Several Complex Variables
MSRI Publications
Volume **37**, 1999

Resolution of Singularities

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Abstract. This article is an exposition of our algorithm for canonical resolution of singularities in characteristic zero (*Invent. Math.* **128** (1997), 207–302), with an essentially complete proof of the theorem in the hypersurface case. We define a local invariant for desingularization whose values are finite sequences that can be compared lexicographically. Our invariant takes only finitely many maximum values (at least locally), and we get an algorithm for canonical desingularization by successively blowing up its maximum loci. The invariant can be described by a local construction that provides equations for the centres of blowing up. Our construction is presented here in parallel with a worked example.

1. Introduction

Resolution of singularities has a long history that goes back to Newton in the case of plane curves. For higher-dimensional singular spaces, the problem was formulated toward the end of the last century, and it was solved in general, for algebraic varieties defined over fields of characteristic zero, by Hironaka in his famous paper [1964]. (That paper includes the case of real-analytic spaces; Hironaka's theorem for complex-analytic spaces is proved in [Hironaka 1974; Aroca et al. 1975; 1977].) But Hironaka's result is highly non-constructive. His proof is one of the longest and hardest in mathematics, and it seems fair to say that only a handful of mathematicians have fully understood it. We are not among them! Resolution of singularities is used in many areas of mathematics, but even certain aspects of the theorem (for example, *canonicity:* see 1.11 below) have remained unclear.

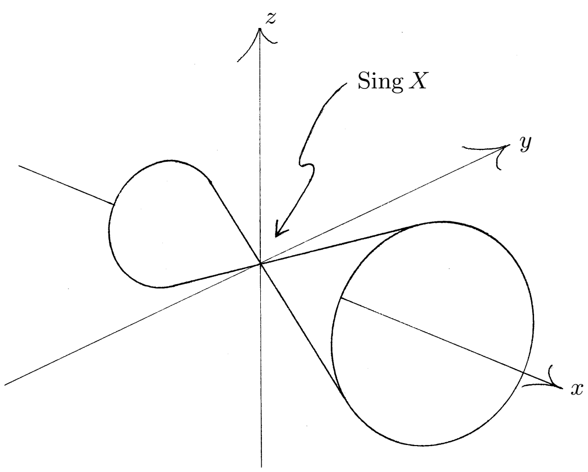
This article is an exposition of an elementary constructive proof of canonical resolution of singularities in characteristic zero. Our proof was sketched in the hypersurface case in [Bierstone and Milman 1991] and is presented in detail in [Bierstone and Milman 1997].

When we started thinking about the subject almost twenty years ago, our aim was simply to understand resolution of singularities. But we soon became convinced that it should be possible to give simple direct proofs of at least those aspects of the theorem that are important in analysis. In [Bierstone and Milman 1988], for example, we published a very simple proof that any real-analytic variety is the image by a proper analytic mapping of a manifold of the same dimension. The latter statement is a real version of a local form of resolution of singularities, called *local uniformization.*

It is the idea of [Bierstone and Milman 1988, Section 4] that we have developed (via [Bierstone and Milman 1989]) to define a new local invariant for desingularization that is the main subject of this exposition. Our invariant $\mathrm{inv}_X(a)$ is a finite sequence (of nonnegative rational numbers and perhaps ∞, in the case of a hypersurface), defined at each point a of our space X. Such sequences can be compared lexicographically. $\mathrm{inv}_X(\,\cdot\,)$ takes only finitely many maximum values (at least locally) and we get an algorithm for canonical resolution of singularities by successively blowing up its maximum loci. Moreover, $\mathrm{inv}_X(\,\cdot\,)$ can be described by local computations that provide equations for the centres of blowing up.

We begin with an example to illustrate the meaning of resolution of singularities:

EXAMPLE 1.1. Let X denote the quadratic cone $x^2 - y^2 - z^2 = 0$ in affine 3-space — the simplest example of a singular surface.



$$X: \; x^2 - y^2 - z^2 = 0$$

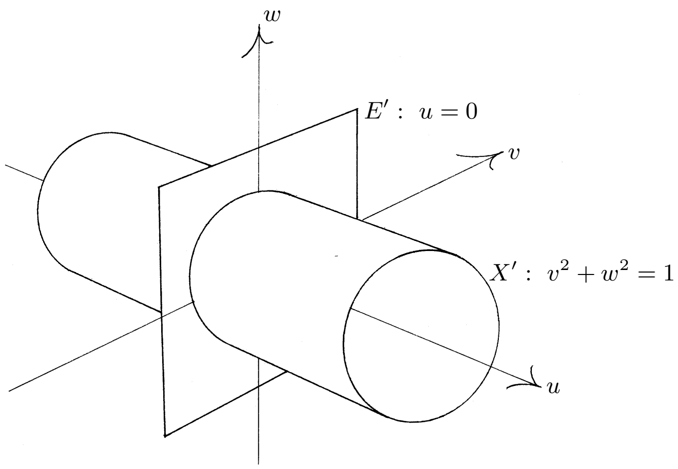
X can be desingularized by making a simple quadratic transformation of the ambient space:

$$\sigma: \quad x = u, \; y = uv, \; z = uw.$$

The inverse image of X by this mapping σ is given by substituting the formulas for x, y and z into the equation of X:

$$\sigma^{-1}(X): \quad u^2(1 - v^2 - w^2) = 0.$$

Thus $\sigma^{-1}(X)$ has two components: The plane $u = 0$ is the set of critical points of the mapping σ; it is called the *exceptional hypersurface.* (Here $E' := \{u = 0\}$ is the inverse image of the singular point of X.) The quotient after completely factoring out the "exceptional divisor" u defines what is called the *strict transform* X' of X by σ. Here X' is the cylinder $v^2 + w^2 = 1$.



In this example, $\sigma|X'$ is a *resolution of singularities* of X: X' is smooth and $\sigma|X'$ is a proper mapping onto X that is an isomorphism outside the singularity. But the example illustrates a stronger statement, called *embedded resolution of singularities*: X is desingularized by making a simple transformation of the ambient space, after which, in addition, the strict transform X' and the exceptional hypersurface E' have only *normal crossings*; this means that each point admits a coordinate neighbourhood with respect to which both X' and E' are coordinate subspaces.

The quadratic transformation σ in Example 1.1 is also called a *blowing-up* with *centre* the origin. (The centre is the set of critical values of σ.) More accurately, the blowing-up of affine 3-space with centre a point is covered in a natural way by three affine coordinate charts, and σ above is the formula for the blowing-up restricted to one chart.

Sequences of quadratic transformations, or point blowings-up, were first used to resolve the singularities of curves by Max Noether in the 1870's [Brill and Noether 1892–93].

The more general statement of "embedded resolution of singularities" seems to have been formulated precisely first by Hironaka. But it is implicit already in the earliest rigorous proofs of local desingularization of surfaces, as a natural generalization prerequisite to the inductive step of a proof by induction on dimension (compare Sections 2 and 3 below). For example, in one of the earliest proofs of local desingularization or uniformization of surfaces, Jung used embedded desingularization of curves by sequences of quadratic transformations (applied to the branch locus of a suitable projection) to prove uniformization for surfaces [Jung 1908]. Similar ideas were used in the first proofs of global resolution of singularities of algebraic surfaces, by Walker [1935] and Zariski [1939]. (The latter was the first algebraic proof, by sequences of normalizations and point blowings-up.)

From the point of view of subsequent work, however, Zariski's breakthrough came in [1943], where he localized the idea of the centre of blowing-up, thus making possible an extension of the notion of quadratic transformation to blowings-up with centres that are not necessarily 0-dimensional. This led him to a version of embedded resolution of singularities of surfaces, and to a weaker (non-embedded) theorem for 3-dimensional algebraic varieties [Zariski 1944]. It was the path that led to Hironaka's great theorem and to most subsequent work in the area, including our own. Among the references not otherwise cited in this article we mention [Abhyankar 1966; 1982; 1988; Bierstone and Milman 1990; Giraud 1974; Hironaka 1977; Lipman 1975; Moh 1992; Villamayor 1989; 1992; Youssin 1990]. (Added in proof: [Encinas and Villamayor 1998].)

From a general viewpoint, some important features of our work in comparison with previous treatments are: (1) It is canonical (see 1.11). (2) We isolate simple local properties of an invariant (Section 4, Theorem B) from which global desingularization is automatic. (3) Our proof in the case of a hypersurface (a space defined locally by a single equation) does not involve passing to higher codimension, as does the inductive procedure of [Hironaka 1964].

Very significant results on resolution of singularities over fields of nonzero characteristic have recently been obtained by de Jong [1996] and have been announced by Spivakovsky.

1.2. Blowing up. We first describe the blowing-up of an open subset W of r-dimensional affine space with centre a point a. (Say $a = 0 \in W$.) The *blowing-up* σ with *centre* 0 is the projection onto W of a space W' that is obtained by replacing the origin by the $(r-1)$-dimensional projective space $\mathbb{P}^{r-1}$ of all lines through 0:

$$W' = \{(x,\lambda) \in W \times \mathbb{P}^{r-1} : x \in \lambda\}$$

and $\sigma\colon W' \to W$ is defined by $\sigma(x,\lambda) = x$. (Outside the origin, a point x belongs to a unique line λ, but $\sigma^{-1}(0) = \mathbb{P}^{r-1}$. Clearly, σ is a proper mapping.) W' has a natural algebraic structure: If we write x in terms of the affine coordinates $x = (x_1, \dots, x_r)$, and λ in the corresponding homogeneous coordinates $\lambda = [\lambda_1, \dots, \lambda_r]$, then the relation $x \in \lambda$ translates into the system of equations $x_i\lambda_j = x_j\lambda_i$, for all i, j.

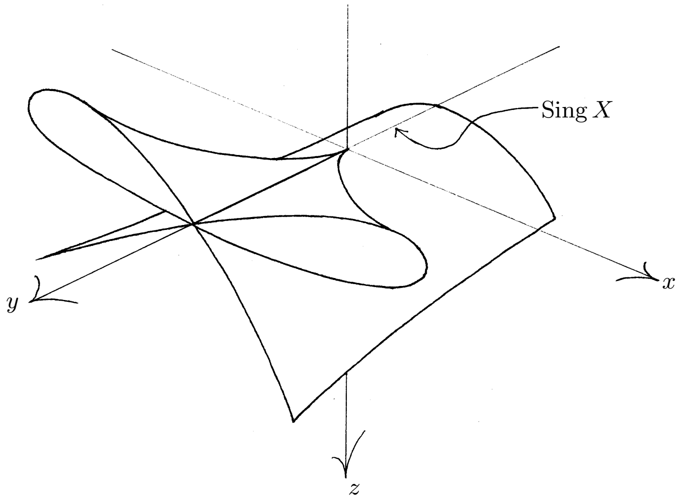
These equations can be used to see that W' has the structure of an algebraic manifold: For each $i = 1, \dots, r$, let W_i' denote the open subset of W' where $\lambda_i \neq 0$. In W_i' we have $x_j = x_i\lambda_j/\lambda_i$, for each $j \neq i$, so we see that W_i' is smooth: it is the graph of a mapping in terms of coordinates $(y_1, \dots, y_r)$ for W_i' defined by $y_i = x_i$ and $y_j = \lambda_j/\lambda_i$ if $j \neq i$. In these coordinates, σ is a quadratic transformation given by the formulas

$$x_i = y_i, \qquad x_j = y_i y_j \text{ for all } j \neq i,$$

as in Example 1.1.

Once blowing up with centre a point has been described as above, it is a simple matter to extend the idea to blowing up a manifold, or smooth space, M with centre an arbitrary smooth closed subspace C of M: Each point of C has a product coordinate neighbourhood $V \times W$ in which $C = V \times \{0\}$; over this neighbourhood, the blowing-up with centre C identifies with $\mathrm{id}_V \times \sigma$: $V \times W' \to V \times W$, where id_V is the identity mapping of V and σ: $W' \to W$ is the blowing-up of W with centre $\{0\}$. The blowing-up $M' \to M$ with centre C is an isomorphism over $M \setminus C$. The preceding conditions determine $M' \to M$ uniquely, up to an isomorphism of M' commuting with the projections to M.

Example 1.3. Let X denote the surface $z^3 - x^2yz - x^4 = 0$.



This surface is particularly interesting in the real case because, as a *subset* of $\mathbb{R}^3$, it is singular only along the nonnegative y-axis. But resolution of singularities is an algebraic process: it applies to spaces that include a functional structure (given here by the equation for X). As a *subspace* of $\mathbb{R}^3$, X is singular along the entire y-axis.

In general, for a hypersurface X — defined locally, say, by an equation $f(x) = 0$ — to say that a point a is *singular* means there are no linear terms in the Taylor expansion of f at a; in other words, the order $\mu_a(f) > 1$. (The *order* or *multiplicity* $\mu_a(f)$ of f at a is the degree of the lowest-order homogeneous part of the Taylor expansion of f at a. We will also call $\mu_a(f)$ the *order* $\nu_{X,a}$ of the hypersurface X at a.)

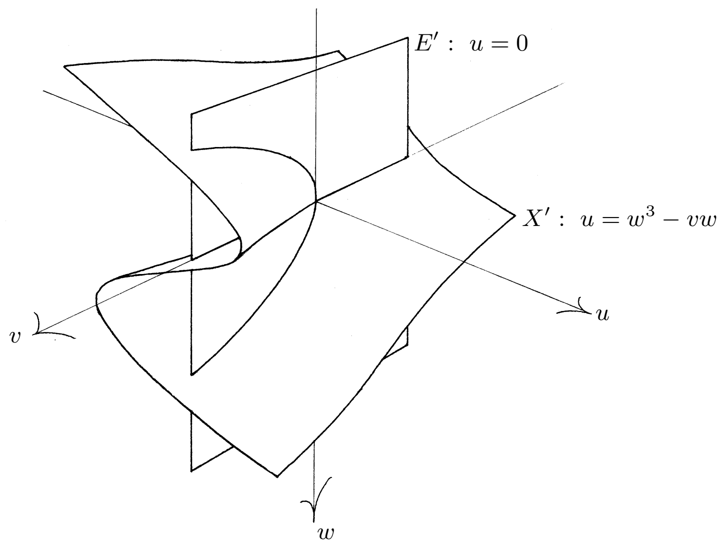
The general philosophy of our approach to desingularization — going back to [Zariski 1944] — is to blow up with smooth centre as large as possible inside the locus of the most singular points. In our example here, X has order 3 at each point of the y-axis. In general, order is not a delicate enough invariant to determine a centre of blowing-up for resolution of singularities, even in the hypersurface case. (We will refine order in our definition of inv_X.) But here let us take the blowing-up σ with centre the y-axis:

$$\sigma: \quad x = u, \; y = v, \; z = uw.$$

(Again, this is the formula for blowing up in one of two coordinate charts required to cover our space. But the strict transform of X in fact lies completely within this chart.) The inverse image of X is

$$\sigma^{-1}(X): \quad u^3(w^3 - vw - u) = 0;$$

$\{u = 0\}$ is the exceptional hypersurface E' (the inverse image of the centre of blowing up) and the strict transform X' is smooth. (It is the graph of a function $u = w^3 - vw$.)



X' is a desingularization of X, but we have not yet achieved an embedded resolution of singularities because X' and E' do not have normal crossings at the origin. Further blowings-up are needed for embedded resolution of singularities.

1.4. Embedded resolution of singularities. Let X denote a (singular) space. We assume, for simplicity, that X is a closed subspace of a smooth ambient space M. (This is always true locally.) The goal of embedded desingularization, in its simplest version, is to find a proper morphism σ from a smooth space M' onto M, in our category, with the following properties:

(1) σ is an isomorphism outside the singular locus $\operatorname{Sing} X$ of X.
(2) The strict transform X' of X by σ is smooth. (See 1.6 below.) X' can be described geometrically (at least if our field $\mathbb{K}$ is algebraically closed: compare [Bierstone and Milman 1997, Remark 3.15]) as the smallest closed subspace of M' that includes $\sigma^{-1}(X \setminus \operatorname{Sing} X)$.
(3) X' and $E' = \sigma^{-1}(\operatorname{Sing} X)$ simultaneously have only normal crossings. This means that, locally, we can choose coordinates with respect to which X' is a coordinate subspace and E' is a collection of coordinate hyperplanes.

We can achieve this goal with σ the composite of a sequence of blowings-up; a finite sequence when our spaces have a compact topology (for example, in an algebraic category), or a locally-finite sequence for non-compact analytic spaces. (A sequence of blowings-up over M is *locally finite* if all but finitely many of the blowings-up are trivial over any compact subset of M. The composite of a locally-finite sequence of blowings-up is a well-defined morphism σ.)

1.5. The category of spaces. Our desingularization theorem applies to the usual spaces of algebraic and analytic geometry over fields $\mathbb{K}$ of characteristic zero — algebraic varieties, schemes of finite type, analytic spaces (over $\mathbb{R}$, $\mathbb{C}$ or any locally compact $\mathbb{K}$) — but in addition to certain categories of spaces intermediate between analytic and C^∞ [Bierstone and Milman 1997]. In any case, we are dealing with a category of local-ringed spaces $X = (|X|, \mathcal{O}_X)$ over $\mathbb{K}$, where $\mathcal{O}_X$ is a coherent sheaf of rings. We are intentionally not specific about the category in this exposition because we want to emphasize the principles involved, and the main requirement for our desingularization algorithm is simply that a smooth space $M = (|M|, \mathcal{O}_M)$ in our category admit a covering by *(regular) coordinate charts* in which we have analogues of the usual operations of calculus of analytic functions; namely:

The coordinates $(x_1, \dots, x_n)$ of a chart U are *regular functions* on U (i.e., each $x_i \in \mathcal{O}_M(U)$) and all partial derivatives $\partial^{|\alpha|}/\partial x^\alpha = \partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ make sense as transformations $\mathcal{O}_M(U) \to \mathcal{O}_M(U)$. Moreover, for each $a \in U$, there is an *injective* "Taylor series homomorphism" T_a: $\mathcal{O}_{M,a} \to \mathbb{F}_a[[X]] = \mathbb{F}_a[[X_1, \dots, X_n]]$, where $\mathbb{F}_a$ denotes the residue field $\mathcal{O}_{M,a}/\underline{m}_{M,a}$, such that T_a induces an isomorphism

$$\widehat{\mathcal{O}}_{M,a} \xrightarrow{\cong} \mathbb{F}_a[[X]]$$

and T_a commutes with differentiation: $T_a \circ (\partial^{|\alpha|}/\partial x^\alpha) = (\partial^{|\alpha|}/\partial X^\alpha) \circ T_a$, for all $\alpha \in \mathbb{N}^n$. ($\underline{m}_{M,a}$ denotes the maximal ideal and $\widehat{\mathcal{O}}_{M,a}$ the completion of $\mathcal{O}_{M,a}$. $\mathbb{N}$ denotes the nonnegative integers.)

In the case of real- or complex-analytic spaces, of course, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathbb{F}_a = \mathbb{K}$ at each point, and "coordinate chart" means the classical notion. Regular coordinate charts for schemes of finite type are introduced in [Bierstone and Milman 1997, Section 3].

Suppose that $M = (|M|, \mathcal{O}_M)$ is a manifold (smooth space) and that $X = (|X|, \mathcal{O}_X)$ is a closed subspace of M. This means there is a coherent sheaf of ideals $\mathcal{I}_X$ in $\mathcal{O}_M$ such that $|X| = \operatorname{supp} \mathcal{O}_M/\mathcal{I}_X$ and $\mathcal{O}_X$ is the restriction to $|X|$ of $\mathcal{O}_M/\mathcal{I}_X$. We say that X is a *hypersurface* in M if $\mathcal{I}_{X,a}$ is a principal ideal, for each $a \in |X|$. Equivalently, for every $a \in |X|$, there is an open neighbourhood U of a in $|M|$ and a regular function $f \in \mathcal{O}_M(U)$ such that $|X|U| = \{x \in U : f(x) = 0\}$ and $\mathcal{I}_X|U$ is the principal ideal (f) generated by f; we write $X|U = V(f)$.

1.6. Strict transform. Let X denote a closed subspace of a manifold M, and let $\sigma\colon M' \to M$ be a blowing-up with smooth centre C. If X is a hypersurface, then the strict transform X' of X by σ is a closed subspace of M' that can be defined as follows: Say that $X = V(f)$ in a neighbourhood of $a \in |X|$. Then, in some neighbourhood of $a' \in \sigma^{-1}(a)$, $X' = V(f')$, where $f' = y_{\mathrm{exc}}^{-d} f \circ \sigma$, y_{exc} denotes a local generator of $\mathcal{I}_{\sigma^{-1}(C)} \subset \mathcal{O}_{M'}$, and $d = \mu_{C,a}(f)$ denotes the *order* of f *along* C at a: $d = \max\{k : (f) \subset \mathcal{I}_{C,a}^k\}$; d is the largest power to which y_{exc} factors from $f \circ \sigma$ at a'.

The strict transform X' of a general closed subspace X of M can be defined locally, at each $a' \in \sigma^{-1}(a)$, as the intersection of all hypersurfaces $V(f')$, for all $f \in \mathcal{I}_{X,a}$. We likewise define the strict transform by a sequence of blowings-up with smooth centres.

Each of the categories listed in 1.5 above is closed under blowing up and strict transform [Bierstone and Milman 1997, Proposition 3.13 ff.]; the latter condition is needed to apply the desingularization algorithm in a given category.

1.7. The invariant. Let X denote a closed subspace of a manifold M. To describe inv_X, we consider a sequence of transformations

$$
\begin{array}{ccccccccc}
\longrightarrow & M_{j+1} & \xrightarrow{\sigma_{j+1}} & M_j & \longrightarrow \cdots \longrightarrow & M_1 & \xrightarrow{\sigma_1} & M_0 & = & M \\
& X_{j+1} & & X_j & & X_1 & & X_0 & = & X \\
& E_{j+1} & & E_j & & E_1 & & E_0 & = & \varnothing
\end{array}
\tag{1.8}
$$

where, for each j, $\sigma_{j+1}\colon M_{j+1} \to M_j$ denotes a blowing-up with smooth centre $C_j \subset M_j$, X_{j+1} is the strict transform of X_j by σ_{j+1}, and E_{j+1} is the set of exceptional hypersurfaces in M_{j+1}; i.e., $E_{j+1} = E_j' \cup \{\sigma_{j+1}^{-1}(C_j)\}$, where E_j' denotes the set of strict transforms by σ_{j+1} of all hypersurfaces in E_j.

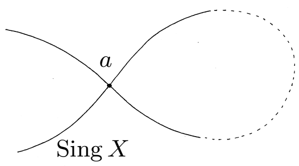
Our invariant $\mathrm{inv}_X(a)$, where $a \in M_j$ and $j = 0, 1, 2, \ldots$, will be defined inductively over the sequence of blowings-up; for each j, the invariant $\mathrm{inv}_X(a)$, for $a \in M_j$, can be defined provided that the centres C_i, $i < j$, are *admissible* (or inv_X-*admissible*) in the sense that

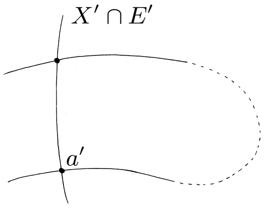
(1) C_i and E_i simultaneously have only normal crossings, and
(2) $\mathrm{inv}_X(\,\cdot\,)$ is locally constant on C_i.

The condition (1) guarantees that E_{i+1} is a collection of smooth hypersurfaces having only normal crossings. We can think of the desingularization algorithm in the following way: $X \subset M$ determines $\mathrm{inv}_X(a)$, for $a \in M$, and thus the first admissible centre of blowing up $C = C_0$; then $\mathrm{inv}_X(a)$ can be defined on M_1 and determines C_1, etc.

The notation $\mathrm{inv}_X(a)$, where $a \in M_j$, indicates a dependence not only on X_j, but also on the original space X. In fact $\mathrm{inv}_X(a)$, for $a \in M_j$, is invariant under local isomorphisms of X_j that preserve $E(a) = \{H \in E_j : H \ni a\}$ and certain subcollections $E^r(a)$ (which will be taken to encode the history of the resolution process). To understand why some dependence on the history should be needed, we consider how, in principle, it might be possible to determine a *global* centre of blowing up using a *local* invariant:

EXAMPLE 1.9. It is easy to find an example of a surface X whose singular locus, in a neighbourhood of a point a, consists of two smooth curves with a normal crossing at a, and where X has the property that, if we blow up with centre $\{a\}$, there are points a' in the fibre $\sigma^{-1}(a)$ where the strict transform X' has the *same* local equation (in suitable coordinates) as that of X at a, or an even more complicated equation (as in Example 3.1 below). This suggests that to simplify the singularities in a neighbourhood of a by blowing up with smooth centre in $\operatorname{Sing} X$, we should choose as centre one of the two smooth curves. But our surface may have the property that neither curve extends to a global smooth centre, as illustrated. So there is no choice but to blow up with centre $\{a\}$, although it seems to accomplish nothing: The figure shows the singular locus of X'; there are two points $a' \in \sigma^{-1}(a)$ where the singularity is the same as or worse than before. But what has changed at each of these points is the status of one of the curves, which is now *exceptional*. The moral is that, although the singularity of X at a has not been simplified in the strict transform, an invariant which takes into account the history of the resolution process as recorded by the accumulating exceptional hypersurfaces might nevertheless measure some improvement.





Consider a sequence of blowings-up as before. For simplicity, *we will assume that $X \subset M$ is a hypersurface*. Then $\mathrm{inv}_X(a)$, for $a \in M_j$, is a finite sequence beginning with the order $\nu_1(a) = \nu_{X_j,a}$ of X_j at a:

$$\mathrm{inv}_X(a) \;=\; \big(\nu_1(a), s_1(a);\, \nu_2(a), s_2(a);\, \ldots,\, s_t(a);\, \nu_{t+1}(a)\big).$$

(In the general case, $\nu_1(a)$ is replaced by a more delicate invariant of X_j at a, the *Hilbert–Samuel function* $H_{X_j,a}$ — see [Bierstone and Milman 1997] — but the remaining entries of $\mathrm{inv}_X(a)$ are still rational numbers (or ∞) as we will describe, and the theorems below are unchanged.) The $s_r(a)$ are nonnegative integers counting exceptional hypersurfaces that accumulate in certain blocks

$E^r(a)$ depending on the history of the resolution process. And the $\nu_r(a)$, for $r \geq 2$, represent certain "higher-order multiplicities" of the equation of X_j at a; $\nu_2(a), \ldots, \nu_t(a)$ are quotients of positive integers whose denominators are bounded in terms of the previous entries of $\mathrm{inv}_X(a)$. (More precisely, we have $e_{r-1}!\nu_r(a) \in \mathbb{N}$ for $r = 1, \ldots, t$, where $e_0 = 1$ and $e_r = \max\{e_{r-1}!, e_{r-1}!\nu_r(a)\}$.) The pairs $\big(\nu_r(a), s_r(a)\big)$ can be defined successively using data that depends on $n - r + 1$ variables (where n is the ambient dimension), so that $t \leq n$ by exhaustion of variables; the final entry $\nu_{t+1}(a)$ is either 0 (the order of a nonvanishing function) or ∞ (the order of the function identically zero).

EXAMPLE 1.10. Let $X \subset \mathbb{K}^n$ be the hypersurface $x_1^{d_1} + x_2^{d_2} + \cdots + x_t^{d_t} = 0$, where $1 < d_1 \leq \cdots \leq d_t$ for $t \leq n$. Then

$$\mathrm{inv}_X(0) = \left(d_1, 0;\ \frac{d_2}{d_1}, 0;\ \ldots;\ \frac{d_t}{d_{t-1}}, 0;\ \infty\right).$$

This is $\mathrm{inv}_X(0)$ in "year zero" (before the first blowing up), so there are no exceptional hypersurfaces.

THEOREM A (Embedded desingularization.) *There is a finite sequence of blowings-up* (1.8) *with smooth* inv_X*-admissible centres* C_j (*or a locally finite sequence, in the case of noncompact analytic spaces*) *such that*:

(1) *For each* j, *either* $C_j \subset \operatorname{Sing} X_j$ *or* X_j *is smooth and* $C_j \subset X_j \cap E_j$.
(2) *Let* X' *and* E' *denote the final strict transform of* X *and exceptional set*, *respectively*. *Then* X' *is smooth and* X', E' *simultaneously have only normal crossings*.

If σ denotes the composite of the sequence of blowings-up σ_j, then E' is the critical locus of σ and $E' = \sigma^{-1}(\operatorname{Sing} X)$. In each of our categories of spaces, $\operatorname{Sing} X$ is closed in the *Zariski topology* of $|X|$ (the topology whose closed sets are of the form $|Y|$, for any closed subspace Y of X; see [Bierstone and Milman 1997, Proposition 10.1]). Theorem A resolves the singularities of X in a meaningful geometric sense provided that $|X| \setminus \operatorname{Sing} X$ is Zariski-dense in $|X|$. (For example, if X is a *reduced* complex-analytic space or a scheme of finite type.) More precise desingularization theorems (for example, for spaces that are not necessarily reduced) are given in [Bierstone and Milman 1997, Chapter IV].

This paper contains an essentially complete proof of Theorem A in the hypersurface case, presented though in a more informal way than in [Bierstone and Milman 1997]. We give a constructive definition of inv_X in Section 3, in parallel with a detailed example. In Section 4, we show that inv_X is indeed an invariant, and we summarize its key properties in Theorem B. (The terms $s_r(a)$ of $\mathrm{inv}_X(a)$ can, in fact, be introduced immediately in an invariant way; see 1.12 below.) It follows from Theorem B(3) that the maximum locus of inv_X has only normal crossings and, moreover, each of its local components extends to a global smooth subspace. (See Remark 3.6.)

The point is that each component is the intersection of the maximum locus of inv_X with those exceptional hypersurfaces containing the component; the exceptional divisors serve as global coordinates. We can obtain Theorem A by successively blowing up with centre given by any component of the maximum locus.

1.11. UNIVERSAL AND CANONICAL DESINGULARIZATION. The exceptional hypersurfaces (the elements of E_j) can be ordered in a natural way (by their "years of birth" in the history of the resolution process). We can use this ordering to extend $\mathrm{inv}_X(a)$ by an additional term $J(a)$ that will have the effect of picking out one component of the maximum locus of $\mathrm{inv}_X(\,\cdot\,)$ in a canonical way; see Remark 3.6. We write $\mathrm{inv}_X^{\mathrm{e}}(\,\cdot\,)$ for the extended invariant $\big(\mathrm{inv}_X(\,\cdot\,); J(\,\cdot\,)\big)$. Then our embedded desingularization theorem A can be obtained as follows:

ALGORITHM. *Choose as each successive centre of blowing up C_j the maximum locus of* $\mathrm{inv}_X^{\mathrm{e}}$ *on X_j.*

The algorithm stops when our space is "resolved" as in the conclusion of Theorem A. In the general (not necessarily hypersurface) case, we choose more precisely as each successive centre C_j the maximum locus of $\mathrm{inv}_X^{\mathrm{e}}$ on the non-resolved locus Z_j of X_j; in general, $\{x : \mathrm{inv}_X(x) = \mathrm{inv}_X(a)\} \subset Z_j$ (as germs at a), so that again each C_j is smooth, by Theorem B(3), and the algorithm stops when $Z_j = \varnothing$.

The algorithm applies to a category of spaces satisfying a compactness assumption (for example, schemes of finite type, restrictions of analytic spaces to relatively compact open subsets), so that $\mathrm{inv}_X(\,\cdot\,)$ has global maxima. Since the centres of blowing up are completely determined by an invariant, our desingularization theorem is automatically *universal* in the following sense: To every X, we associate a morphism of resolution of singularities $\sigma_X\colon\ X' \to X$ such that any local isomorphism $X|U \to Y|V$ (over open subsets U of $|X|$ and V of $|Y|$) lifts to an isomorphism $X'|\sigma_X^{-1}(U) \to Y'|\sigma_Y^{-1}(V)$ (in fact, lifts to isomorphisms throughout the entire towers of blowings-up).

For analytic spaces that are not necessarily compact, we can use an exhaustion by relatively compact open sets to deduce *canonical* resolution of singularities: Given X, there is a morphism of desingularization $\sigma_X\colon\ X' \to X$ such that any local isomorphism $X|U \to X|V$ (over open subsets of $|X|$) lifts to an isomorphism $X'|\sigma_X^{-1}(U) \to X'|\sigma_X^{-1}(V)$. See [Bierstone and Milman 1997, Section 13].

1.12. THE TERMS $s_r(a)$. The entries $s_1(a)$, $\nu_2(a)$, $s_2(a)$, ... of $\mathrm{inv}_X(a) = \big(\nu_1(a), s_1(a); \ldots, s_t(a); \nu_{t+1}(a)\big)$ will themselves be defined recursively. We write inv_r for inv_X truncated after s_r (with the convention that $\mathrm{inv}_r(a) = \mathrm{inv}_X(a)$ if $r > t$). We also write $\mathrm{inv}_{r+1/2} = (\mathrm{inv}_r; \nu_{r+1})$ (with the same convention), so that $\mathrm{inv}_{1/2}(a)$ means $\nu_1(a) = \nu_{X_j,a}$ (in the hypersurface case, or $H_{X_j,a}$ in general). For each r, the entries s_r, ν_{r+1} of inv_X can be defined over a sequence of blowings-up (1.8) whose centres C_i are $(r-\frac{1}{2})$-*admissible* (or $\mathrm{inv}_{r-1/2}$-*admissible*) in the sense that:

(1) C_i and E_i simultaneously have only normal crossings.

(2) $\mathrm{inv}_{r-1/2}(\,\cdot\,)$ is locally constant on C_i.

The terms $s_r(a)$ can be introduced immediately, as follows: Write $\pi_{ij} = \sigma_{i+1} \circ \cdots \circ \sigma_j$, for $i = 0, \ldots, j-1$, and $\pi_{jj} =$ identity. If $a \in M_j$, set $a_i = \pi_{ij}(a)$, for $i = 0, \ldots, j$. First consider a sequence of blowings-up (1.8) with $\frac{1}{2}$-admissible centres. ($\mathrm{inv}_{1/2} = \nu_1$ can only decrease over such a sequence; see, for example, Section 2 following.) Suppose $a \in M_j$. Let i denote the "earliest year" k such that $\nu_1(a) = \nu_1(a_k)$, and set $E^1(a) = \{H \in E(a) : H$ is the strict transform of some hypersurface in $E(a_i)\}$. We define $s_1(a) = \#E^1(a)$.

The block of exceptional hypersurfaces $E^1(a)$ intervenes in our desingularization algorithm in a way that can be thought of intuitively as follows. (The idea will be made precise in Sections 2 and 3.) The exceptional hypersurfaces passing through a but not in $E^1(a)$ have accumulated during the recent part of our history, when the order ν_1 has not changed; we have good control over these hypersurfaces. But those in $E^1(a)$ accumulated long ago; we have forgotten a lot about them in the form of our equations (for example, if we restrict the equations of X to these hypersurfaces, their orders might increase) and we recall them using $s_1(a)$.

In general, consider a sequence of blowings-up (1.8) with $(r+\frac{1}{2})$-admissible centres. ($\mathrm{inv}_{r+1/2}$ can only decrease over such a sequence; see Section 3 and Theorem B.) Suppose that i is the smallest index k such that $\mathrm{inv}_{r+1/2}(a) = \mathrm{inv}_{r+1/2}(a_k)$. Let $E^{r+1}(a) = \{H \in E(a) \setminus \bigcup_{q\le r} E^q(a) : H$ is transformed from $E(a_i)\}$. We define $s_{r+1}(a) = \#E^{r+1}(a)$.

It is less straightforward to define the multiplicities $\nu_2(a), \nu_3(a), \ldots$ and to show they are invariants. Our definition depends on a construction in local coordinates that we present in Section 3. But we first try to convey the idea by describing the origin of our algorithm.

2. The Origin of our Approach

Consider a hypersurface X, defined locally by an equation $f(x) = 0$. Let $a \in X$ and let $d = d(a)$ denote the order of X (or of f) at a; i.e., $d = \nu_1(a) = \mu_a(f)$. We can choose local coordinates $(x_1, \ldots, x_n)$ in which $a = 0$ and $(\partial^d f/\partial x_n^d)(a) \neq 0$; then we can write

$$f(x) = c_0(\tilde{x}) + c_1(\tilde{x})x_n + \cdots + c_{d-1}(\tilde{x})x_n^{d-1} + c_d(x)x_n^d$$

in a neighbourhood of a, where $c_d(x)$ does not vanish. ($\tilde{x}$ means $(x_1, \ldots, x_{n-1})$.) Assume for simplicity that $c_d(x) \equiv 1$ (for example, by the Weierstrass preparation theorem, but see Remark 2.3 below). We can also assume that $c_{d-1}(\tilde{x}) \equiv 0$, by "completing the d-th power" (i.e., by the coordinate change $x_n' = x_n +$

$c_{d-1}(\tilde{x})/d)$; thus

$$f(x) = c_0(\tilde{x}) + \cdots + c_{d-2}(\tilde{x})x_n^{d-2} + x_n^d. \tag{2.1}$$

Our aim is to simplify f by blowing up with smooth centre in the *equimultiple locus* of $a = 0$; i.e., in the locus of points of order d,

$$S_{(f,d)} = \{x : \mu_x(f) = d\}.$$

The representation (2.1) makes it clear that the equimultiple locus lies in a smooth subspace of codimension 1; in fact, by elementary calculus,

$$S_{(f,d)} = \{x : x_n = 0 \text{ and } \mu_{\tilde{x}}(c_q) \geq d - q,\ q = 0, \ldots, d-2\}. \tag{2.2}$$

The idea now is that the given data $\big(f(x), d\big)$ involving n variables should be equivalent, in some sense, to the data $\mathcal{H}_1(a) = \big\{\big(c_q(\tilde{x}), d-q\big)\big\}$ in $n-1$ variables, thus making possible an induction on the number of variables. (Here in year zero, before we begin to blow up, $\nu_2(a) = \min_q \mu_a(c_q)/(d-q)$.)

Remark 2.3. For the global desingularization algorithm, the Weierstrass preparation theorem must be avoided for two important reasons: (1) It may take us outside the given category (for example, in the algebraic case). (2) Even in the complex-analytic case, we need to prove that inv_X is semicontinuous in the sense that any point admits a coordinate neighbourhood V such that, given $a \in V$, $\{x \in V : \mathrm{inv}_X(x) \leq \mathrm{inv}_X(a)\}$ is Zariski-open in V (i.e., is the complement of a closed analytic subset). We therefore need a representation like (2.2) that is valid in a Zariski-open neighbourhood of a in V. This can be achieved in the following simple way that involves neither making $c_d(x) \equiv 1$ nor explicitly completing the d-th power: By a linear coordinate change, we can assume that $(\partial^d f/\partial x_n^d)(a) \neq 0$. Then in the Zariski-open neighbourhood of a where $(\partial^d f/\partial x_n^d)(x) \neq 0$, we let $N_1 = N_1(a)$ denote the submanifold of codimension one (in our category) defined by $z = 0$, where $z = \partial^{d-1} f/\partial x_n^{d-1}$, and we take $\mathcal{H}_1(a) = \big\{\big((\partial^q f/\partial x_n^q)|N_1,\ d-q\big)\big\}$. As before, we have $S_{(f,d)} = \{x : x \in N_1$ and $\mu_x(h) \geq \mu_h$, for all $(h, \mu_h) = \big((\partial^q f/\partial x_n^q)|N_1,\ d-q\big) \in \mathcal{H}_1(a)\}$.

We now consider the effect of a blowing-up σ with smooth centre $C \subset S_{(f,d)}$. By a transformation of the variables $(x_1, \ldots, x_{n-1})$, we can assume that in our local coordinate neighbourhood U of a, C has the form

$$Z_I = \{x : x_n = 0 \text{ and } x_i = 0,\ i \in I\}, \tag{2.4}$$

where $I \subset \{1, \ldots, n-1\}$. According to 1.2 above, $U' = \sigma^{-1}(U)$ is covered by coordinate charts U'_i, for $i \in I \cup \{n\}$, where each U'_i has coordinates $y = (y_1, \ldots, y_n)$ in which σ is given by

$$\begin{aligned} x_i &= y_i, \\ x_j &= y_i y_j, \quad \text{if } j \in (I \cup \{n\}) \setminus \{i\}, \\ x_j &= y_j\ , \quad \text{if } j \notin I \cup \{n\}. \end{aligned}$$

In each U'_i, we can write $f\big(\sigma(y)\big) = y_i^d f'(y)$; the strict transform X' of X by σ is defined in U'_i by the equation $f'(y) = 0$. (To be as simple as possible, we continue to assume $c_d(x) \equiv 1$, though we could just as well work with the set-up of Remark 2.3; see [Bierstone and Milman 1997, Proposition 4.12].) By (2.1), if $i \in I$, then

$$f'(y) = c'_0(\tilde{y}) + \cdots + c'_{d-2}(\tilde{y}) y_n^{d-2} + y_n^d, \tag{2.5}$$

where

$$c'_q(\tilde{y}) = y_i^{-(d-q)} c_q\big(\tilde{\sigma}(\tilde{y})\big), \qquad \text{for } q = 0, \ldots, d-2. \tag{2.6}$$

The analogous formula for the strict transform in the chart U'_n shows that f' is invertible at every point of $U'_n \setminus \bigcup_{i \in I} U'_i = \{y \in U'_n : y_i = 0,\ i \in I\}$; in other words, $X' \cap U' \subset \bigcup_{i \in I} U'_i$.

The formula for $f'(y)$ above shows that the representation (2.2) of the equimultiple locus (or that of Remark 2.3) is stable under ν_1-admissible blowing up when the order does not decrease; i.e., at a point $a' \in U'_i$ where $d(a') = d$,

$$S_{(f',d)} = \{y : y_n = 0 \text{ and } \mu_{\tilde{y}}(c'_q) \geq d - q \text{ for } q = 0, \ldots, d-2\},$$

where $N_1(a') = \{y_n = 0\}$ is the strict transform of $N_1(a) = \{x_n = 0\}$ and the c'_q are given by the transformation law (2.6). The latter is not strict transform, but something intermediate between strict and total transform $c_q \circ \sigma$. It is essentially for this reason that some form of embedded desingularization will be needed for the coefficients c_q (i.e., in the inductive step) even to prove a weaker form of resolution of singularities for f.

$N_1(a)$ is called a smooth hypersurface of *maximal contact* with X; this means a smooth hypersurface that contains the equimultiple locus of a, stably (i.e., even after admissible blowings-up as above). The existence of $N_1(a)$ depends on characteristic zero. A maximal contact hypersurface is crucial to our construction by increasing codimension. (In 1.12 above, $E^1(a)$ is the block of exceptional hypersurfaces that do not necessarily have normal crossings with respect to a maximal contact hypersurface; the term $s_1(a)$ in $\mathrm{inv}_X(a)$ is needed to deal with these exceptional divisors.)

We will now make a simplifying assumption on the coefficients c_q: we assume that one of these functions is a monomial (times an invertible factor) that divides all the others, but in a way that respects the different "multiplicities" $d - q$ associated with the transformation law (2.6). In other words, we make the monomial assumption on the $c_q^{1/(d-q)}$ (to equalize the "assigned multiplicities" $d-q$) or on the $c_q^{d!/(d-q)}$ (to avoid fractional powers). We assume, then, that

$$c_q(\tilde{x})^{d!/(d-q)} = (\tilde{x}^\Omega)^{d!} c_q^*(\tilde{x}), \qquad \text{for } q = 0, \ldots, d-2, \tag{2.7}$$

where $\Omega = (\Omega_1, \ldots, \Omega_{n-1})$ with $d!\Omega_i \in \mathbb{N}$ for each i, $\tilde{x}^\Omega = x_1^{\Omega_1} \cdots x_{n-1}^{\Omega_{n-1}}$, and the c_q^* are regular functions on $\{x_n = 0\}$ such that $c_q^*(a) \neq 0$ for some q. We also write $\Omega = \Omega(a)$.

We can regard (2.7) provisionally as an assumption made to see what happens in a simple test case, but in fact we can reduce to this case by a suitable induction on dimension (as we will see below). (Assuming (2.7) in year zero, $\nu_2(a) = |\Omega|$, where $|\Omega| = \Omega_1 + \cdots + \Omega_{n-1}$. But from the viewpoint of our algorithm for canonical desingularization as presented in Section 3, the argument following is analogous to a situation where the variables x_i occurring in $\tilde{x}^\Omega$ are exceptional divisors in $E(a)\backslash E^1(a)$; in this context, $|\Omega|$ is an invariant we call $\mu_2(a)$ (Definition 3.2) and $\nu_2(a) = 0$.)

Now, by (2.2) and (2.7),

$$S_{(f,d)} = \{x : x_n = 0 \text{ and } \mu_{\tilde{x}}(\tilde{x}^\Omega) \geq 1\}.$$

(The order of a monomial with rational exponents has the obvious meaning.) Therefore (using the notation (2.4)), $S_{(f,d)} = \bigcup Z_I$, where I runs over the *minimal* subsets of $\{1, \ldots, n-1\}$ such that $\sum_{j \in I} \Omega_j \geq 1$; i.e., where I runs over the subsets of $\{1, \ldots, n-1\}$ such that

$$0 \leq \sum_{j \in I} \Omega_j - 1 < \Omega_i, \qquad \text{for all } i \in I. \tag{2.8}$$

Consider the blowing-up σ with centre $C = Z_I$, for one such I. By (2.7), in the chart U_i' we have

$$c_q'(\tilde{y})^{d!/(d-q)} = \left(y_1^{\Omega_1} \cdots y_i^{\sum_I \Omega_j - 1} \cdots y_{n-1}^{\Omega_{n-1}}\right)^{d!} c_q^*\left(\tilde{\sigma}(\tilde{y})\right), \tag{2.9}$$

$q = 0, \ldots, d-2$. Suppose $a' \in \sigma^{-1}(a) \cap U_i'$. By (2.5), $d(a') \leq d(a)$. Moreover, if $d(a') = d(a)$, then by (2.8) and (2.9), $1 \leq |\Omega(a')| < |\Omega(a)|$. In particular, the order d must decrease after at most $d!|\Omega|$ such blowings-up.

The question then is whether we can reduce to the hypothesis (2.7) by induction on dimension, replacing (f, d) in some sense by the collection $\mathcal{H}_1(a) = \{(c_q, d-q)\}$ on the submanifold $N_1 = \{x_n = 0\}$. To set up the induction, we would have to treat from the start a collection $\mathcal{F}_1 = \{(f, \mu_f)\}$ rather than a single pair (f, d). (A general X is, in any case, defined locally by several equations.) Moreover, since the transformation law (2.6) is not strict transform, we would have to reformulate the original problem to not only desingularize X: $f(x) = 0$, but also make its total transform normal crossings. To this end, suppose that $f(x) = 0$ actually represents the strict transform of our original hypersurface in that year in the history of the blowings-up involved where the order at a first becomes d. (We are following the transforms of the hypersurface at a sequence of points "a" over some original point.) Suppose there are $s = s(a)$ accumulated exceptional hypersurfaces H_p passing through a; as above, we can also assume that H_p is defined near a by an equation

$$x_n + b_p(\tilde{x}) = 0,$$

$1 \leq p \leq s$. (Each $\mu_a(b_p) \geq 1$.) The transformation law for the b_p analogous to (2.6) is

$$b'_p(\tilde{y}) = y_i^{-1} b_p\big(\tilde{\sigma}(\tilde{y})\big), \qquad \text{for } p = 1, \ldots, s.$$

Suppose now that in (2.7) we also have

$$b_p(\tilde{x})^{d!} = (\tilde{x}^{\Omega})^{d!} b_p^*(\tilde{x}), \qquad \text{for } p = 1, \ldots, s \tag{2.10}$$

(and assume that either some $c_q^*(a) \neq 0$ or some $b_p^*(a) \neq 0$). Then the argument above shows that $\big(d(a'), s(a')\big) \leq \big(d(a), s(a)\big)$ (with respect to the lexicographic ordering of pairs), and that if $\big(d(a'), s(a')\big) = \big(d(a), s(a)\big)$ then $1 \leq |\Omega(a')| < |\Omega(a)|$. ($s(a'$) counts the exceptional hypersurfaces H'_p passing through a'. As long as d does not drop, the new exceptional hypersurfaces accumulate simply as $y_i = 0$ for certain $i = 1, \ldots, n-1$, in suitable coordinates $(y_1, \ldots, y_{n-1})$ for the strict transform $N' = \{y_n = 0\}$ of $N = \{x_n = 0\}$.)

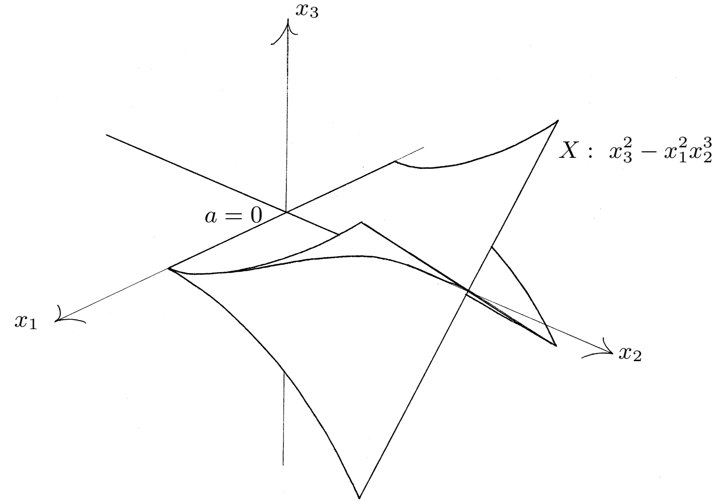
The induction on dimension can be realized in various ways. The simplest — the method of [Bierstone and Milman 1988, Section 4] — is to apply the inductive hypothesis within a coordinate chart to the function of $n-1$ variables given by the product of all nonzero $c_q^{d!/(d-q)}$, all nonzero $b_p^{d!}$, and all their nonzero differences. The result is (2.7) and (2.10) (with $c_q^*(a) \neq 0$ or $b_p^*(a) \neq 0$ for some q or p; see [Bierstone and Milman 1988, Lemma 4.7]). Pullback of the coefficients c_q by a blowing-up of N with smooth centre C, say of the form (2.4) above, corresponds to strict transform of f by the blowing-up with centre $\{x_i = 0 : i \in I\}$. Thus we sacrifice the condition that each centre lie in the equimultiple locus (or even in X!). But we do get a very simple proof of local uniformization. In fact, we get the conclusion (2) of our desingularization theorem A, using a mapping σ: $M' \to M$ which is a composite of mappings that are either blowings-up with smooth centres or surjections of the form $\coprod_j U_j \to \bigcup_j U_j$, where the latter is a locally-finite open covering of a manifold and $\coprod$ means disjoint union.

To prove our canonical desingularization theorem, we repeat the construction above in increasing codimension to obtain $\mathrm{inv}_X(a) = \big(\nu_1(a), s_1(a); \nu_2(a), \ldots\big)$ — here $\big(\nu_1(a), s_1(a)\big)$ is $\big(d(a), s(a)\big)$ above — together with a corresponding local "presentation". The latter means a local description of the locus of constant values of the invariant in terms of regular functions with assigned multiplicities, that survives certain blowings-up. ($N_1(a), \mathcal{H}_1(a)$ above is a presentation of ν_1 at a.)

3. The Desingularization Algorithm

In this section we give a constructive definition of inv_X together with a corresponding presentation (in the hypersurface case). We illustrate the construction by applying the desingularization algorithm to an example — a surface whose desingularization involves all the features of the general hypersurface case. We will use horizontal lines to separate from the example the general considerations that are needed at each step.

EXAMPLE 3.1. Let $X \subset \mathbb{K}^3$ denote the hypersurface $g(x) = 0$, where $g(x) = x_3^2 - x_1^2 x_2^3$:



Let $a = 0$. Then $\nu_1(a) = \mu_a(g) = 2$. Of course, $E(a) = \varnothing$, so that $s_1(a) = 0$. (This is "year zero"; there are no exceptional hypersurfaces.) Thus $\mathrm{inv}_1(a) = \big(\nu_1(a), s_1(a)\big) = (2, 0)$. Let $\mathcal{G}_1(a) = \{(x_3^2 - x_1^2 x_2^3, 2)\}$. We say that $\mathcal{G}_1(a)$ is a *codimension* 0 *presentation of* $\mathrm{inv}_{1/2} = \nu_1$ *at* a. (Here where $s_1(a) = 0$, we can also say that $\mathcal{G}_1(a)$ is a codimension 0 presentation of $\mathrm{inv}_1 = (\nu_1, s_1)$ at a.)

In general, consider a hypersurface $X \subset M$. Let $a \in M$ and let $S_{\mathrm{inv}_{1/2}}(a)$ denote the germ at a of $\{x : \mathrm{inv}_{1/2}(x) \geq \mathrm{inv}_{1/2}(a)\}$, which coincides with the germ at a of $\{x : \mathrm{inv}_{1/2}(x) = \mathrm{inv}_{1/2}(a)\}$. If $g \in \mathcal{O}_{M,a}$ generates the local ideal $\mathcal{I}_{X,a}$ of X and $d = \nu_1(a) = \mu_a(g)$, then $\mathcal{G}_1(a) = \{(g, d)\}$ is a codimension 0 presentation of $\mathrm{inv}_{1/2} = \nu_1$ at a. This means $S_{\mathrm{inv}_{1/2}}(a)$ coincides with the germ of the "equimultiple locus"

$$S_{\mathcal{G}_1(a)} = \{x : \mu_x(g) = d\},$$

and that the latter condition survives certain transformations.

More generally, suppose that $\mathcal{G}_1(a)$ is a finite collection of pairs $\{(g, \mu_g)\}$, where each g is a germ at a of a regular function (i.e., $g \in \mathcal{O}_{M,a}$) with an "assigned multiplicity" $\mu_g \in \mathbb{Q}$, and where we assume that $\mu_a(g) \geq \mu_g$ for every g. Set

$$S_{\mathcal{G}_1(a)} \;=\; \{x : \mu_x(g) \geq \mu_g,\ \text{for all } (g, \mu_g) \in \mathcal{G}_1(a)\};$$

$S_{\mathcal{G}_1(a)}$ is well-defined as a germ at a. To say that $\mathcal{G}_1(a)$ is a *codimension* 0 *presentation of* $\mathrm{inv}_{1/2}$ *at* a means that

$$S_{\mathrm{inv}_{1/2}}(a) = S_{\mathcal{G}_1(a)}$$

and that this condition survives certain transformations:

To be precise, we will consider triples of the form $\big(N = N(a),\, \mathcal{H}(a),\, \mathcal{E}(a)\big)$, where:

N is a germ of a submanifold of codimension p at a (for some $p \geq 0$).

$\mathcal{H}(a) = \{(h, \mu_h)\}$ is a finite collection of pairs (h, μ_h), where $h \in \mathcal{O}_{N,a}$, $\mu_h \in \mathbb{Q}$ and $\mu_a(h) \geq \mu_h$.

$\mathcal{E}(a)$ is a finite set of smooth (exceptional) hyperplanes containing a, such that N and $\mathcal{E}(a)$ simultaneously have normal crossings and $N \not\subset H$, for all $H \in \mathcal{E}(a)$.

A *local blowing-up* σ: $M' \to M$ over a neighbourhood W of a, with smooth centre C, means the composite of a blowing-up $M' \to W$ with centre C, and the inclusion $W \hookrightarrow M$.

Definition 3.2. We say that $\big(N(a), \mathcal{H}(a), \mathcal{E}(a)\big)$ is a *codimension p presentation of* $\mathrm{inv}_{1/2}$ *at* a if:

(1) $S_{\mathrm{inv}_{1/2}}(a) = S_{\mathcal{H}(a)}$, where $S_{\mathcal{H}(a)} = \{x \in N : \mu_x(h) \geq \mu_h$, for all $(h, \mu_h) \in \mathcal{H}(a)\}$ (as a germ at a).
(2) Suppose that σ is a $\frac{1}{2}$-admissible local blowing-up at a (with smooth centre C). Let $a' \in \sigma^{-1}(a)$. Then $\mathrm{inv}_{1/2}(a') = \mathrm{inv}_{1/2}(a)$ if and only if $a' \in N'$ (where $N' = N(a')$ denotes the strict transform of N) and $\mu_{a'}(h') \geq \mu_{h'}$ for all $(h, \mu_h) \in \mathcal{H}(a)$, where $h' = y_{\mathrm{exc}}^{-\mu_h} h \circ \sigma$ and $\mu_{h'} = \mu_h$. (y_{exc} denotes a local generator of $\mathcal{I}_{\sigma^{-1}(C)}$.) In this case, we will write $\mathcal{H}(a') = \{(h', \mu_{h'}) : (h, \mu_h) \in \mathcal{H}(a)\}$ and $\mathcal{E}(a') = \{H' : H \in \mathcal{E}(a)\} \cup \{\sigma^{-1}(C)\}$.
(3) Conditions (1) and (2) continue to hold for the transforms X' and $\big(N(a'), \mathcal{H}(a'), \mathcal{E}(a')\big)$ of our data by sequences of morphisms of the following three types, at points a' in the fibre of a (to be also specified).

The three types of morphisms allowed are the following. (Types (ii) and (iii) are not used in the actual desingularization algorithm. They are needed to prove invariance of the terms $\nu_2(a), \nu_3(a), \ldots$ of $\mathrm{inv}_X(a)$ by making certain sequences of "test blowings-up", as we will explain in Section 4; they are not explicitly needed in this section.)

(i) $\frac{1}{2}$-*admissible local blowing-up* σ, and $a' \in \sigma^{-1}(a)$ such that $\mathrm{inv}_{1/2}(a') = \mathrm{inv}_{1/2}(a)$.
(ii) *Product with a line.* σ is a projection $M' = W \times \mathbb{K} \to W \hookrightarrow M$, where W is a neighbourhood of a, and $a' = (a, 0)$.
(iii) *Exceptional blowing-up.* σ is a local blowing-up $M' \to W \hookrightarrow M$ over a neighbourhood W of a, with centre $H_0 \cap H_1$, where $H_0, H_1 \in \mathcal{E}(a)$, and a' is the unique point of $\sigma^{-1}(a) \cap H_1'$.

The data is transformed to a' in each case above, as follows:

(i): $X' =$ strict transform of X; $\big(N(a'), \mathcal{H}(a'), \mathcal{E}(a')\big)$ as defined in 3.2(2) above.
(ii) and (iii): $X' = \sigma^{-1}(X)$, $N(a') = \sigma^{-1}(N)$, $\mathcal{H}(a') = \{(h \circ \sigma, \mu_h)\}$.

$$\mathcal{E}(a') = \begin{cases} \{\sigma^{-1}(H) : H \in \mathcal{E}(a)\} \cup \{W \times 0\} & \text{in case (ii);} \\ \{H' : H \in \mathcal{E}(a),\ a' \in H'\} \cup \{\sigma^{-1}(C)\} & \text{in case (iii).} \end{cases}$$

If $(N(a), \mathcal{H}(a), \mathcal{E}(a))$ is a presentation of $\mathrm{inv}_{1/2}$ at a, then $N(a)$ is called a subspace of *maximal contact*; compare Section 2.

Suppose now that $\mathcal{G}_1(a)$ is a codimension 0 presentation of $\mathrm{inv}_{1/2}$ at a. (Implicitly, $N(a) = M$ and $\mathcal{E}(a) = \varnothing$.) Assume, moreover, that there exists $(g, \mu_g) = (g_*, \mu_{g_*}) \in \mathcal{G}_1(a)$ such that $\mu_a(g_*) = \mu_{g_*}$ (as in Example 3.1).

We can always assume that each $\mu_g \in \mathbb{N}$, and even that all μ_g coincide: Simply replace each (g, μ_g) by $(g^{e/\mu_g}, e)$, for suitable $e \in \mathbb{N}$.

Then, after a linear coordinate change if necessary, we can assume that $(\partial^d g_*/\partial x_n^d)(a) \neq 0$, where $d = \mu_{g_*}$. Set

$$z = \frac{\partial^{d-1} g_*}{\partial x_n^{d-1}} \in \mathcal{O}_{M,a}, \qquad N_1 = N_1(a) = \{z = 0\},$$

$$\mathcal{H}_1(a) = \left\{ \left(\frac{\partial^q g}{\partial x_n^q}\Big|_{N_1}, \mu_g - q \right) : 0 \le q < \mu_g,\ (g, \mu_g) \in \mathcal{G}_1(a) \right\}.$$

Then $(N_1(a), \mathcal{H}_1(a), \mathcal{E}_1(a) = \varnothing)$ is a codimension 1 presentation of $\mathrm{inv}_{1/2}$ at a: This is an assertion about the way our data transform under sequences of morphisms of types (i), (ii) and (iii) above. The effect of a transformation of type (i) is essentially described by the calculation in Section 2. The effect of a transformation of type (ii) is trivial, and that for type (iii) can be understood in a similar way to (i): see [Bierstone and Milman 1997, Propositions 4.12 and 4.19] for details.

Definition 3.3. We define

$$\mu_2(a) = \min_{\mathcal{H}_1(a)} \frac{\mu_a(h)}{\mu_h}.$$

Then $1 \le \mu_2(a) \le \infty$. If $\mathcal{E}(a) = \varnothing$ (as in year zero), we set

$$\nu_2(a) = \mu_2(a)$$

and $\mathrm{inv}_{1 1/2}(a) = (\mathrm{inv}_1(a); \nu_2(a))$. Then $\nu_2(a) \le \infty$. Moreover, $\nu_2(a) = \infty$ if and only if $\mathcal{G}_1(a) \sim \{(z, 1)\}$. (This means that the latter is also a presentation of $\mathrm{inv}_{1/2}$ at a.) If $\nu_2(a) = \infty$, then we set $\mathrm{inv}_X(a) = \mathrm{inv}_{1 1/2}(a)$. $\mathrm{inv}_X(a) = (d, 0, \infty)$ if and only if X is defined (near a) by the equation $z^d = 0$; in this case, the desingularization algorithm can do no more, unless we blow-up with centre $|X|$!

In Example 3.1, $\mu_a(g) = 2 = \mu_g$, and by the construction above we get the following codimension 1 presentation of $\mathrm{inv}_{1/2}$ (or inv_1) at a:

$$N_1(a) = \{x_3 = 0\}, \qquad \mathcal{H}_1(a) = \{(x_1^2 x_2^3, 2)\}.$$

Thus $\nu_2(a) = \mu_2(a) = \frac{5}{2}$. As a codimension 1 presentation of $\mathrm{inv}_{1 1/2}$ (or inv_2) at a, we can take

$$N_1(a), \qquad \mathcal{G}_2(a) = \{(x_1^2 x_2^3, 5)\}.$$

In general, "presentation of inv_r" (or "of $\mathrm{inv}_{r+1/2}$") means the analogue of "presentation of $\mathrm{inv}_{1/2}$" above. Suppose that $(N_1(a), \mathcal{H}_1(a))$ is a codimension 1 presentation of inv_1 at a $(\mathcal{E}_1(a) = \varnothing)$. Assume that $1 \le \nu_2(a) < \infty$. (In year zero, we always have $\nu_2(a) = \mu_2(a) \ge 1$.) Let

$$\mathcal{G}_2(a) = \{(h, \nu_2(a)\mu_h) : (h, \mu_h) \in \mathcal{H}_1(a)\}.$$

Then $(N_1(a), \mathcal{G}_2(a))$ is a codimension 1 presentation of $\mathrm{inv}_{11/2}$ at a (or of inv_2 at a, when $s_2(a) = 0$ as here). Clearly, there exists $(g_*, \mu_{g_*}) \in \mathcal{G}_2(a)$ such that $\mu_a(g_*) = \mu_{g_*}$.

This completes a cycle in the recursive definition of inv_X, and we can now repeat the above constructions: Let $d = \mu_{g_*}$. After a linear transformation of the coordinates $(x_1, \dots, x_{n-1})$ of $N_1(a)$, we can assume that $(\partial^d g_*/\partial x_{n-1}^d)(a) \neq 0$. We get a codimension 2 presentation of inv_2 at a by taking

$$N_2(a) = \left\{ x \in N_1(a) : \frac{\partial^{d-1} g_*}{\partial x_{n-1}^{d-1}}(x) = 0 \right\},$$

$$\mathcal{H}_2(a) = \left\{ \left(\frac{\partial^q g}{\partial x_{n-1}^q}\Big|_{N_2(a)}, \mu_g - q \right) : 0 \le q < \mu_g,\ (g, \mu_g) \in \mathcal{G}_2(a) \right\}.$$

In our example, the calculation of a codimension 2 presentation can be simplified by the following useful observation: Suppose there is $(g, \mu_g) \in \mathcal{G}_2(a)$ with $\mu_a(g) = \mu_g$ and $g = \prod g_i^{m_i}$. If we replace (g, μ_g) in $\mathcal{G}_2(a)$ by the collection of (g_i, μ_{g_i}), where each $\mu_{g_i} = \mu_a(g_i)$, then we obtain an (equivalent) presentation of inv_2.

In our example, therefore,

$$N_1(a) = \{x_3 = 0\}, \qquad \mathcal{G}_2(a) = \{(x_1, 1), (x_2, 1)\}$$

is a codimension 1 presentation of inv_2 at a. It follows immediately that

$$N_2(a) = \{x_2 = x_3 = 0\}, \qquad \mathcal{H}_2(a) = \{(x_1, 1)\}$$

is a codimension 2 presentation of inv_2 at a. Then $\nu_3(a) = \mu_3(a) = 1$ and, as a codimension 3 presentation of $\mathrm{inv}_{21/2}$ (or of inv_3) at a, we can take

$$N_3(a) = \{x_1 = x_2 = x_3 = 0\}, \qquad \mathcal{H}_3(a) = \varnothing.$$

We put $\nu_4(a) = \mu_4(a) = \infty$. Thus we have

$$\mathrm{inv}_X(a) = (2, 0;\ \tfrac{5}{2}, 0;\ 1, 0;\ \infty)$$

and $S_{\mathrm{inv}_X}(a) = S_{\mathrm{inv}_3}(a) = N_3(a) = \{a\}$. The latter is the centre C_0 of our first blowing-up σ_1: $M_1 \to M_0 = \mathbb{K}^3$; M_1 can be covered by three coordinate charts U_i, $i = 1, 2, 3$, where each U_i is the complement in M_1 of the strict transform of the hyperplane $\{x_i = 0\}$. The strict transform $X_1 = X'$ of X lies in $U_1 \cup U_2$. To illustrate the algorithm, we will follow our construction at a sequence of points over a, choosing after each blowing-up a point in the fibre where inv_X has a maximum value in a given coordinate chart.

Year one. U_1 has a coordinate system (y_1, y_2, y_3) in which σ_1 is given by the transformation

$$x_1 = y_1, \quad x_2 = y_1 y_2, \quad x_3 = y_1 y_3.$$

Then $X_1 \cap U_1 = V(g_1)$, where

$$g_1 = y_1^{-2} g \circ \sigma_1 = y_3^2 - y_1^3 y_2^3.$$

Consider $b = 0$. Then $E(b) = \{H_1\}$, where H_1 is the exceptional hypersurface $H_1 = \sigma_1^{-1}(a) = \{y_1 = 0\}$. Now, $\nu_1(b) = 2 = \nu_1(a)$. Therefore $E^1(b) = \varnothing$ and $s_1(b) = 0$. We write $\mathcal{E}_1(b) = E(b) \setminus E^1(b)$, so that $\mathcal{E}_1(b) = E(b)$ here. Let $\mathcal{F}_1(b) = \mathcal{G}_1(b) = \{(g_1, 2)\}$. Then $\big(N_0(b) = M_1, \mathcal{F}_1(b), \mathcal{E}_1(b)\big)$ is a codimension 0 presentation of inv_1 at b. Set

$$N_1(b) = \{y_3 = 0\} = N_1(a)', \qquad \mathcal{H}_1(b) = \{(y_1^3 y_2^3, 2)\};$$

$\big(N_1(b), \mathcal{H}_1(b), \mathcal{E}_1(b)\big)$ is a codimension 1 presentation of inv_1 at b. As before,

$$\mu_2(b) = \min_{\mathcal{H}_1(b)} \frac{\mu_b(h)}{\mu_h} = \tfrac{6}{2} = 3.$$

But, here, in the presence of nontrivial $\mathcal{E}_1(b)$, $\nu_2(b)$ will involve first factoring from the $h \in \mathcal{H}_1(b)$ the exceptional divisors in $\mathcal{E}_1(b)$ (taking, in a sense, "internal strict transforms" at b of the elements of $\mathcal{H}_1(a)$).

In general, we define

$$\mathcal{F}_1(b) = \mathcal{G}_1(b) \cup \big(E^1(b), 1\big),$$

where $\big(E^1(b), 1\big)$ denotes $\{(y_H, 1) : H \in E^1(b)\}$, and y_H means a local generator of the ideal of H. Then $\big(N_0(b), \mathcal{F}_1(b), \mathcal{E}_1(b)\big)$ is a codimension 0 presentation of $\mathrm{inv}_1 = (\nu_1, s_1)$ at b, and there is a codimension 1 presentation $\big(N_1(b), \mathcal{H}_1(b), \mathcal{E}_1(b)\big)$ as before. The construction of Section 2 above shows that we can choose the coordinates $(y_1, \ldots, y_{n-1})$ of $N_1(b)$ so that each $H \in \mathcal{E}_1(b) = E(b) \setminus E^1(b)$ is $\{y_i = 0\}$, for some $i = 1, \ldots, n-1$; we again write $y_H = y_i$. (In other words, $\mathcal{E}_1(b)$ and $N_1(b)$ simultaneously have normal crossings, and $N_1(b) \not\subset H$, for all $H \in \mathcal{E}_1(b)$.)

Definition 3.4. For each $H \in \mathcal{E}_1(b)$, we set

$$\mu_{2H}(b) = \min_{(h,\mu_h) \in \mathcal{H}_1(b)} \frac{\mu_{H,b}(h)}{\mu_h},$$

where $\mu_{H,b}(h)$ denotes the *order* of h *along* H at b; i.e., the order to which y_H factors from $h \in \mathcal{O}_{N,b}$, $N = N_1(b)$, or $\max\{k : h \in \mathcal{I}^k_{H,b}\}$, where $\mathcal{I}_{H,b}$ is the ideal of $H \cap N$ in $\mathcal{O}_{N,b}$. We define

$$\nu_2(b) = \mu_2(b) - \sum_{H \in \mathcal{E}_1(b)} \mu_{2H}(b).$$

In our example,

$$\nu_2(b) = \mu_2(b) - \mu_{2H_1}(b) = 3 - \tfrac{3}{2} = \tfrac{3}{2}.$$

Write

$$D_2(b) = \prod_{H \in \mathcal{E}_1(b)} y_H^{\mu_{2H}(b)}.$$

Suppose, as before, that all μ_h are equal: say all $\mu_h = d \in \mathbb{N}$. Then $D^d = D_2(b)^d$ is the greatest common divisor of the h that is a monomial in the exceptional coordinates y_H, $H \in \mathcal{E}_1(b)$. For each $h \in \mathcal{H}_1(b)$, write $h = D^d g$ and set $\mu_g = d\nu_2(b)$; then $\mu_b(g) \geq \mu_g$. Clearly, $\nu_2(b) = \min_g \mu_b(g)/d$. Moreover, $0 \leq \nu_2(b) \leq \infty$, and $\nu_2(b) = \infty$ if and only if $\mu_2(b) = \infty$.

If $\nu_2(b) = 0$ or ∞, we put $\mathrm{inv}_X(b) = \mathrm{inv}_{1\frac{1}{2}}(b)$. If $\nu_2(b) = \infty$, then $S_{\mathrm{inv}_X}(b) = N_1(b)$. If $\nu_2(b) = 0$ and we set $\mathcal{G}_2(b) = \{(D_2(b), 1)\}$, then $(N_1(b), \mathcal{G}_2(b), \mathcal{E}_1(b))$ is a codimension 1 presentation of inv_X at b; in particular,

$$S_{\mathrm{inv}_X}(b) = \{y \in N_1(b) : \mu_y(D_2(b)) \geq 1\}$$

(compare Section 2).

Consider the case that $0 < \nu_2(b) < \infty$. Let $\mathcal{G}_2(b)$ denote the collection of pairs $(g, \mu_g) = (g, d\nu_2(b))$ for all $(h, \mu_h) = (h, d)$, as above, together with the pair $(D_2(b)^d, (1 - \nu_2(b))d)$ *provided that* $\nu_2(b) < 1$. Then $(N_1(b), \mathcal{G}_2(b), \mathcal{E}_1(b))$ is a codimension 1 presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at b.

In the latter case, we introduce $E^2(b) \subset \mathcal{E}_1(b)$ as in 1.12, and we set $s_2(b) = \#E^2(b)$, $\mathcal{E}_2(b) = \mathcal{E}_1(b) \setminus E^2(b)$. Set

$$\mathcal{F}_2(b) = \mathcal{G}_2(b) \cup (E^2(b), 1).$$

Then $(N_1(b), \mathcal{F}_2(b), \mathcal{E}_1(b))$ is a codimension 1 presentation of inv_2 at b, and we can pass to a codimension 2 presentation $(N_2(b), \mathcal{H}_2(b), \mathcal{E}_2(b))$. Here it is important to replace $\mathcal{E}_1(b)$ by the subset $\mathcal{E}_2(b)$, to have the property that $\mathcal{E}_2(b)$, $N_2(b)$ simultaneously have normal crossings and $N_2(b) \not\subset H$, for all $H \in \mathcal{E}_2(b)$. (Again, the main rôle of $\mathcal{E}$ in a presentation is to prove invariance of the $\mu_{2H}(\cdot)$ and in general of the $\mu_{3H}(\cdot), \ldots$, as in Section 4.)

Returning to our example (in year one), we have $\mathcal{H}_1(b) = \{(y_1^3 y_2^3, 2)\}$, so that $D_2(b) = y_1^{3/2}$. We can take $\mathcal{G}_2(b) = \{(y_2^3, 3)\}$ or, equivalently, $\mathcal{G}_2(b) = \{(y_2, 1)\}$ to get a codimension 1 presentation $\big(N_1(b), \mathcal{G}_2(b), \mathcal{E}_1(b)\big)$ of $\mathrm{inv}_{1 1/2}$ at b.

Now, $E^2(b) = \{H_1\}$, so that $s_2(b) = 1$. We set

$$\mathcal{F}_2(b) = \mathcal{G}_2(b) \cup \big(E^2(b), 1\big) = \{(y_1, 1), (y_2, 1)\}$$

and $\mathcal{E}_2(b) = \mathcal{E}_1(b) \setminus E^2(b) = \varnothing$. Then $\big(N_1(b), \mathcal{F}_2(b), \mathcal{E}_1(b)\big)$ is a codimension 1 presentation of inv_2 at b, and we can get a codimension 2 presentation $\big(N_2(b), \mathcal{H}_2(b), \mathcal{E}_2(b)\big)$ of inv_2 at b by taking $N_2(b) = \{y_2 = y_3 = 0\}$ and $\mathcal{H}_2(b) = \{(y_1, 1)\}$.

It follows that $\nu_3(b) = 1$. Since $E^3(b) = \varnothing$, $s_3(b) = 0$. We get a codimension 3 presentation of inv_3 at b by taking

$$N_3(b) = \{y_1 = y_2 = y_3 = 0\} = \{b\}, \qquad H_3(b) = \varnothing.$$

Therefore,

$$\mathrm{inv}_X(b) = \big(2, 0;\ \tfrac{3}{2}, 1;\ 1, 0;\ \infty\big)$$

and $S_{\mathrm{inv}_X}(b) = S_{\mathrm{inv}_2}(b) = \{b\}$. The latter is the centre of the next blowing-up σ_2. The set $\sigma_2^{-1}(U_1)$ is covered by three coordinate charts

$$U_{1i} = \sigma_2^{-1}(U_1) \setminus \{y_i = 0\}', \qquad i = 1, 2, 3.$$

For example, U_{12} has coordinates (z_1, z_2, z_3) with respect to which σ_2 is given by

$$y_1 = z_1 z_2, \qquad y_2 = z_2, \qquad y_3 = z_2 z_3.$$

REMARK 3.5. *Zariski-semicontinuity of the invariant.* Each point of M_j, for $j = 0, 1, \ldots$, admits a coordinate neighbourhood U such that, for all $x_0 \in U$, $\{x \in U : \mathrm{inv}_{\cdot}(x) \le \mathrm{inv}_{\cdot}(x_0)\}$ is Zariski-open in U (i.e., the complement of a Zariski-closed subset of U): For $\mathrm{inv}_{1/2}$, this is just Zariski-semicontinuity of the order of a regular function g (a local generator of the ideal of X). For inv_1, the result is a consequence of the following semicontinuity assertion for $E^1(x)$: There is a Zariski-open neighbourhood of x_0 in U, in which $E^1(x) = E(x) \cap E^1(x_0)$, for all $x \in S_{\mathrm{inv}_{1/2}}(x_0) = \{x \in U : \mathrm{inv}_{1/2}(x) \ge \mathrm{inv}_{1/2}(x_0)\}$. (See [Bierstone and Milman 1997, Proposition 6.6] for a simple proof.)

For $\mathrm{inv}_{1 1/2}$: Suppose that $\mu_k = d \in \mathbb{N}$, for all $(h, \mu_h) \in \mathcal{H}_1(x_0)$, as above. Then, in a Zariski-open neighbourhood of x_0 where $S_{\mathrm{inv}_1(x_0)} = \{x : \mathrm{inv}_1(x) = \mathrm{inv}_1(x_0)\}$, we have

$$d\nu_2(x) = \min_{\mathcal{H}_1(x_0)} \mu_x\left(\frac{h}{D_2(x_0)^d}\right), \qquad \text{for } x \in S_{\mathrm{inv}_1(x_0)}.$$

Semicontinuity of $\nu_2(x)$ is thus a consequence of semicontinuity of the order of an element $g = h/D_2(x_0)^d$ such that $\mu_{x_0}(g) = d\nu_2(x_0)$.

Likewise for inv_2, $\mathrm{inv}_{2^1\!/_2}$, ...

Year two. Let X_2 denote the strict transform X_1' of X_1 by σ_2. Then $X_2 \cap U_{12} = V(g_2)$, where

$$g_2 = z_2^{-2} g_1 \circ \sigma_2 = z_3^2 - z_1^3 z_2^4.$$

Let c be the origin of U_{12}. Then $E(c) = \{H_1, H_2\}$ where

$$\begin{aligned} H_1 &= \{y_1 = 0\}' = \{z_1 = 0\}, \\ H_2 &= \sigma_2^{-1}(b) = \{z_2 = 0\}. \end{aligned}$$

We have $\nu_1(c) = 2 = \nu_1(a)$. Therefore, $E^1(c) = \varnothing$, $s_1(c) = 0$, $\mathcal{E}_1(c) = E(c)$. $\mathcal{F}_1(c) = \mathcal{G}_1(c) = \{(g_2, 2)\}$ provides a codimension 0 presentation of inv_1 at c, and we get a codimension 1 presentation by taking

$$N_1(c) = \{z_3 = 0\}, \qquad \mathcal{H}_1(c) = \{(z_1^3 z_2^4, 2)\}.$$

Therefore $\mu_2(c) = \frac{7}{2}$, $\mu_{2H_1}(c) = \frac{3}{2}$ and $\mu_{2H_2}(c) = \frac{4}{2} = 2$, so that $\nu_2(c) = 0$ and

$$\mathrm{inv}_X(c) = (2, 0; 0).$$

Moreover, $D_2(c) = z_1^{3/2} z_2^2$, and we get a codimension 1 presentation of $\mathrm{inv}_X = \mathrm{inv}_{1^1\!/_2}$ at c using

$$N_1(c) = \{z_3 = 0\}, \qquad \mathcal{G}_2(c) = \{(z_1^{3/2} z_2^2, 1)\}.$$

Therefore,

$$S_{\mathrm{inv}_X}(c) = S_{\mathrm{inv}_{1^1\!/_2}}(c) = \{z_1 = z_3 = 0\} \cup \{z_2 = z_3 = 0\};$$

of course, $\{z_1 = z_3 = 0\} = S_{\mathrm{inv}_X}(c) \cap H_1$ and $\{z_2 = z_3 = 0\} = S_{\mathrm{inv}_X}(c) \cap H_2$.

REMARK 3.6. In general, suppose that $\mathrm{inv}_X(c) = \mathrm{inv}_{t+1/2}(c)$ and $v_{t+1}(c) = 0$. (We assume $c \in M_j$, for some $j = 1, 2, \ldots$.) Then inv_X has a codimension t presentation at c: $N_t(c) = \{z_{n-t+1} = \cdots = z_n = 0\}$, $\mathcal{G}_{t+1}(c) = \{(D_{t+1}(c), 1)\}$, where $D_{t+1}(c)$ is a monomial with rational exponents in the exceptional divisors z_H, $H \in \mathcal{E}_t(c)$; $N_t(c)$ has coordinates $(z_1, \ldots, z_{n-t})$ in which each such $z_H = z_i$, for some $i = 1, \ldots, n-t$. It follows that each component Z of $S_{\mathrm{inv}_X}(c)$ has the form

$$Z = S_{\mathrm{inv}_X}(c) \cap \bigcap \{H \in E(c) : Z \subset H\};$$

we will write $Z = Z_I$, where $I = \{H \in E(c) : Z \subset H\}$. It follows that, if U is any open neighbourhood of c on which $\mathrm{inv}_X(c)$ is a maximum value of inv_X, then every component Z_I of $S_{\mathrm{inv}_X}(c)$ extends to a global smooth closed subset of U:

First consider any total order on $\{I : I \subset E_j\}$. For any $c \in M_j$, set

$$J(c) = \max\{I : Z_I \text{ is a component of } S_{\mathrm{inv}_X}(c)\},$$
$$\mathrm{inv}^{\mathrm{e}}_X(c) = \big(\mathrm{inv}_X(c);\ J(c)\big).$$

Then $\mathrm{inv}^{\mathrm{e}}_X$ is Zariski-semicontinuous (again comparing values of $\mathrm{inv}^{\mathrm{e}}_X$ lexicographically), and its locus of maximum values on any given open subset of M_j is smooth.

Of course, given $c \in M_j$ and a component Z_I of $S_{\mathrm{inv}_X}(c)$, we can choose the ordering of $\{J : J \subset E_j\}$ so that $I = J(c) = \max\{J : J \subset E_j\}$. It follows that, if U is any open neighbourhood of c on which $\mathrm{inv}_X(c)$ is a maximum value of inv_X, then Z_I extends to a smooth closed subset of U.

To obtain an algorithm for canonical desingularization, we can choose as each successive centre of blowing up the maximum locus of

$$\mathrm{inv}^{\mathrm{e}}_X(\,\cdot\,) = \big(\mathrm{inv}_X(\,\cdot\,), J(\,\cdot\,)\big),$$

where J is defined as above using the following total ordering of the subsets of E_j: Write $E_j = \{H^j_1, \ldots, H^j_j\}$, where each H^j_i is the strict transform in M_j of the exceptional hypersurface $H^i_i = \sigma_i^{-1}(C_{i-1}) \subset M_i$, $i = 1, \ldots, j$. We can order $\{I : I \subset E_j\}$ by associating to each subset I the lexicographic order of the sequence $(\delta_1, \ldots, \delta_j)$, where $\delta_i = 0$ if $H^j_i \notin I$ and $\delta_i = 1$ if $H^j_i \in I$.

In our example, year two, we have

$$S_{\mathrm{inv}_X}(c) = \big(S_{\mathrm{inv}_X}(c) \cap H_1\big) \cup \big(S_{\mathrm{inv}_X}(c) \cap H_2\big).$$

(Each H_i is H^2_i in the notation preceding.) The order of H_1 (respectively, H_2) is $(1, 0)$ (respectively, $(0, 1)$), so that $J(c) = \{H_1\}$ and the centre of the third blowing-up σ_3 is $C_2 = S_{\mathrm{inv}_X}(c) \cap H_1 = \{z_1 = z_3 = 0\}$.

Thus $\sigma_3^{-1}(U_{12}) = U_{121} \cup U_{123}$, where $U_{12i} = \sigma_3^{-1}(U_{12}) \backslash \{z_i = 0\}'$, $i = 1, 3$. The strict transform of $X_2 \cap U_{12}$ lies in U_{121}; the latter has coordinates (w_1, w_2, w_3) in which σ_3 can be written

$$z_1 = w_1, \qquad z_2 = w_2, \qquad z_3 = w_1 w_3.$$

Year three. Let X_3 denote the strict transform of X_2 by σ_3. Then $X_3 \cap U_{121} = V(g_3)$, where $g_3(w) = w_3^2 - w_1 w_2^4$. Let $d = 0$ in U_{121}. There are three exceptional hypersurfaces $H_1 = \{z_1 = 0\}'$, $H_2 = \{z_2 = 0\}' = \{w_2 = 0\}$ and $H_3 = \sigma_3^{-1}(C_2) = \{w_1 = 0\}$; since $H_1 \not\ni d$, $E(d) = \{H_2, H_3\}$. We have $\nu_1(d) = 2 = \nu_1(a)$. Therefore, $E^1(d) = \varnothing$, $s_1(d) = 0$ and $\mathcal{E}_1(d) = E(d)$. $\mathcal{F}_1(d) = \mathcal{G}_1(d) = \{(g_3, 2)\}$ provides a codimension 0 presentation of inv_1 at d, and we get a codimension 1 presentation by taking

$$N_1(d) = \{w_3 = 0\}, \qquad \mathcal{H}_1(d) = \{(w_1 w_2^4, 2)\}.$$

Therefore, $\mu_2(c) = \frac{5}{2}$ and $D_2(d) = w_1^{1/2}w_2^2$, so that $\nu_2(d) = 0$ and

$$\mathrm{inv}_X(d) = (2, 0, 0) = \mathrm{inv}_X(c)!$$

However,

$$\mu_2(d) = \tfrac{5}{2} < \tfrac{7}{2} = \mu_2(c);$$

i.e., $1 \le \mu_X(d) < \mu_X(c)$, where $\mu_X = \mu_2$ (compare (2.8) and following). We get a codimension 1 presentation of $\mathrm{inv}_X = \mathrm{inv}_{1\,1/2}$ at d by taking

$$N_1(d) = \{w_3 = 0\}, \qquad \mathcal{G}_2(d) = \big\{\big(D_2(d), 1\big)\big\}.$$

Therefore,

$$S_{\mathrm{inv}_X}(d) = S_{\mathrm{inv}_1}(d) = \{w_2 = w_3 = 0\},$$

so we let σ_4 be the blowing-up with centre $C_3 = \{w_2 = w_3 = 0\}$. Then $\sigma_4^{-1}(U_{121}) = U_{1212} \cup U_{1213}$, where $U_{121i} = \sigma_4^{-1}(U_{121}) \setminus \{w_i = 0\}'$, $i = 2, 3$; U_{1212} has coordinates (v_1, v_2, v_3) in which σ_4 is given by

$$w_1 = v_1, \qquad w_2 = v_2, \qquad w_3 = v_2 v_3.$$

Year four. Let X_4 be the strict transform of X_3. Then $X_4 \cap U_{1212} = V(g_4)$, where $g_4(v) = v_3^2 - v_1 v_2^2$. Let $e = 0$ in U_{1212}. Then $E(e) = \{H_3, H_4\}$, where $H_3 = \{w_1 = 0\}' = \{v_1 = 0\}$ and $H_4 = \sigma_4^{-1}(C_3) = \{v_2 = 0\}$. Again $\nu_1(e) = 2 = \nu_1(a)$, so that $E^1(e) = \varnothing$, $s_1(e) = 0$ and $\mathcal{E}_1(e) = E(e)$. Calculating as above, we obtain $\mu_2(e) = \frac{3}{2}$ and $D_2(e) = v_1^{1/2}v_2$, so that $\nu_2(e) = 0$ and $\mathrm{inv}_X(e) = (2, 0; 0)$ again. But now $\mu_X(e) = \mu_2(e) = \frac{3}{2}$. Our invariant inv_X is presented at e by

$$N_1(e) = \{v_3 = 0\}, \qquad \mathcal{G}_2(e) = \{(v_1^{1/2}v_2, 1)\}.$$

Therefore, $S_{\mathrm{inv}_X}(e) = \{v_2 = v_3 = 0\}$. Taking as σ_5 the blowing-up with centre $C_4 = S_{\mathrm{inv}_X}(e)$, the strict transform X_5 becomes smooth (over U_{1212}). ($\mu_2(e) - 1 < 1$, so $\nu_1(\,\cdot\,)$ must decrease over C_4.)

Further blowings-up are still needed to obtain the stronger assertion of embedded resolution of singularities.

Remark 3.7. The hypersurface $V(g_4)$ in year four above is called *Whitney's umbrella*. Consider the same hypersurface $X = \{x_3^2 - x_1 x_2^2 = 0\}$ but without a history of blowings-up; i.e., $E(\,\cdot\,) = \varnothing$. Let $a = 0$. In this case, $\mathrm{inv}_{1\,1/2}(a) = (2, 0; \frac{3}{2})$, and we get a codimension 1 presentation of $\mathrm{inv}_{1\,1/2}$ at a using

$$N_1(a) = \{x_3 = 0\}, \qquad \mathcal{G}_2(a) = \{(x_1 x_2^2, 3)\}$$

or, equivalently, $\mathcal{G}_2(a) = \{(x_1, 1), (x_2, 1)\}$, as in year zero of Example 3.1. Therefore,

$$\mathrm{inv}_X(a) = \big(2, 0;\ \tfrac{3}{2}, 0;\ 1, 0;\ \infty\big).$$

As a centre of blowing up we would choose $C = S_{\mathrm{inv}_X}(a) = \{a\}$ — not the x_1-axis as in year four above, although the singularity is the same!

4. Key Properties of the Invariant

Our main goal in this section is to explain why $\mathrm{inv}_X(a)$ is indeed an invariant. Once we establish invariance, the Embedded Desingularization Theorem A follows directly from local properties of inv_X. The crucial properties have already been explained in Section 3 above; we summarize them in the following theorem.

THEOREM B [Bierstone and Milman 1997, Theorem 1.14]. *Consider any sequence of* inv_X*-admissible* (*local*) *blowings-up* (1.8). *Then the following properties hold*:

(1) Semicontinuity:

(i) *For each* j, *every point of* M_j *admits a neighbourhood* U *such that* inv_X *takes only finitely many values in* U *and*, *for all* $a \in U$, $\{x \in U : \mathrm{inv}_X(x) \leq \mathrm{inv}_X(a)\}$ *is Zariski-open in* U.

(ii) inv_X *is* infinitesimally upper-semicontinuous *in the sense that* $\mathrm{inv}_X(a) \leq \mathrm{inv}_X\big(\sigma_j(a)\big)$ *for all* $a \in M_j$, $j \geq 1$.

(2) Stabilization: *Given* $a_j \in M_j$ *such that* $a_j = \sigma_{j+1}(a_{j+1})$, $j = 0, 1, 2, \dots$, *there exists* j_0 *such that* $\mathrm{inv}_X(a_j) = \mathrm{inv}_X(a_{j+1})$ *when* $j \geq j_0$. (*In fact*, *any nonincreasing sequence in the value set of* inv_X *stabilizes*.)

(3) Normal crossings: *Let* $a \in M_j$. *Then* $S_{\mathrm{inv}_X}(a)$ *and* $E(a)$ *simultaneously have only normal crossings*. *Suppose* $\mathrm{inv}_X(a) = \big(\dots; \nu_{t+1}(a)\big)$. *If* $\nu_{t+1}(a) = \infty$, *then* $S_{\mathrm{inv}_X}(a)$ *is smooth*. *If* $\nu_{t+1}(a) = 0$ *and* Z *denotes any irreducible component of* $S_{\mathrm{inv}_X}(a)$, *then*

$$Z = S_{\mathrm{inv}_X}(a) \cap \bigcap \{H \in E(a) : Z \subset H\}.$$

(4) Decrease: *Let* $a \in M_j$ *and suppose* $\mathrm{inv}_X(a) = \big(\dots; \nu_{t+1}(a)\big)$. *If* $\nu_{t+1}(a) = \infty$ *and* σ *is the local blowing-up of* M_j *with centre* $S_{\mathrm{inv}_X}(a)$, *then* $\mathrm{inv}_X(a') < \mathrm{inv}_X(a)$ *for all* $a' \in \sigma^{-1}(a)$. *If* $\nu_{t+1}(a) = 0$, *then there is an additional invariant* $\mu_X(a) = \mu_{t+1}(a) \geq 1$ *such that*, *if* Z *is an irreducible component of* $S_{\mathrm{inv}_X}(a)$ *and* σ *is the local blowing-up with centre* Z, *then*

$$\big(\mathrm{inv}_X(a'), \mu_X(a')\big) < \big(\mathrm{inv}_X(a), \mu_X(a)\big)$$

for all $a' \in \sigma^{-1}(a)$. (*We have* $e_t!\mu_X(a) \in \mathbb{N}$, *where* e_t *is defined as in Section* 1 *or in the proof following*.)

PROOF. The semicontinuity property (1)(i) has been explained in Remark 3.5. Infinitesimal upper-semicontinuity (1)(ii) is immediate from the definition of the $s_r(a)$ and from infinitesimal upper-semicontinuity of the order of a function on blowing up locally with smooth centre in its equimultiple locus. (The latter property is an elementary Taylor series computation, and is also clear from the calculation in Section 2 above.)

The stabilization property (2) for $\mathrm{inv}_{1/2}$ is obvious in the hypersurface case because then $\mathrm{inv}_{1/2}(a) = \nu_1(a) \in \mathbb{N}$. (In the general case, we need to begin with stabiization of the Hilbert–Samuel function; see [Bierstone and Milman 1989, Theorem 5.2.1] for a very simple proof of this result due originally to Bennett [1970].) The stabilization assertion for inv_X follows from that for $\mathrm{inv}_{1/2}$ and from infinitesimal semicontinuity because, although $\nu_{r+1}(a)$, for each $r > 0$, is perhaps only rational, our construction in Section 3 shows that $e_r!\nu_{r+1}(a) \in \mathbb{N}$, where $e_1 = \nu_1(a)$ and $e_{r+1} = \max\{e_r!, e_r!\nu_{r+1}(a)\}$, $r > 0$. (In the general case, the Hilbert–Samuel function $H_{X_j,a}(l)$ coincides with a polynomial if $l \geq k$, for k large enough, and we can take as e_1 the least such k.)

The normal crossings condition (3) has also been explained in Section 3; see Remark 3.6, in particular, for the case that $\nu_{t+1}(a) = 0$. The calculation in Section 2 then gives the property of decrease (4), as is evident also in the example of Section 3. □

When our spaces satisfy a compactness assumption (so that inv_X takes maximum values), it follows from Theorem B that we can obtain the Embedded Desingularization Theorem A by simply applying the algorithm of 1.11 above, stopping when inv_X becomes (locally) constant. To be more precise, let $\mathrm{inv}_X^{\mathrm{e}}$ denote the extended invariant for canonical desingularization introduced in Remark 3.6. Consider a sequence of blowings-up (1.8) with inv_X-admissible centres. Note that if X_j is not smooth and $a \in \mathrm{Sing}\, X_j$, then $S_{\mathrm{inv}_X}(a) \subset \mathrm{Sing}\, X_j$ because ν_1 (or, in general, $H_{X_j,a}$) already distinguishes between smooth and singular points. Since $\mathrm{Sing}\, X_j$ is Zariski-closed, it follows that if C_j denotes the locus of maximum values of $\mathrm{inv}_X^{\mathrm{e}}$ on $\mathrm{Sing}\, X_j$, then C_j is smooth. By Theorem B, there is a finite sequence of blowings-up with such centres, after which X_j is smooth.

On the other hand, if X_j is smooth and $a \in S_j$, where $S_j = \{x \in X_j : s_1(x) > 0\}$, then $S_{\mathrm{inv}_X}(a) \subset S_j$. Since S_j is Zariski-closed, it follows that if C_j denotes the locus of maximum values of $\mathrm{inv}_X^{\mathrm{e}}$ on S_j, then C_j is smooth. Therefore, after finitely many further blowings-up $\sigma_{j+1}, \ldots, \sigma_k$ with such centres, $S_k = \varnothing$. It is clear from the definition of s_1 that, if X_k is smooth and $S_k = \varnothing$, then each $H \in E_k$ which intersects X_k is the strict transform in M_k of $\sigma_{i+1}^{-1}(C_i)$, for some i such that X_i is smooth along C_i; therefore, X_k and E_k simultaneously have only normal crossings, and we have Theorem A.

We will prove invariance of inv_X using the idea of a "presentation" introduced in Section 3 above. It will be convenient to consider "presentation" in an abstract sense, rather than associated to a particular invariant: Let M denote a manifold and let $a \in M$.

Definitions 4.1. An abstract (*infinitesimal*) *presentation of codimension* p at a means simply a triple $(N = N_p(a),\ \mathcal{H}(a),\ \mathcal{E}(a))$ as in Section 3; namely: N is a germ of a submanifold of codimension p at a, $\mathcal{H}(a)$ is a finite collection of pairs (h, μ_h), where $h \in \mathcal{O}_{N,a}$, $\mu_h \in \mathbb{Q}$ and $\mu_a(h) \geq \mu_h$, and $\mathcal{E}(a)$ is a finite set of smooth hypersurfaces containing a, such that N and $\mathcal{E}(a)$ simultaneously have normal crossings and $N \not\subset H$, for all $H \in \mathcal{E}(a)$.

A local blowing-up σ with centre $C \ni a$ will be called *admissible* (for an infinitesimal presentation as above) if $C \subset S_{\mathcal{H}(a)} = \{x \in N : \mu_x(h) \geq \mu_h$, for all $(h, \mu_h) \in \mathcal{H}(a)\}$.

Definition 4.2. We will say that two infinitesimal presentations ($N = N_p(a)$, $\mathcal{H}(a)$, $\mathcal{E}(a)$) and ($P = P_q(a)$, $\mathcal{F}(a)$, $\mathcal{E}(a)$) with given $\mathcal{E}(a)$, but not necessarily of the same codimension, are *equivalent* if (in analogy with Definition 3.2):

(1) $S_{\mathcal{H}(a)} = S_{\mathcal{F}(a)}$, as germs at a in M.
(2) If σ is an admissible local blowing-up and $a' \in \sigma^{-1}(a)$, then $a' \in N'$ and $\mu_{a'}(y_{\mathrm{exc}}^{-\mu_h} h \circ \sigma) \geq \mu_h$ for all $(h, \mu_h) \in \mathcal{H}(a)$ if and only if $a' \in P'$ and

$$\mu_{a'}(y_{\mathrm{exc}}^{-\mu_f} f \circ \sigma) \geq \mu_f \qquad \text{for all } (f, \mu_f) \in \mathcal{F}(a).$$

(3) Conditions (1) and (2) continue to hold for the transforms ($N_p(a')$, $\mathcal{H}(a')$, $\mathcal{E}(a')$) and ($P_q(a')$, $\mathcal{F}(a')$, $\mathcal{E}(a')$) of our data by sequences of morphisms of types (i), (ii) and (iii) as in Definition 3.2.

We will, in fact, impose a further condition on the way that exceptional blowings-up (iii) are allowed to occur in a sequence of transformations in condition (3) above; see Definition 4.5 below.

Our proof of invariance of inv_X follows the constructive definition outlined in Section 3. Let X denote a hypersurface in M, and consider any sequence of blowings-up (or local blowings-up) (1.8), where we assume (at first) that the centres of blowing up are $\frac{1}{2}$-admissible. Let $a \in M_j$, for some $j = 0, 1, 2, \ldots$. Suppose that $g \in \mathcal{O}_{M_j,a}$ generates the local ideal $\mathcal{I}_{X_j,a}$ of X_j at a, and let $\mu_g = \mu_a(g)$. Then, as in Section 3, $\mathcal{G}_1(a) = \{(g, \mu_g)\}$ determines a codimension zero presentation ($N_0(a)$, $\mathcal{G}_1(a)$, $\mathcal{E}_0(a)$) of $\mathrm{inv}_{1/2} = \nu_1$ at a, where $N_0(a)$ is the germ of M_j at a, and $\mathcal{E}_0(a) = \varnothing$. In particular, the equivalence class of ($N_0(a)$, $\mathcal{G}_1(a)$, $\mathcal{E}_0(a)$) in the sense of Definition 4.2 depends only on the local isomorphism class of (M_j, X_j) at a.

We introduce $E^1(a)$ as in 1.12 above, and let $s_1(a) = \#E^1(a)$, $\mathcal{E}_1(a) = E(a) \backslash E^1(a)$. Let

$$\mathcal{F}_1(a) = \mathcal{G}_1(a) \cup \big(E^1(a), 1\big),$$

where $\big(E^1(a), 1\big)$ denotes $\{(x_H, 1) : H \in E^1(a)\}$ and x_H means a local generator of the ideal of H. Then ($N_0(a)$, $\mathcal{F}_1(a)$, $\mathcal{E}_1(a)$) is a codimension zero presentation of $\mathrm{inv}_1 = (\nu_1, s_1)$ at a. Clearly, the equivalence class of ($N_0(a)$, $\mathcal{F}_1(a)$, $\mathcal{E}_1(a)$) depends only on the local isomorphism class of (M_j, X_j, E_j, $E^1(a)$). Moreover, ($N_0(a)$, $\mathcal{F}_1(a)$, $\mathcal{E}_1(a)$) has an equivalent codimension one presentation ($N_1(a)$, $\mathcal{H}_1(a)$, $\mathcal{E}_1(a)$) as described in Section 3. For example, let $a_k = \pi_{kj}(a)$, for $k = 0, \ldots, j$, as in 1.12, and let i denote the "earliest year" k such that $\mathrm{inv}_{1/2}(a) = \mathrm{inv}_{1/2}(a_k)$. Then $\mathcal{E}_1(a_i) = \varnothing$. As in Section 3, we can take $N_1(a_i) =$ any hypersurface of maximal contact for X_i at a_i. If $(x_1, \ldots, x_n)$ are local coordinates for M_i with respect to which $N_1(a_i) = \{x_n = 0\}$, then we can take

$$\mathcal{H}_1(a_i) = \left\{ \left(\frac{\partial^q f}{\partial x_n^q}\Big|_{N_1(a_i)},\ \mu_f - q \right) : 0 \le q < \mu_f,\ (f, \mu_f) \in \mathcal{F}_1(a_i) \right\}.$$

A codimension one presentation $(N_1(a),\ \mathcal{H}_1(a),\ \mathcal{E}_1(a))$ of inv_1 at a can be obtained by transforming $(N_1(a_i),\ \mathcal{H}_1(a_i),\ \mathcal{E}_1(a_i))$ to a. The condition that $N_1(a)$ and $\mathcal{E}_1(a)$ simultaneously have normal crossings and $N_1(a) \not\subset H$ for all $H \in \mathcal{E}_1(a)$ is a consequence of the effect of blowing with smooth centre of codimension at least 1 in $N(a_k)$, $i \le k < j$ (as in the calculation in Section 2).

Say that $\mathcal{H}_1(a) = \{(h, \mu_h)\}$; each $h \in \mathcal{O}_{N_1(a),a}$ and $\mu_h \le \mu_a(h)$. Recall that we define

$$\mu_2(a) = \min_{\mathcal{H}_1(a)} \frac{\mu_a(h)}{\mu_h},$$

$$\mu_{2H}(a) = \min_{\mathcal{H}_1(a)} \frac{\mu_{H,a}(h)}{\mu_h}, \qquad H \in \mathcal{E}_1(a),$$

$$\text{and} \qquad \nu_2(a) = \mu_2(a) - \sum_{H \in \mathcal{E}_1(a)} \mu_{2H}(a)$$

(Definitions 3.2, 3.4). Propositions 4.4 and 4.6 below show that each of $\mu_2(a)$ and $\mu_{2H}(a)$, $H \in \mathcal{E}_1(a)$, depends only on the equivalence class of $(N_1(a),\ \mathcal{H}_1(a),\ \mathcal{E}_1(a))$, and thus only on the local isomorphism class of $(M_j,\ X_j,\ E_j,\ E^1(a))$.

If $\nu_2(a) = 0$ or ∞, then we set $\mathrm{inv}_X(a) = \mathrm{inv}_{1^1\!/_2}(a)$. If $0 < \nu_2(a) < \infty$, then we construct a codimension one presentation $(N_1(a),\ \mathcal{G}_2(a),\ \mathcal{E}_1(a))$ of $\mathrm{inv}_{1^1\!/_2}$ at a, as in Section 3. From the construction, it is not hard to see that the equivalence class of $(N_1(a),\ \mathcal{G}_2(a),\ \mathcal{E}_1(a))$ depends only on that of $(N_1(a),\ \mathcal{H}_1(a),\ \mathcal{E}_1(a))$. (See [Bierstone and Milman 1997, 4.23 and 4.24] as well as Proposition 4.6 ff. below.)

This completes a cycle in the inductive definition of inv_X. Assume now that the centres of the blowings-up in (1.8) are $1\frac{1}{2}$-admissible. We introduce $E^2(a)$ as in 1.12, and let $s_2(a) = \#E^2(a)$, $\mathcal{E}_2(a) = \mathcal{E}_1(a) \backslash E^2(a)$. If $\mathcal{F}_2(a) = \mathcal{G}_2(a) \cup \big(E^2(a), 1\big)$, where $\big(E^2(a), 1\big)$ denotes $\{(x_H|_{N_1(a)}, 1) : H \in E_2(a)\}$, then $(N_1(a), \mathcal{F}_2(a), \mathcal{E}_2(a))$ is a codimension one presentation of $\mathrm{inv}_2 = (\mathrm{inv}_{1^1\!/_2}, s_2)$ at a, whose equivalence class depends only on the local isomorphism class of $(M_j,\ X_j,\ E_j,\ E^1(a),\ E^2(a))$. It is clear from the construction of $\mathcal{G}_2(a)$ that $\mu_{\mathcal{G}_2(a)} = 1$, where

$$\mu_{\mathcal{G}_2(a)} = \min_{(g,\mu_g) \in \mathcal{G}_2(a)} \frac{\mu_a(g)}{\mu_g}.$$

Therefore $(N_1(a),\ \mathcal{F}_2(a),\ \mathcal{E}_2(a))$ admits an equivalent codimension two presentation $(N_2(a),\ \mathcal{H}_2(a),\ \mathcal{E}_2(a))$, and we define $\nu_3(a) = \mu_3(a) - \sum_{H \in \mathcal{E}_2(a)} \mu_{3H}(a)$, as above. By Propositions 4.4 and 4.6, $\mu_3(a)$ and each $\mu_{3H}(a)$ depend only on the equivalence class of $(N_2(a),\ \mathcal{H}_2(a),\ \mathcal{E}_2(a))$, ... We continue until $\nu_{t+1}(a) = 0$ or ∞ for some t, and then take $\mathrm{inv}_X(a) = \mathrm{inv}_{t+1/2}(a)$.

Invariance of inv_X thus follows from Propositions 4.4 and 4.6 below, which are formulated purely in terms of an abstract infinitesimal presentation.

Let M be a manifold, and let $(N(a),\, \mathcal{H}(a),\, \mathcal{E}(a))$ be an infinitesimal presentation of codimension $r \geq 0$ at a point $a \in M$. We write $\mathcal{H}(a) = \{(h, \mu_h)\}$, where $\mu_a(h) \geq \mu_h$ for all (h, μ_h).

Definitions 4.3. We define $\mu(a) = \mu_{\mathcal{H}(a)}$ as

$$\mu_{\mathcal{H}(a)} = \min_{\mathcal{H}(a)} \frac{\mu_a(h)}{\mu_h}.$$

Thus $1 \leq \mu(a) \leq \infty$. If $\mu(a) < \infty$, then we define $\mu_H(a) = \mu_{\mathcal{H}(a),H}$, for each $H \in \mathcal{E}(a)$, as

$$\mu_{\mathcal{H}(a),H} = \min_{\mathcal{H}(a)} \frac{\mu_{H,a}(h)}{\mu_h}.$$

We will show that each of $\mu(a)$ and the $\mu_H(a)$ depends only on the equivalence class of $(N(a),\, \mathcal{H}(a),\, \mathcal{E}(a))$ (where we consider only *presentations of the same codimension* r). The main point is that $\mu(a)$ and the $\mu_H(a)$ can be detected by "test blowings-up" (test transformations of the form (i), (ii), (iii) as allowed by the definition 4.2 of equivalence).

For $\mu(a)$, we show in fact that if $(N^i(a),\, \mathcal{H}^i(a),\, \mathcal{E}(a))$, for $i = 1, 2$, are two infinitesimal presentations of the same codimension r, then $\mu_{\mathcal{H}^1(a)} = \mu_{\mathcal{H}^2(a)}$ if the presentations are equivalent with respect to transformations of types (i) and (ii) alone (i.e., where we allow only transformations of types (i) and (ii) in Definition 4.2). This ia a stronger condition than invariance under equivalence in the sense of Definition 4.2 (using all three types of transformations) because the equivalence class with respect to transformations of types (i) and (ii) alone is, of course, larger than the equivalence class with respect to transformations of all three types (i), (ii) and (iii).

Proposition 4.4. [Bierstone and Milman 1997, Proposition 4.8]. $\mu(a)$ *depends only on the equivalence class of* $(N(a),\, \mathcal{H}(a),\, \mathcal{E}(a))$ (*among presentations of the same codimension* r) *with respect to transformations of types* (*i*) *and* (*ii*).

Proof. Clearly, $\mu(a) = \infty$ if and only if $S_{\mathcal{H}(a)} = N(a)$; i.e., if and only if $S_{\mathcal{H}(a)}$ is (a germ of) a submanifold of codimension r in M.

Suppose that $\mu(a) < \infty$. We can assume that $\mathcal{H}(a) = \{(h, \mu_h)\}$ where all $\mu_h = e$, for some $e \in \mathbb{N}$. Let σ_0: $P_0 = M \times \mathbb{K} \to M$ be the projection from the product with a line (i.e., a morphism of type (ii)) and let $(N(c_0),\, \mathcal{H}(c_0),\, \mathcal{E}(c_0))$ denote the transform of $(N(a),\, \mathcal{H}(a),\, \mathcal{E}(a))$ at $c_0 = (a, 0) \in P_0$; i.e., $N(c_0) = N(a) \times \mathbb{K}$, $\mathcal{E}(c_0) = \{H \times \mathbb{K}$, for all $H \in \mathcal{E}(a)$, and $M \times \{0\}\}$ and $\mathcal{H}(c_0) = \{(h \circ \sigma_0, \mu_h)\colon (h, \mu_h) \in \mathcal{H}(a)\}$. We follow σ_0 by a sequence of admissible blowings-up (morphisms of type (i)),

$$\longrightarrow P_{\beta+1} \xrightarrow{\sigma_{\beta+1}} P_\beta \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\sigma_1} P_0,$$

where each $\sigma_{\beta+1}$ is a blowing-up with centre a point $c_\beta \in P_\beta$ determined as follows: Let γ_0 denote the arc in P_0 given by $\gamma_0(t) = (a, t)$. For $\beta \geq 1$, define $\gamma_{\beta+1}$ inductively as the lifting of γ_β to $P_{\beta+1}$, and set $c_{\beta+1} = \gamma_{\beta+1}(0)$.

We can choose local coordinates $(x_1, \ldots, x_n)$ for M at a, in which $a = 0$ and $N(a) = \{x_{n-r+1} = \cdots = x_n = 0\}$. Write $(x, t) = (x_1, \ldots, x_{n-r}, t)$ for the corresponding coordinate system of $N(c_0)$. In P_1, the strict transform $N(c_1)$ of $N(c_0)$ has a local coordinate system $(x, t) = (x_1, \ldots, x_{n-r}, t)$ at c_1 with respect to which $\sigma_1(x, t) = (tx, t)$, and $\gamma_1(t) = (0, t)$ in this coordinate chart; moreover, $\mathcal{H}(c_1) = \{(t^{-e}h(tx), e)$, for all $(h, \mu_h) = (h, e) \in \mathcal{H}(a)\}$. After β blowings-up as above, $N(c_\beta)$ has a local coordinate system $(x, t) = (x_1, \ldots, x_{n-r}, t)$ with respect to which $\sigma_1 \circ \cdots \circ \sigma_\beta$ is given by $(x, t) \mapsto (t^\beta x, t)$, $\gamma_\beta(t) = (0, t)$ and $\mathcal{H}(c_\beta) = \{(h', \mu_{h'} = e)\}$, where

$$h' = t^{-\beta e} h(t^\beta x),$$

for all $(h, \mu_h) = (h, e) \in \mathcal{H}(a)$. By the definition of $\mu(a)$, each

$$h(t^\beta x) = t^{\beta\mu(a)e} \tilde{h}'(x, t),$$

where the $\tilde{h}'(x, t)$ do not admit t as a common divisor; for each $(h, \mu_h) \in \mathcal{H}(a)$, we have

$$h' = t^{\beta(\mu(a)-1)e} \tilde{h}'.$$

We now introduce a subset S of $\mathbb{N} \times \mathbb{N}$ depending only on the equivalence class of $(N(a),\ \mathcal{H}(a),\ \mathcal{E}(a))$ (with respect to transformations of types (i) and (ii)) as follows: First, we say that $(\beta, 0) \in S$, for $\beta \geq 1$, if after β blowings-up as above, there exists (a germ of) a submanifold W_0 of codimension r in the exceptional hypersurface $H_\beta = \sigma_\beta^{-1}(c_{\beta-1})$ such that $W_0 \subset S_{\mathcal{H}(c_\beta)}$. If so, then necessarily $W_0 = H_\beta \cap N(c_\beta) = \{t = 0\}$, and the condition that $W_0 \subset S_{\mathcal{H}(c_\beta)}$ means precisely that $\mu_{W_0, c_\beta}(h') \geq e$, for all h'; i.e., that $\beta\big(\mu(a) - 1\big)e \geq e$, or $\beta\big(\mu(a) - 1\big) \geq 1$. (In particular, since $\mu(a) \geq 1$, $(\beta, 0) \notin S$ for all $\beta \geq 1$ if and only if $\mu(a) = 1$.)

Suppose that $(\beta, 0) \in S$, for some $\beta \geq 1$, as above. Then we can blow up P_β locally with centre W_0. Set $Q_0 = P_\beta$, $d_0 = c_\beta$ and $\delta_0 = \gamma_\beta$. Let τ_1: $Q_1 \to Q_0$ denote the local blowing-up with centre W_0, and let $d_1 = \delta_1(0)$, where δ_1 denotes the lifting of δ_0 to Q_1. (Then $\tau_1|N(d_1)$: $N(d_1) \to N(d_0)$ is the identity.) We say that $(\beta, 1) \in S$ if there exists a submanifold W_1 of codimension r in the hypersurface $H_1 = \tau_1^{-1}(W_0)$ such that $W_1 \subset S_{\mathcal{H}(d_1)}$. If so, then again necessarily $W_0 = H_1 \cap N(d_1) = \{t = 0\}$. Since $\mathcal{H}(d_1) = \{(h', e)\}$, where each $h' = t^{\beta(\mu(a)-1)e-e}\tilde{h}'$ and the $\tilde{h}'$ do not admit t as a common factor, it follows that $(\beta, 1) \in S$ if and only if $\beta\big(\mu(a) - 1\big)e - e \geq e$.

We continue inductively: If $\alpha \geq 1$ and $(\beta, \alpha - 1) \in S$, let τ_α: $Q_\alpha \to Q_{\alpha-1}$ denote the local blowing-up with centre $W_{\alpha-1}$, and let $d_\alpha = \delta_\alpha(0)$, where δ_α is the lifting of $\delta_{\alpha-1}$ to Q_α. We say that $(\beta, \alpha) \in S$ if there exists (a germ of) a submanifold W_α of codimension r in the exceptional hypersurface $H_\alpha =$

$\tau_\alpha^{-1}(W_{\alpha-1})$ such that $W_\alpha \subset S_{\mathcal{H}(d_\alpha)}$. Since $\mathcal{H}(d_\alpha) = \{(h', e)\}$, where each $h' = t^{\beta(\mu(a)-1)e-\alpha e}\tilde{h}'$ and the $\tilde{h}'$ do not admit t as a common factor, it follows as before that $(\beta, \alpha) \in S$ if and only if $\beta\big(\mu(a)-1\big) - \alpha \geq 1$.

Now S, by its definition, depends only on the equivalence class of $(N(a), \mathcal{H}(a), \mathcal{E}(a))$ (with respect to transformations of types (i) and (ii)). On the other hand, we have proved that $S = \varnothing$ if and only if $\mu(a) = 1$, and, if $S \neq \varnothing$, then

$$S = \big\{(\beta, \alpha) \in \mathbb{N} \times \mathbb{N} : \beta\big(\mu(a)-1\big) - \alpha \geq 1\big\}.$$

Our proposition follows since $\mu(a)$ is uniquely determined by S; in the case that $S \neq \varnothing$,

$$\mu(a) = 1 + \sup_{(\beta,\alpha)\in S} \frac{\alpha+1}{\beta}. \qquad \square$$

Suppose that $\mu(a) < \infty$. Then we can also use test blowings-up to prove invariance of $\mu_H(a) = \mu_{\mathcal{H}(a),H}$, $H \in \mathcal{E}(a)$: Fix $H \in \mathcal{E}(a)$. As before we begin with the projection σ_0: $P_0 = M \times \mathbb{K} \to M$ from the product with a line. Let $(N(a_0)$, $\mathcal{H}(a_0)$, $\mathcal{E}(a_0))$ denote the transform of $(N(a), \mathcal{H}(a), \mathcal{E}(a))$ at $a_0 = (a, 0) \in P_0$ by the morphism σ_0 (of type (ii)), and let $H_0^0 = M \times \{0\}$, $H_1^0 = \sigma_0^{-1}(H) = H \times \mathbb{K}$. Thus $H_0^0, H_1^0 \in \mathcal{E}(a_0)$. We follow σ_0 by a sequence of exceptional blowings-up (morphisms of type (iii)),

$$\longrightarrow P_{j+1} \xrightarrow{\sigma_{j+1}} P_j \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\sigma_1} P_0,$$

where each σ_{j+1}, for $j \geq 0$, has centre $C_j = H_0^j \cap H_1^j$ and $H_0^{j+1} = \sigma_{j+1}^{-1}(C_j)$, $H_1^{j+1} =$ the strict transform of H_1^j by σ_{j+1}. Let a_{j+1} denote the unique intersection point of C_{j+1} and $\sigma_{j+1}^{-1}(a_j)$, for $j \geq 0$. (Thus $a_{j+1} = \gamma_{j+1}(0)$, where γ_0 denotes the arc $\gamma_0(t) = (a, t)$ in P_0 and γ_{j+1} denotes the lifting of γ_j by σ_{j+1}, for $j \geq 0$.)

We can choose local coordinates $(x_1, \ldots, x_n)$ for M at a, in which $a = 0$, $N(a) = \{x_{n-r+1} = \cdots = x_n = 0\}$, and each $K \in \mathcal{E}(a)$ is given by $x_i = 0$, for some $i = 1, \ldots, n-r$. (Set $x_i = x_K$.) Write $(x, t) = (x_1, \ldots, x_m, t)$, where $m = n - r$, for the corresponding coordinate system of $N(a_0) = N(a) \times \mathbb{K}$.

We can assume that $x_H = x_1$. In P_1, the strict transform $N(a_1)$ of $N(a_0)$ has a chart with coordinates $(x, t) = (x_1, \ldots, x_m, t)$ in which σ_1 is given by $\sigma_1(x, t) = (tx_1, x_2, \ldots, x_m, t)$ and in which $a_1 = (0, 0)$, $\gamma_1(t) = (0, t)$ and $x_1 = x_H$. (x_H now means $x_{H_1^1}$.) Proceeding inductively, for each j, $N(a_j)$ has a coordinate system $(x, t) = (x_1, \ldots, x_m, t)$ in which $a_j = (0, 0)$ and $\sigma_1 \circ \cdots \circ \sigma_j$: $N(a_j) \to N(a_0)$ is given by

$$(x, t) \mapsto (t^j x_1, x_2, \ldots, x_m, t).$$

We can assume that $\mu_h = e \in \mathbb{N}$, for all $(h, \mu_h) \in \mathcal{H}(a)$. Set

$$D = \prod_{K \in \mathcal{E}(a)} x_K^{\mu_K(a)}.$$

Thus D^e is a monomial in the coordinates $(x_1, \dots, x_m)$ of $N(a)$ with exponents in $\mathbb{N}$, and D^e is the greatest common divisor of the h in $\mathcal{H}(a)$ which is a monomial in x_K, $K \in \mathcal{E}(a)$ (by Definitions 4.3). In particular, for some $h = D^e g$ in $\mathcal{H}(a)$, $g = g_H$ is not divisible by $x_1 = x_H$. Therefore, there exists $i \geq 1$ such that

$$\mu_{a_j}(g_H \circ \pi_j) = \mu_{a_i}(g_H \circ \pi_i),$$

for all $j \geq i$, where $\pi_j = \sigma_0 \circ \sigma_1 \circ \dots \circ \sigma_j$. (We can simply take i to be the least order of a monomial not involving x_H in the Taylor expansion of g_H.)

On the other hand, for each $h = D^e g$ in $\mathcal{H}(a)$, $\mu_{a_j}(g \circ \pi_j)$ increases as $j \to \infty$ unless g is not divisible by x_H. Therefore, we can choose $h = D^e g_H$, as above, and i large enough so that we also have $\mu(a_j) = \mu_{a_j}(h \circ \pi_j)/e$, for all $j \geq i$. Clearly, if $j \geq i$, then

$$\mu_H(a) = \mu(a_{j+1}) - \mu(a_j).$$

Since $\mu(a)$ depends only on the equivalence class of $(N(a), \mathcal{H}(a), \mathcal{E}(a))$ among presentations of the same codimension r, as defined by 4.2, the preceding argument shows that each $\mu_H(a)$, $H \in \mathcal{E}(a)$, is also an invariant of this equivalence class. But the argument shows more precisely that the $\mu_H(a)$ depend only on a larger equivalence class obtained by allowing in Definition 4.2 only certain sequences of morphisms of types (i), (ii) and (iii):

Definition 4.5. We weaken the notion of equivalence in Definition 4.2 by allowing only the transforms induced by certain sequences of morphisms of types (i), (ii) and (iii); namely,

$$\begin{array}{ccccccccc} \longrightarrow M_j & \xrightarrow{\sigma_j} & M_{j-1} & \longrightarrow \cdots & \xrightarrow{\sigma_{i+1}} & M_i & \longrightarrow \cdots \longrightarrow & M_0 = M \\ \mathcal{E}(a_j) & & \mathcal{E}(a_{j-1}) & & & \mathcal{E}(a_i) & & \mathcal{E}(a_0) = \mathcal{E}(a) \end{array}$$

where, if $\sigma_{i+1}, \dots, \sigma_j$ are exceptional blowings-up (iii), then $i \geq 1$ and σ_i is of either type (iii) or (ii). In the latter case, σ_i: $M_i = M_{i-1} \times \mathbb{K} \to M_{i-1}$ is the projection, each σ_{k+1}, $k = i, \dots, j-1$, is the blowing-up with centre $C_k = H_0^k \cap H_1^k$ where H_0^k, $H_1^k \in \mathcal{E}(a_k)$, $a_{k+1} = \sigma_{k+1}^{-1}(a_k) \cap H_1^{k+1}$, and we require that the H_0^k, H_1^k be determined by some fixed $H \in \mathcal{E}(a_{i-1})$ inductively in the following way: $H_0^i = M_{i-1} \times \{0\}$, $H_1^i = \sigma_i^{-1}(H)$, and, for $k = i+1, \dots, j-1$, $H_0^k = \sigma_k^{-1}(C_{k-1})$, $H_1^k =$ the strict transform of H_1^{k-1} by σ_k.

In other words, with this notion of equivalence, we have proved:

Proposition 4.6. [Bierstone and Milman 1997, Proposition 4.11]. *Each $\mu_H(a)$, $H \in \mathcal{E}(a)$, and therefore also $\nu(a) = \mu(a) - \Sigma\mu_H(a)$ depends only on the equivalence class of $(N(a), \mathcal{H}(a), \mathcal{E}(a))$ (among presentations of the same codimension).*

Recall that in the r-th cycle of our recursive definition of inv_X, we use a codimension r presentation $(N_r(a), \mathcal{H}_r(a), \mathcal{E}_r(a))$ of inv_r at a to construct a codimension r presentation $(N_r(a), \mathcal{G}_{r+1}(a), \mathcal{E}_r(a))$ of $\mathrm{inv}_{r+1/2}$ at a. The construction involved survives transformations as allowed by Definition 4.5, but perhaps not an arbitrary sequence of transformations of types (i), (ii) and (iii) (compare [Bierstone and Milman 1997, 4.23 and 4.24]; in other words, we show only that the equivalence class of $(N_r(a), \mathcal{G}_{r+1}(a), \mathcal{E}_r(a))$ as given by Definition 4.5 depends only on that of $(N_r(a), \mathcal{H}_r(a), \mathcal{E}_r(a))$. It is for this reason that we need Proposition 4.6 as stated.

Acknowledgement

We are happy to thank Paul Centore for the line drawings in this paper.

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