Several Complex Variables
MSRI Publications
Volume **37**, 1999

How to Use the Cycle Space in Complex Geometry

DANIEL BARLET

Abstract. In complex geometry, the use of n-convexity and the use of ampleness of the normal bundle of a d-codimensional submanifold are quite difficult for $n > 0$ and $d > 1$. The aim of this paper is to explain how some constructions on the cycle space (the Chow variety in the quasiprojective setting) allows one to pass from the n-convexity of Z to the 0-convexity of $C_n(Z)$ and from a $(n+1)$-codimensional submanifold of Z having an ample normal bundle to a Cartier divisor of $C_n(Z)$ having the same property. We illustrate the use of these tools with some applications.

1. Basic Definitions

Let Z be a complex manifold; recall that an *n-cycle* in Z is a locally finite sum

$$X = \sum_{j \in J} n_j X_j,$$

where the X_j are distinct nonempty closed irreducible n-dimensional analytic subsets of Z, and where $n_j \in \mathbb{N}^*$ for any $j \in J$. The *support* of the cycle X is the closed analytic set $|X| = \bigcup_{j \in J} X_j$ of pure dimension n. The integer n_j is the *multiplicity* of the irreducible component X_j of $|X|$ in the cycle X. The cycle X is *compact* if and only if each X_j is compact and J is finite. We shall consider mainly compact cycles, but to understand problems which are of local nature on cycles it will be better to drop this assumption from time to time. We shall make it explicit when the cycles are assumed to be compact.

Topology of the cycle space. For simplicity we assume here that cycles are compact. The continuity of a family of cycles $(C_s)_{s \in S}$ consists of two conditions:

– Geometric continuity of the supports: This is the fact that $\{s \in S / |C_s| \subset U\}$ is open in S when U is an open set in Z.

This text is an expanded version of a series of two lectures of the same title given at MSRI at the end of March 1996.

- Continuity of the volume: For any choice of a continuous positive hermitian $(1,1)$ form on Z, the volume function

$$\mathrm{vol}_h(C_s) = \int_{C_s} h^{\wedge n}$$

is continuous on S.

It is not quite obvious that, when the first condition is fulfilled, the second one can be expressed in the following way:

- If Y is a locally closed submanifold of codimension n in Z such that, for $s_0 \in S$, $\partial Y := \overline{Y} - Y$ does not intersect C_{s_0} and such that $C_{s_0} \cap Y$ has exactly k points (counting multiplicities[1]), then for s near enough to s_0, we have again $\sharp\,(C_s \cap Y) = k$ (counting multiplicities) and the intersection map $s \to (C_s \cap Y)$ with value in the symmetric product $\mathrm{Sym}^k Y$ is continuous near s_0.

For more information on the relationship between volume and intersection multiplicities, see [Barlet 1980c].

A main tool in the topological study of cycles is E. Bishop's compactness theorem (see [Bishop 1964; Barlet 1978a; Lieberman 1978; Fujiki 1978; SGAN 1982]):

THEOREM 1. *Let Z a complex analytic space and $C_n(Z)$ the* (*topological*) *space of compact n-cycles of Z. A subset $\mathcal{A}$ of $C_n(Z)$ is relatively compact if and only if*

(1) *there is a compact subset K compact of Z such that $|C| \subset K$ for every $C \in \mathcal{A}$, and*
(2) *there is a positive definite hermitian metric of class C^0 in Z and $\Gamma = \Gamma(h, \mathcal{A})$ such that*

$$\mathrm{vol}_h(C) = \int_C h^{\wedge n} \le \Gamma \quad \textit{for all } C \in \mathcal{A}. \qquad \square$$

REMARK. If Z is a Kähler manifold and if we choose h to be the Kähler metric on Z, the function vol_h is locally constant on $C_n(Z)$ so the condition (2) is satisfied for any connected set $\mathcal{A}$ in $C_n(Z)$. See [Barlet 1978a, Prop. 1].

[1]Multiplicities are counted as follows: locally we can assume that $Z \simeq U \times Y$ where U and Y are open polydiscs in $\mathbb{C}^n$ and $\mathbb{C}^p$, such that $|C_{s_0}| \cap \overline{U} \times \partial Y = \varnothing$, because $|C_{s_0}| \cap Y$ is finite (compare to the definition of "écaille adapté" in [Barlet 1975, Chapter 1]). Then C_{s_0} defines a branched coverings of U via the projection $U \times Y \to U$ and we have the following classification theorem for degree k branched coverings in such a situation [Barlet 1975, Chapter 0]: *There exists a natural bijection between degree k branched coverings of U in $U \times Y$ and holomorphic maps $f : U \to \mathrm{Sym}^k Y$.* So if C_{s_0} corresponds to f and Y is $\{t_0\} \times Y$ in Z, the intersection $C_{s_0} \cap Y$ is the k-uple $f(t_0)$.

Analytic families of cycles. Consider a family of compact n-dimensional cycles $(C_s)_{s\in S}$ of the complex manifold Z parametrized by a *reduced* complex space S. Assume that this family is continuous and let Y be a locally closed complex submanifold of Z such that in an open neighbourhood S' of $s_0 \in S$ we have

$$|C_s| \cap \partial Y = \varnothing \quad \text{and} \quad \sharp\,(Y \cap C_s) = k.$$

Then we require that the intersection map

$$I_Y : S' \to \operatorname{Sym}^k Y$$

be holomorphic, where $\operatorname{Sym}^k Y$, the k-th symmetric product of Y, is endowed with the normal complex-space structure given by the quotient Y^k/σ_k. We say that $(C_s)_{s\in S}$ is analytic near $s_0 \in S$ if, for *any* such choice of Y, the map I_Y is analytic near s_0.

For an analytic family $(C_s)_{s\in S}$ the graph

$$|G| = \{(s,z) \in S \times Z/z \in |C_s|\}$$

is a closed analytic subset of $S \times Z$ which is proper and n-equidimensional over S by the first projection.

Though it is quite hard to prove that a given family $(C_s)_{s\in S}$ is analytic using our definition, for normal S we have the following very simple criterion:

THEOREM 2. *Let Z a complex manifold and S a normal complex space. Let $G \subset S \times Z$ a analytic set which is proper and n-equidimensional over S. Then there is a unique analytic family of compact n-dimensional cycles $(C_s)_{s\in S}$ of Z satisfying these conditions*:

(i) *For s generic in S, we have $C_s = |C_s|$ (so all multiplicities are equal to one).*
(ii) *For all $s \in S$, we have $\{s\} \times |C_s| = G \cap (\{s\} \times Z)$ (as sets).* □

REMARKS. (i) Of course for nongeneric $s \in S$ in the theorem, we could have $C_s \neq |C_s|$, and one point in the proof is to explain what are the multiplicities on the irreducible components on $|C_s|$ we have to choose. The answer comes in fact from the continuity property of the intersection with a codimension n submanifold Y as explained before.

(ii) In fact the notion of analytic family of cycles is invariant by local embeddings of Z, so it is possible to extend our definition to singular Z by using a local embedding in a manifold. Then, the previous theorem extends to any Z.

(iii) To decide if a family of cycles is analytic, when S has wild singularities, could be delicate (see for instance the example in [Barlet 1975, p. 44]).

(iv) A flat family of compact n-dimensional *subspaces* of Z gives rise to an analytic family of n-cycles. More precisely, if $G \subset S \times Z$ is a S-flat and S-proper *subspace* of $S \times Z$ (with S reduced) which is n-equidimensional on S, then the family of cycles of Z associated to $\pi^{-1}(s)$ where $\pi : G \to S$ is the first projection (and $\pi^{-1}(s)$ is a *subspace* of $\{s\} \times Z$) is an analytic family of cycles [Barlet 1975, Chapter 5]. For cycles of higher codimension one has to take care that for each cycle C there exists a lot of *subspaces* of Z such the associated cycle is C.

To conclude this section, recall that the functor

$$S \to \{\text{analytic families of } n\text{-compact cycles of } Z\}$$

is representable in the category of finite-dimensional reduced complex analytic spaces [Barlet 1975, Chapter 3]. This means that it is possible to endow the (topological) space $C_n(Z)$ with a reduced locally finite-dimensional complex analytic structure in such a way that we get a natural bijective correspondence between holomorphic maps $f : S \to C_n(Z)$ and analytic families of compact n-cycles of Z parametrized by S (any reduced complex space). This correspondence is given by the pull back of the (so called) universal family on $C_n(Z)$ (each compact n-cycle of Z is parametrized by the corresponding point in $C_n(Z)$).

2. Holomorphic Functions on $C_n(Z)$

The idea for building holomorphic functions on $C_n(Z)$ by means of integration of cohomology classes in $H^n(Z, \Omega^n_Z)$ comes from the pioneering work [Andreotti and Norguet 1967]. It was motivated by the following question, which comes up after the famous paper [Andreotti and Grauert 1962]: vanishing (or finiteness) theorems for $H^{n+1}(Z, \mathcal{F})$ for any coherent sheaf $\mathcal{F}$ on Z allow one to produce cohomology classes in $H^n(X, \mathcal{F})$. But what to do with such cohomology classes when $n > 0$?

The answer given in [Andreotti and Norguet 1967] is: produce a lot of holomorphic functions on $C_n(Z)$ in order to prove the holomorphic convexity of the components of $C_n(Z)$.

If we assume Z smooth and allow us to normalize $C_n(Z)$, the following theorem is an easy consequence of Stokes theorem.

THEOREM 3. *There exists a natural linear map*

$$\rho : H^n(Z, \Omega^n_Z) \to H^0\big(C_n(Z), \mathcal{O}\big)$$

given by $\rho(\omega))(C) = \int_C \widetilde{\omega}$, *where* $\widetilde{\omega}$ *is a Dolbeault representative* (*so a* (n, n) C^∞ *form on* $Z, \bar{\partial}$ *closed*) *of* $\omega \in H^n(Z, \Omega^n_Z)$. □

For nonnormal parameter space, this result is much deeper and is proved in [Barlet 1980b]. For general Z (not necessarily smooth) and general S (reduced) this result was proved later, in [Barlet and Varouchas 1989].

Let me sketch now the main idea in [Andreotti and Norguet 1967] (in a simplified way). Assume that Z is a n-complete manifold. (In this terminology from Andreotti and Norguet, 0-complete is equivalent to Stein, so the n-completeness of Z implies that $H^{n+1}(Z, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on Z.) Let $C_1 \neq C_2$

two compact n-dimensional cycles in Z. Let $X = |C_1| \cup |C_2|$; it is easy to find $\omega \in H^n(X, \Omega^n_X)$ (X is compact n-dimensional) such that

$$\int_{C_1} \omega \neq \int_{C_2} \omega.$$

The long exact sequence of cohomology for

$$0 \to \mathcal{F} \to \Omega^n_Z \to \Omega^n_X \to 0$$

and the vanishing of $H^{n+1}(Z, \mathcal{F})$ give an $\Omega \in H^n(Z, \Omega^n_Z)$ inducing ω on X. Then the global holomorphic function F on $C_n(Z)$ defined by

$$F(C) = \int_C \Omega$$

satisfies $F(C_1) \neq F(C_2)$ and $C_n(Z)$ is holomorphically separable!

Proving the next theorem, which is an improvement of [Andreotti and Norguet 1967] and [Norguet and Siu 1977] obtained in [Barlet 1978a], requires much more work.

Theorem 4. *Let Z a strongly n-convex analytic space. Assume that the exceptional compact set* (*that is*, *the compact set where the exhaustion may fail to be n-convex*) *has a kählerian neighbourhood. Then $C_n(Z)$ is holomorphically convex.*

If Z is compact, Z is strongly n-convex but the conclusion may be false if Z is not Kähler; see [Barlet 1978a, Example 1].

3. Construction of Plurisubharmonic Functions on $C_n(Z)$

One way to pass directly from the n-convexity of Z to the 0-convexity of $C_n(Z)$ is to build up a strictly plurisubharmonic function on $C_n(Z)$ from the given n-convex exhaustion of Z. One important tool for that purpose is the following:

Theorem 5 see [Barlet 1978a, Theorem 3]. *Let Z be an analytic space and φ a real differential form on Z of class C^2 and type (n,n). Assume $i\partial\bar\partial\varphi \geq 0$ on Z and $i\partial\bar\partial\varphi \gg 0$ on the open set U* (*positivity is here in the sens of Lelong*; *it means positivity on totally decomposed vectors of $\Lambda^n T_Z$ for smooth Z*). *Then the function F_φ defined on $C_n(Z)$ by*

$$F_\varphi(C) = \int_C \varphi$$

is continuous and plurisubharmonic on $C_n(Z)$.

Moreover, *when each irreducible component of the cycle C_0 meets U*, *F_φ is strongly plurisubharmonic near C_0* (*that is*, *it stays plurisubharmonic after any small local C^2 perturbation*). ☐

The strong plurisubharmonic conclusion is sharp: such a property is not stable by base change, so the conclusion can only be true in the cycle space itself!

As a consequence, we obtained the following nice, but not very usefull, result:

THEOREM 6 [Barlet 1978a]. *If Z is a n-complete space, $C_n(Z)$ is a 0-complete space (i.e., Stein).* □

In fact it is possible to give some intermediate statement between Theorem 4 and Theorem 6 in order to obtain the following application:

THEOREM 7 [Barlet 1983]. *Let V be a compact connected Kähler manifold and let $F \to V$ a vector bundle on V such that*

(1) *F is a n-convex space, and*
(2) *through each point in F passes a compact n-dimension analytic subset of F.*

Then the algebraic dimension $a(V)$ of V (that is, the transcendance degree over $\mathbb{C}$ of the field of meromorphic function on V) satisfies $a(V) \geq \dim_{\mathbb{C}} V - n$. □

For $n = 0$ this reduce to a variant of Kodaira's projectivity theorem.

Note that if V is a compact Kähler manifold admitting a smooth fibration with n-dimensional fibers on a projective manifold X, say $f : V \to X$, we can choose $F = f^*L$ where L is a positive line bundle on X to satisfy the hypothesis in the previous theorem.

To give an idea of how the meromorphic functions on V are built, I merely indicate that, in an holomorphically convex space which is a proper modification of its Remmert reduction, any compact analytic subspace is Moišezon (this is a consequence of Hironaka's flattening theorem [1975]). But Theorem 5 gives a way to show that an irreducible component Γ of $C_n(Z)$ is a proper modification of its Remmert reduction: it is enough to have a plurisubharmonic function on Γ that is strongly plurisubharmonic at one point.

4. Construction of a Kähler Metric on $C_n(Z)$

As an illustration of the idea presented in the previous paragraph, I will explain the following beautifull result of J. Varouchas (see [Varouchas 1984], [Varouchas 1989] + [Barlet and Varouchas 1989]):

THEOREM 8. *If Z is a Kähler space, $C_n(Z)$ is also a Kähler space.* □

REMARK. Here "Kähler space" is being used in the strong sense: there exists an open covering $(U_\alpha)_{\alpha \in A}$ of Z and $\varphi_\alpha \in C^\infty(U_\alpha)$ such that φ_α is strongly plurisubharmonic and $\varphi_\alpha - \varphi_\beta = \mathrm{Re}(f_{\alpha\beta})$ on $U_\alpha \cap U_\beta$ with $f_{\alpha\beta} \in H^0(U_\alpha \cap U_\beta, \mathcal{O}_Z)$. An important fact, proved by J. Varouchas [1984] using Richberg's Lemma [1968], is that you obtain an equivalent definition by assuming that the φ_α are only continuous and strongly plurisubharmonic.

To give the idea of the construction, assume that Z is smooth and fix a compact n-cycle C_0 in Z. Let ω be the given Kähler form on Z. The first step is to explain that, in an open neighbourhood U of $|C_0|$ one can write $\omega^{\wedge n+1} = i\,\partial\bar{\partial}\alpha$, where α is a real C^∞ (n,n)-form on U. This is acheived by using the following result:

Theorem 9 [Barlet 1980a]. *Let Z a complex space and let C a n-dimensional compact analytic set in Z. Then C admits a basis of open neighbourhoods that are n-complete.* $\square$

The next step is to use the Theorem 5 to get the strict plurisubharmonicity of the continuous function $C \to \int_C \alpha$ using the strong Lelong positivity of $\omega^{\wedge n+1}$. The third step is then to prove that the difference of two such local strongly plurisubharmonic continuous functions on $C_n(Z)$ is the real part of an holomorphic function. This is delicate and uses the integration Theorem 3.

A very nice corollary of this result is the following theorem, which explains that Fujiki's class $\mathcal{C}$ (consisting of holomorphic images of compact complex Kähler manifolds: see [Fujiki 1980]) is the class of compact complex spaces which are bimeromorphic to compact Kähler manifolds.

Theorem 10 [Varouchas 1989]. *Let Z a compact connected Kähler manifold and let $\pi : Z \to X$ a surjective map on a complex space X. Then there exists a compact Kähler manifold W and a surjective modification $\tau : W \to X$.*

Proof. Sketch of proof Denote by n the dimension of the generic fiber of π and let $\Sigma \subset X$ a nowhere dense closed analytic subset such that

$$Z - \pi^{-1}(\Sigma) \to X - \Sigma$$

is n-equidimensional with $X - \Sigma$ smooth. By Theorem 2 the fibers of π restricted to $\big(Z - \pi^{-1}(\Sigma)\big)$ give an analytic family of n cycles of Z parametrized by $X - \Sigma$. So we have an holomorphic map $f : X - \Sigma \to C_n(Z)$.

In fact, f is meromorphic along Σ [Barlet 1980c]. Let $Y \subset X \times C_n(Z)$ the graph of this meromorphic map. Then $Y \to X$ is a surjective modification (along Σ) and $Y \to C_n(Z)$ is generically injective.

Let $\tilde{Y}$ the image of Y in $C_n(Z)$. This is a compact Kähler space (being a closed subspace of Kähler space) and we have a diagram of modifications:

$$\begin{tikzcd} & Y \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X & & \tilde{Y} \end{tikzcd}$$

Using [Hironaka 1964] we can find a projective modification $W \xrightarrow{\gamma} \tilde{Y}$ such that W is smooth (and Kähler because γ is projective and $\tilde{Y}$ is Kähler) with a commutative diagram

$$\begin{tikzcd}
& & W \arrow[dl, "\delta"'] \arrow[ddr, "\gamma"] & \\
& Y \arrow[dl, "\alpha"'] \arrow[drr, "\beta"] & & \\
X & & & \tilde{Y}
\end{tikzcd}$$

Then $\alpha \circ \delta$ is a modification and the theorem is proved! □

5. Higher Integration

Already in [Andreotti and Norguet 1967] there appears the idea of considering "higher integration" maps

$$\rho^{p,q} : H^{n+q}(Z, \Omega_Z^{n+p}) \to H^q\big(C_n(Z), \Omega^p_{C_n(Z)}\big). \tag{1}$$

For a family of compact n-cycles in a smooth Z parametrized by a smooth S, it is easy to deduce such a map from the usual direct image of currents and the Dolbeault–Grothendieck lemma.

First remark that the case $p = 0$ is a rather standard consequence (in full generality) of Theorem 3.

But it is clear that the case $p \geq 1$, $q = 0$ allows one to hope for a way to build up holomorphic p-forms on $C_n(Z)$. Some relationship between intermediate Jacobian of Z and Picard groups of components of $C_n(Z)$ looks very interesting!

But this is not so simple: M. Kaddar [1995] has shown that such a map $\rho^{p,0}$ does not exist in general for $p \geq 1$. But if one replaces the sheaf $\Omega^p_{C_n(Z)}$ by the sheaf $\omega^p_{C_n(Z)}$ of $\bar\partial$ closed $(p, 0)$ currents on $C_n(Z)$ (modulo torsion), the existence of

$$\rho^{p,q} : H^{n+q}(Z, \Omega_Z^{n+p}) \to H^q\big(C_n(Z), \omega^p_{C_n(Z)}\big). \tag{2}$$

is proved in the same reference.

For a reduced pure dimensional space X, the sheaf ω^p_X has been introduced in [Barlet 1978b]. It is a coherent sheaf, it satisfies the analytic extension property in codimension 2 and coincides with Grothendieck sheaf in maximal degree (which is the dualizing sheaf for X Gorenstein).

But, of course, something is lost in this higher integration process because we begin with $\Omega^{\cdot}_Z$ and we end with $\omega^{\cdot}_{C_n(Z)}$. Again M. Kaddar has given an example to show that the map (2) does not factorize by $H^{n+q}(Z, \omega_Z^{n+p})$.

The good point of view is to work with L_2 p-holomorphic forms (a meromorphic p-form on X is L_2 if and only if its pull back in a desingularization of X is holomorphic; this is independent on the chosen desingularization).

The sheaf L_2^p is again a coherent sheaf without torsion on any reduced space X and we have natural inclusions of coherent sheaves (for any $p \geq 0$)

$$\Omega^p_{X/\mathrm{torsion}} \hookrightarrow L_2^p \hookrightarrow \omega_X^p,$$

which coincide on the regular part of X.

Kaddar [1996b] also proved the following result:

THEOREM 11. *The higher integration map $\rho^{p,q}$ can be factorized through a natural map*

$$R^{p,q} : H^{n+q}(Z, L_2^{n+p}) \to H^q\big(C_n(Z), L_2^p\big). \qquad \square$$

The main difficulty in this "final version" of the higher integration map is to prove that the $L_2^{\cdot}$ holomorphic forms can be restricted to subspaces (in a natural way). Of course the bad case is when the subspace is included in the singular set of the ambient space. To handle this difficulty, the idea is again to use higher integration via the map (2), to define, at generic points first, the desired restriction from a suitable desingularization. Of course one has to show that it satisfies the L_2-condition that this does not depend on choices; then Kaddar shows that this construction has nice functorial properties.

6. An Application

In this section I shall present a famous conjecture of R. Hartshorne [1970] which is a typical problem where the reduction of convexity gives a nice strategy to solve the problem. Unfortunately, in the general case, it is not known how to build up a convenient family of compact cycles in order to reach the contradiction.

This is related to the following difficult problem:

PROBLEM. Let X a projective manifold and $A \subset X$ a compact submanifold of dimension $d \geq 2$ with an ample (or positive) normal bundle. Is it possible to find an irreducible analytic family of $(d-1)$-cycles in X which fills up X and such at least one member of the family is contained in A (as a set)?

Even for $\dim A = 2$ and $\dim X = 4$ I do not know if this is possible in general (though the particular case where $\dim X = 4$ and X is, in a neighbourhood of A, the normal bundle of the surface A follows from [Barlet et al. 1990, Theorem (1.1)].) The easy but interesting case I know is when X is an hypersurface of an homogeneous manifold W: it is enough to use the family $gA \cap X$ where $g \in \mathrm{Aut}_0(W)$.

Let me recall a transcendental variant of Hartshorne's conjecture:

CONJECTURE (H). *Let X be a compact connected Kähler manifold. Let A and B two compact submanifolds of X with positive normal bundles. Assume $\dim_{\mathbb{C}} A + \dim_{\mathbb{C}} B \geq \dim_{\mathbb{C}} X$. Then $A \cap B \neq \varnothing$.*

Before explaining the general strategy used in [Barlet 1987] (and [Barlet et al. 1990]), let me give a proof in the case where A is a curve and B is a divisor:

By the positivity of the normal bundle of B in X we know (from [Schneider 1973]) that $X - B$ is strongly 0-convex. So $H^0(X{-}B,\, \mathcal{O}_X)$ is an infinite-dimensional vector space. Using again [Schneider 1973] the positivity of the normal bundle of A in X we can find arbitrary small open neighbourhoods of A in X which are $(\dim X - 1)$-concave. This implies $\dim_{\mathbb{C}} H^0(\mathcal{U}, \mathcal{O}_X) < +\infty$ by [Andreotti and Grauert 1962] for any such $\mathcal{U}$. Assume now $A \cap B = \varnothing$ and choose $\mathcal{U} \subset X - B$. Now, by analytic continuation, the restriction map: $H^0(X{-}B,\, \mathcal{O}_X) \to H^0(\mathcal{U}, \mathcal{O}_X)$ is injective and this gives a contradiction.

The main idea to understand what is going on in this proof is to observe that a point can get out of A and go to reach B to make the analytic continuation.

In the general case, assume $\dim A + \dim B = \dim X$ (to simplify notations) and that we get an irreducible analytic family $(C_s)_{s\in S}$ of compact n-cycles in X such that

(1) $n = \dim_{\mathbb{C}} A - 1$,
(2) there exists $s_0 \in S$ such that $|C_{s_0}| \subset A$, and
(3) there exists $s_\infty \in S$ such that any component of $|C_{s_\infty}|$ meets B.

Then we argue along the same lines:

Assume $A \cap B = \varnothing$ and let $S_\infty = \{s \in S / |C_s| \cap B \neq \varnothing\}$. Then S_∞ is a nowhere dense, closed analytic subset of S.

Using [Schneider 1973] we get: $X - B$ is strongly n-convex; so integration of cohomology classes in $H^n(X{-}B,\, \Omega^n_X)$ will produce enough holomorphic functions on $S - S_\infty$ to separate points near infinity in the Remmert reduction of $S - S_\infty$.

There exists again an $\big(\dim X - (n{+}1)\big)$-concave open set $\mathcal{U} \supset A$ contained in $X - B$. Then by [Andreotti and Grauert 1962] we have $\dim_{\mathbb{C}} H^n(\mathcal{U}, \Omega^n_X) < +\infty$.

Now, because of the irreducibility of S, any holomorphic function on $S - S_\infty$ is uniquely determined by its restriction to the (nonempty) open set $V = \{s \in S - S_\infty / |C_s| \subset \mathcal{U}\}$. Now we have the following commutative diagram:

$$\begin{array}{ccc} H^n(X{-}B,\, \Omega^n_X) & \xrightarrow{\text{res}} & H^n(\mathcal{U}, \Omega^n_X) \\ \Big\downarrow{\scriptstyle\int} & & \Big\downarrow{\scriptstyle\int} \\ H^0(S - S_\infty, \mathcal{O}) & \xrightarrow{\text{res}} & H^0(V, \mathcal{O}). \end{array}$$

This does not yet give the contradiction.

To obtain one, we have to consider the family of cycles parametrized by $\mathrm{Sym}^k(S) \simeq S^k/\sigma_k$ defined by

$$(s_1 \dots s_k) \to \sum_{i=1}^{k} C_{s_i}.$$

When $k \to +\infty$ the dimension of the Remmert'reduction of $\mathrm{Sym}^k(S - S_\infty)$ goes to $+\infty$, but the dimension of $H^n(\mathcal{U}, \Omega_X^n)$ does not change and that gives the contradiction.

THEOREM 12 [Barlet 1987]. *Hartshorne's conjecture* (H) *is true for X a compact connected Kähler smooth hypersurface of an homogeneous complex manifold.* $\square$

We now discuss how to algebrize this strategy in order to reach the initial formulation of R. Harshorne.

CONJECTURE (H). *Let X a smooth projective compact connected variety and let A and B two submanifolds with ample normal bundles such that*

$$\dim_{\mathbb{C}} A + \dim_{\mathbb{C}} B \geq \dim_{\mathbb{C}} X.$$

Then $A \cap B \neq \varnothing$.

Now the ampleness assumption does not imply the positivity (it is not yet known if ampleness implies positivity for rank ≥ 2) and so the convexity and concavity fail in the previous proof. The concavity part will be replaced by the following theorem:

THEOREM 13 [Barlet et al. 1994]. *Let X a complex manifold and $(C_s)_{s\in S}$ an analytic family of n-cycles. Fix $s_0 \in S$ and let $|C_{s_0}| = Y$. Then there exists an increasing sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \alpha_k = +\infty$ and if $\omega \in H^n(X, \Omega_X^n)$ is in the kernel of the restriction map*

$$H^n(X, \Omega_X^n) \to H^n(Y, \Omega_X^n / \mathcal{I}_Y^k \Omega_X^n)$$

then $\rho(\omega)_{s_0} \in \mathcal{M}_{S,s_0}^{\alpha_k}$, where $\mathcal{M}_{S,s_0}$ is the maximal ideal of $\mathcal{O}_{S,s_0}$. $\square$

Here $\mathcal{I}_Y$ is the defining ideal sheaf of Y and $\rho(\omega)(s) = \int_{C_s} \omega$.

So this shows that the $(\alpha_k - 1)$-jet at s_0 of the function $s \to \int_{C_s} \omega$ is determined by the restriction of ω in $H^n(Y, \Omega_X^n / \mathcal{I}_Y^k \Omega_X^n)$.

Now if $Y \subset A$, where A is a submanifold with an ample normal bundle, the previous restriction will factorize through $H^n(A, \Omega_X^n / \mathcal{I}_A^k \Omega_X^n)$; and the ampleness of $N_{A/X}$ gives the fact that this vector space stabilizes for $k \gg 1$.

This shows that the image of

$$\tilde{\rho} : H^n(X, \Omega_X^n) \to \mathcal{O}_{S,s_0}$$

is finite-dimensional. Again the irreducibility of S allows to conclude that $\rho\big(H^n(X, \Omega_X^n)\big) \subset H^0(S, \mathcal{O}_S)$ is finite-dimensional.

7. Construction of Meromorphic Functions on $C_n(X)$

To replace the convexity part we introduce a new idea:

Let B be a smooth manifold of codimension $n+1$ in X and consider the algebraic analogue of the exact sequence

$$\cdots \to H^n(X, \Omega_X^n) \to H^n(X{-}B,\, \Omega_X^n) \to H_B^{n+1}(X, \Omega_X^n) \to \cdots$$

denoted by

$$\cdots \to H^n(X, \Omega_X^n) \to\ H_{\mathrm{alg}}^n(X{-}B,\, \Omega_X^n) \to H_{[B]}^{n+1}(X, \Omega_X^n) \to \cdots.$$

So $H_{\mathrm{alg}}^n(X{-}B,\, \Omega_X^n)$ is the subspace of $H^n(X{-}B,\, \Omega_X^n)$ of cohomology classes having a meromorphic singularity along B,

$$\big(H_{[B]}^{n+1}(X, \Omega_X^n) := \varinjlim_k \mathrm{Ext}^{n+1}(\mathcal{O}_X/\mathcal{I}_B^k, \Omega_X^n)\big).$$

Let $(C_s)_{s\in S}$ be a family of compact n-cycles in X and set

$$S_\infty = \{s \in S/|C_s| \cap B \neq \varnothing\}.$$

We want to investigate the behaviour of the function $s \to \int_{C_s} \omega$ when $s \to S_\infty$ assuming that

$$\omega \in H_{\mathrm{alg}}^n(X{-}B,\, \Omega_X^n).$$

This question is solved in [Barlet and Magnusson 1998] in a rather general context. Here I give a simpler statement.

Theorem 14 [Barlet and Magnusson 1998]. *Let X be a complex manifold and B a submanifold of codimension $n+1$. Let $(C_s)_{s\in S}$ an analytic family of compact n-cycles in X parametrized by the reduced space S. Let*

$$|G| = \{(s, x) \in S \times X/x \in |C_s|\}$$

be the graph of the family and denote by p_1 and p_2 the projection of $|G|$ on S and X respectively. Assume that $p_2^*(B)$ is proper over S by p_1 and denote by $\Sigma = (p_1)_* p_2^*(B)$ as a complex subspace of S. Then there exists a natural integration map

$$\sigma : H_B^{n+1}(X, \Omega_X^n) \to H_\Sigma^1(S, \mathcal{O}_S)$$

which induces a filtered map

$$\sigma : H_{[B]}^{n+1}(X, \Omega_X^n) \to H_{[\Sigma]}^1(S, \mathcal{O}_S)$$

compatible with the usual integration map, so that the following diagram is commutative:

$$
\begin{tikzcd}
H^n_{\mathrm{alg}}(X-B,\, \Omega^n_X) \arrow[rr] \arrow[dr] \arrow[dd, "\rho"'] & & H^{n+1}_{[B]}(X, \Omega^n_X) \arrow[dr] \arrow[dd, "\sigma"] & \\
& H^n(X-B,\, \Omega^n_X) \arrow[rr, "\mathrm{res}"] \arrow[dd, "\rho"'] & & H^{n+1}_B(X, \Omega^n_X) \arrow[dd, "\sigma"] \\
H^0_{\mathrm{alg}}(S-\Sigma,\, \mathcal{O}_S) \arrow[rr] \arrow[dr] & & H^1_{[\Sigma]}(S, \mathcal{O}_S) \arrow[dr] & \\
& H^0(S-\Sigma,\, \mathcal{O}_S) \arrow[rr, "\mathrm{res}"] & & H^1_\Sigma(S, \mathcal{O}_S)
\end{tikzcd}
$$

This map σ has a nice functorial behaviour and can be sheafified in a filtered sheaf map

$$(p_1)_* p_2^* \big(\underline{H}^{n+1}_{[B]}(\Omega^n_X) \big) \to \underline{H}^1_{[\Sigma]}(\mathcal{O}_S). \qquad \square$$

Remarks. (1) This result asserts that a cohomology class ω in $H^n_{\mathrm{alg}}(X-B,\, \Omega^n_X)$ having a pole of order $\le q$ along B $\big($i.e., $\mathcal{I}^q_B.\omega$ is locally zero in $\underline{H}^{n+1}_{[B]}(\Omega^n_X)\big)$ will give a meromorphic function on S with an order $\le q$ pole along Σ $\big($where m has an order $\le q$ pole along Σ means that $\mathcal{I}^q_\Sigma.m$ is locally zero in $H^1_{[\Sigma]}(\mathcal{O}_S)\big)$. Of course the ideal $\mathcal{I}_\Sigma$ is associated to $(p_1)_* p_2^*(B) = \Sigma$ as a subspace of S (B is reduced).

(2) The compactness of cycles is not important in the previous result, but of course keeping the assumption that $p_2^* B$ is proper over S.

In the noncompact case, we have to add a family of support on X in order to have an integration map $\rho : H^n_\Phi(X-B,\, \Omega^n_X) \to H^0(S-\Sigma, \mathcal{O}_S)$ where $F \in \Phi$ if F is closed in X and F is proper over S; so $B \in \Phi$ and we have again compatibility between ρ and σ via the commutative diagram

$$
\begin{tikzcd}
H^n_\Phi(X-B,\, \Omega^n_X) \arrow[r, "\rho"] \arrow[d, "\mathrm{res}"] & H^0(S-\Sigma, \mathcal{O}_S) \arrow[d, "\mathrm{res}"] \\
H^{n+1}_B(X, \Omega^n_X) \arrow[r, "\sigma"] & H^1_\Sigma(S, \mathcal{O}_S)
\end{tikzcd}
$$

Our next result is to show that, in fact, the closed analytic subset $|\Sigma|$ can be endowed with a natural "Cartier divisor" structure in S (remember that S is any reduced space; or to say that in an other way: we have no control on the singularities of $C_n(X)$!).

Theorem 15 [Barlet and Magnusson 1998]. *Let Z be a complex manifold and $Y \subset Z$ a closed analytic subspace of Z of codimension $n+1$ which is locally a complete intersection in Z. Let $(C_s)_{s\in S}$ be an analytic family of n-cycles in Z (not necessarily compact) such that we have the following property*:

Let $|G| \subset X \times Z$ the graph of the family $(C_s)_{s\in S}$ and denote by p_1 and p_2 the projections of $|G|$ on S and Z respectively. Assume that

$$p_1 : p_2^{-1}(|Y|) \to S$$

is finite. Then there exists a natural Cartier divisor structure Σ_Y on the closed analytic set $|\Sigma| = p_1(p_2^{-1}|Y|)$, that is, a locally principal $\mathcal{I}_{\Sigma_Y}$ ideal of $\mathcal{O}_S$ defining $|\Sigma|$.

This Cartier structure is characterized by the following properties:

(1) *Let $s_0 \in |\Sigma|$ and set $|C_{s_0}| \cap |Y| = \{y_1, \dots, y_l\}$, where $y_i \neq y_j$ when $i \neq j$. Let $U_1 \dots U_l$ be disjoint open sets in Z such that $y_j \in U_i$ for $j \in [1, l]$. Let $\mathcal{I}_j$ be the Cartier structure on $|\Sigma_j| = p_1\big(p_2^{-1}(|Y| \cap U_j)\big)$ near s_0. Then $\mathcal{I}_{\Sigma_Y} \prod_{j=1}^{l} \mathcal{I}_j$ near s_0.*

(2) *Let $s_0 \in |\Sigma|$ and let U be an open set in Z such that $|C_{s_0}| \cap |Y| \subset U$ and such that $\mathcal{I}_Y = \pi^*(m_{\mathbb{C}^{n+1}}, 0)$ where $\pi : U \to \mathbb{C}^{n+1}$ is an holomorphic flat map (that is, $\pi := (z_0 \dots z_n)$ where $z_0 \dots z_n$ is a generator of $\mathcal{I}_Y$ on U). Then a generator of $\mathcal{I}_{\Sigma_Y}$ near s_0 is given by the holomorphic function*

$$s \to N(z_0)\big(C_s \cap (z_1 = \dots = z_n = 0)\big),$$

where $C_s \cap (z_1 = \dots = z_n = 0)$ in $\operatorname{Sym}^k Z$ is defined via [Barlet 1975, Theorem 6 (local)] *and where $N(z_0) : \operatorname{Sym}^k U \to \mathbb{C}$ is the norm of the holomorphic function $z_0 : U \to \mathbb{C}$ (so $N(z_0)(x_1 \dots x_k) = \prod_{j=1}^{k} z_0(x_j)$).*

(3) *The construction of Σ_Y is compatible with base change*: *so if $\tau : T \to S$ is holomorphic (with T reduced) the Cartier structure associated to the family $(C_{\tau(t)})_{t\in T}$ and $Y \subset Z$ is the Cartier divisor $\tau^*(\Sigma_Y)$ in T.*

Now the filtration by the order of poles in the Theorem 14 is related to the Cartier divisor structure of Theorem 15 by this result:

PROPOSITION. *In the situation of the Theorem* 15 *we have $(p_1)_*\big(p_2^*(Y)\big)$ is a subspace of Σ_Y. So the morphism of Theorem* 14 *gives a filtered sheaf map*

$$(\rho_1)_*\big(\underline{H}^{n+1}_{[p_2^* Y]}(p_2^* \Omega^n_Z)\big) \to \underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S).$$

This means that the order of poles along $|\Sigma|$ for meromorphic function on S, obtained by integration, is now defined by the "natural" equation of $|\Sigma|$ given by the Theorem 15.

REMARK. Let $Z = \mathbb{P}_N(\mathbb{C})$ and Y be any codimension $n+1$ cycle in Z (in this special case we don't need a local complete intersection subspace!); take S to be the Grassmann manifold of n-planes of $\mathbb{P}_N(\mathbb{C})$. Then we find here Chow and Van der Waerden construction, the Cartier divisor on S gives, in a Plücker embedding, the Cayley form of the cycle Y.

Now we come to the main result in [Barlet and Magnusson 1999], which asserts that, assuming moreover that Z is n-convex and that Y is a compact submanifold of codimension $n+1$ with ample normal bundle, the line bundle associated to the Cartier divisor Σ_Y of S is ample near Σ when the family $(C_s)_{s\in S}$ is sufficiently nice. So it transfers in this case the ampleness from $N_{Y/Z}$ to $N_{\Sigma/S}$.

THEOREM 16 [Barlet and Magnusson 1999]. *Let Z be a complex manifold which is n-convex. Let $(Y, \mathcal{I}_Y)$ a compact subspace of Z which is locally a complete intersection of codimension $n+1$ in Z and such that $N_{Y/Z}$ is ample. Let S a reduced analytic space and $(C_s)_{s\in S}$ an analytic family of cycles and let $|G| \subset S\times Z$ the graph of this family, with projections p_1 and p_2 on S and Z respectively. Assume that*

(1) *$p_1 : p_2^{-1}(|Y|) \to S$ is proper and injective;*
(2) *for all $s \in |\Sigma|$, C_s and Y are smooth and transverse at z_s, where $z_s = C_s \cap Y$; and*
(3) *there exists a closed analytic set $\Theta \subset |\Sigma| \times |\Sigma|$, symmetric, finite on $|\Sigma|$, and such that for all $(s, s') \notin \Theta$, we have either $z_s = z_{s'}$ or $T_{C_s,z_s} \neq T_{C_{s'},z_s}$, where $T_{C,z}$ is the tangent space to C at z (C has to be smooth at z!), when $z_s = z_{s'}$.*

Then, denoting by Σ_Y the Cartier divisor structure on $|\Sigma| = p_1\big(p_2^{-1}(|Y|)\big)$ given by Theorem 15, the line bundle $[\Sigma_Y]$ is locally ample in S, that is, there exists $\nu \in \mathbb{N}$ such that $E_\nu = H^0(S, [\Sigma_Y]^\nu)$ gives an holomorphic map $S \to \mathbb{P}(E_\nu^*)$, finite in a neighbourhood of Σ. □

To conclude, I will quote Kaddar's application [1996a] of his construction of a relative fundamental class in Deligne cohomology for an analytic family of cycles in a complex manifold Z. Using this class, he can associate to a codimension $n+1$ cycle Y in Z a line bundle on S (the parameter space) by integration at the level of Deligne cohomology. Moreover he proved that this gives, say in a projective setting, an holomorphic map from cycles of codimension $n+1$ to the Picard group of the cycle space of n-cycles. This was a first motivation for me to prove Theorem 15 which produces also in a rather wide context a Cartier divisor on S, and so a line bundle.

The idea that a result such as Theorem 16 was possible goes back to F. Campana's work [1980; 1981] on algebraicity of the cycle space. He notices that for a compact analytic subset S of the cycle space, the analytic subset Σ of these cycles which meet a given Moišezon subspace Y in Z is again Moišezon. So this transfer of algebraicity is a rough basis for Theorem16 where we transfer the ampleness of the normal bundle of Y in Z to the ampleness of the normal bundle of Σ in S. This of course gives not only algebraicity on Σ but also information on S: we build up enough meromorphic functions on S to prove that S is Moišezon (when S is compact) in fact with a weaker assumption than stated here (see the weak version of Theorem 16 in [Barlet and Magnusson 1999]) and we also describe the line bundle which gives meromorphic functions on S.

Acknowledgments

I thank MSRI for its hospitality and the organizers of the Special Year in Several Complex Variables for inviting me. I also thank Silvio Levy for his remarkable work in preparing this article for publication.

The commutative diagrams in this article were made using Paul Taylor's diagrams.sty package.

References

[Andreotti and Grauert 1962] A. Andreotti and H. Grauert, "Théorème de finitude pour la cohomologie des espaces complexes", *Bull. Soc. Math. France* **90** (1962), 193–259.

[Andreotti and Norguet 1967] A. Andreotti and F. Norguet, "La convexité holomorphe dans l'espace analytique des cycles d'une variété algébrique", *Ann. Scuola Norm. Sup. Pisa* (3) **21** (1967), 31–82.

[Barlet 1975] D. Barlet, "Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie", pp. 1–158 in *Fonctions de plusieurs variables complexes II*, edited by F. Norguet, Lecture Notes in Math. **482**, Springer, Berlin, 1975.

[Barlet 1978a] D. Barlet, "Convexité de l'espace des cycles", *Bull. Soc. Math. France* **106**:4 (1978), 373–397.

[Barlet 1978b] D. Barlet, "Le faisceau $\omega_X^{\cdot}$ sur un espace analytique X de dimension pure", pp. 187–204 in *Fonctions de plusieurs variables complexes III*, edited by F. Norguet, Lecture Notes in Math. **670**, Springer, Berlin, 1978.

[Barlet 1980a] D. Barlet, "Convexité au voisinage d'un cycle", pp. 102–121 in *Fonctions de plusieurs variables complexes IV*, edited by F. Norguet, Lecture Notes in Math. **807**, Springer, Berlin, 1980.

[Barlet 1980b] D. Barlet, "Familles analytiques de cycles et classes fondamentales relatives", pp. 1–24 in *Fonctions de plusieurs variables complexes IV*, edited by F. Norguet, Lecture Notes in Math. **807**, Springer, Berlin, 1980.

[Barlet 1980c] D. Barlet, "Majoration du volume des fibres génériques et forme géométrique du théorème d'aplatissement", pp. 1–17 in *Séminaire Pierre Lelong — Henri Skoda* (*Analyse*) 1978/79, edited by P. Lelong and H. Skoda, Lecture Notes in Math. **822**, Springer, Berlin, 1980.

[Barlet 1983] D. Barlet, "Un théorème d'algébricité intermédiaire", *Invent. Math.* **71**:3 (1983), 655–660.

[Barlet 1987] D. Barlet, "À propos d'une conjecture de R. Hartshorne", *J. Reine Angew. Math.* **374** (1987), 214–220.

[Barlet and Magnusson 1998] D. Barlet and J. Magnusson, "Intégration de classes de cohomologie méromorphes et diviseurs d'incidence", *Ann. Sci. École Norm. Sup.* (4) **31**:6 (1998), 811–842.

[Barlet and Magnusson 1999] D. Barlet and J. Magnusson, "Transfert de l'amplitude du fibré normal au diviseur d'incidence", *J. reine angew. Math.* **513** (1999). to appear.

[Barlet and Varouchas 1989] D. Barlet and J. Varouchas, "Fonctions holomorphes sur l'espace des cycles", *Bull. Soc. Math. France* **117**:3 (1989), 327–341.

[Barlet et al. 1990] D. Barlet, T. Peternell, and M. Schneider, "On two conjectures of Hartshorne's", *Math. Ann.* **286**:1-3 (1990), 13–25.

[Barlet et al. 1994] D. Barlet, L. Doustaing, and J. Magnússon, "La conjecture de R. Hartshorne pour les hypersurfaces lisses de $\mathbf{P}^n$", *J. Reine Angew. Math.* **457** (1994), 189–202.

[Bishop 1964] E. Bishop, "Conditions for the analyticity of certain sets", *Michigan Math. J.* **11** (1964), 289–304.

[Campana 1980] F. Campana, "Algébricité et compacité dans l'espace des cycles d'un espace analytique complexe", *Math. Ann.* **251**:1 (1980), 7–18.

[Campana 1981] F. Campana, "Coréduction algébrique d'un espace analytique faiblement kählérien compact", *Invent. Math.* **63**:2 (1981), 187–223.

[Fujiki 1978] A. Fujiki, "Closedness of the Douady spaces of compact Kähler spaces", *Publ. Res. Inst. Math. Sci.* **14**:1 (1978), 1–52.

[Fujiki 1980] A. Fujiki, "Some results in the classification theory of compact complex manifolds in $\mathcal{C}$", *Proc. Japan Acad. Ser. A Math. Sci.* **56**:7 (1980), 324–327.

[Hartshorne 1970] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Math. **156**, Springer, Berlin, 1970. Notes written in collaboration with C. Musili.

[Hironaka 1964] H. Hironaka, "Resolution of singularities of an algebraic variety over a field of characteristic zero", *Ann. of Math.* (2) **79** (1964), 109–203, 205–326.

[Hironaka 1975] H. Hironaka, "Flattening theorem in complex-analytic geometry", *Amer. J. Math.* **97** (1975), 503–547.

[Kaddar 1995] M. Kaddar, "Intégration partielle sur les cycles", *C. R. Acad. Sci. Paris Sér. I Math.* **320**:12 (1995), 1513–1516.

[Kaddar 1996a] M. Kaddar, "Classe fondamentale relative en cohomologie de Deligne et application", *Math. Ann.* **306**:2 (1996), 285–322.

[Kaddar 1996b] M. Kaddar, "Intégration sur les cycles et formes de type L^2", *C. R. Acad. Sci. Paris Sér. I Math.* **322**:7 (1996), 663–668.

[Lieberman 1978] D. I. Lieberman, "Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds", pp. 140–186 in *Fonctions de plusieurs variables complexes III*, Lecture Notes in Math. **670**, Springer, Berlin, 1978.

[Norguet and Siu 1977] F. Norguet and Y. T. Siu, "Holomorphic convexity of spaces of analytic cycles", *Bull. Soc. Math. France* **105**:2 (1977), 191–223.

[Richberg 1968] R. Richberg, "Stetige streng pseudokonvexe Funktionen", *Math. Ann.* **175** (1968), 257–286.

[Schneider 1973] M. Schneider, "Über eine Vermutung von Hartshorne", *Math. Ann.* **201** (1973), 221–229.

[SGAN 1982] *Séminaire: Géométrie analytique* (2$^{\text{eme}}$ partie), Rev. Inst. Élie Cartan **5**, 1982.

[Varouchas 1984] J. Varouchas, “Stabilité de la classe des variétés kählériennes par certains morphismes propres”, *Invent. Math.* **77**:1 (1984), 117–127.

[Varouchas 1989] J. Varouchas, “Kähler spaces and proper open morphisms”, *Math. Ann.* **283**:1 (1989), 13–52.

Daniel Barlet
Institut Élie Cartan (Nancy)
CNRS/INRIA/UHP, UMR 7502
Université Henri Poincaré (Nancy I) et Institut Universitaire de France
Faculté des Sciences, B.P. 239
54506 Vandoeuvre-les-Nancy Cedex
France
barlet@iecn.u-nancy.fr