\alpha|\le 2k_0}=\Lambda_0$, we have

$$H(z,Q(z,\chi,0))\equiv\Phi(z,\chi,\Lambda_0). \tag{5.2}$$

Again here the fact that Φ does not depend on H follows from a close analysis of the proof of (2.3).

STEP 3. We begin with the following lemma.

LEMMA 5.3. *Assume $(M,p),(M',p')$ are as above. There exists a $\mathbb{C}^N$-valued function $F(z,t,\Lambda)$ holomorphic in a neighborhood of $0\times 0\times G_0^{2k_0}$ in $\mathbb{C}^n\times\mathbb{C}\times G_0^{2k_0}$ and a germ at 0 of a nontrivial holomorphic function $B(z)$, such that for a fixed $\Lambda_0\in G_0^{2k_0}$ there exists $H\in\mathcal{F}(M,p;M',p')$ with*

$$(\partial^\alpha H(0))_{|\alpha|\le 2k_0}=\Lambda_0 \tag{5.4}$$

if and only if all three following conditions hold:

(i) $(z,w)\mapsto F(z,w/B(z),\Lambda_0)$ *extends to a function $K_{\Lambda_0}(z,w)$ holomorphic in a full neighborhood of 0 in $\mathbb{C}^N$.*
(ii) $(\partial^\alpha K_{\Lambda_0}(0))_{|\alpha|\le 2k_0}=\Lambda_0$.
(iii) $K_{\Lambda_0}(M)\subset M'$.

If (i), (ii), (iii) *hold, then the unique mapping in $\mathcal{F}(M,p;M',p')$ satisfying* (5.4) *is given by $H(Z)=K_{\Lambda_0}(Z)$.*

Proof. From the k_0-nondegeneracy, we have $Q_{\chi_1}(z,0,0) \not\equiv 0$ and we set

$$A(z) = Q_{\chi_1}(z,0,0). \tag{5.5}$$

We write $\chi = (\chi_1, \chi')$; we shall solve the equation

$$w = Q(z,(\chi_1,0),0) \tag{5.6}$$

for χ_1 as a function of (z,w) and analyze the solution as z and w approach 0. We have

$$Q(z,(\chi_1,0),0) = \sum_{j=1}^{\infty} A_j(z)\chi_1^j, \tag{5.7}$$

with $A_1(z) = A(z)$ and $A_j(0) = 0$, for $j = 1, \ldots$. Dividing (5.6) by $A(z)^2$, we obtain

$$\frac{w}{A(z)^2} = \frac{\chi_1}{A(z)} + \sum_{j=2}^{\infty} A_j(z)\frac{\chi_1^j}{A(z)^2}.$$

We set $C_j(z) = A_j(z)A(z)^{j-2}$, with $j \geq 2$, and let

$$\psi(z,t) = t + \sum_{j=2}^{\infty} v_j(z)t^j \tag{5.8}$$

be the solution in u given by the implicit function theorem of the equation $t = u + \sum_{j=2}^{\infty} C_j(z)u^j$, with $\psi(0,0) = 0$. The functions ψ and v_j are then holomorphic at 0 and $v_j(0) = 0$. A solution for χ_1 in (5.6) is then given by

$$\chi_1 = \theta(z,w) = A(z)\psi\Big(z, \frac{w}{A(z)^2}\Big). \tag{5.9}$$

The function $\theta(z,w)$ is holomorphic in an open set in $\mathbb{C}^{n+1}$ having the origin as a limit point.

Now define F by

$$F(z,t,\Lambda) = \Phi(z, (A(z)\psi(z,t),0), \Lambda), \tag{5.10}$$

where Φ is given by Step 2, and let $B(z) = A(z)^2$, with $A(z)$ given by (5.5). Then (i) follows from Step 2. The rest of the proof of the lemma is now easy and is left to the reader. □

It follows from Lemma 5.3 and its proof that if $H(z,w)$ is a biholomorphic mapping taking (M,p) into (M',p') with $\Lambda = \eta(H)$ then

$$H(z,w) = F\Big(z, \frac{w}{A(z)^2}, \Lambda\Big), \tag{5.11}$$

where $A(z)$ is defined by (5.5), and $F(z,t,\Lambda)$ is defined by (5.10). Note again here that F and A are independent of H.

Step 4. In this last step, the following lemma and its proof give the construction of the real analytic functions defining $\Sigma_{p,p'} = \eta(\mathcal{F})$.

Lemma 5.12. *Under the hypotheses of Theorem* 4 *there exist functions* b_j, *for* $j = 1, 2, \dots$, *holomorphic in* $G_0^{2k_0} \times G_0^{2k_0}$ *such that there is* $H \in \mathcal{F}(M, p; M', p')$ *satisfying* (5.4) *if and only if* $b_j(\Lambda_0, \overline{\Lambda_0}) = 0$, *for* $j = 1, 2, \dots$.

The proof of Lemma 5.12 will actually give an algorithm for the construction of the functions b_j from the defining equations of M and M'.

Proof. We first construct a function $K(Z, \Lambda)$ holomorphic in a neighborhood of $0 \times G_0^{2k_0}$ in $\mathbb{C}^N \times G_0^{2k_0}$ such that (i) of Lemma 5.3 holds for a fixed $\Lambda_0 \in G_0^{2k_0}$ if and only if $F(z, w/B(z), \Lambda_0) \equiv K(z, w, \Lambda_0)$. Recall that $F(z, t, \Lambda)$ is holomorphic in a neighborhood of $0 \times 0 \times G_0^{2k_0}$ in $\mathbb{C}^n \times \mathbb{C} \times G_0^{2k_0}$. Hence we can write

$$F(z, t, \Lambda) = \sum_{\alpha, j} F_{\alpha j}(\Lambda) z^\alpha t^j, \tag{5.13}$$

with $F_{\alpha j}$ holomorphic in $G_0^{2k_0}$. For each compact subset $L \subset G_0^{2k_0}$ there exists $C > 0$ such that the series $(z, t) \mapsto \sum_{\alpha, j} F_{\alpha j}(\Lambda) z^\alpha t^j$ converges uniformly for $|z|, |t| \le C$ and for each fixed $\Lambda \in L$. For $|z| \le C$ and $\big|w/B(z)\big| \le C$ we have

$$F\Big(z, \frac{w}{B(z)}, \Lambda\Big) = \sum_{j=0}^{\infty} \frac{F_j(z, \Lambda)}{B(z)^j} w^j \tag{5.14}$$

with $F_j(z, \Lambda) = \sum_\alpha F_{\alpha, j}(\Lambda) z^\alpha$. After a linear change of holomorphic coordinates if necessary, and putting $z = (z_1, z')$, we may assume, by using the Weierstrass Preparation Theorem, that

$$B(z)^j = U_j(z) \bigg(z_1^{K_j} + \sum_{p=0}^{K_j - 1} a_{jp}(z') z_1^p \bigg),$$

with $U_j(0) \neq 0$ and $a_{jp}(0) = 0$. By the Weierstrass Division Theorem we have the unique decomposition

$$F_j(z, \Lambda) = Q_j(z, \Lambda) B(z)^j + \sum_{p=0}^{K_j - 1} r_{jp}(z', \Lambda) z_1^p, \tag{5.15}$$

where $Q_j(z, \Lambda)$ and $r_{jp}(z', \Lambda)$ are holomorphic in a neighborhood of $0 \times G_0^{2k_0}$ in $\mathbb{C}^n \times G_0^{2k_0}$. It then suffices to take

$$K(z, w, \Lambda) = \sum_j Q_j(z, \Lambda) w^j. \tag{5.16}$$

Moreover, (i) of Lemma 5.3 holds if and only if $z' \mapsto r_{jp}(z', \Lambda_0)$ vanishes identically for all j, p. By taking the coefficients of the Taylor expansion of the r_{jp} with respect to z' we conclude that there exist functions c_j, $j = 1, 2, \dots$, holomorphic in $G_0^{2k_0}$ such that (i) holds if and only if $c_j(\Lambda_0) = 0$, $j = 1, 2, \dots$.

It follows from the above that we have $K_{\Lambda_0}(Z) \equiv K(Z, \Lambda_0)$, where $K_{\Lambda_0}(Z)$ is given by Lemma 5.3. By taking $d_j(\Lambda)$, for $1 \le j \le J$, as the components of

$(\partial_Z^\alpha K(0,\Lambda))_{|\alpha|\le 2k_0} - \Lambda$, we find that if (i) is satisfied then (ii) holds if and only if $d_j(\Lambda_0)=0$, for $1\le j\le J$. Similarly, we note that (iii) is equivalent to

$$\rho'\big(K(z,w,\Lambda_0),\ \overline{K}(\chi,\overline{Q}(\chi,z,w),\ \overline{\Lambda_0})\big)\equiv 0, \tag{5.17}$$

where ρ' is a defining function for M'. By expanding the left hand side of (5.17) as a series in z,w,χ with coefficients which are holomorphic functions of $\Lambda_0,\overline{\Lambda_0}$, we conclude that there exist functions e_j, for $j=1,2,\ldots$, holomorphic in $G_0^{2k_0}\times G_0^{2k_0}$ such that if (i) is satisfied then (iii) holds if and only if $e_j(\Lambda_0,\overline{\Lambda_0})=0$ for $j=1,2,\ldots$. □

The main points in the proof of Theorem 4 follow from Steps 1–4 above. The proof that $\Sigma_{p,p'}$ is a manifold is first reduced to the case where $M=M',\ p=p'$; for that case one uses the fact that a closed subgroup of a Lie group is again a Lie group; see [Varadarajan 1974], for example. We shall omit the rest of the details of the proof.

Remark 5.18. We have stated Theorems 4–6 only for hypersurfaces. The proofs of these results do not generalize to CR manifolds of higher codimension. In fact, the proofs given here are based on an analysis of the behavior of the Segre set N_2 near the origin; see (5.7)–(5.9). Such a precise analysis for higher codimension seems much more complicated. It would be interesting to have analogues of Theorems 4–6 in higher codimension.

6. Holomorphic Mappings Between Real Algebraic Sets

In this section we shall consider the case where the submanifolds M and M' are algebraic. Recall that a subset $A\subset\mathbb{C}^N$ is a *real algebraic set* if it is defined by the vanishing of real valued polynomials in $2N$ real variables; we shall always assume that A is irreducible. By A_{reg} we mean the regular points of A: see [Hodge and Pedoe 1947–53], for example. Recall that A_{reg} is a real submanifold of $\mathbb{C}^N$, all points of which have the same dimension. We write $\dim A=\dim_{\mathbb{R}} A$ for the dimension of the real submanifold A_{reg}. A germ of a holomorphic function f at a point $p_0\in\mathbb{C}^N$ is called *algebraic* if it satisfies a polynomial equation of the form $a_K(Z)f^K(Z)+\cdots+a_1(Z)f(Z)+a_0(Z)\equiv 0$, where the $a_j(Z)$ are holomorphic polynomials in N complex variables with $a_K(Z)\not\equiv 0$. In this section we give conditions under which a germ of a holomorphic map in $\mathbb{C}^N$, mapping an irreducible real algebraic set A into another such set of the same dimension, is actually algebraic, that is, all its components are algebraic functions.

The first result deals with biholomorphic mappings between algebraic hypersurfaces. The following theorem gives a necessary and sufficient condition for algebraicity of mappings in this case.

Theorem 7 [Baouendi and Rothschild 1995]. *Let M,M' be two connected, real algebraic, hypersurfaces in $\mathbb{C}^N$. If M is holomorphically nondegenerate and H is a biholomorphic mapping defined in an open neighborhood in $\mathbb{C}^N$ of a point*

$p_0 \in M$ satisfying $H(M) \subset M'$, then H is algebraic. Conversely, if M is holomorphically degenerate, then for every $p_0 \in M$ there exists a germ H of a nonalgebraic biholomorphism of $\mathbb{C}^N$ at p_0 with $H(M) \subset M$ and $H(p_0) = p_0$.

For generic manifolds of higher codimension, holomorphic nondegeneracy is no longer sufficient for the algebraicity of local biholomorphisms. For example, one can take $M = M'$ to be the generic submanifold of $\mathbb{C}^3$ given by $\operatorname{Im} Z_2 = |Z_1|^2, \operatorname{Im} Z_3 = 0$, then the biholomorphism $Z \mapsto (Z_1, Z_2, e^{Z_3})$ maps M into itself, but is not algebraic. Here M is holomorphically nondegenerate, but nowhere minimal. However:

Theorem 8 [Baouendi et al. 1996a]. *Let $M, M' \subset \mathbb{C}^N$ be two real algebraic, holomorphically nondegenerate, generic submanifolds of the same dimension. Assume there exists $p \in M$, such that M is minimal at p. If H is a biholomorphic mapping defined in an open neighborhood in $\mathbb{C}^N$ of a point $p_0 \in M$ satisfying $H(M) \subset M'$ then H is algebraic.*

The conditions for Theorem 8 are almost necessary, as is shown by the following converse.

Theorem 9 [Baouendi et al. 1996a]. *Let $M \subset \mathbb{C}^N$ be a connected real algebraic generic submanifold. If M is holomorphically degenerate then for every $p_0 \in M$ there exists a germ of a nonalgebraic biholomorphism H of $\mathbb{C}^N$ at p_0 mapping M into itself with $H(p_0) = p_0$. When M is defined by weighted homogeneous real-valued polynomials, the existence of such a nonalgebraic mapping also holds if M is not minimal at any point (even if M is holomorphically nondegenerate).*

We do not give here the details of the proofs of Theorems 7–9. The main ingredient in proving the algebraicity of H in Theorems 7 and 8 is the fact that the closure of each of the Segre sets N_j (described in Section 1) of a generic real algebraic manifold is actually a complex algebraic set in $\mathbb{C}^N$. In particular, the following result is a consequence of Theorems 1 and 8 and the algebraicity of the maximal Segre set N_{j_0}.

Corollary 6.1 [Baouendi et al. 1996a]. *Let M be a real algebraic CR submanifold of $\mathbb{C}^N$ and $p_0 \in M$. Then the CR orbit of p_0 is a real algebraic submanifold of M and its intrinsic complexification, X, is a complex algebraic submanifold of $\mathbb{C}^N$. For any germ H of a biholomorphism at p_0 of $\mathbb{C}^N$ into itself mapping M into another real algebraic manifold of the same dimension as that of M, the restriction of H to X is algebraic.*

It is perhaps worth mentioning here that the CR orbit of p_0 is the Nagano leaf passing through p_0 ([Nagano 1966]) and hence can be obtained by solving systems of ODE's. In general, the solution of such a system is not algebraic, even when the coefficients of the differential equations are algebraic.

Example 6.2. Consider the algebraic holomorphically nondegenerate generic submanifold $M \subset \mathbb{C}^4$ given by

$$\operatorname{Im} w_1 = |z|^2 + \operatorname{Re} w_2 |z|^2, \quad \operatorname{Im} w_2 = \operatorname{Re} w_3 |z|^4, \quad \operatorname{Im} w_3 = 0. \tag{6.3}$$

Here M is holomorphically nondegenerate, but nowhere minimal. For all $0 \neq r \in \mathbb{R}$ the orbit of the point $(0,0,0,r)$ is the leaf $M \cap \{(z,w) : w_3 = r\}$, and its intrinsic complexification is $\{(z,w), w_3 = r\}$. By Corollary 6.1, if H is a germ of a biholomorphism at $0 \in \mathbb{C}^4$ mapping M into an algebraic submanifold of $\mathbb{C}^4$ of dimension 5, then $(z, w_1, w_2) \mapsto H(z, w_1, w_2, r)$ is algebraic for all $r \neq 0$ and small. The orbit of the point $0 \in \mathbb{C}^4$ is $M \cap \{z,w) : w_2 = w_3 = 0\}$ and its intrinsic complexification is $\{(z,w) : w_2 = w_3 = 0\}$ and hence again by Corollary 6.1, the mapping $(z, w_1) \mapsto H(z, w_1, 0, 0)$ is algebraic. By further results in [Baouendi et al. 1996a] on propagation of algebraicity, one can also show that the mapping $(z, w_1, w_2) \mapsto H(z,w)$ is algebraic for all fixed $w_3 \in \mathbb{C}$, sufficiently small. This result is optimal. Indeed, the nonalgebraic mapping $H : \mathbb{C}^4 \mapsto \mathbb{C}^4$, defined by

$$H(z, w_1, w_2, w_3) = (ze^{iw_3}, w_1, w_2, w_3),$$

is a biholomorphism near the origin, and maps M into itself.

The following statement extends Theorem 8 to more general real algebraic sets.

Theorem 10 [Baouendi et al. 1996a]. *Let $A \subset \mathbb{C}^N$ be an irreducible real algebraic set, and p_0 a point in $\overline{A_{\mathrm{reg}}}$, the closure of A_{reg} in $\mathbb{C}^N$. Suppose the following two conditions hold.*

(i) *The submanifold A_{reg} is holomorphically nondegenerate.*
(ii) *If f is a germ, at a point in A, of a holomorphic algebraic function in $\mathbb{C}^N$ such that the restriction of f to A is real valued, then f is constant.*

Then if H is a holomorphic map from an open neighborhood in $\mathbb{C}^N$ of p_0 into $\mathbb{C}^N$, with Jac $H \not\equiv 0$, *and mapping A into another real algebraic set A' with* dim $A' =$ dim A, *necessarily the map H is algebraic.*

For further results on algebraicity and partial algebraicity, see [Baouendi et al. 1996a].

We will end with a brief history of some previous work on the algebraicity of holomorphic mappings between real algebraic sets. Early in this century Poincaré [Poincaré 1907] proved that if a biholomorphism defined in an open set in $\mathbb{C}^2$ maps an open piece of a sphere into another, it is necessarily a rational map. This result was extended by Tanaka [1962] to spheres in higher dimensions. Webster [1977a] proved a general result for algebraic, Levi-nondegenerate real hypersurfaces in $\mathbb{C}^N$; he proved that any biholomorphism mapping such a hypersurface into another is algebraic. Later, Webster's result was extended in some cases to Levi-nondegenerate hypersurfaces in complex spaces of different dimensions (see, for example, [Webster 1979; Forstnerič 1989; Huang 1994] and their references). See also [Bedford and Bell 1985] for other related results. We also refer the reader to [Tumanov and Khenkin 1983; Tumanov 1988b], which contain results on mappings of higher codimensional quadratic manifolds. See also related results of Sharipov and Sukhov [1996] using Levi form criteria; some of these results are special cases of Theorem 8.

Added in Proof. Since this paper was submitted, the authors jointly with Ebenfelt, have published a book [Baouendi et al. 1999]. The reader is referred to this book for background material as well as further results related to the subject of this article.

References

[Baouendi and Rothschild 1995] M. S. Baouendi and L. P. Rothschild, "Mappings of real algebraic hypersurfaces", *J. Amer. Math. Soc.* **8**:4 (1995), 997–1015.

[Baouendi et al. 1985] M. S. Baouendi, H. Jacobowitz, and F. Treves, "On the analyticity of CR mappings", *Ann. of Math.* (2) **122**:2 (1985), 365–400.

[Baouendi et al. 1996a] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, "Algebraicity of holomorphic mappings between real algebraic sets in $\mathbb{C}^n$", *Acta Math.* **177**:2 (1996), 225–273.

[Baouendi et al. 1996b] M. S. Baouendi, X. Huang, and L. Preiss Rothschild, "Regularity of CR mappings between algebraic hypersurfaces", *Invent. Math.* **125**:1 (1996), 13–36.

[Baouendi et al. 1997] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, "Parametrization of local biholomorphisms of real analytic hypersurfaces", *Asian J. Math.* **1**:1 (1997), 1–16.

[Baouendi et al. 1998] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, "CR automorphisms of real analytic manifolds in complex space", *Comm. Anal. Geom.* **6**:2 (1998), 291–315.

[Baouendi et al. 1999] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, *Real submanifolds in complex space and their mappings*, Press, Princeton Math. Series **47**, Princeton Univ. Press, Princeton, NJ, 1999.

[Bedford and Bell 1985] E. Bedford and S. Bell, "Extension of proper holomorphic mappings past the boundary", *Manuscripta Math.* **50** (1985), 1–10.

[Belošapka 1979] V. K. Belošapka, "On the dimension of the group of automorphisms of an analytic hypersurface", *Izv. Akad. Nauk SSSR Ser. Mat.* **43**:2 (1979), 243–266, 479. In Russian; translated in *Math USSR Izv.* **14** (1980), 223–245.

[Bloom and Graham 1977] T. Bloom and I. Graham, "On "type" conditions for generic real submanifolds of C^n", *Invent. Math.* **40**:3 (1977), 217–243.

[Burns and Shnider 1977] D. Burns, Jr. and S. Shnider, "Real hypersurfaces in complex manifolds", pp. 141–168 in *Several complex variables,* part 2 (Williamstown, MA, 1975), edited by R. O. Wells, Jr., Proc. Sympos. Pure Math. **30**, Amer. Math. Soc., Providence, 1977.

[Chern and Ji 1995] S. S. Chern and S. Ji, "Projective geometry and Riemann's mapping problem", *Math. Ann.* **302**:3 (1995), 581–600.

[Chern and Moser 1974] S. S. Chern and J. K. Moser, "Real hypersurfaces in complex manifolds", *Acta Math.* **133** (1974), 219–271. Erratum in **150**:3–4 (1983), p. 297.

[Diederich and Fornæss 1988] K. Diederich and J. E. Fornæss, "Proper holomorphic mappings between real-analytic pseudoconvex domains in C^n", *Math. Ann.* **282**:4 (1988), 681–700.

[Diederich and Webster 1980] K. Diederich and S. M. Webster, "A reflection principle for degenerate real hypersurfaces", *Duke Math. J.* **47**:4 (1980), 835–843.

[Ebenfelt 1998] P. Ebenfelt, "Holomorphic mappings between real analytic submanifolds in complex space", pp. 35–69 in *Integral geometry, Radon transforms and complex analysis* (Venice, 1996), edited by E. Casadio Tarabusi et al., Lecture Notes in Math. **1684**, Springer, Berlin, 1998.

[Forstnerič 1989] F. Forstnerič, "Extending proper holomorphic mappings of positive codimension", *Invent. Math.* **95**:1 (1989), 31–61.

[Golubitsky and Guillemin 1973] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Math. **14**, Springer, New York, 1973.

[Hodge and Pedoe 1947–53] W. H. D. Hodge and D. Pedoe, *Methods of algebraic geometry*, Cambridge University Press, Cambridge, 1947–53.

[Huang 1994] X. Huang, "On the mapping problem for algebraic real hypersurfaces in the complex spaces of different dimensions", *Ann. Inst. Fourier* (*Grenoble*) **44**:2 (1994), 433–463.

[Kobayashi 1972] S. Kobayashi, *Transformation groups in differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete **70**, Springer, New York, 1972.

[Kruzhilin 1987] N. G. Kruzhilin, "Description of the local automorphism groups of real hypersurfaces", pp. 749–758 in *Proceedings of the International Congress of Mathematicians* (Berkeley, 1986), vol. 1, Amer. Math. Soc., Providence, RI, 1987.

[Loboda 1981] A. V. Loboda, "Local automorphisms of real-analytic hypersurfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **45**:3 (1981), 620–645. In Russian; translated in *Math. USSR Izv.* **18** (1982), 537–559.

[Malgrange 1967] B. Malgrange, *Ideals of differentiable functions*, Tata Research Studies in Mathematics **3**, Oxford University Press and Tata Institute, Bombay, 1967.

[Nagano 1966] T. Nagano, "Linear differential systems with singularities and an application to transitive Lie algebras", *J. Math. Soc. Japan* **18** (1966), 398–404.

[Poincaré 1907] H. Poincaré, "Les fonctions analytiques de deux variables et la représentation conforme", *Rend. Circ. Mat. Palermo, II Ser.* **23** (1907), 185–220.

[Segre 1931] B. Segre, "Intorno al problem di Poincaré della rappresentazione pseudoconforme", *Rend. Acc. Lincei* **13** (1931), 676–683.

[Sharipov and Sukhov 1996] R. Sharipov and A. Sukhov, "On CR-mappings between algebraic Cauchy-Riemann manifolds and separate algebraicity for holomorphic functions", *Trans. Amer. Math. Soc.* **348**:2 (1996), 767–780.

[Stanton 1995] N. K. Stanton, "Infinitesimal CR automorphisms of rigid hypersurfaces", *Amer. J. Math.* **117**:1 (1995), 141–167.

[Stanton 1996] N. K. Stanton, "Infinitesimal CR automorphisms of real hypersurfaces", *Amer. J. Math.* **118**:1 (1996), 209–233.

[Tanaka 1962] N. Tanaka, "On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables", *J. Math. Soc. Japan* **14** (1962), 397–429.

[Tumanov 1988a] A. E. Tumanov, "Extending CR functions on manifolds of finite type to a wedge", *Mat. Sb. (N.S.)* **136(178)**:1 (1988), 128–139. In Russian; translated in *Math. USSR Sb.* **64**:1 (1989), 129–140.

[Tumanov 1988b] A. E. Tumanov, "Finite-dimensionality of the group of CR-automorphisms of a standard CR-manifold, and characteristic holomorphic mappings of Siegel domains", *Izv. Akad. Nauk SSSR Ser. Mat.* **52**:3 (1988), 651–659, 672. In Russian; translated in *Math. USSR Izvestia*, **32** (1989), 655–662.

[Tumanov and Khenkin 1983] A. E. Tumanov and G. M. Khenkin, "Local characterization of holomorphic automorphisms of Siegel domains", *Funktsional. Anal. i Prilozhen.* **17**:4 (1983), 49–61. In Russian; translated in *Functional Anal. Appl.* **17** (1983), 285–294.

[Varadarajan 1974] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Prentice-Hall, Englewood Cliffs, N.J., 1974. Reprinted as Graduate Texts in Mathematics **102**, Springer, New York, 1984.

[Vitushkin 1985] A. G. Vitushkin, "Holomorphic mappings and the geometry of hypersurfaces", pp. 159–214 in *Several Complex Variables I*, edited by A. G. Vitushkin, Encycl. Math. Sci. **7**, Springer, Berlin, 1985.

[Webster 1977a] S. M. Webster, "On the mapping problem for algebraic real hypersurfaces", *Invent. Math.* **43**:1 (1977), 53–68.

[Webster 1977b] S. M. Webster, "On the transformation group of a real hypersurface", *Trans. Amer. Math. Soc.* **231**:1 (1977), 179–190.

[Webster 1979] S. M. Webster, "On mapping an n-ball into an $(n+1)$-ball in complex spaces", *Pacific J. Math.* **81**:1 (1979), 267–272.

M. Salah Baouendi
Department of Mathematics 0112
University of California, San Diego
La Jolla, CA 92093-0112
United States
sbaouendi@ucsd.edu

Linda Preiss Rothschild
Department of Mathematics 0112
University of California, San Diego
La Jolla, CA 92093-0112
United States
lrothschild@ucsd.edu