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Global Regularity of the $\bar{\partial}$-Neumann Problem: A Survey of the L^2-Sobolev Theory

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Abstract. The fundamental boundary value problem in the function theory of several complex variables is the $\bar{\partial}$-Neumann problem. The L^2 existence theory on bounded pseudoconvex domains and the C^∞ regularity of solutions up to the boundary on smooth, bounded, strongly pseudoconvex domains were proved in the 1960s. On the other hand, it was discovered quite recently that global regularity up to the boundary fails in some smooth, bounded, weakly pseudoconvex domains.

We survey the global regularity theory of the $\bar{\partial}$-Neumann problem in the setting of L^2 Sobolev spaces on bounded pseudoconvex domains, beginning with the classical results and continuing up to the frontiers of current research. We also briefly discuss the related global regularity theory of the Bergman projection.

Contents

1. Introduction

The $\bar{\partial}$-Neumann problem is a natural example of a boundary-value problem with an elliptic operator but with non-coercive boundary conditions. It is also a prototype (in the case of finite-type domains) of a subelliptic boundary-value

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problem, in much the same way that the Dirichlet problem is the archetypal elliptic boundary-value problem. In this survey, we discuss global regularity of the $\bar\partial$-Neumann problem in the L^2-Sobolev spaces $W^s(\Omega)$ for all non-negative s and also in the space $C^\infty(\overline{\Omega})$. For estimates in other function spaces, such as Hölder spaces and L^p-Sobolev spaces, see [Beals et al. 1987; Berndtsson 1994; Chang et al. 1992; Cho 1995; Christ 1991; Fefferman 1995; Fefferman and Kohn 1988; Fefferman et al. 1990; Greiner and Stein 1977; Kerzman 1971; Krantz 1979; Lieb 1993; McNeal 1991; McNeal and Stein 1994; Nagel et al. 1989; Straube 1995]; for questions of real analytic regularity, see, for example, [Chen 1988; Christ 1996b; Derridj and Tartakoff 1976; Komatsu 1976; Tartakoff 1978; 1980; Tolli 1996; Treves 1978] and [Christ 1999, Section 10] in this volume.

We also discuss the closely related question of global regularity of the Bergman projection operator. This question is intimately connected with the boundary regularity of holomorphic mappings; see, for example, [Bedford 1984; Bell 1981; 1984; 1990; Bell and Ligocka 1980; Bell and Catlin 1982; Diederich and Fornæss 1982; Forstnerič 1993].

For an overview of techniques of partial differential equations in complex analysis, see [Folland and Kohn 1972; Hörmander 1965; 1990; 1994; Kohn 1977; Krantz 1992].

2. The L^2 Existence Theory

Throughout the paper, Ω denotes a bounded domain in $\mathbb{C}^n$, where $n > 1$. We say that Ω has class C^k boundary if $\Omega = \{z : \rho(z) < 0\}$, where ρ is a k times continuously differentiable real-valued function in a neighborhood of the closure $\overline{\Omega}$ whose gradient is normalized to length 1 on the boundary $b\Omega$. We denote the standard L^2-Sobolev space of order s by $W^s(\Omega)$; see, for example, [Adams 1975; Lions and Magenes 1972; Treves 1975]. The space of $(0,q)$ forms with coefficients in $W^s(\Omega)$ is written $W^s_{(0,q)}(\Omega)$, the norm being defined by

$$\Big\| \sum_J{}' a_J \, d\bar z_J \Big\|_s^2 = \sum_J{}' \|a_J\|_s^2, \tag{2–1}$$

where $d\bar z_J$ means $d\bar z_{j_1} \wedge d\bar z_{j_2} \wedge \cdots \wedge d\bar z_{j_q}$, and the prime indicates that the sum is taken over strictly increasing q-tuples J. We will consider the coefficients a_J, originally defined only for increasing multi-indices J, to be defined for other J so as to be antisymmetric functions of the indices. For economy of notation, we restrict attention to $(0,q)$ forms; modifications for (p,q) forms are simple (because the $\bar\partial$ operator does not see the dz differentials).

The $\bar\partial$ operator acts as usual on a $(0,q)$ form via

$$\bar\partial\Big(\sum_J{}' a_J \, d\bar z_J \Big) = \sum_{j=1}^n \sum_J{}' \frac{\partial a_J}{\partial \bar z_j} d\bar z_{jJ}. \tag{2–2}$$

The domain of $\bar\partial : L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ consists of those forms u for which $\bar\partial u$, defined in the sense of distributions, belongs to $L^2_{(0,q+1)}(\Omega)$. It is routine to check that $\bar\partial$ is a closed, densely defined operator from $L^2_{(0,q)}(\Omega)$ to $L^2_{(0,q+1)}(\Omega)$. Consequently, the Hilbert-space adjoint $\bar\partial^*$ also exists and defines a closed, densely defined operator from $L^2_{(0,q+1)}(\Omega)$ to $L^2_{(0,q)}(\Omega)$.

Suppose $u = \sum'_J u_J \, d\bar z_J$ is continuously differentiable on the closure $\overline{\Omega}$, and ψ is a smooth test form. If the boundary $b\Omega$ is sufficiently smooth, then pairing u with $\bar\partial\psi$ and integrating by parts gives

$$(u, \bar\partial\psi) = \left(-\sum_{k=1}^n \sum_K{}' \frac{\partial u_{kK}}{\partial z_k}\, d\bar z_K, \psi \right) + \sum_K{}' \int_{b\Omega} \overline{\psi}_K \sum_{k=1}^n u_{kK} \frac{\partial \rho}{\partial z_k}\, d\sigma. \tag{2–3}$$

The same calculation with a compactly supported ψ shows (without any boundary smoothness hypothesis) that if u is a square-integrable form in the domain of $\bar\partial^*$, then $\bar\partial^* u = \vartheta u$, where the *formal adjoint* ϑ is given by the equation

$$\vartheta u = -\sum_{k=1}^n \sum_K{}' \frac{\partial u_{kK}}{\partial z_k}\, d\bar z_K. \tag{2–4}$$

It follows that a continuously differentiable form u is in the domain of $\bar\partial^*$ if and only if

$$\sum_{k=1}^n u_{kK} \frac{\partial \rho}{\partial z_k} \bigg|_{b\Omega} = 0 \text{ for every } K. \tag{2–5}$$

The method of Friedrichs mollifiers shows that forms which are continuously differentiable on the closure $\overline{\Omega}$ are dense in the intersection of the domains of $\bar\partial$ and $\bar\partial^*$ with respect to the graph norm

$$\bigl(\|u\|^2 + \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2\bigr)^{1/2}$$

when the boundary $b\Omega$ is sufficiently smooth; see, for instance, [Hörmander 1965, § 1.2 and Prop. 2.1.1]. Also, forms that are continuously differentiable on the closure are dense in the domain of $\bar\partial$ with respect to the graph norm

$$(\|u\|^2 + \|\bar\partial u\|^2)^{1/2}.$$

The fundamental L^2 existence theorem for the $\bar\partial$-Neumann problem is due to Hörmander [1965]. One version of the result is the following.

THEOREM 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, where $n \ge 2$. Let D denote the diameter of Ω, and suppose $1 \le q \le n$.*

(i) *The complex Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ is an unbounded, self-adjoint, surjective operator from $L^2_{(0,q)}(\Omega)$ to itself having a bounded inverse N_q (the $\bar\partial$-Neumann operator).*

(ii) *For all u in $L^2_{(0,q)}(\Omega)$, we have the estimates*

$$\|N_q u\| \le \left(\frac{D^2 e}{q}\right)\|u\|, \quad \|\bar\partial^* N_q u\| \le \left(\frac{D^2 e}{q}\right)^{1/2}\|u\|, \quad \|\bar\partial N_q u\| \le \left(\frac{D^2 e}{q}\right)^{1/2}\|u\|. \tag{2–6}$$

(iii) *If f is a $\bar\partial$-closed $(0,q)$ form, then the canonical solution of the equation $\bar\partial u = f$ (the solution orthogonal to the kernel of $\bar\partial$) is given by $u = \bar\partial^* N_q f$; if f is a $\bar\partial^*$-closed $(0,q)$ form, then the canonical solution of the equation $\bar\partial^* u = f$ (the solution orthogonal to the kernel of $\bar\partial^*$) is given by $u = \bar\partial N_q f$.*

The Hilbert space method for proving Theorem 1 is based on estimating the norm of a form u in terms of the norms of $\bar\partial u$ and $\bar\partial^* u$. Hörmander discovered that it is advantageous to introduce weighted spaces $L^2(\Omega, e^{-\varphi})$, even for studying the unweighted problem. We denote the norm in the weighted space by $\|u\|_\varphi = \|ue^{-\varphi/2}\|$ and the adjoint of $\bar\partial$ with respect to the weighted inner product by $\bar\partial^*_\varphi(\,\cdot\,) = e^{\varphi}\,\bar\partial^*(\,\cdot\, e^{-\varphi})$. More generally, one can choose different exponential weights in $L^2_{(0,q-1)}$, $L^2_{(0,q)}$, and $L^2_{(0,q+1)}$; see [Hörmander 1990] for this method and applications.

The following identity is the basic starting point. The proof involves integrating by parts and manipulating the boundary integrals with the aid of the boundary condition (2–5) for membership in the domain of $\bar\partial^*$. The idea of introducing a second auxiliary function a was originally used in [Ohsawa and Takegoshi 1987; Ohsawa 1988], which deal with extending square-integrable holomorphic functions from submanifolds. The formulation given below comes from [Siu 1996; McNeal 1996]. In these papers (see also [Diederich and Herbort 1992]) the freedom to manipulate both the weight factor φ and the twisting factor a is essential.

Proposition 2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with class C^2 boundary; let u be a $(0,q)$ form (where $1 \le q \le n$) that is in the domain of $\bar\partial^*$ and that is continuously differentiable on the closure $\overline{\Omega}$; and let a and φ be real functions that are twice continuously differentiable on $\overline{\Omega}$, with $a \ge 0$. Then*

$$\begin{aligned}
&\|\sqrt{a}\,\bar\partial u\|^2_\varphi + \|\sqrt{a}\,\bar\partial^*_\varphi u\|^2_\varphi \\
&\qquad = {\sum_K}' \sum_{j,k=1}^n \int_{b\Omega} a \frac{\partial^2\rho}{\partial z_j \partial \bar z_k} u_{jK} \bar u_{kK} e^{-\varphi}\, d\sigma \\
&\qquad\quad + {\sum_J}' \sum_{j=1}^n \int_\Omega a \left|\frac{\partial u_J}{\partial \bar z_j}\right|^2 e^{-\varphi}\, dV + 2\,\mathrm{Re}\left({\sum_K}' \sum_{j=1}^n u_{jK} \frac{\partial a}{\partial z_j}\, d\bar z_K, \bar\partial^*_\varphi u\right)_\varphi \\
&\qquad\quad + {\sum_K}' \sum_{j,k=1}^n \int_\Omega \left(a \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} - \frac{\partial^2 a}{\partial z_j \partial \bar z_k}\right) u_{jK} \bar u_{kK} e^{-\varphi}\, dV.
\end{aligned} \tag{2–7}$$

For $a \equiv 1$ see [Hörmander 1965]; the case $a \equiv 1$ and $\varphi \equiv 0$ is the classical Kohn–Morrey formula [Kohn 1963; 1964; Morrey 1958]; see also [Ash 1964]. The usual proof of the L^2 existence theorem is based on a variant of (2–7) with $a \equiv 1$ and with different exponential weights φ in the different $L^2_{(0,q)}$ spaces; see [Catlin 1984b] for an elegant implementation of this approach. Here we will give an argument that has not appeared explicitly in the literature: we take $\varphi \equiv 0$ and make a good choice of a.

Suppose that Ω is a pseudoconvex domain: this means that the complex Hessian of the defining function ρ is a non-negative form on the vectors in the complex tangent space. Consequently, the boundary integral in (2–7) is non-negative. In particular, taking a to be identically equal to 1 gives

$$\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 \geq {\sum_J}' \sum_{j=1}^{n} \left\| \frac{\partial u_J}{\partial \bar{z}_j} \right\|^2, \tag{2–8}$$

so the bar derivatives of u are always under control.

If we replace a by $1 - e^b$, where b is an arbitrary twice continuously differentiable non-positive function, then after applying the Cauchy–Schwarz inequality to the term in (2–7) involving first derivatives of a, we find

$$\|\sqrt{a}\,\bar{\partial} u\|^2 + \|\sqrt{a}\,\bar{\partial}^* u\|^2 \geq {\sum_K}' \sum_{j,k=1}^{n} \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} u_{jK} \bar{u}_{kK}\, dV - \|e^{b/2} \bar{\partial}^* u\|^2. \tag{2–9}$$

Since $a + e^b = 1$ and $a \leq 1$, it follows that

$$\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 \geq {\sum_K}' \sum_{j,k=1}^{n} \int_\Omega e^b \frac{\partial^2 b}{\partial z_j \partial \bar{z}_k} u_{jK} \bar{u}_{kK}\, dV \tag{2–10}$$

for every twice continuously differentiable non-positive function b. Notice that this inequality becomes a strong one if there happens to exist a bounded plurisubharmonic function b whose complex Hessian has large eigenvalues. (This theme will recur later on: see the discussion after Theorem 10 and the discussion of property (P) in Section 5.)

In particular, let p be a point of Ω, and set $b(z) = -1 + |z-p|^2/D^2$, where D is the diameter of the bounded domain Ω. The preceding inequality then implies the fundamental estimate

$$\|u\|^2 \leq \frac{D^2 e}{q} \left(\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 \right). \tag{2–11}$$

Although this estimate was derived under the assumption that u is continuously differentiable on the closure $\overline{\Omega}$, it holds by density for all square-integrable forms u that are in the intersection of the domains of $\bar{\partial}$ and $\bar{\partial}^*$. We also assumed that the boundary of Ω is smooth enough to permit integration by parts. Estimate (2–11) is equivalent to every form in $L^2_{(0,q)}(\Omega)$ admitting a representation as $\bar{\partial} v + \bar{\partial}^* w$ with $\|v\|^2 + \|w\|^2 \leq (D^2 e/q)\|u\|^2$. The latter property carries over to arbitrary bounded pseudoconvex domains by exhausting a nonsmooth Ω by smooth ones, and therefore so does inequality (2–11).

Once estimate (2–11) is in hand, the proof of Theorem 1 follows from standard Hilbert space arguments; see, for example, [Catlin 1983, pp. 164–165] or [Shaw 1992, § 2]. The latter paper also shows the existence of the $\bar{\partial}$-Neumann operator N_0 on $(\ker \bar{\partial})^{\perp}$.

3. Regularity on General Pseudoconvex Domains

A basic question is whether one can improve Theorem 1 to get regularity estimates in Sobolev norms: $\|Nu\|_s \le C\|u\|_s$, $\|\bar{\partial}^* Nu\|_s \le C\|u\|_s$, $\|\bar{\partial} Nu\|_s \le C\|u\|_s$. If such estimates were to hold for all positive s, then Sobolev's lemma would imply that the $\bar{\partial}$-Neumann operator N (together with $\bar{\partial}^* N$ and $\bar{\partial} N$) is continuous in the space $C^\infty(\overline{\Omega})$ of functions smooth up to the boundary.

At first sight, it appears that one ought to be able to generalize the fundamental L^2 estimate (2–11) directly to an estimate of the form $\|u\|_s \le C(\|\bar{\partial} u\|_s + \|\bar{\partial}^* u\|_s)$, simply by replacing u by a derivative of u. This naive expectation is erroneous: the difficulty is that not every derivative of a form u in the domain of $\bar{\partial}^*$ is again in the domain of $\bar{\partial}^*$. The usual attempt to overcome this difficulty is to cover the boundary of Ω with special boundary charts [Folland and Kohn 1972, p. 33] in each of which one can take a frame of tangential vector fields that do preserve the domain of $\bar{\partial}^*$. Since such vector fields have variable coefficients, they do not commute with either $\bar{\partial}$ or $\bar{\partial}^*$, and so one needs to handle error terms that arise from the commutators.

In subsequent sections, we will discuss various hypotheses on the domain Ω that yield regularity estimates in Sobolev norms. In this section, we discuss firstly some completely general results on smoothly bounded pseudoconvex domains and secondly some counterexamples.

It is an observation of J. J. Kohn and his school that the $\bar{\partial}$-Neumann problem is always regular in $W^\varepsilon(\Omega)$ for a sufficiently small positive ε.

Proposition 3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. There exist positive ε and C (both depending on Ω) such that $\|Nu\|_\varepsilon \le C\|u\|_\varepsilon$, $\|\bar{\partial}^* Nu\|_\varepsilon \le C\|u\|_\varepsilon$, and $\|\bar{\partial} Nu\|_\varepsilon \le C\|u\|_\varepsilon$ for every $(0,q)$ form u (where $1 \le q \le n$).*

The idea of the proof is very simple. Since the commutator of a differential operator of order ε with $\bar{\partial}$ or $\bar{\partial}^*$ is again an operator of order ε, but with a coefficient bounded by a constant times ε, error terms can be absorbed into the main term when ε is sufficiently small.

Theorem 4. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. Fix a positive s. There exists a T (depending on s and Ω) such that for every t larger than T, the weighted $\bar{\partial}$-Neumann problem for the space $L^2_{(0,q)}(\Omega, e^{-t|z|^2}\, dV(z))$ is regular in $W^s(\Omega)$. In other words, N_t, $\bar{\partial}^*_t N_t$, and $\bar{\partial} N_t$ are continuous in $W^s(\Omega)$.*

Moreover, if f is a $\bar\partial$-closed $(0,q)$ form with coefficients in $C^\infty(\overline{\Omega})$, then there exists a form u with coefficients in $C^\infty(\overline{\Omega})$ such that $\bar\partial u = f$.

This fundamental result on continuity of the weighted operators is due to Kohn [1973]. It says that one can always have regularity for the $\bar\partial$-Neumann problem up to a certain number of derivatives if one is willing to change the measure with respect to which the problem is defined. The idea of the proof is to apply Proposition 2 with $a \equiv 1$ and $\varphi(z) = t|z|^2$ to obtain

$$\|e^{-t|z|^2/2}u\|^2 \le Ct^{-1}\bigl(\|e^{-t|z|^2/2}\,\bar\partial u\|^2 + \|e^{-t|z|^2/2}\,\bar\partial_t^* u\|^2\bigr).$$

When t is sufficiently large, the factor t^{-1} makes it possible to absorb error terms coming from commutators (see the sketch of the proof of Theorem 12 below for the ideas of the technique). The resulting *a priori* estimates are valid under the assumption that the left-hand sides of the inequalities are known to be finite; Kohn completed the proof by applying the method of elliptic regularization [Kohn and Nirenberg 1965] (see also the remarks after Theorem 7 below).

By means of a Mittag-Leffler argument ([Kohn 1977, p. 230], argument attributed to Hörmander), one can deduce solvability of the equation $\bar\partial u = f$ in the space $C^\infty(\overline{\Omega})$ (but the solution will not be the canonical solution orthogonal to the kernel of $\bar\partial$). With some extra care, the solution operator can be made linear, and also continuous from $W^{s+\varepsilon}_{(0,q+1)}(\Omega) \cap \ker\bar\partial$ to $W^{s}_{(0,q)}(\Omega)$ for every positive s and ε [Sibony 1990]. It is unknown whether or not there exists a linear solution operator for $\bar\partial$ that breaks even at every level in the Sobolev scale. Solvability with Sobolev estimates (with a loss of three derivatives) has recently been obtained for domains with only C^4 boundary by S. L. Yie [1995].

Given any solution of the equation $\bar\partial u = f$, one obtains the canonical solution by subtracting from u its projection onto the kernel of $\bar\partial$. In view of Kohn's result above, it is natural to study the regularity properties of the projection mapping. We denote the orthogonal projection from $L^2_{(0,q)}(\Omega)$ onto $\ker\bar\partial$ by P_q; when $q = 0$, this operator is the *Bergman projection*. A direct relation between the Bergman projection and the $\bar\partial$-Neumann operator is given by Kohn's formula $P_q = \mathrm{Id} - \bar\partial^* N_{q+1}\bar\partial$ for $0 \le q \le n$. It is evident that if the $\bar\partial$-Neumann operator N_{q+1} is continuous in $C^\infty(\overline{\Omega})$, then so is P_q. The exact relationship between regularity properties of the $\bar\partial$-Neumann operators and the Bergman projections was determined in [Boas and Straube 1990].

THEOREM 5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. Fix an integer q such that $1 \le q \le n$. Then the $\bar\partial$-Neumann operator N_q is continuous on $C^\infty_{(0,q)}(\overline{\Omega})$ if and only if the projection operators P_{q-1}, P_q, and P_{q+1} are continuous on the corresponding $C^\infty(\overline{\Omega})$ spaces. The analogous statement holds with the Sobolev space $W^s(\Omega)$ in place of $C^\infty(\overline{\Omega})$.*

In view of the implications for boundary regularity of biholomorphic and proper holomorphic mappings [Bedford 1984; Bell 1981; 1984; 1990; Bell and Ligocka

1980; Bell and Catlin 1982; Diederich and Fornæss 1982; Forstnerič 1993], regularity in $C^\infty(\overline{\Omega})$ is a key issue.

For some years there was uncertainty over whether the Bergman projection operator P_0 of every bounded domain in $\mathbb{C}^n$ with C^∞ smooth boundary might be regular in the space $C^\infty(\overline{\Omega})$. Barrett [1984] found the first counterexample, motivated by the so-called "worm domains" of Diederich and Fornæss [1977a]. In his example, for every $p > 2$ there is a smooth, compactly supported function whose Bergman projection is not in $L^p(\Omega)$. Barrett and Fornæss [1986] constructed a counterexample even more closely related to the worm domains. Although the worm domains are smoothly bounded pseudoconvex domains in $\mathbb{C}^2$, these counterexamples are not pseudoconvex. Subsequently, Kiselman [1991] showed that pseudoconvex, but nonsmooth, truncated versions of the worm domains have irregular Bergman projections.

Later Barrett [1992] (see [Barrett 1998] for a generalization) used a scaling argument together with computations on piecewise Levi-flat model domains to show that the Bergman projection of a worm domain must fail to preserve the space $W^s(\Omega)$ when s is sufficiently large. In view of Theorem 5, the $\bar\partial$-Neumann operator N_1 also fails to preserve $W^s_{(0,1)}(\Omega)$. This left open the possibility of regularity in $C^\infty(\overline{\Omega})$. Finally the question was resolved by Christ [1996a], as follows.

THEOREM 6. *For every worm domain, the Bergman projection operator P_0 and the $\bar\partial$-Neumann operator N_1 fail to be continuous on $C^\infty(\overline{\Omega})$ and $C^\infty_{(0,1)}(\overline{\Omega})$.*

Christ's proof is delicate and indirect. Roughly speaking, he shows that the $\bar\partial$-Neumann operator does satisfy for most values of s an estimate of the form $\|N_1u\|_s \le C\|u\|_s$ for all u for which N_1u is known a priori to lie in $C^\infty_{(0,1)}(\overline{\Omega})$. If N_1 were to preserve $C^\infty_{(0,1)}(\overline{\Omega})$, then density of $C^\infty_{(0,1)}(\overline{\Omega})$ in $W^s_{(0,1)}(\Omega)$ would imply continuity of N_1 in $W^s_{(0,1)}(\Omega)$, contradicting Barrett's result.

The obstruction to continuity in $W^s(\Omega)$ for every s on the worm domains is a global one: namely, the nonvanishing of a certain class in the first De Rham cohomology of the annulus of weakly pseudoconvex boundary points (this class measures the twisting of the boundary at the annulus; for details, see Theorem 15). For smoothly bounded domains Ω, it is known that for each fixed s there is no local obstruction in the boundary to continuity in $W^s(\Omega)$ [Barrett 1986; Chen 1991b].

For all domains Ω where continuity in $C^\infty(\overline{\Omega})$ is known, one can actually prove continuity in $W^s(\Omega)$ for all positive s. This intriguing phenomenon is not understood at present. (The corresponding phenomenon does not hold for partial differential operators in general: see [Christ 1999, Section 3] in this volume.)

Although regularity of the $\bar\partial$-Neumann problem in $C^\infty(\overline{\Omega})$ is known in large classes of pseudoconvex domains (see sections 4–6), the example of the worm domains shows that regularity sometimes fails. At present, necessary and sufficient conditions for global regularity of the $\bar\partial$-Neumann operator and of the Bergman projection are not known.

4. Domains of Finite Type

Historically, the first major development on the $\bar\partial$-Neumann problem was its solution by Kohn [1963; 1964] for strictly pseudoconvex domains. A strictly pseudoconvex domain can be defined by a strictly plurisubharmonic function, so by taking $a \equiv 1$ and $\varphi \equiv 0$ in (2–7) and keeping the boundary term we find that $\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 \geq C\|u\|^2_{L^2(b\Omega)}$. Roughly speaking, this inequality says that we have gained half a derivative, since the restriction map $W^{s+1/2}(\Omega) \to W^s(b\Omega)$ is continuous when $s > 0$. This gain is half of what occurs for an ordinary elliptic boundary-value problem, so we have a "subelliptic estimate."

THEOREM 7. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. If $1 \leq q \leq n$, then for each non-negative s there is a constant C such that the following estimates hold for every $(0,q)$ form u*:

$$\begin{aligned} \|u\|_{s+1/2} &\leq C\bigl(\|\bar\partial u\|_s + \|\bar\partial^* u\|_s\bigr) \qquad \textit{if } u \in \operatorname{dom}\bar\partial \cap \operatorname{dom}\bar\partial^*, \\ \|N_q u\|_{s+1} &\leq C\|u\|_s, \qquad \|\bar\partial N_q u\|_{s+1/2} + \|\bar\partial^* N_q u\|_{s+1/2} \leq C\|u\|_s. \end{aligned} \tag{4–1}$$

The standard reference for the proof of this result is [Folland and Kohn 1972], where the theory is developed for almost complex manifolds; see also [Krantz 1992]. The estimates can be localized, as in Theorem 8 below.

A key technical point in the proof of Theorem 7 is that after establishing the estimates under the assumption that the left-hand side is a priori finite, one then has to convert the a priori estimates into genuine estimates, in the sense that the left-hand side is finite when the right-hand side is finite. Kohn's original approach was considerably simplified in [Kohn and Nirenberg 1965] in a very general framework, via the elegant device of "elliptic regularization." The idea of the method is to add to $\Box$ an elliptic operator times ε (thereby obtaining a standard elliptic problem), to prove estimates independent of ε, and to let ε go to zero. (The analysis used in [Christ 1996a] to prove Theorem 6 shows that indeed a priori estimates cannot always be converted into genuine estimates. For this phenomenon in the context of the Bergman projection, see [Boas and Straube 1992a].) Another interesting approach to the proof of Theorem 7 was indicated by Morrey [1963; 1964].

A number of authors (see [Beals et al. 1987; Greiner and Stein 1977] and their references) have refined the results for strictly pseudoconvex domains in various ways, such as estimates in other function spaces and anisotropic estimates. In particular, N gains two derivatives in complex tangential directions; this gain results from the bar derivatives always being under control (see (2–8)). Integral kernel methods have also been developed successfully on strictly pseudoconvex domains; see [Grauert and Lieb 1970; Henkin 1969; 1970; Lieb 1993; Lieb and Range 1987; Ramirez 1970; Range 1986; 1987] and their references.

The gain of one derivative for the $\bar\partial$-Neumann operator N_1 in Theorem 7 is sharp, and the domain is necessarily strictly pseudoconvex if this estimate holds.

For discussion of this point, see [Catlin 1983; Folland and Kohn 1972, § III.2; Hörmander 1965, § 3.2; Krantz 1979, § 4].

More generally, one can ask when the $\bar\partial$-Neumann operator gains some fractional derivative. One says that a subelliptic estimate of order ε holds for the $\bar\partial$-Neumann problem on $(0,q)$ forms in a neighborhood U of a boundary point z_0 of a pseudoconvex domain in $\mathbb{C}^n$ if there is a constant C such that

$$\|u\|_\varepsilon^2 \le C\bigl(\|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2\bigr) \tag{4–2}$$

for every smooth $(0,q)$ form u that is supported in $U\cap\bar\Omega$ and that is in the domain of $\bar\partial^*$. The systematic study of subelliptic estimates in [Kohn and Nirenberg 1965] provides the following "pseudolocal estimates."

THEOREM 8. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. Suppose that a subelliptic estimate* (4–2) *holds in a neighborhood U of a boundary point z_0. Let χ_1 and χ_2 be smooth cutoff functions supported in U with χ_2 identically equal to* 1 *in a neighborhood of the support of χ_1. For every non-negative s, there is a constant C such that the $\bar\partial$-Neumann operator N_q and the Bergman projection P_q satisfy the estimates*

$$\begin{aligned} \|\chi_1 N_q u\|_{s+2\varepsilon} &\le C(\|\chi_2 u\|_s + \|u\|_0), \quad \textit{for } 1\le q\le n,\\ \|\chi_1\bar\partial^* N_q u\|_{s+\varepsilon} + \|\chi_1\bar\partial N_q u\|_{s+\varepsilon} &\le C(\|\chi_2 u\|_s + \|u\|_0), \quad \textit{for } 1\le q\le n,\\ \|\chi_1 P_q u\|_s &\le C(\|\chi_2 u\|_s + \|u\|_0), \quad \textit{for } 0\le q\le n. \end{aligned} \tag{4–3}$$

Consequently, if a subelliptic estimate (4–2) holds in a neighborhood of every boundary point of a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$, then the Bergman projection is continuous from $W^s_{(0,q)}(\Omega)$ to itself, and the $\bar\partial$-Neumann operator is continuous from $W^s_{(0,q)}(\Omega)$ to $W^{s+2\varepsilon}_{(0,q)}(\Omega)$.

In a sequence of papers [D'Angelo 1979; 1980; 1982; Catlin 1983; 1984a; 1987b], Kohn's students David Catlin and John D'Angelo resolved the question of when subelliptic estimates hold in a neighborhood of a boundary point of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. The necessary and sufficient condition is that the point have "finite type" in an appropriate sense. We briefly sketch this work; for details, consult the above papers as well as [Catlin 1987a; D'Angelo 1993, 1995; Diederich and Lieb 1981; Greiner 1974; Kohn 1979a; 1979b; 1981; 1984] and the survey [D'Angelo and Kohn 1999] in this volume.

The simplest obstruction to a subelliptic estimate is the presence of a germ of an analytic variety in the boundary of a domain. Indeed, examples show that local regularity of the $\bar\partial$-Neumann problem fails when there are complex varieties in the boundary; see [Catlin 1981; Diederich and Pflug 1981]. If the boundary is real-analytic near a point, then the absence of germs of q-dimensional complex-analytic varieties in the boundary near the point is necessary and sufficient for the existence of a subelliptic estimate on $(0,q)$-forms [Kohn 1979b]. This was first proved by combining a sufficient condition from Kohn's theory of ideals of subelliptic multipliers [1979b] with a theorem of Diederich and Fornæss [1978] on analytic varieties. Moreover, Diederich and Fornæss showed that a compact real-analytic manifold contains no germs of complex-analytic varieties of positive dimension, so subelliptic estimates hold for every bounded pseudoconvex domain in $\mathbb{C}^n$ with real-analytic boundary.

The first positive results in the C^∞ category were established in dimension two. A boundary point of a domain in $\mathbb{C}^2$ is of finite type if the boundary has finite order of contact with complex manifolds through the point; equivalently, if some finite-order commutator of complex tangential vector fields has a component that is transverse to the complex tangent space to the boundary. If m is an upper bound for the order of contact of complex manifolds with the boundary, then a subelliptic estimate (4–2) holds with $\varepsilon = 1/m$. For these results, see [Greiner 1974; Kohn 1979b]; for the equivalence of the two notions of finite type, see [Bloom and Graham 1977]. For pseudoconvex domains of finite type in dimension two, sharp estimates for the $\bar\partial$-Neumann problem are now known in many function spaces; see [Chang et al. 1992; Christ 1991] and their references.

In higher dimensions, it is no longer the case that all reasonable notions of finite type agree; for relations among them, see [D'Angelo 1987a]. D'Angelo's notion of finite type has turned out to be the right one for characterizing subelliptic estimates for the $\bar\partial$-Neumann problem. His idea to measure the order of contact of varieties with a real hypersurface M in $\mathbb{C}^n$ at a point z_0 is to fix a defining function ρ for M and to consider the order of vanishing at the origin of $\rho\circ f$, where f is a nonconstant holomorphic mapping from a neighborhood of the origin in $\mathbb{C}$ to $\mathbb{C}^n$ with $f(0) = z_0$. Since the variety that is the image of f may be singular, it is necessary to normalize by dividing by the order of vanishing at the origin of $f(\,\cdot\,) - z_0$. The supremum over all f of this normalized order of contact of germs of varieties with M is the D'Angelo 1-type of z_0.

THEOREM 9. *The set of points of finite* 1*-type of a smooth real hypersurface M in $\mathbb{C}^n$ is an open subset of M, and the* 1*-type is a locally bounded function on M.*

This fundamental result of D'Angelo [1982] is remarkable, because the 1-type may fail to be an upper semi-continuous function (see [D'Angelo 1993, p. 136] for a simple example). The theorem implies that if every point of a bounded domain in $\mathbb{C}^n$ is of finite 1-type, then there is a global upper bound on the 1-type.

For higher-dimensional varieties, there is no canonical way that serves all purposes to define the order of contact with a hypersurface. Catlin [1987b] defined a quantity $D_q(z_0)$ that measures the order of contact of q-dimensional varieties in "generic" directions (and D_1 agrees with D'Angelo's 1-type). Catlin's fundamental result is the following.

THEOREM 10. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. A subelliptic estimate for the $\bar\partial$-Neumann problem on $(0,q)$ forms holds in a neighborhood of a boundary point z_0 if and only if $D_q(z_0)$ is finite. The ε in the subelliptic estimate* (4–2) *satisfies $\varepsilon \le 1/D_q(z_0)$.*

The necessity of finite order of contact, together with the upper bound on ε, was proved in [Catlin 1983] (see also [Catlin 1981]), and the sufficiency in [Catlin 1987b]. Catlin's proof of sufficiency has two parts. His theory of multitypes [1984a] implies the existence of a stratification of the set of weakly pseudoconvex boundary points. The stratification is used to construct families of bounded plurisubharmonic functions whose complex Hessians in neighborhoods of the boundary have eigenvalues that blow up like inverse powers of the thickness of the neighborhoods. Such powers heuristically act like derivatives, and so it should be plausible that the basic inequality (2–10) leads to a subelliptic estimate (4–2).

It is unknown in general how to determine the optimal value of ε in a subelliptic estimate in terms of boundary data. For convex domains of finite type in $\mathbb{C}^n$, the optimal ε in a subelliptic estimate for $(0,1)$ forms is the reciprocal of the D'Angelo 1-type [Fornæss and Sibony 1989; McNeal 1992]; this is shown by a direct construction of bounded plurisubharmonic functions with suitable Hessians near the boundary. McNeal [1992] proved that for convex domains, the D'Angelo 1-type can be computed simply as the maximal order of contact of the boundary with complex lines. (There is an elementary geometric proof of McNeal's result in [Boas and Straube 1992b] and an analogue for Reinhardt domains in [Fu et al. 1996].) It is clear that in general, the best ε cannot equal the reciprocal of the type, simply because the type is not necessarily upper semi-continuous. For more about this subtle issue, see [D'Angelo 1993; 1995; D'Angelo and Kohn 1999; Diederich and Herbort 1993].

5. Compactness

A subelliptic estimate (4–2) implies, in particular, that the $\bar\partial$-Neumann operator is compact as an operator from $L^2_{(0,q)}(\Omega)$ to itself. This follows because the embedding from $W^\varepsilon_{(0,q)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ is compact when Ω is bounded with reasonable boundary, by the Rellich–Kondrashov theorem; see, for example, [Treves 1975, Prop. 25.5]. One might think of compactness in the $\bar\partial$-Neumann problem as a limiting case of subellipticity as $\varepsilon \to 0$.

The following lemma reformulates the compactness condition.

LEMMA 11. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, and suppose that $1 \le q \le n$. The following statements are equivalent.*

(i) *The $\bar\partial$-Neumann operator N_q is compact from $L^2_{(0,q)}(\Omega)$ to itself.*

(ii) *The embedding of the space $\operatorname{dom}\bar\partial \cap \operatorname{dom}\bar\partial^*$, provided with the graph norm $u \mapsto \|\bar\partial u\|_0 + \|\bar\partial^* u\|_0$, into $L^2_{(0,q)}(\Omega)$ is compact.*

(iii) *For every positive ε there exists C_ε such that*

$$\|u\|_0^2 \le \varepsilon\big(\|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2\big) + C_\varepsilon \|u\|_{-1}^2 \tag{5–1}$$

when $u \in \operatorname{dom}\bar\partial \cap \operatorname{dom}\bar\partial^*$.

Statement (c) is called a *compactness estimate* for the $\bar\partial$-Neumann problem. Its equivalence with statement (b) is in [Kohn and Nirenberg 1965, Lemma 1.1]. The equivalence of statement (a) with statements (b) and (c) follows easily from the L^2 theory discussed in Section 2 and the compactness of the embedding $L^2_{(0,q)}(\Omega) \to W^{-1}_{(0,q)}(\Omega)$.

In view of Theorem 8, it is a reasonable guess that compactness in the $\bar\partial$-Neumann problem implies global regularity of the $\bar\partial$-Neumann operator in the sense that N_q maps $W^s_{(0,q)}(\Omega)$ into itself. Work of Kohn and Nirenberg [1965] shows that this conjecture is correct.

THEOREM 12. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary*, *and suppose* $1 \le q \le n$. *If a compactness estimate* (5–1) *holds for the $\bar\partial$-Neumann problem on* $(0,q)$ *forms*, *then the $\bar\partial$-Neumann operator N_q is a compact* (*in particular*, *continuous*) *operator from $W^s_{(0,q)}(\Omega)$ into itself for every non-negative s.*

It suffices to prove the result for integral s; the intermediate cases follow from standard interpolation theorems [Bergh and Löfström 1976; Persson 1964]. We sketch the argument for $s = 1$, which illustrates the method. To prove the compactness of the $\bar\partial$-Neumann operator in $W^1_{(0,q)}(\Omega)$, we will establish the (a priori) estimate $\|N_q u\|_1^2 \le \varepsilon \|u\|_1^2 + C_\varepsilon \|u\|_0^2$ for arbitrary positive ε under the assumption that u and $N_q u$ are both in $C^\infty(\bar\Omega)$.

First we show that the compactness estimate (5–1) lifts to 1-norms: namely,

$$\|u\|_1^2 \le \varepsilon\big(\|\bar\partial u\|_1^2 + \|\bar\partial^* u\|_1^2\big) + C_\varepsilon \|u\|_{-1}^2$$

for smooth forms u in $\operatorname{dom}\bar\partial^*$ (with a new constant C_ε). In a neighborhood of a boundary point, we complete $\bar\partial\rho$ to an orthogonal basis of $(0,1)$ forms and choose dual vector fields. To estimate tangential derivatives of u, we apply (5–1) to these derivatives (valid since they preserve the domain of $\bar\partial^*$). We then commute the derivatives with $\bar\partial$ and $\bar\partial^*$, which gives an error term that is of the same order as the quantity on the left-hand side that we are trying to estimate, but multiplied by a factor of ε. We also need to estimate the normal derivative of u, but since the boundary is noncharacteristic for the elliptic complex $\bar\partial \oplus \bar\partial^*$, the normal derivative of u can be expressed in terms of $\bar\partial u$, $\bar\partial^* u$, and tangential derivatives of u. Summing over a collection of special boundary charts that cover the boundary, and using interior elliptic regularity to estimate the norm on a compact set, we obtain an inequality of the form

$$\|u\|_1^2 \le A\varepsilon\big(\|\bar\partial u\|_1^2 + \|\bar\partial^* u\|_1^2 + \|u\|_1^2\big) + B\big(\|\bar\partial u\|_0^2 + \|\bar\partial^* u\|_0^2 + \|u\|_0^2\big),$$

where the constants A and B are independent of ε. We can use the standard interpolation inequality $\|f\|_s \le \varepsilon\|f\|_{s+1} + C_\varepsilon\|f\|_{s-1}$ to absorb terms into the left-hand side when ε is sufficiently small.

The lifted compactness estimate together with the L^2 boundedness of the $\bar\partial$-Neumann operator implies

$$\|N_q u\|_1^2 \le \varepsilon\big(\|\bar\partial N_q u\|_1^2 + \|\bar\partial^* N_q u\|_1^2\big) + C_\varepsilon \|u\|_0^2. \tag{5–2}$$

Working as before in special boundary charts, we commute derivatives and integrate by parts on the right-hand side to make $\bar\partial\bar\partial^* N_q u + \bar\partial^*\bar\partial N_q u = u$ appear; see [Kohn 1984, p. 140; Boas and Straube 1990, p. 31]. Keeping track of commutator error terms and applying the Cauchy–Schwarz inequality, we find

$$\begin{aligned}\|\bar\partial N_q u\|_1^2 &+ \|\bar\partial^* N_q u\|_1^2 \\ &\le A\big(\|N_q u\|_1\|u\|_1 + \|u\|_0^2 + (\|\bar\partial N_q u\|_1 + \|\bar\partial^* N_q u\|_1)\|N_q u\|_1\big)\end{aligned} \tag{5–3}$$

for some constant A. Consequently $\|\bar\partial N_q u\|_1^2 + \|\bar\partial^* N_q u\|_1^2 \le B(\|N_q u\|_1^2 + \|u\|_1^2)$ for some constant B. Combining this with (5–2) gives the required a priori estimate

$$\|N_q u\|_1^2 \le \varepsilon\|u\|_1^2 + C_\varepsilon\|u\|_0^2. \tag{5–4}$$

Kohn and Nirenberg [1965] developed the method of elliptic regularization (described above after Theorem 7) to convert these a priori estimates into genuine ones.

There is a large class of domains for which the $\bar\partial$-Neumann operator is compact [Catlin 1984b; Sibony 1987]. Catlin [1984b] introduced "property (P)" and showed that it implies a compactness estimate (5–1) for the $\bar\partial$-Neumann problem. A domain Ω has property (P) if for every positive number M there exists a plurisubharmonic function λ in $C^\infty(\bar\Omega)$, bounded between 0 and 1, whose complex Hessian has all its eigenvalues bounded below by M on $b\Omega$:

$$\sum_{j,k=1}^{n} \frac{\partial^2\lambda}{\partial z_j \partial \bar z_k}(z) w_j \bar w_k \ge M|w|^2, \qquad \text{for } z \in b\Omega,\ w \in \mathbb{C}^n. \tag{5–5}$$

That property (P) implies a compactness estimate (5–1) follows directly from (2–10) and interior elliptic regularity.

It is easy to see that the existence of a strictly plurisubharmonic defining function implies property (P), so strictly pseudoconvex domains satisfy property (P). So do pseudoconvex domains of finite type: this was proved by Catlin [1984b] as a consequence of the analysis of finite type boundaries in [Catlin 1984a; D'Angelo 1982].

Property (P) is, however, much more general than the condition of finite type. For instance, it is easy to see that a domain that is strictly pseudoconvex except for one infinitely flat boundary point must have property (P). More generally, property (P) holds if the set of weakly pseudoconvex boundary points has Hausdorff two-dimensional measure equal to zero [Boas 1988; Sibony 1987]. This latter reference contains a systematic study of the property (under the name of "B-regularity"). In particular, Sibony found examples of B-regular domains whose boundary points of infinite type form a set of positive measure.

It is folklore that an analytic disc in the boundary of a pseudoconvex domain in $\mathbb{C}^2$ obstructs compactness of the $\bar{\partial}$-Neumann problem: this can be proved by an adaptation of the argument used in [Catlin 1981; Diederich and Pflug 1981] to show (in any dimension) that analytic discs in the boundary preclude hypoellipticity of $\bar{\partial}$. In higher dimensions, tamely embedded analytic discs in the boundary obstruct compactness, but the general situation seems not to be understood; see [Krantz 1988; Ligocka 1985] for a discussion of some interesting examples. Salinas found an obstruction to compactness phrased in terms of the C^*-algebra generated by the operators of multiplication by coordinate functions; see the survey [Salinas 1991] and its references.

In view of the maximum principle, property (P) excludes analytic structure from the boundary: in particular, the boundary cannot contain analytic discs. However, the absence of analytic discs in the boundary does not guarantee property (P) [Sibony 1987, p. 310], although it does in the special cases of convex domains and complete Reinhardt domains [Sibony 1987, Prop. 2.4].

It is not yet understood how much room there is between property (P) and compactness. Having necessary and sufficient conditions on the boundary of a domain for compactness of the $\bar{\partial}$-Neumann problem would shed considerable light on the interactions among complex geometry, pluripotential theory, and partial differential equations.

6. The Vector Field Method

In the preceding section, we saw that the $\bar{\partial}$-Neumann problem is globally regular in domains that support bounded plurisubharmonic functions with arbitrarily large complex Hessian at the boundary. Now we will discuss a method that applies, for example, to domains admitting defining functions that are plurisubharmonic on the boundary. The method is based on the construction of certain vector fields that almost commute with $\bar{\partial}$.

We begin with some general remarks about proving a priori estimates of the form $\|N_q u\|_s \le C\|u\|_s$ and $\|P_q u\|_s \le C\|u\|_s$ in Sobolev spaces for the $\bar{\partial}$-Neumann operator and the Bergman projection. Firstly, all the action is near the boundary. This is clear for the Bergman projection on functions, because the mean-value property shows that every Sobolev norm of a holomorphic function on a compact subset of a domain is dominated by a weak norm on the whole domain (for instance, the L^2 norm). The corresponding property holds for the $\bar{\partial}$-Neumann operator due to interior elliptic regularity.

Secondly, the conjugate holomorphic derivatives $\partial/\partial\bar{z}_j$ are always under control. This is obvious for the case of the Bergman projection P_0 on functions (since holomorphic functions are annihilated by anti-holomorphic derivatives), and the inequality (2–8) shows that anti-holomorphic derivatives are tame for the $\bar{\partial}$-Neumann problem.

Thirdly, differentiation by vector fields whose restrictions to the boundary lie in the complex tangent space is also innocuous. Indeed, integrating by parts turns tangential vector fields of type $(1,0)$ into vector fields of type $(0,1)$, which are tame, plus terms that can be absorbed [Boas and Straube 1991, formula (3)].

Thus, we only need to estimate derivatives in the complex normal direction near the boundary. Moreover, since the bar derivatives are free, it will do to estimate either the real part or the imaginary part of the complex normal derivative. That is, we can get by with estimating either the real normal derivative, or a tangential derivative that is transverse to the complex tangent space.

A simple application of these ideas shows, for example, that the Bergman projection P_0 on functions for every bounded Reinhardt domain Ω in $\mathbb{C}^n$ with class C^∞ boundary is continuous from $W^s(\Omega)$ to itself for every positive integer s [Boas 1984; Straube 1986]. Indeed, the domain is invariant under rotations in each variable, so the Bergman projection commutes with each angular derivative $\partial/\partial\theta_j$. At every boundary point, at least one of these derivatives is transverse to the complex tangent space, so

$$\|P_0u\|_1 \le C\textstyle\sum_{j=1}^n \|(\partial/\partial\theta_j)P_0u\|_0 = C\sum_{j=1}^n \|P_0(\partial u/\partial\theta_j)\|_0 \le C'\|u\|_1.$$

Higher derivatives are handled analogously. A similar technique proves global regularity of the $\bar\partial$-Neumann operator on bounded pseudoconvex Reinhardt domains [Boas et al. 1988; Chen 1989].

Thus, the nicest situation for proving estimates in Sobolev norms for the $\bar\partial$-Neumann operator is to have a tangential vector field, transverse to the complex tangent space, that commutes with the $\bar\partial$-Neumann operator, or what is nearly the same thing, that commutes with $\bar\partial$ and $\bar\partial^*$. (This method is classical: see [Derridj 1978; Derridj and Tartakoff 1976; Komatsu 1976].) Actually, it would be enough for the commutator with each anti-holomorphic derivative $\partial/\partial\bar z_j$ to have vanishing $(1,0)$ component in the complex normal direction. However, [Derridj 1991, Théorème 2.6 and the remark following it] shows that no such field can exist in general.

If we have a real tangential vector field T, transverse to the complex tangent space, whose commutator with each $\partial/\partial\bar z_j$ has $(1,0)$ component in the complex normal direction of modulus less than ε, then we get an estimate of the form

$$\|T^sN_qu\|_0 \le A_s(\|u\|_s + \varepsilon\|N_qu\|_s) + C_{s,T}\|u\|_0.$$

If the field T is normalized so that its coefficients and its angle with the complex tangent space are bounded away from zero, then $\|T^sN_qu\|_0$ controls $\|N_qu\|_s$ (independently of ε), so we get global regularity of N_q up to a certain level in the Sobolev scale. (By making estimates uniformly on a sequence of interior approximating strongly pseudoconvex domains, we can convert the a priori estimates to genuine ones.) Moreover, it suffices if T is approximately tangential in the sense that its normal component is of order ε. (This idea comes from [Barrett 1986]; see the proof of Theorem 16.) If we can find a sequence of such normalized vector fields corresponding to progressively smaller values of ε, then the $\bar{\partial}$-Neumann problem is globally regular at every level in the Sobolev scale. Because of the local regularity at points of finite type, the vector fields need exist only in (progressively smaller) neighborhoods of the boundary points of infinite type. In other words, we have the following result (where the imaginary parts of the X_ε correspond to the vector fields described above) [Boas and Straube 1991; Boas and Straube 1993].

THEOREM 13. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary and defining function ρ. Suppose there is a positive constant C such that for every positive ε there exists a vector field X_ε of type $(1,0)$ whose coefficients are smooth in a neighborhood U_ε in $\mathbb{C}^n$ of the set of boundary points of Ω of infinite type and such that*

(i) *$|\arg X_\varepsilon \rho| < \varepsilon$ on U_ε, and moreover $C^{-1} < |X_\varepsilon \rho| < C$ on U_ε, and*
(ii) *when $1 \le j \le n$, the form $\partial\rho$ applied to the commutator $[X_\varepsilon, \partial/\partial\bar{z}_j]$ has modulus less than ε on U_ε.*

Then the $\bar{\partial}$-Neumann operators N_q (for $1 \le q \le n$) and the Bergman projections P_q (for $0 \le q \le n$) are continuous on the Sobolev space $W^s_{(0,q)}(\Omega)$ when $s \ge 0$.

For a simple example in which the hypothesis of this theorem can be verified, consider a ball with a cap sliced off by a real hyperplane, and the edges rounded. The normal direction to the hyperplane will serve as X_ε (the U_ε being shrinking neighborhoods of the flat part of the boundary), so the $\bar{\partial}$-Neumann operator for this domain is continuous at every level in the Sobolev scale.

Indeed, the hypothesis of Theorem 13 can be verified for all convex domains. (The regularity of the $\bar{\partial}$-Neumann problem for convex domains in dimension two was obtained independently by Chen [1991a] using related ideas.) More generally, the theorem applies to domains admitting a defining function that is plurisubharmonic on the boundary [Boas and Straube 1991]. We state this as a separate result and sketch the proof. (Continuity in $W^{1/2}(\Omega)$ in the presence of a plurisubharmonic defining function was obtained earlier by Bonami and Charpentier [1988; 1990].)

THEOREM 14. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. Suppose that Ω has a C^∞ defining function ρ that is plurisubharmonic on the boundary: $\sum_{j,k=1}^n (\partial^2\rho/\partial z_j \partial\bar{z}_k) w_j \bar{w}_k \ge 0$ for all $z \in b\Omega$ and all $w \in \mathbb{C}^n$. Then for every positive s there exists a constant C such that for all $u \in W^s_{(0,q)}(\Omega)$ we have*

$$\|N_q u\|_s \le C\|u\|_s, \quad \textit{for } 1 \le q \le n,$$
$$\|P_q u\|_s \le C\|u\|_s, \quad \textit{for } 0 \le q \le n.$$

Pseudoconvexity says that on the boundary, $\sum_{j,k=1}^n (\partial^2\rho/\partial z_j \partial\bar{z}_k) w_j \bar{w}_k \ge 0$ for vectors w in the complex tangent space: that is, those vectors for which

$\sum_{j=1}^{n}(\partial\rho/\partial z_j)w_j = 0$. The hypothesis of the theorem is that on the boundary, the complex Hessian of ρ is non-negative on all vectors, not just complex tangent vectors. (There are examples of pseudoconvex domains, even with real-analytic boundary, that do not admit such a defining function even locally [Behrens 1984; 1985; Fornæss 1979.) We now sketch how this extra information can be used to construct the special vector fields needed to invoke Theorem 13.

The key observation is that for each j, derivatives of $\partial\rho/\partial z_j$ of type $(0,1)$ in directions that lie in the null space of the Levi form must vanish. Indeed, if $\partial/\partial\bar{z}_1$ (say) is in the null space of the Levi form at a boundary point p, then $\partial^2\rho/\partial z_1\partial\bar{z}_1(p) = 0$, but since the matrix $\partial^2\rho/\partial z_j\partial\bar{z}_k(p)$ is positive semidefinite, its whole first column must vanish. (It was earlier observed by Noell [1991] that the unit normal to the boundary of a convex domain is constant along Levi-null curves.)

To construct the required global vector field, it will suffice to construct a vector field whose commutator with each complex tangential field of type $(1,0)$ has vanishing component in the complex normal direction at a specified boundary point p. Indeed, these components will be bounded by ε in a neighborhood of p by continuity, and we can use a partition of unity to patch local fields into a global field. (Terms in the commutator coming from derivatives of the partition of unity cause no difficulty because they are complex tangential.) It is easy to extend the field from the boundary to the inside of the domain to prescribe the proper commutator with the complex normal direction.

Suppose that $\partial\rho/\partial z_n(p) \neq 0$. We want to correct the field $(\partial\rho/\partial z_n)^{-1}(\partial/\partial z_n)$ by subtracting a linear combination of complex tangential vector fields so as to adjust the commutators. Since the Levi form may have some zero eigenvalues at p, we need a compatibility condition to solve the resulting linear system. The observation above that type $(0,1)$ derivatives in Levi-null directions annihilate $\partial\rho/\partial z_n$ at p is precisely the condition needed for solvability. For details of the proof, see [Boas and Straube 1991].

Kohn [1998] has found a new proof and generalization of Theorem 14. By a theorem of Diederich and Fornæss [1977b] (see also [Range 1981]), a smooth bounded pseudoconvex domain admits a defining function such that some (small) positive power δ of its absolute value is plurisuperharmonic inside Ω. Kohn shows that the $\bar{\partial}$-Neumann problem is regular in $W^s(\Omega)$ for s up to a level depending, roughly speaking, on δ. (More generally, the power of the defining function need only be "approximately plurisuperharmonic"; compare [Kohn 1998, Remark 5.1].)

Theorem 13 applies to other situations besides the one described in Theorem 14. For instance, it is possible to construct the vector fields on pseudoconvex domains that are regular in the sense of Diederich and Fornæss [1977c] and Catlin [1984b]. (This gives no new theorem, however, since the $\bar{\partial}$-Neumann problem is known to be compact on such domains [Catlin 1984b]; nor does it give a simplified proof of global regularity in the finite type case, since the construction of the vector fields still requires Catlin's stratification [1984a] of the set of weakly pseudoconvex points.)

As mentioned in Section 3, global regularity for the $\bar{\partial}$-Neumann problem breaks down on the Diederich–Fornæss worm domains. On those domains, the set of weakly pseudoconvex boundary points is precisely an annulus, and it is possible to compute directly that the vector fields specified in Theorem 13 cannot exist on this annulus.

For domains of this kind, where the boundary points of infinite type form a nice submanifold of the boundary, there is a natural condition that guarantees the existence of the vector fields needed to apply Theorem 13. Following the notation of [D'Angelo 1987b; 1993], we let η denote a purely imaginary, non-vanishing one-form on the boundary $b\Omega$ that annihilates the complex tangent space and its conjugate. Let T denote the purely imaginary tangential vector field on $b\Omega$ orthogonal to the complex tangent space and its conjugate and such that $\eta(T) \equiv 1$. Up to sign, the Levi form of two complex tangential vector fields X and Y is $\eta([X, \overline{Y}])$. The (real) one-form α is defined to be minus the Lie derivative of η in the direction of T:

$$\alpha = -\mathcal{L}_T \eta. \tag{6–1}$$

One can show [Boas and Straube 1993, § 2] that if M is a submanifold of the boundary whose real tangent space is contained in the null space of the Levi form, then the restriction of the form α to M is closed, and hence represents a cohomology class in the first De Rham cohomology $H^1(M)$. (In the special case when M is a complex submanifold, this closedness corresponds to the pluriharmonicity of certain argument functions, as in [Barrett and Fornæss 1988; Bedford and Fornæss 1978, Prop. 3.1; Bedford and Fornæss 1981, Lemma 1; Diederich and Fornæss 1977a, p. 290].) This class is independent of the choice of η. If this cohomology class vanishes on such a submanifold M, and if M contains the points of infinite type, then the vector fields described in Theorem 13 do exist. Thus, we have the following result [Boas and Straube 1993].

THEOREM 15. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. Suppose there is a smooth real submanifold M (with or without boundary) of $b\Omega$ that contains all the points of infinite type of $b\Omega$ and whose real tangent space at each point is contained in the null space of the Levi form at that point (under the usual identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$). If the $H^1(M)$ cohomology class $[\alpha|_M]$ is zero, then the $\bar{\partial}$-Neumann operators N_q (for $1 \le q \le n$) and the Bergman projections P_q (for $0 \le q \le n$) are continuous on the Sobolev space $W^s_{(0,q)}(\Omega)$ when $s \ge 0$.*

On the worm domains, one can compute directly that the class $[\alpha|_M]$ is not zero. The appearance of this cohomology class explains, in particular, why an analytic annulus in the boundary of the worm domains is bad for Sobolev estimates, while an annulus in the boundary of other domains may be innocuous [Boas and

Straube 1992a], and an analytic disc is always benign (same reference). In the special case that $n = 2$ and M is a bordered Riemann surface, Barrett has shown that there is a pluripolar subset of $H^1(M)$ such that estimates in $W^k(\Omega)$ fail for sufficiently large k if $[\alpha|_M]$ lies outside this subset [Barrett 1998]. When M is a complex submanifold of the boundary, $[\alpha|_M]$ has a geometric interpretation as a measure of the winding of the boundary of Ω around M (equivalently, the winding of the vector normal to the boundary). For details, see [Bedford and Fornæss 1978]. In the context of Hartogs domains in $\mathbb{C}^2$, see also [Boas and Straube 1992a].

The constructions of the vector fields (needed to apply Theorem 13) in the proofs of Theorems 14 and 15 are more closely related than appears at first glance. The vector fields can be written locally in the form

$$e^h L_n + \sum_{j=1}^{n-1} a_j L_j,$$

where $L_1, \dots, L_{n-1}$ form a local basis for the tangential vector fields of type $(1,0)$, L_n is the normal field of type $(1,0)$, and h and the a_j are smooth functions. The commutator conditions in Theorem 13 in directions not in the null space of the Levi form can *always* be satisfied by using the a_j to correct the commutators. Computing the commutators in the remaining directions leads to the equation $dh|_{\mathcal{N}(p)} = \alpha|_{\mathcal{N}(p)}$ at points p of infinite type (where $\mathcal{N}(p)$ is the null space of the Levi form at p). The above proof of Theorem 14 amounts to showing that $\alpha|_{\mathcal{N}(p)} = 0$ when there is a defining function that is plurisubharmonic on the boundary, whence $h \equiv 0$ gives a solution. In Theorem 15, the hypothesis of the vanishing of the cohomology class of α on M allows us to solve for h (on M).

In general, the points of infinite type need not lie in a "nice" submanifold of the boundary. It is not known what should play the role of the cohomology class $[\alpha|_M]$ in the general situation. (Note that the analogue of the property that $\alpha|_M$ is closed holds in general: $d\alpha|_{\mathcal{N}(p)} = 0$; see [Boas and Straube 1993, § 2].) Furthermore, it is not understood how to combine the ideas of this section with the pluripotential theoretic methods discussed in Section 5, such as B-regularity and property (P).

7. The Bergman Projection on General Domains

In pseudoconvex domains, global regularity of the $\bar\partial$-Neumann problem is essentially equivalent to global regularity of the Bergman projection [Boas and Straube 1990]. In nonpseudoconvex domains, the $\bar\partial$-Neumann operator may not exist, yet the Bergman projection is still well defined. Since global regularity of the Bergman projection on functions is intimately connected to the boundary regularity of biholomorphic and proper holomorphic mappings (see [Bedford 1984; Bell 1981; 1984; 1990; Bell and Ligocka 1980; Bell and Catlin 1982;

Diederich and Fornæss 1982; Forstnerič 1993]), it is interesting to analyze the Bergman projection directly, without recourse to the $\bar\partial$-Neumann problem. Even very weak regularity properties of the Bergman projection can be exploited in the study of biholomorphic mappings [Barrett 1986; Lempert 1986].

In this section, we survey the theory of global regularity of the Bergman projection on general (that is, not necessarily pseudoconvex) domains.

The first regularity results for the Bergman projection that were obtained without the help of the $\bar\partial$-Neumann theory are in [Bell and Boas 1981], where it is shown that the Bergman projection P on functions maps the space $C^\infty(\overline{\Omega})$ of functions smooth up to the boundary continuously into itself when Ω is a bounded complete Reinhardt domain with C^∞ smooth boundary.

This result was generalized in [Barrett 1982] to domains with "transverse symmetries." A domain Ω is said to have transverse symmetries if it admits a Lie group G of holomorphic automorphisms acting transversely in the sense that the map $G\times\Omega \to \Omega$ taking (g, z) to $g(z)$ extends to a smooth map $G\times\overline{\Omega} \to \overline{\Omega}$, and for each point $z_0 \in b\Omega$ the map $g \mapsto g(z_0)$ of G to $b\Omega$ induces a map on tangent spaces $T_{\mathrm{Id}}G \to T^{\mathbb{R}}_{z_0}(b\Omega)$ whose image is not contained in the complex tangent space to $b\Omega$ at z_0. In other words, there exists for each boundary point z_0 a one-parameter family of automorphisms of $\overline{\Omega}$ whose infinitesimal generator is transverse to the tangent space at z_0. This class of domains includes many Cartan domains as well as all smooth bounded Reinhardt domains; in both cases, suitable Lie groups of rotations provide the transverse symmetries [Barrett 1982]. For domains with transverse symmetries, it was observed in [Straube 1986] that the Bergman projection not only maps the space $C^\infty(\overline{\Omega})$ into itself, but actually preserves the Sobolev spaces.

More generally, one can obtain regularity results in the presence of a transverse vector field of type $(1, 0)$ with holomorphic coefficients, even if it does not come from a family of automorphisms. David Barrett obtained the following result [Barrett 1986].

THEOREM 16. *Let Ω be a bounded domain in $\mathbb{C}^n$ with class C^∞ boundary and defining function ρ. Suppose there is a vector field X of type $(1,0)$ with holomorphic coefficients in $C^\infty(\overline{\Omega})$ that is nowhere tangent to the boundary of Ω and such that $|\arg X\rho| < \pi/4k$ for some positive integer k. Then the Bergman projection on functions maps the Sobolev space $W^k(\Omega)$ continuously into itself.*

In particular, Theorem 16 implies that there are no local obstructions to W^k regularity of the Bergman projection. In other words, any sufficiently small piece of C^∞ boundary can be a piece of the boundary of a domain G whose Bergman projection is continuous in $W^k(G)$: indeed, G can be taken to be a small perturbation of a ball, and then the radial field satisfies the hypothesis of the theorem.

Theorem 16 also applies when $k = 1/2$ and the boundary is only Lipschitz smooth. For example, the hypothesis holds for $k = 1/2$ when the domain is strictly star-shaped. Lempert [1986] has exploited this weak regularity property to prove a Hölder regularity theorem for biholomorphic mappings between star-shaped domains with real-analytic boundaries.

The first step in the proof of Theorem 16 is one we have seen before in Section 6: namely, it suffices to estimate derivatives of holomorphic functions in a direction transverse to the boundary. Thus, to bound $\|Pf\|_k$ it suffices to bound $\|X^k Pf\|_0$. However, the inner product $\langle X^k Pf, X^k Pf\rangle$ is bounded above by a constant times $|\langle \varphi^k X^k Pf, X^k Pf\rangle|$ when $\operatorname{Re}\varphi^k$ is bounded away from zero. By the hypothesis of the theorem, we can take φ to be a smooth function that equals $\overline{X}\rho/X\rho$ near the boundary. We then replace $\varphi^k X^k$ on the left-hand side of the inner product by $(\varphi X - \overline{X})^k$, making a lower-order error (since $\overline{X}$ annihilates holomorphic functions). The point is that $(\varphi X - \overline{X})$ is tangential at the boundary, so we can integrate by parts without boundary terms, obtaining $|\langle Pf, X^{2k} Pf\rangle|$ plus lower-order terms. Since X is a holomorphic field, we can remove the Bergman projection operator from the left-hand side of the inner product, integrate by parts, and apply the Cauchy–Schwarz inequality to get an upper bound of the form $C\|f\|_k\|Pf\|_k$. This gives an a priori estimate $\|Pf\|_k \le C\|f\|_k$. The estimate can be converted into a genuine estimate via an argument involving the resolvent of the semigroup generated by the real part of X. For details of the proof, see [Barrett 1986].

It is possible to combine such methods with techniques based on pseudoconvexity. Estimates for the Bergman projection and the $\bar\partial$-Neumann operator on pseudoconvex domains that have transverse symmetries on the complement of a compact subset of the boundary consisting of points of finite type were obtained in [Chen 1987] and [Boas et al. 1988].

A domain in $\mathbb{C}^2$ is called a Hartogs domain if, with each of its points (z, w), it contains the circle $\{(z, \lambda w) : |\lambda| = 1\}$; it is complete if it also contains the disc $\{(z, \lambda w) : |\lambda| \le 1\}$. The (pseudoconvex) worm domains [Diederich and Fornæss 1977a] and the (nonpseudoconvex) counterexample domains in [Barrett 1984; Barrett and Fornæss 1986] with irregular Bergman projections are incomplete Hartogs domains in $\mathbb{C}^2$. It is easy to see that when a Hartogs domain in $\mathbb{C}^2$ is complete, the obstruction to regularity identified in Section 6 cannot occur; see [Boas and Straube 1992a, § 1]. Actually, completeness guarantees that the Bergman projection is regular whether or not the domain is pseudoconvex [Boas and Straube 1989]. (See [Boas and Straube 1992a] for a systematic study of the Bergman projection on Hartogs domains in $\mathbb{C}^2$.)

THEOREM 17. *Let Ω be a bounded complete Hartogs domain in $\mathbb{C}^2$ with class C^∞ boundary. The Bergman projection maps the Sobolev space $W^s(\Omega)$ continuously into itself when $s \ge 0$.*

The proof again uses different arguments on different parts of the boundary. An interesting new twist occurs in that the $\bar\partial$-Neumann operator of the envelope of holomorphy of the domain (which is still a complete Hartogs domain) is exploited.

The Bergman projection is known to preserve the Sobolev spaces $W^s(\Omega)$ in all cases in which it is known to preserve the space $C^\infty(\overline{\Omega})$ of functions smooth up to the boundary (as is the case for the $\bar{\partial}$-Neumann operator on pseudoconvex domains). It is an intriguing question whether this is a general phenomenon.

We now turn to the connection between the regularity theory of the Bergman projection and the duality theory of holomorphic function spaces, which originates with Bell [1982b]. When k is an integer, let $A^k(\Omega)$ denote the subspace of the Sobolev space $W^k(\Omega)$ consisting of holomorphic functions, and let $A^\infty(\Omega)$ denote the subspace of $C^\infty(\overline{\Omega})$ consisting of holomorphic functions. We may view the Fréchet space $A^\infty(\Omega)$ as the projective limit of the Hilbert spaces $A^k(\Omega)$, and we introduce the notation $A^{-\infty}(\Omega)$ for the space $\bigcup_{k=1}^\infty A^{-k}(\Omega)$, provided with the inductive limit topology.

For discussion of some of the technical properties of these spaces of holomorphic functions, see [Bell and Boas 1984; Straube 1984]. In particular, $A^{-\infty}(\Omega)$ is a Montel space, and subsets of $A^{-\infty}(\Omega)$ are bounded if and only if they are contained and bounded in some $A^{-k}(\Omega)$. The inductive limit structure on $A^{-\infty}(\Omega)$ turns out to be "nice" because the embeddings $A^{-k}(\Omega) \to A^{-k-1}(\Omega)$ are compact (as a consequence of Rellich's lemma). Functions in $A^{-\infty}(\Omega)$ can be characterized in two equivalent ways: they have growth near the boundary of Ω that is at most polynomial in the reciprocal of the distance to the boundary, and their traces on interior approximating surfaces $b\Omega_\varepsilon$ converge in the sense of distributions on $b\Omega$. See [Straube 1984] for an elementary discussion of these facts.

The L^2 inner product extends to a more general pairing. Harmonic functions are a natural setting for this extension. We use the notations $h^\infty(\Omega)$ and $h^{-\infty}(\Omega)$ for the spaces of harmonic functions analogous to $A^\infty(\Omega)$ and $A^{-\infty}(\Omega)$.

PROPOSITION 18. *Let Ω be a bounded domain in $\mathbb{C}^n$ with class C^∞ boundary. For each positive integer k there is a constant C_k such that for every square-integrable harmonic function f, and every $g \in C^\infty(\overline{\Omega})$, we have the inequality*

$$\left| \int_\Omega f\bar{g} \right| \le C_k \|f\|_{-k} \|g\|_k. \tag{7–1}$$

The proof of Proposition 18 follows from the observation that for every $g \in C^\infty(\overline{\Omega})$, there is a function g_1 vanishing to high order at the boundary of Ω such that the difference $g - g_1$ is orthogonal to the harmonic functions. See [Bell 1982c; Boas 1987, Appendix B; and Ligocka 1986] for details; the root idea originates with Bell [1979] in the context of holomorphic functions. Alternatively, Proposition 18 can be derived from elementary facts about the Dirichlet problem for the Laplace operator [Straube 1984].

Because the square-integrable harmonic functions are dense in $h^{-\infty}(\Omega)$, it follows from (7–1) that the L^2 pairing extends by continuity to a pairing $\langle f, g\rangle$ on $h^{-\infty}(\Omega) \times C^\infty(\overline{\Omega})$. In particular, this pairing is well defined and separately continuous on $A_{\mathrm{cl}}^{-\infty}(\Omega) \times A^\infty(\Omega)$, where $A_{\mathrm{cl}}^{-\infty}(\Omega)$ denotes the closure of $A^0(\Omega)$ in $A^{-\infty}(\Omega)$.

PROPOSITION 19. *Let Ω be a bounded domain in $\mathbb{C}^n$ with class C^∞ boundary. The following statements are equivalent.*

(i) *The Bergman projection P maps the space $C^\infty(\overline{\Omega})$ continuously into itself.*

(ii) *The spaces $A^{-\infty}_{\mathrm{cl}}(\Omega)$ and $A^\infty(\Omega)$ of holomorphic functions are mutually dual via the extended pairing $\langle\ ,\ \rangle$.*

Proposition 19 is from [Bell and Boas 1984; Komatsu 1984]; the case of a strictly pseudoconvex domain is in [Bell 1982b], and duality of spaces of harmonic functions is studied in [Bell 1982a; Ligocka 1986]. Once Proposition 18 is in hand, Proposition 19 is easily proved. For example, suppose that the Bergman projection is known to preserve the space $C^\infty(\overline{\Omega})$, and let τ be a continuous linear functional on the space $A^{-\infty}_{\mathrm{cl}}(\Omega)$. Because τ extends to a continuous linear functional on the inductive limit $W^{-\infty}(\Omega)$ of the ordinary Sobolev spaces, it is represented by pairing with a function g in the space $W^\infty_0(\Omega)$ of functions vanishing to infinite order at the boundary. On $A^0(\Omega)$, and hence on $A^{-\infty}_{\mathrm{cl}}(\Omega)$, pairing with g is the same as pairing with Pg since, by hypothesis, $Pg \in A^\infty(\Omega)$. Therefore τ is indeed represented by an element of $A^\infty(\Omega)$.

It is nontrivial that $A^{-\infty}_{\mathrm{cl}}(\Omega) = A^{-\infty}(\Omega)$ when Ω is pseudoconvex. Examples show that density properties fail dramatically in the nonpseudoconvex case [Barrett 1984; Barrett and Fornæss 1986]. The arguments in these papers can be adapted to show that $A^{-\infty}_{\mathrm{cl}}(\Omega) \neq A^{-\infty}(\Omega)$ for these examples.

THEOREM 20. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. Then the space $A^\infty(\Omega)$ of holomorphic functions is dense both in $A^k(\Omega)$ and in $A^{-k}(\Omega)$ for each non-negative integer k.*

The first part is in [Catlin 1980], the second in [Bell and Boas 1984]. In particular, the Bergman projection is globally regular on a pseudoconvex domain Ω if and only if the spaces $A^{-\infty}(\Omega)$ and $A^\infty(\Omega)$ are mutually dual via the pairing $\langle\ ,\ \rangle$.

Here is a typical application of Proposition 19 to the theory of the Bergman kernel function $K(w, z)$.

COROLLARY 21. *Let Ω be a bounded domain in $\mathbb{C}^n$ with class C^∞ boundary. Suppose that the Bergman projection maps the space $C^\infty(\overline{\Omega})$ into itself. If S is a set of determinacy for holomorphic functions on Ω, then $\{K(\cdot\,, z) : z \in S\}$ has dense linear span in $A^\infty(\Omega)$.*

Indeed, global regularity of the Bergman projection P implies that $K(\,\cdot\,, z) \in A^\infty(\Omega)$ for each z in Ω, since $K(\,\cdot\,, z)$ is the projection of a smooth, radially symmetric bump function (this idea originates in [Kerzman 1972]). Now if a linear functional τ on $A^\infty(\Omega)$ vanishes on each $K(\,\cdot\,, z)$ for $z \in S$, then $\tau(z) = 0$ on S, whence $\tau \equiv 0$. (Note that since $\tau \in A^{-\infty}_{\mathrm{cl}}(\Omega)$, the Bergman kernel does reproduce τ, because evaluation at an interior point is continuous in the topology of $A^{-\infty}(\Omega)$.)

Corollary 21 is due to Bell [1979; 1982b]. It is the key to certain non-vanishing properties of the Bergman kernel function that are essential in the approach to boundary regularity of holomorphic mappings developed in [Bell and Ligocka 1980; Bell 1981; 1984; Ligocka 1980; 1981; Webster 1979].

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