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Recent Developments in the Classification Theory of Compact Kähler Manifolds

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Abstract. We review some of the major recent developments in global complex geometry, specifically:

1. Mori theory, rational curves and the structure of Fano manifolds.
2. Non-splitting families of rational curves and the structure of compact Kähler threefolds.
3. Topology of compact Kähler manifolds: topological versus analytic isomorphism.
4. Topology of compact Kähler manifolds: the fundamental group.
5. Biregular classification: curvature and manifolds with nef tangent/anticanonical bundles.

Introduction

This article reports some of the recent developments in the classification theory of compact complex Kähler manifolds with special emphasis on manifolds of non-positive Kodaira dimension (vaguely: semipositively curved manifolds). In the introduction we want to give some general comments on classification theory concerning main principles, objectives and methods. Of course one could ask more generally for a classification theory of arbitrary compact manifolds but this seems hopeless as most of the techniques available break down in the "general" case (such as Hodge theory). Also there are a lot of pathologies which tell us to introduce some reasonable assumptions. From an algebraic point of view one will restrict to projective manifolds but from a more complex-analytic viewpoint, the Kähler condition is the most natural. Clearly manifolds which are only bimeromorphic to a projective or Kähler manifold are interesting, too, but these will be mainly ignored in this article and might occur only as intermediate products. The most basic questions in classification theory are the following.

(A) Which topological or differentiable manifolds carry a complex (algebraic or Kähler) structure? If a topological manifold carries a complex structure, try to describe them (moduli spaces, deformations, invariants).

(B) Birational or bimeromorphic classification: describe manifolds up to bimeromorphic equivalence and try to find nice models in every class.
(C) Biregular theory: try to describe manifolds up to biholomorphic equivalence; this is only possible with additional assumptions (such as curvature), study their properties and invariants.

There are also intermediate questions such as: What happens to the bimeromorphic class of a manifold in a deformation?

We give some more explanations to the single problems and relate then to the content of this article.

(A) We will mainly ignore the existence problem, which has not been of central interest in the past except for low dimensions. As to the moduli problem for complex structures, the first thing is to look for invariants, in particular the Kodaira dimension. For surfaces there is a big difference between topological and differentiable isomorphy: the Kodaira dimension is a diffeomorphic invariant but not a topological one (Donaldson). In dimension 3 the difference between topological and differentiable equivalence vanishes and therefore 3-folds which are diffeomorphic need not have the same Kodaira dimension. Nevertheless one can still ask to "classify" all complex Kähler structures for a given Kähler manifold and in particular to determine as many invariants as possible. The strongest assertions one can look for would predict that for restricted classes of manifolds topological equivalence already implies biholomorphic or at least deformation equivalence — for example, for Fano manifolds with $b_2(X) = 1$. This is a very difficult question and even in dimension 3 it is known only for a few examples such as projective space. The problem gets still more difficult if one looks for all complex structures; then we are far from giving the answer even for projective 3-space. One main difficulty is the lacking of a new topological invariant, such as the holomorphic Euler characteristic, in dimension 3. These and related questions will be discussed in section 4. One of the most subtle topological invariants of compact Kähler manifolds is certainly the fundamental group which has attracted much interest in the last few years. We discuss this in section 5.

(B) The most important birational (bimeromorphic) invariant is the Kodaira dimension. Therefore one wants to study the structure of the particular classes of manifolds X of a given Kodaira dimension $\kappa = \kappa(X)$. The most interesting cases are $\kappa = -\infty$, 0 and $\dim X$, while the cases $1 \leq \kappa \leq \dim X - 1$ are "interpolations" of these (in lower dimension) in terms of fiber spaces. We concentrate on the class of varieties with negative κ; it is studied in detail in Section 1. In the context of birational geometry two varieties are considered equal, if they coincide after some birational surgery such as blow-ups. Therefore one is looking for good birational models. The construction of such models in dimension 3 in the projective case, the so-called Mori theory, is discussed in Section 2. It depends on a numerical theory of the canonical bundle on X. The theory lives from projective techniques but the results should hold in the Kähler case, too. Some results in this direction are discussed in Section 3; it seems that a general theory needs a new, analytic way to construct rational curves.

(C) Here we want to study a single manifold as individual or a specific class of manifolds. A typical problem: classify manifolds with certain curvature conditions: for example, semipositive holomorphic bisectional or Ricci curvature. This is discussed in Section 6. Most of this type of problems deal with manifolds which are not of general or of non-positive Kodaira dimension because these have a richer geometry (and are fewer, hence more rigid.)

Sections 1, 2 and 5 have mainly written up by Campana, while Peternell is responsible for the rest. Both authors had the opportunity to spend a significant period at MSRI during the special year on complex analysis 1995/96. They would like to thank the institute for the support and the excellent working conditions.

1. Birational Classification. The Kodaira Dimension.

In this section we introduce the Kodaira dimension of compact (Kähler) manifolds and discuss the class of projective manifolds with negative Kodaira dimension. Furthermore a refined version of the Kodaira dimension is introduced.

Let X be a compact Kähler manifold with canonical bundle K_X and L a line bundle on X. Let $n = \dim X$.

DEFINITION 1.1. The **Iitaka dimension** of L is defined by

$$\kappa(X,L) := \overline{\lim_{\substack{m>0\\ m\to\infty}}} \Big(\frac{\log h^0(X,mL)}{\log m}\Big).$$

The **Kodaira dimension** of X is

$$\kappa(X) := \kappa(X,K_X).$$

This definition is short but not very illuminating. It can be more concretely described as follows:

$$\kappa(X,L) = \begin{cases} -\infty & \text{if and only if } h^0(X,mL) = 0 \text{ for all } m > 0, \\ 0 & \text{if and only if } h^0(X,mL) \le 1 \text{ (and not always 0).} \end{cases}$$

In fact Iitaka showed that if $\kappa(X,L) \ge 0$ there exist $d \in \{0,\dots,n\}$, a number $m_0 > 0$ and constants $0 < A < B$ such that

$$Am^d \le h^0(X,mL) \le Am^d \quad \text{for all } m \in \mathbb{N} \text{ divisible by } m_0.$$

In geometrical terms we have

$$d = \max\dim \Phi_m(X),$$

where $\Phi_m : X \rightharpoonup \mathbb{P}\big(H^0(X, mL)^*\big)$ is defined by the linear system $|mL|$ (when $\kappa(X) \geq 0$).

EXAMPLES 1.2. (1) If $\dim X = 1$, then:

$$\kappa(X) = \begin{cases} -\infty & \text{if and only if } X = \mathbb{P}_1, \\ 0 & \text{if and only if } g(X) = 1 \ \ (X \text{ is an elliptic curve}), \\ 1 & \text{if and only if } g(X) \geq 2. \end{cases}$$

(2) If $\dim X = 2$, we have the Kodaira–Enriques classification of algebraic surfaces with invariants $\kappa = \kappa(X)$ and $q = h^1(X, \mathcal{O}_X)$, which are both birational invariants. The following table gives the values of κ and q when X is bimeromorphic to a surface of the specified type:

$\kappa = -\infty$	$q = 0$	$\mathbb{P}_2$
$\kappa = -\infty$	$q \geq 1$	$\mathbb{P}_1 \times C$, where C is a smooth curve with $g(C) = q$
$\kappa = 0$	$q = 0$	a K3 or Enriques surface ($q = 0$; $K = \mathcal{O}_X$ or $2K_X = \mathcal{O}_X$)
$\kappa = 0$	$q = 1$	a bielliptic surface
$\kappa = 0$	$q = 2$	an abelian surface
$\kappa = 1$		an elliptic fibration given by $\lvert mK_X\rvert$
$\kappa = 2$		general type: $\lvert mK_X\rvert$ gives a birational map (for $m \geq 5$)

(3) $n = \dim X$ arbitrary. Assume $\kappa(X) \geq 0$. The linear system $|mK_X|$ defines a rational dominant map

$$\Phi_m : X \rightharpoonup Y$$

with connected fibers, called the **Iitaka fibration** of X [Ueno 1975] and $\dim Y = \kappa(X)$. The general fiber X_y of Φ_m has $\kappa(X_y) = 0$. This map is a birational invariant of X via the birational invariance of the plurigenera $P_m := h^0(X, mK_X)$.

The class of projective (smooth) n-folds thus falls into $n + 1$ classes, according to the value of κ. There are 3 "new" classes in each dimension n:

(a) $\kappa(X) = -\infty$: The linear systems $|mK_X|$ do not give any information.

(b) $\kappa(X) = 0$.

(c) $\kappa(X) = n$ (X is said to be of "general type").

Indeed, for the classes $1 \leq \kappa(X) \leq n - 1$, the Iitaka fibration expresses X as a fibration over a lower-dimensional manifold, and with fibers having $\kappa = 0$. This reduces largely the structure of X to lower-dimensional cases.

As we can see from the case of curves and surfaces, the 3 classes above differ completely: the special ones have $\kappa = -\infty$ or $\kappa = 0$, whereas the general ones have $\kappa = 2$ (hence the name).

We now discuss manifolds with $\kappa = -\infty$. Here we have a standard conjecture:

CONJECTURE 1.3. Let X be a projective manifold. Then $\kappa = -\infty$ if and only if X is uniruled.

Recall that X is said to be **uniruled** if there exists a dominant rational map $\psi : \mathbb{P}_1 \times T \dashrightarrow X$ with $\dim T = \dim X - 1$. In other words: there exists a rational curve going through the general point of X.

The "only if" part is an easy exercise. The converse is known for $n \leq 3$. For $n = 2$ this is the Enriques classification and results from the famous "Castelnuovo criterion"; for $n = 3$, this is a deep theorem proved by Y. Miyaoka using results of Kawamata and Mori (see Section 2).

There is however a big difference (from the birational point of view) between $\mathbb{P}_1 \times \mathbb{P}_1$ say and $\mathbb{P}_1 \times C$ where C is a curve with $g(C) \geq 1$. In fact $\mathbb{P}_1 \times C$ has much less rational curves than $\mathbb{P}_1 \times \mathbb{P}_1$. We try to make this more precise:

DEFINITION 1.4. Let X be a projective manifold.

(1) X is **rationally generated** if for any dominant map $\varphi : X \dashrightarrow Y$, the variety Y is uniruled.
(2) X is **rationally connected** if any two generic points on X can be joined by a **rational chain** C (that is, a connected curve with all its irreducible components rational).
(2′) X is **strongly rationally connected** if moreover the chain C in (2) can be chosen to be irreducible.
(3) X is **unirational** if there is a dominant rational map $\varphi : \mathbb{P}_n \dashrightarrow X$. If moreover φ is birational we say that X is **rational**.

Notice the following obvious implications:

$$
\begin{aligned}
&X \text{ rational} \Rightarrow X \text{ unirational } \Rightarrow X \text{ strongly rationally connected} \Rightarrow \\
&\qquad \Rightarrow X \text{ rationally connected} \Rightarrow X \text{ rationally generated} \Rightarrow X \text{ uniruled.}
\end{aligned}
$$

COMMENTS. When $n = 1$, all these properties are equivalent.

When $n = 2$, rationality is equivalent to rational generatedness but of course uniruledness is weaker then the other properties.

When $n = 3$, rational connectedness is equivalent to strong rational connectedness by [Kollár et al. 1992a], and unirationality is distinct from rationality (as shown in [Clemens and Griffiths 1972], the cubic hypersurface in $\mathbb{P}_4$ is non-rational, but unirational). It is also unknown whether rational connectedness implies unirationality; this is in fact doubtfull: the general quartic hypersurfaces in $\mathbb{P}_4$ are rationally connected but expected not to be unirational.

The only known reverse implication for arbitrary dimension is that rational connectedness implies strong rational connectedness; see [Kollár et al. 1992a] (the smoothness assumption is essential here: consider the cone over an elliptic curve!). This is based on relative deformation theory of maps.

1.5. We now discuss the difference between rational connectedness and rational generatedness in a special case: let $\varphi : X \to \mathbb{P}_1$ be a (regular) map with generic fiber X_λ rationally connected. Then X is obviously rationally generated. But it is rationally connected if and only if there is a rational curve C in X such that $\varphi(C) = \mathbb{P}_1$. It is in fact sufficient to check the equivalence of rational generatedness and rational connectedness in this special case in order to prove that the two notions coincide in general. However this equivalence might very well be a low-dimensional phenomenon.

An important example of rationally connected manifolds are the Fano manifolds (see Section 2). Conversely, we ask:

REMARK 1.6. It is unlikely that every rationally connected manifold is birational to some Fano variety. In fact, there are infinitely many birationally inequivalent families of conic bundles over surfaces, whereas it is expected that there are only finitely many families of Fano 3-folds. This last fact is known — as explained later — in the smooth case and it is also known in the singular case if $b_2(X) = 1$ (Kawamata).

THEOREM 1.7 [Campana 1992; Kollár et al. 1992a]. *Let X be a smooth projective n-fold. There exists a unique dominant rational map $\rho : X \dashrightarrow X_1$ such that, for x "**general**" in X, the fiber of ρ through x consists of the points $x' \in X$ which can be joined to x by some rational chain C. Moreover, ρ is a "**quasi-fibration**", so that its generic fiber is smooth and rationally connected. It is characterized by the following property: for any dominant $\rho' : X \dashrightarrow Y$ with generic fiber rationally connected, ρ' dominates ρ (that is, there exists $\psi : Y \dashrightarrow X_1$ such that $\psi \circ \rho' = \rho$). The map ρ is a birational invariant of X.*

Recall that $x \in X$ is **general** if it lies outside a countable union of Zariski closed subsets with empty interior, and that $\rho : X \to X_1$ is a **quasi-fibration** if there exist Zariski open nonempty subsets V of Y_1, and U of X such that the restriction of ρ to U is regular, maps U to V and $\rho : U \to V$ is proper. (In other words: the indeterminacy locus of ρ is **not** mapped **onto** X_1).

The map ρ above is called the maximal rationally connected fibration in [Kollár et al. 1992a] and the rational quotient in [Campana 1992].

Notice that this construction holds also for X compact Kähler [Campana 1992] and for X only normal. But in the normal case it is in general no longer a birational invariant.

Now it might happen that X_1 is again uniruled (this would happen precisely if the general fiber of $\rho_1 \circ \rho$ is rationally generated but not rationally connected, ρ_1 being explained in the next sentence). So X_1 has a rational quotient $\rho_1 : X_1 \to X_2$ as well. Proceeding this way, the dimension decreases by at least one at each step until it finally stops. Therefore we can state:

COROLLARY 1.8 [Campana 1995a]. *There exists a (unique) rational dominant map $\sigma : X \dashrightarrow S(X)$ to a non-uniruled variety $S(X)$ and with generic rationally generated fiber. It dominates any other $\sigma' : X \dashrightarrow Y$ with Y non-uniruled, and* *is dominated by any $\sigma' : X \dashrightarrow Y$ with generic rational generated fiber. This map σ is a quasi-fibration and a birational invariant (for X smooth).*

We call σ the LNU-quotient of X (for "largest non-uniruled"), or the MRG-fibration of X (for "maximal rationally generated").

Notice that, by convention, **a point is not uniruled** in case X itself is rationally generated, and that $X = S(X)$ if X is not uniruled.

We now introduce, after [Campana 1995a], a refined Kodaira dimension which should (at least conjecturally) calculate $\kappa\big(S(X)\big)$, and plays an essential role in Section 5; it leads also to refinements of Conjecture 1.3 above.

DEFINITION 1.9 [Campana 1995a]. Let X be a compact complex manifold. We define

$$\kappa_+(X) := \max\{\kappa(Y) \mid \text{there exists } \varphi : X \rightharpoonup Y \text{ dominant}\}$$

and

$$\kappa^+(X) := \max\big\{\kappa(X, \det \mathcal{F}) \,|\, \mathcal{F} \neq 0 \text{ is a coherent subsheaf of } \Omega_X^p, \text{ for some } p > 0\big\}.$$

Here $\det(\mathcal{F})$ is the saturation of $\det \mathcal{F} \subset \bigwedge^r \Omega_X^p$ if $r = \mathrm{rk}(\mathcal{F})$.

We have the following easy properties of κ_+ and κ^+. Here $a(X)$ denotes the algebraic dimension; see Theorem and Definition 3.1.

PROPOSITION 1.10 [Campana 1995a]. (1) *κ_+ and κ^+ are birational invariants.*
(2) *If $\varphi : X \to Y$ is dominant, then $\kappa^+(X) \geq \kappa^+(Y)$ (and similarly of course for κ_+).*
(3) $\dim(X) \geq a(X) \geq \kappa^+(X) \geq \kappa_+(X) \geq \kappa(X) \geq -\infty$.
(4) *If $\varphi : X \to Y$ has a generic fiber which is rationally generated, then $\kappa^+(X) = \kappa^+(Y)$ and $\kappa_+(X) = \kappa_+(Y)$.*
(4′) *If X is rationally generated, then $\kappa^+(X) = \kappa_+(X) = -\infty$.*

EXAMPLES 1.11 (curves and surfaces). (1) If $\dim X = 1$, then $\kappa^+ = \kappa_+ = \kappa$.
(2) If $\dim X = 2$, the situation is more interesting:

- (a) $\kappa^+(X) = \kappa_+(X) = \kappa(X)$ if $\kappa(X) \geq 0$ (use for example the Castelnuovo–de Franchis theorem). Thus only when $\kappa(X) = -\infty$ we get more information on X from κ^+ than from κ.
- (b) If $\kappa(X) = -\infty$ then $\kappa^+(X) = \kappa_+(X) = -\infty$ if and only if X is rational; and $\kappa^+(X) = \kappa_+(X) = 0$ (respectively 1) if and only if X is birational to $\mathbb{P}_1 \times B$, where B is a curve of genus $g = 1$ (respectively $g \geq 2$).

CONJECTURE 1.12 [Campana 1995a]. Let X be a projective (or compact Kähler) manifold. Then:

(a) $\kappa^+(X) = \kappa_+(X) = \kappa(X)$ if $\kappa(X) \geq 0$.
(b) $\kappa^+(X) = \kappa_+(X) = \kappa\big(S(X)\big)$ if $S(X)$ is the LNU-quotient of X (see paragraph after Corollary 1.8), unless X is rationally generated (that is, $S(X)$ is a point).

(c) $\kappa^+(X) = -\infty$ if and only if X is rationally generated.

This conjecture is in fact a consequence of Conjecture 1.3 and of standard conjectures in the Minimal Model Program. More precisely: 1.11 holds if 1.3 holds and if every projective (or compact Kähler) manifold with $\kappa(X) = 0$ is bimeromorphic to a variety X' with only $\mathbb{Q}$-factorial terminal singularities and such that $K_{X'} \equiv 0$ (or $c_1(X') = 0$). See Section 2 for the terminology.

The reduction of 1.11 to these other conjectures rests in the projective case on Miyaoka's generic semipositivity theorem (Theorem 2.7), and thus on characteristic $p > 0$ methods.

Observe finally that the class of rationally generated manifolds is invariant under deformations, and that all sections of tensor bundles vanish for manifolds in that class. This makes these manifolds difficult to distinguish from rationally connected or unirational manifolds. Should the properties "rationally connected" and "rationally generated" be different, the right class to consider (characterized by $\kappa^+ = -\infty$) is the class of rationally generated manifolds.

We shall see in Section 5 that π_1 vanishes for these, too.

Examples 1.13. We give some instances in which Conjecture 1.11 holds.

(1) If $n \leq 3$ and if X is projective, the conjecture holds:
  (a) For $n = 1$, this is obvious since $\kappa^+ = \kappa$ for curves.
  (b) For $n = 2$, this is easy, too, because the only non-trivial sheaves $\mathcal{F} \subsetneq \Omega^p_X$ appearing are of rank one in Ω^1_X. The Castelnuovo–de Franchis theorem then applies and solves the problem (in the Kähler case as well).
  (c) For $n = 3$, this is a consequence of the fact that the Minimal Model Program and Abundance conjecture have been solved in that dimension by the Japanese School (Kawamata, Miyaoka, Mori). See Section 2 for more details.

(2) If $c_1(X) = 0$, the conjecture holds (that is, $\kappa^+(X) = \kappa(X) = 0$). This is proved in [Campana 1995a] using Miyaoka's generic semipositivity theorem (our Theorem 2.7) if X is projective. In the Kähler case, this is an easy consequence of the existence of Ricci-flat Kähler metrics: holomorphic tensors are parallel.

(3) If K_X is nef and $\kappa^+(X) = n$, then $\kappa(X) = n$, too. The proof involves Miyaoka's generic semipositivity theorem again, but is more involved.

2. Numerical Theory: The Minimal Model Program

This section gives a short introduction to the minimal model theory or Mori theory of projective manifolds. It consists of two parts:

(a) producing a "contraction" of X when the canonical bundle K_X of a projective manifold X is not nef;

(b) giving a structure theorem in case K_X is nef, namely that mK_X is generated by global sections.

Let X be a projective manifold, and L a line bundle on X. Recall that L is **nef** if $L.C \geq 0$ for any effective curve C in X, and ample if the linear system $|mL|$ provides an embedding for some $m > 0$. If mL is generated by global sections for some $m > 0$, then L is nef (the converse is not true in general).

There is also a relative version: if $\varphi : X \to Y$ is a morphism, then L is φ-nef if $L.C \geq 0$ for any curve C in X contained in some fiber of φ; and L is φ-ample if the natural evaluation map $\varphi^*\varphi_*(mL) \to mL$ is surjective for some $m > 0$ and defines an embedding of X over Y in $\mathbb{P}\big(\varphi^*\varphi_*(mL)\big)$.

We denote by $\equiv$ numerical equivalence of Cartier divisors.

2.1. Introduction. As already seen, projective n-folds X fall into 2 classes, according to their value of κ:

(1) $\kappa(X) = -\infty$.
(2) $\kappa(X) \geq 0$: the Iitaka fibration $I_X : X \rightharpoonup Y$ reduces the structure of X to that of $I(X)$ and its general fiber X_y, which has $\kappa(X_y) = 0$. One is thus largely reduced to lower-dimensional varieties, except in the two extreme cases: $\kappa(X) = 0$ and $\kappa(X) = n$.

Classes 1 and 2 above contain their numerical analogues $(1')$ and $(2')$ defined as follows:

$(1')$ X is a **Fano fibration** (that is, there exists a map $\varphi : X \to Y$ such that K_X^{-1} is φ-ample). An extreme case is when φ is constant (that is, K_X^{-1} is ample). In this case by definition X is said to be **Fano** (or **del Pezzo** when $n = 2$).
$(2')$ K_X is nef and mK_X is generated by global sections for some $m \gg 0$.

In this case, $I_X : X \to Y$ is a morphism defined by the linear system $|m'K_X|$, for a suitable m'. In the special case $\kappa(X) = 0$ condition $(2')$ means that K_X is **torsion**, and in the case $\kappa(X) = n$ it means that K_X is **ample**. Observe that K_{X_y} is torsion for the generic fiber X_y of I_X.

A natural question is whether conversely any X has a (birational) **minimal model** X' in one of the classes $(1')$ or $(2')$ above, with mild singularities.

As we shall see below, the answer is yes for $n \leq 3$ (and conjecturally for all n).

The interest in dealing with varieties in classes $(1')$ and $(2')$ is twofold:

(a) a precise biregular classification of X' can be expected by the study of Fano manifolds in case $(1')$ and the study of the linear systems $|mK_X|$ for the class $(2')$.
(b) The knowledge of the numerical invariants of X' — in contrary to the birational ones — allows the use of Riemann–Roch formula and vanishing theorems.

We look first to the case of dimension 2, where the situation is classically understood, although not from that point of view. Later on, we shall describe what happens for $n = 3$, where new phenomena occur, discovered mainly by S. Mori in 1980–1988.

THEOREM 2.2. *Let X be a smooth projective surface. Exactly one of the following possibilities occurs*:

(1) K_X *is nef*.
(2) X *contains a* (-1)*-curve*.
(3) X *is ruled, that is, it admits a* $\mathbb{P}_1$*-bundle structure* $\rho : X \to B$ *over a curve* B.
(4) $X \cong \mathbb{P}_2$.

Recall that a (-1)-curve is a curve C such that $C \simeq \mathbb{P}_1$ and $N_{C|X} \simeq \mathcal{O}(-1)$. Such curves are numerically characterized by: $K.C < 0$ and $C^2 < 0$; see [Barth et al. 1984], for example. Every (-1)-curve is the exceptional divisor of a contraction: $\gamma : X \to X_1$, where X_1 is a smooth surface, $\gamma(C) = x_1 \in X_1$ is a point and γ is the blow-up of this point in X_1, with $C = \gamma^{-1}(x_1)$. Such a contraction decreases b_2 by one. Thus, after contracting finitely many (-1)-curves, one gets a smooth surface X', birational to X, such that either (1), (3) or (4) holds for X'. In case (1) we say that X' is a **minimal model for** X (it is in fact unique in that case, so that its numerical invariants are birational invariants for X). In case (3) and (4), we get a Fano-fibration for X' (the map ρ might be the constant map).

We say a few words about a possible proof of Theorem 2.2: Assume that (1) and (2) do not hold. Then $-K_X.C > 0$ for some curve C, and $C^2 \geq 0$ for any such C. The all point is then to show that C can be chosen to be **rational**. This can be shown easily when $q(X) := h^1(\mathcal{O}_X) = 0$ by using the arguments of ([Barth et al. 1984]), and similar ones when $q(X) > 0$ (after introducing the Albanese map whose image is a curve in that case; the point is just to show that this is the desired ruling).

The second step to conclude the program above is then:

THEOREM 2.3. *Assume that K_X is nef. Then mK_X is generated by global sections for some $m > 0$.*

Here again, the proof can be divided into cases (we know that $K_X^2 \geq 0$ since K_X is nef):

(1) $K_X^2 > 0$. This case is easy — no special property of K is needed.
(2) $K_X^2 = 0$, but $K_X \not\equiv 0$. One just has to show that $P_m := h^0(K_X^m) \geq 2$ for some $m > 0$. We thus only need to consider the special case of surfaces with $p_g := h^0(K_X) = 0, 1$.
(3) $K_X \equiv 0$. One has to show that: $h^0(mK_X) = P_m > 0$ for some m. Again one has to consider the special case $p_g = 0$.

By the Noether formula, $\chi(\mathcal{O}_X) \geq 0$ and $q \leq 2$ in all the special cases — with equality only if the Albanese image is an abelian surface. The situations in 2 and 3 can then be classified (one can use the arguments of [Beauville 1978, VI and VII], for example).

Theorem 2.2 generalizes to the 3-dimensional case as follows:

THEOREM 2.4 [Mori 1982]. *Let X be a smooth projective 3-fold. Exactly one of the following situations occurs*:

(1) *K_X is nef.*

(2) *X contains a rational curve C such that $-4 \leq K_X.C \leq -1$, and there exists a unique morphism $\varphi : X \to Y$ with connected fibers to a projective normal variety Y such that K_X^{-1} is φ-ample, $\rho(X) = 1 + \rho(Y)$ and φ maps to points exactly the curves C' which are numerically proportional to C. The map φ, called an* ***extremal contraction***, *is of one of the following types*:

  (2a) *φ is birational. It then contracts an irreducible divisor E to either a point of a curve. There are five possible situations*:

    (2a1) *$\varphi(E)$ is a smooth curve of Y blown-up by φ.*

    (2a2) *$E \simeq \mathbb{P}_2$; $N_{E|X} \simeq \mathcal{O}(-1)$; $y = \varphi(E)$ is a smooth point blown-up by φ.*

    (2a3) *$E = \mathbb{P}_2$; $N_{E|X} \simeq \mathcal{O}(-2)$; $y = \varphi(E)$ is singular.*

    (2a4) *$E \simeq \mathbb{P}_1 \times \mathbb{P}_1$; $N_{E|X} \simeq \mathcal{O}(-1,-1)$; $\varphi(E) = y$ is a ordinary double point.*

    (2a5) *E is a quadric cone in $\mathbb{P}_3$; $N_{E|X} \simeq \mathcal{O}_{\mathbb{P}_3(-1)}$; $\varphi(E) = y$ is analytically $u^2 + v^2 + w^2 + t^3 = 0$.*

  (2b) *Y is a surface*; *then K_X^{-1} is φ-ample and φ is a conic bundle.*

  (2c) *Y is a curve*; *then K_X^{-1} is φ-ample and one says φ is a del Pezzo fibration.*

  (2d) *Y is a point*; *then X is Fano with $\rho(X) = 1$.*

This result also shows the non-apparent relationship between the cases (2), (3), (4) of Theorem 2.2.

We say a few words of the proof of Theorem 2.4: it is very different from the proof of Theorem 2.2, which proceeds by classical methods using linear systems and Riemann–Roch. These methods are not available in higher dimensions since the curves are no longer divisors. (The proof of S. Mori about the existence of a rational curve C such that $0 < K_X^{-1}.C < n+1$ and about the existence of φ works in every dimension $n = \dim X$.) Instead, the proof of Theorem 2.4 is based on deformation theory (of maps) and uses in an essential way the Frobenius morphisms in characteristic $p > 0$. The curve C is first constructed by reduction (mod p) in characteristic p and then lifted in characteristic zero. The existence (and list) of the extremal contraction φ is deduced from a detailed study of the deformations of C (with $0 < K_X^{-1}.C \leq 4$ taken as small as possible).

There is another approach, cohomological, to the existence of extremal contractions. It has been developed essentially by Kawamata, and works for varieties with only $\mathbb{Q}$-factorial, terminal singularities (see below). It does not give in general the existence of a rational curve C as above.

One of the main differences between Theorems 2.2 and 2.4 is that $Y = X_1$ is no longer smooth in general, so that the operation can a priori not be iterated.

So the next step is whether there is a reasonable class of singularities to allow for which the elementary contractions can be defined without leaving that class.

There are two guiding principles for conditions to be imposed on the singularities:

(1) K_X being nef (or φ-ample) should have a meaning in terms of intersection numbers, that is, $K_X \cdot C$ must have a meaning for every curve $C \subset X$. This is true if K_X is not only a Weil, but a Cartier divisor. However K_Y is not Cartier in case (2a3) of Theorem 2.4. But at least, some multiple mK_Y of K_Y becomes Cartier; so that one can define $K_Y.C := (1/m)(mK_Y.C)$ with the usual properties. One therefore says that Y has only **$\mathbb{Q}$-factorial singularities** if any Weil divisor D on Y is **$\mathbb{Q}$-Cartier** (that is, mY is Cartier for some $m \neq 0$). See [Reid 1983; 1987] for a detailed introduction to these questions.

(2) The second property one can ask is that the singularities do not effect the plurigenera. In other words: if $\tilde{Y} \xrightarrow{\delta} Y$ is any resolution of Y and mK_Y is Cartier, then $H^0(\tilde{Y}, mK_{\tilde{Y}}) = \delta^* H^0(Y, mK_Y)$. This does not depend on the resolution and is certainly guaranteed if

$$K_Y = \delta_* K_{\tilde{Y}}.$$

To understand this condition, write

$$K_{\tilde{Y}} = \delta^* K_Y + \Sigma \delta_i E_i,$$

where the E_i's are the divisors contracted by δ, where $\delta_i \in \mathbb{Z}$. Then the equality above holds precisely when $\delta_i \geq 0$ for any i. Such singularities are called **canonical**. Since however we are mostly interested in birational contractions $\varphi : X \to Y$ for which K_X^{-1} is φ-ample, it is natural to impose a more restrictive condition, namely that $\delta_i > 0$ for all i. We then say that Y has only **terminal singularities** if this is the case.

It turns out that the class of normal varieties Y "with only $\mathbb{Q}$-factorial terminal singularities" seems to be precisely the right one to consider: extremal contractions still exist and if $\varphi : X \to Y$ is such a contraction which is birational and contracting some divisor, then Y is again in the same class. Moreover terminal surface singularities are smooth points and, more generally, terminal singularities occur only in codimension at least 3. In particular, they are isolated in dimension 3.

Fortunately, if Y is a projective 3-fold with at most $\mathbb{Q}$-factorial terminal singularities with K_Y not nef, then a rational curve with $-K_X.C > 0$ still exists, and also an extremal contraction $\varphi : X \to Y$ still exists. However, unlike in the case X is smooth, this contraction may be **small**. This means that φ is birational, but contracts only finitely many curves (and not divisors) C_i's necessarily rational.

In this case, Y acquires a bad singularity at the image points since K_Y is no longer $\mathbb{Q}$-Cartier. Indeed, if K_Y is $\mathbb{Q}$-Cartier, then $K_X = \varphi^* K_Y$ since the exceptional locus has codimension 2. But $0 > K_X.C_i = \varphi^* K_Y.C_i = 0$.

To proceed with the construction of a minimal model, another birational transformation called a **flip** has been introduced; see [Kawamata et al. 1987; Mori 1988], for example. A flip is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & Y & \end{tikzcd}$$

where

(1) X^+ is $\mathbb{Q}$-factorial with only terminal singularities,
(2) f is isomorphic outside the indeterminacy locus of φ and φ^+ which are both at least 2-codimensional, and
(3) $K_{X^+}.C^+ > 0$ for any curve $C^+ \subset X^+$ such that $\dim \varphi^+(C^+) = 0$.

In particular, $\rho(X) = \rho(X^+)$ and $K_{X^+} = f_* K_X$. (Notice that small contractions do not exist in dimension 2).

The existence of flips was established in dimension 3 by S. Mori [1988]; this is the deepest part of the minimal model program in dimension 3. It uses classification of all "extremal neighborhoods" of irreducible curves (necessarily smooth rational) contracted by a small contraction. The existence of flips is unknown when $n \geq 4$; a proof would presumably require new methods since a similar classification does not seem to be possible in higher dimension.

The final result is:

THEOREM 2.5. *Let X be a projective $\mathbb{Q}$-factorial 3-fold with only terminal singularities. Then there exists a birational map $\psi : X \rightharpoonup X'$ which is a finite sequence of extremal contractions and flips, and such that X' is again $\mathbb{Q}$-factorial with only terminal singularities, and either*

(1) *$K_{X'}$ is nef, or*
(2) *there exists an extremal contraction $\varphi : X' \to Y'$ such that $\dim Y' \leq 2$, and of course $-K_{X'}$ is φ-ample* (similar to cases (2), (3) or (4) of Theorem 2.2).

We call X' a **minimal model of** X in case 1.

The second part of the program — the so-called "Abundance Conjecture" — has been solved by Y. Kawamata and Y. Miyaoka (and by E. Viehweg when $q(X) > 0$ and $\kappa(X) \geq 0$ using the solution of Iitaka's conjecture in dimension 3).

THEOREM 2.6. *Let X' be $\mathbb{Q}$-factorial with only terminal singularities and assume that $K_{X'}$ is nef. Then $mK_{X'}$ is generated by global sections for suitable $m > 0$ such that $mK_{X'}$ is Cartier. Hence the induced map is "the" Iitaka reduction of X'.*

This result says that $\kappa(X') = \nu(K_{X'})$, where $\nu(K_{X'})$ is the **numerical Kodaira dimension**, defined as $\min\{0 \leq d \leq n \mid K_{X'}^d \not\equiv 0\}$ if $\dim(X') = n$.

As for $n = 2$, the proof distinguishes the 4 cases:

(1) $K_X^3 > 0$ ($\nu = 3$). This case is easy (even if $n > 3$).
(2) $K_X^3 = 0$ but $K_X^2 \not\equiv 0$ ($\nu = 2$).
(3) $K_X^2 \equiv 0$; $K_X \not\equiv 0$ ($\nu = 1$).
(4) $K_X \equiv 0$ ($\nu = 0$).

To deal with the remaining cases (2), (3) and (4), a very delicate analysis of the elements in $|mK_{X'}|$ (which is assumed to be non-empty) is necessary. However, the very first step is to prove non-emptyness for some $m > 0$. In other words, one has to show that $\kappa(X') \geq 0$ if $K_{X'}$ is nef. This is especially hard when $q(X') = 0$, otherwise the Albanese map can be used. This step is easy when $n = 2$, because if X is a smooth surface with $q = p_g = 0$, then

$$h^0(2K) + h^0(-K) \geq \chi(2K) = K^2 + \chi(\mathcal{O}_X) \geq \chi(\mathcal{O}_X) = 1 - q + p_g = 1.$$

But K_X being nef also implies $h^0(-K) = 0$, so $P_2 > 0$.

But in dimension 3, a new approach has to be found. It was discovered by Y. Miyaoka, who gave criteria for uniruledness in arbitrary dimension n. As in S. Mori's approach, the method of reduction to characteristic $p > 0$ is used in an essential way. But this time, not only the numerical properties of K_X^{-1}, but also those of T_X, come into the game.

THEOREM 2.7 (Y. Miyaoka). *Let X be a smooth projective n-fold, H an ample divisor on X and C a complete intersection curve cut out by general elements in $|mH|$ for $m \gg 0$. If X is not uniruled, then $\Omega_X^1|C$ is semi-positive (or, equivalently, nef); that is, all rank-one quotient sheaves have non-negative degree.*

This result is known as the generic semi-positivity theorem.

COROLLARY 2.8 (Y. Miyaoka). *Let X be a normal projective n-fold with singularities in codimension at least 3 (this is the case if X is $\mathbb{Q}$-factorial with only terminal singularities). Assume that X is not uniruled. Then*

$$K_X^2.H^{n-2} \leq 3c_2(T_X).H^{n-2}$$

for H nef (and $c_2(T_X)$ being the direct image of the corresponding c_2 of any smooth model of X).

In dimension 3, we have the following important consequence:

THEOREM 2.9. *Let X' be a $\mathbb{Q}$-factorial projective 3-fold with at most terminal singularities. Assume $K_{X'}$ nef. Then X' is uniruled if and only if $\kappa(X') = -\infty$.*

We give the argument in case X' is Gorenstein; in the non-Gorenstein case one needs additional arguments: see [Miyaoka 1988]. First notice that if X' is Gorenstein we have

$$\chi(X', \mathcal{O}_{X'}) = -\frac{1}{24} K_{X'} \cdot c_2(X').$$

In general, this is false; see [Reid 1987]. From Corollary 2.8 we thus get the inequality $\chi(X', \mathcal{O}) \leq 0$. If $q(X') = 0$, we deduce $h^{3,0} > 0$, so that $\kappa(X') \geq 0$. (Recall that canonical (and so terminal) singularities are rational, so $h^{3,0}(X') = h^{3,0}(X)$ if X is any smooth model of X'). If $q(X') > 0$, we can use the Albanese map and various versions of $C_{n,m}$ to conclude, using results of Viehweg.

Notice that X' is uniruled if $\chi(\mathcal{O}_X) > 0$.

2.10. We now turn to **Fano manifolds**, the building blocks for the Fano fibrations. Recall that a $\mathbb{Q}$-factorial projective variety X with only terminal singularities is said to be Fano (in full, $\mathbb{Q}$-Fano) if $-K_X$ is ample in an obvious sense (replace $-K_X$ by $-mK_X$).

Consider first the cases $n = 1, 2, 3$:

- $n = 1$: There is a single one: $\mathbb{P}_1$.
- $n = 2$: There are 10 deformation families: $\mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1$ and $\mathbb{P}_2$ blown-up in $1 \leq d \leq 8$ points in general position (not 3 on a line; 6 on a conic or 7 on a singular cubic, this singular point being one of them). Recall that terminal means smooth for surfaces.
- $n = 3$ and X is smooth: There are then 104 deformation families. Seventeen of them have $b_2 = 1$ and were classified by Iskovskih and Shokurov using linear systems. The basic invariant in this classification is the **index of** X, defined as $r(X) := \max\{S > 0 \mid -K_X = S.H \text{ for some } H \text{ in } \mathrm{Pic}(X)\}$. One has $1 \leq r(X) \leq n+1$. If $r = n+1$ then $X = \mathbb{P}_n$, and if $r = n$ then $X = Q_n$ the n-dimensional quadric [Kobayashi and Ochiai 1973]. The other 87 families have been classified by Mori and Mukai using Theorem 2.4 (see Section 2.14 below). Note that since $b_2 \geq 2$ the variety X has at least 2 different extremal contractions.

With one exception, all these families are obtained by standard methods, which we now describe (in any dimension n):

- Take $X = \mathbb{P}_n$, $X = Q_n$ (the n-dimensional quadric in $\mathbb{P}_{n+1}$), or X rational homogeneous (it is easy to see that these are Fano).

- Take smooth blow-ups of X along submanifolds Y. (Y has to be of small anticanonical degree. It may happen that no such Y can be blown-up so that the result is Fano: if X is $\mathbb{P}_2$ already blown-up in 8 points for example).
- Take complete intersections of hypersurfaces of small (anticanonical) degree: this works (by adjunction formula) and gives examples if the index r of X is large. For example, if $X = \mathbb{P}_{n+1}$, its smooth hypersurfaces of degree $d \leq n+1$ are Fano (of index $n+2-d$).
- Take double (or cyclic) coverings of X, branched along smooth hypersurfaces of small degree. Again this gives examples (by adjunction formula) if r is large. Double coverings of $\mathbb{P}_n$ branched along hypersurfaces of degree $2d \leq 2n$ are Fano, of index $(n+1-d)$.

In all these constructions it is easy by counting dimensions to see the existence of many rational curves on the manifolds obtained. This is a general phenomenon, moreover the study of rational curves on Fano manifolds leads to essential results concerning their structure. This is illustrated by the following result:

Theorem 2.11 [Campana 1992; Kollár et al. 1992a]. *Let X be a smooth Fano n-fold. Then X is rationally connected.*

It was implicitly shown in [Mori 1979] that X is uniruled. The proof rests on ideas similar to those found in that reference.

Theorem 2.11 implies that $\pi_1(X) = \{1\}$ if X is Fano; see Section 6.

The study of the birational structure of Fano n folds is very difficult, but interesting, already in dimension 3:

- Cubic hypersurfaces in $\mathbb{P}_4$ are non-rational, but unirational (by a result of Clemens and Griffiths) and birationally distinct if non-isomorphic. (So the set of birational classes of Fano threefolds is not countable).
- Quadric hypersurfaces in $\mathbb{P}_4$ are non-rational; some are unirational (of degree 24) (Iskovskih–Manin). It is unknown whether or not the general one is unirational.

Question 2.12. Assume X is rationally connected. Under which conditions is X birational to some $\mathbb{Q}$-Fano n-fold ($\mathbb{Q}$-factorial with at most terminal singularities)?

Theorem 2.13 [Kollár et al. 1992b]. *There exists an explicit constant $A(n)$ such that every smooth Fano n-fold X satisfies*

$$c_1(X)^n \leq A(n)^n.$$

The big Matsusaka theorem then implies that the family of smooth Fano n-folds is bounded (that is, they can all be embedded in $\mathbb{P}_{N(n)}$ with degree less than $d(n)$, where $d(n)$ and $N(n)$ are explicit constants). In particular, there are only finitely many deformation families (and diffeomorphism types) of Fano n-folds.

We emphasize that the existence of a bound in Theorem 2.13 rests in an essential way on Theorem 2.11, and hence on the study of rational curves. The idea is in fact to join two general points on a Fano n-fold X by an **irreducible** (rational; this is not essential, but these are the objects that one can produce) curve of **anticanonical** degree $\delta \leq A(n)$. The "gluing Lemma" of [Kollár et al. 1992a] is used here. An easy ingenious argument (due to F. Fano) then gives the bound as in Theorem 2.13.

2.14. References. We now give references for the proofs, and further studies. We try to cite books and introductory papers; references to the original proofs may be found there.

(1) Canonical and terminal singularities: [Reid 1983; 1987; Clemens et al. 1988].
(2) Minimal model program (cone theorem): [Kawamata et al. 1987; Clemens et al. 1988; Kollár 1989; Miyaoka and Peternell 1997].
(3) Flips: [Mori 1988; Clemens et al. 1988; Kollár 1992; Miyaoka and Peternell 1997].
(4) Abundance conjecture: [Kollár 1992; Miyaoka and Peternell 1997].
(5) Fano manifolds: [Iskovskih 1977; 1978; 1989; Shokurov 1979; Mori and Mukai 1983 (for 3-folds); Campana 1992; Kollár et al. 1992a; Kollár 1996].

3. Compact Kähler Manifolds

In this section we want to discuss the global structure of (connected) compact Kähler manifolds. First we measure how far a compact manifold is from being algebraic.

Theorem and Definition 3.1. *Let X be a compact complex manifold of dimension n. Let $\mathcal{M}(X)$ denote its field of meromorphic functions. Let $a(X)$ be the transcendence degree of $\mathcal{M}(X)$. Then*

$$0 \leq a(X) \leq n.$$

Moreover $\mathcal{M}(X)$ is an algebraic function field, that is, there is a projective manifold Y with $\dim Y = a(X)$, such that $\mathcal{M}(X) \simeq \mathcal{M}(Y)$. The number $a(X)$ is called the algebraic dimension of X.

If $a(X) = n$, the manifold X is called ***Moishezon****.*

This theorem is due to Siegel. For this and for the elementary theory of the algebraic dimension as well as algebraic reductions which we are going to define next, we refer to [Ueno 1975], or, for a less detailed and shorter presentation, [Grauert et al. 1994]. Most prominent examples of non-algebraic compact (Kähler) manifolds are of course general tori and general K3-surfaces. To define and construct algebraic reductions, fix a compact manifold X and take Y as in Theorem 3.1. Then there is a meromorphic map $f : X \rightharpoonup Y$ such that $f^*(\mathcal{M}(Y)) = \mathcal{M}(X)$. Of course there is no unique choice of Y (unless $a(X) = 0, 1$ and unless we agree to choose Y normal). Every Y' bimeromorphic to Y does the same job.

Definition 3.2. Let X be a compact manifold (or irreducible reduced compact complex space). Let Y be a normal projective variety. A meromorphic map $f : X \rightharpoonup Y$ is called an **algebraic reduction** of X if

$$\mathcal{M}(X) = f^*(\mathcal{M}(Y)).$$

The extreme, and often the most difficult, case is $a(X) = 0$. Then one knows that there are only finitely many irreducible hypersurfaces in X and possibly none. One can say that the more algebraic X is the more compact subvarieties it has. This can be made precise in the following way.

Definition 3.3. A compact manifold X is **algebraically connected** if

(a) every irreducible component of $\mathcal{C}_1(X)$, the cycle space or Barlet space of 1-cycles, is compact, and
(b) every two general points in X can be joined by a connected compact complex curve.

In a moment we will comment on the cycle space or Barlet space (Chow scheme in the algebraic case); the compactness is fulfilled if X is Kähler, for example. Compactness allows one to take limits of families of cycles. The importance of the notion of algebraic connectedness is demonstrated by the following result [Campana 1981].

Theorem 3.4. *Let X be an algebraically connected compact* (*Kähler*) *manifold. Then X is Moishezon.*

The converse of Theorem 3.4 is obvious. There are many counterexamples (twistor spaces) to Theorem 3.4 if one drops condition (a) in Definition 3.3.

Instead of assuming the existence of many curves one might think of supposing the existence of a "big" submanifold forcing X to be algebraic. For example, if $Y \subset X$ is a hypersurface with ample normal bundle, then X is Moishezon.

Problem 3.5. Let X be a compact Kähler manifold and $Y \subset X$ a compact submanifold with ample normal bundle. Is X Moishezon (hence projective)?

3.6. One cannot expect a reasonable structure theory for arbitrary compact complex manifolds. Pathologies will be given in Section 4. The most reasonable assumption (without assuming projectivity) is the Kähler assumption, which we will choose, or, slightly more generally, the assumption that manifolds should be bimeromorphic to a Kähler manifold; such manifolds form the so-called class $\mathcal{C}$.

Here we collect some major tools for the investigation of compact Kähler manifolds.

(1) The Albanese map $X \to \mathrm{Alb}(X)$ to the Albanese torus $\mathrm{Alb}(X)$ given by integration of d-closed 1-forms. This map exists in general for compact manifolds, however in the Kähler case every 1-form is d-closed, hence contributes to the Albanese, which in general is false. See Section 6 for some application of the Albanese in classification theory.

(2) Hodge decomposition (or better Hodge theory). This is completely false for general compact manifolds.
(3) The cycle space or Barlet space $\mathcal{C}(X)$. This is the analogue of the Chow scheme in complex geometry. $\mathcal{C}_q(X)$ parametrises q-cycles Z, that is,

$$Z = \sum n_i Z_i,$$

where $n_i \geq 0$ and Z_i are irreducible reduced compact subspaces of dimension q, the sum of course being finite. One of the most basic results is, as already mentioned, the compactness of every irreducible component of the cycle space, if X is compact Kähler (or in class $\mathcal{C}$). In algebraic geometry varieties are usually studied via ample line bundles, vanishing theorems, linear systems etc. These concepts do not work on general compact Kähler manifolds. In some sense the substitute should be cycles, in particular curves, as we shall see later in this section. For an overview of the theory of cycle spaces and applications as well as references, see [Grauert et al. 1994, Chapter 8].

How far is a compact Kähler manifold from being projective? There is a basic criterion for projectivity, due to Kodaira (see [Morrow and Kodaira 1971], for example):

Theorem 3.7. *Let X be a compact Kähler manifold with $H^2(X, \mathcal{O}_X) = 0$. Then X is projective.*

This indicates that 2-forms should play an important role in the theory of non-algebraic compact Kähler. There is a "conjecture", due to Kodaira (or Andreotti), concerning the vague question posed above.

Definition 3.8. Let X be a compact n-dimensional Kähler manifold. We say that X can be **approximated algebraically** if the following condition holds. There is a complex manifold $\mathcal{X}$ and a proper surjective submersion

$$\pi : \mathcal{X} \to \Delta = \{z \in \mathbb{C}^n \mid \|z\| < 1\},$$

such that, putting $X_t = \pi^{-1}(t)$, we have:

(a) $X_0 \simeq X$;
(b) there is a sequence (t_ν) converging to 0 and such that all the X_{t_ν} are projective.

We call $\pi : \mathcal{X} \to \Delta$ a family of compact manifolds and we often denote it by (X_t).

Problem 3.9. Can every compact Kähler manifold be approximated algebraically?

This is true for surfaces, but it is proved in a rather indirect way, via the Kodaira–Enriques classification. And this is the only evidence we have. Certainly it would be very interesting to find a conceptual proof for surfaces.

3.10. Our main intention is now to try to understand compact Kähler manifolds according to their Kodaira dimension. More precisely we ask whether there is a Mori theory in the Kähler case. This means:

(1) proving that a compact Kähler manifold X has $\kappa(X) = -\infty$ if and only if it is uniruled (one direction being clear) and trying to find a birational model which has a Fano fibration as described in Section 2;
(2) if $\kappa(X) \geq 0$, finding a minimal model X', that is, X' has only terminal singularities and $K_{X'}$ is nef;
(3) if K_X is nef, then mK_X is generated by global sections for some m (abundance).

But Mori theory is somewhat more: it predicts how to find a minimal model and a Fano fibration. Namely, if K_X is not nef, then there should be a "canonical" contraction, the contraction of an extremal ray in the algebraic category. So we first have to explain what "nef" means in the Kähler case.

Definition 3.11. Let X be a compact complex manifold and L a line bundle on X. Fix a positive (not necessarily closed) $(1,1)$-form ω on X. Then L is **nef** if for every $\varepsilon > 0$ there exists a hermitian metric h_ε on L with curvature

$$\Theta_{h_\varepsilon} \geq -\varepsilon\omega.$$

Remarks 3.12. (1) Obviously the definition is independent on the choice of ω.
(2) If X is projective, then L is nef if and only if $L \cdot C \geq 0$ for all curves $C \subset X$. For this and many more basic properties of nef line bundles we refer to [Demailly et al. 1994].
(3) Call L **algebraically nef** if $L \cdot C \geq 0$ for all curves C. Then in general "algebraically nef" does not imply "nef". For an example take any compact Kähler surface X with $a(X) = 1$. Then the algebraic reduction is an elliptic fibration $f : X \to B$. Any curve in X is contained in some fiber of f. Now take an ample line bundle G on B and put $L = f^*(G^*)$. Then clearly L is not nef (because its dual has a section with zeroes!) but it is algebraically nef.
(4) There are examples of nef line bundles which do not admit a metric of semi-positive curvature [Demailly et al. 1994]. So we really need to work with ε in the definition 3.11.
(5) Assume X is Kähler. Let $KC(X)$ denote the (closed) Kähler cone of X. This is the closed cone inside $H^{1,1}(X) \cap H^2(X,\mathbb{R})$ generated by the classes of the Kähler forms. Let L be a line bundle on X. Then L is nef if and only if $c_1(L) \in KC(X)$. See [Peternell 1998b].

3.13. The first basic question for a Mori theory in the Kähler case is therefore the following: given a compact Kähler manifold X with K_X not nef, is there a curve C such that $K_X \cdot C < 0$? If yes, can we choose C rational? We have seen that for general L the answer is no, but K_X of course has special properties. We will give a positive answer in some cases below. A general method to attack the problem would be to deform the complex structure to a generic almost complex structure and then to try to use the theory of J-holomorphic curves. However for the approach one would need the following openness property (we state it only in the holomorphic category).

PROBLEM 3.14. Let $\mathcal{X} \to \Delta$ be a family of compact Kähler manifolds. Assume that K_{X_0} is not nef. Is then K_{X_t} not nef for all (small) t?

This is unknown even in the projective case. See [Andreatta and Peternell 1997] for some results in this direction.

The standard approach to Mori theory in the projective case is as follows. Assume K_X not nef. Fix an ample line bundle H. Let r be the uniquely determined positive number such that $K_X + rH$ is on the boundary of the ample cone, which is to say it is nef but not ample. Then r is rational. Now $m(K_X + rH)$ is generated by global sections and the associated morphism gives a contraction we are looking for. Needless to say that the approach completely breaks down in the Kähler case. The substitute should be the theory of non-splitting families of rational curves, this allows to avoid thoroughly to speak about line bundles (except the canonical bundle, of course), sections and linear systems. It was Kollár [1991a] who reconstructed contractions for smooth threefolds using this method. We are now going to explain the geometry of non-splitting families of rational curves.

DEFINITION 3.15. A **non-splitting family** $(C_t)_{t\in T}$ of (rational) curves is a family of curves (C_t) such that the parameter space T is compact irreducible and C_t is an irreducible reduced (rational) curve for every $t \in T$. It is described by its graph $\mathcal{C}$ with projections $p : \mathcal{C} \to X$ and $q : \mathcal{C} \to T$ such that $C_t = p(q^{-1}(t))$.

3.16. Mori's breaking lemma [1979] is an indispensable tool in dealing with families of rational curves. It holds on every compact complex manifold X for which condition 3.3(a) holds and states that if (C_t) is a family of rational curves (with compact and irreducible T as usual, of course) and if there are points $p, q \in X, p \neq q$, such that $p, q \in C_t$ for all $t \in T$, then (C_t) has to split (ig $\dim T > 0$). Moreover Mori proved that if $\dim X = n$ and if (C_t) is non-splitting, then $K_X \cdot C_t \geq -n-1$. Equality holds for the family of lines on the projective space $\mathbb{P}_n$ and it is conjectured that this is the only example.

Now we describe the structure of non-splitting families of rational curves in compact Kähler threefolds as given in [Campana and Peternell 1997].

THEOREM 3.17. *Let X be a compact Kähler threefold and $(C_t)_{t\in T}$ a non-splitting family of rational curves.*

(1) *If $K_X \cdot C_t = -4$ and if $\dim T = 4$, then $X \simeq \mathbb{P}_3$.*
(2) *If $K_X \cdot C_t = -3$, and if $\dim T = 3$, then either $X \simeq Q_3$, the 3-dimensional quadric, or X is a $\mathbb{P}_2$-bundle over a smooth curve.*
(3) *Assume $K_X \cdot C_t = -2$ and $\dim T = 2$.*

(3a) *If X is non-algebraic and if the C_t fill up a surface $S \subset X$, then $S \simeq \mathbb{P}_2$ with normal bundle $N_{S|X} = \mathcal{O}(-1)$. The same holds for X projective if S is normal.*

(3b) *If X is covered by the C_t, we are in one of the following cases.*

(3b1) *X is Fano with $b_2(X) = 1$ and index 2.*

(3b2) *X is a quadric bundle over a smooth curve with C_t contained in fibers.*

(3b3) *X is a $\mathbb{P}_1$-bundle over a surface, the C_t being the fibers.*

(3b4) *X is the blow-up of a $\mathbb{P}_2$-bundle over a curve along a section. Here the C_t are the strict transforms of the lines in the $\mathbb{P}_2'$s meeting the section.*

(4) *Let $K_X \cdot C_t = -1$ and $\dim T = 1$. Then the C_t fill up a surface S. Assume that S is non-algebraic.*

(4a) *If S is normal, we are in one of the following cases.*

(4a1) *$S = \mathbb{P}_2$ with $N_S = \mathcal{O}(-2)$.*

(4a2) *$S = \mathbb{P}_1 \times \mathbb{P}_1$ with $N_S = \mathcal{O}(-1, -1)$.*

(4a3) *$S = Q_0$, the quadric cone, with $N_S = \mathcal{O}(-1)$.*

(4a4) *S is a ruled surface over a smooth curve and X is the blow-up of a smooth threefold along C such that S is the exceptional divisor.*

(4b) *Let S be non-normal. Then $\kappa(X) = -\infty$. If moreover X can be approximated algebraically, then we have $a(X) = 1$, and under some further (necessary and sufficient) condition* [Peternell 1998b, 5.2], *X is a conic bundle over a surface Y with $a(Y) = 1$. The surface S consists of the reducible conics and the C_t are the irreducible components of the reducible conics.*

The essential content of the theorem can be rephrased as follows. Assume that C is a rational curve with $K_X \cdot C = k$, where $-1 \geq k \geq 4$. If no deformation of C splits, the conclusion of the theorem states that the C_t give rise to a special geometric situation. There are basically two different situations in the theorem. Either the C_t fill up X, then one can consider the "rational quotient" with respect to that family [Campana 1992; Kollár et al. 1992a], which is a priori only meromorphic, and investigate its structure. The results are just the fibrations one has in the algebraic case in the Mori theory. Or the C_t fill up a surface S. Now one has to study in great detail the structure of S. As result, in the normal case and at least if X is non-algebraic, one can blow down S to obtain a birational contraction $X \to Y$ of the same type as in the Mori theory, however in general Y will not be Kähler. We will come to this point later. In the normal case, with some extra assumption we get conic bundles. For all details of proof we refer to [Campana and Peternell 1997]. Of course it would be interesting to prove something along the lines of Theorem 3.17 also in the higher-dimensional or singular case.

Theorem 3.17 was used in [Peternell 1998b] to prove the following result:

THEOREM 3.18. *Let X be a non-algebraic compact Kähler threefold satisfying one of the following conditions.*

(I) *X can be approximated algebraically.*
(II) $\kappa(X) = 2$.
(III) *X has a good minimal model (that is, $mK_{X'}$ is generated by global sections).*

Assume that K_X is not nef. Then

(1) *X contains a rational curve C with $K_X \cdot C < 0$;*
(2) *there exists a surjective holomorphic map $\varphi : X \to Y$ to a normal complex space Y with $\varphi_*(\mathcal{O}_X) = \mathcal{O}_Y$ of one of the following types.*
   (2a) *φ is a $\mathbb{P}_1$- bundle or a conic bundle over a non-algebraic surface. (This can only happen in case (1).)*
   (2b) *φ is bimeromorphic contracting an irreducible divisor E to a point, and E together with its normal bundle N is one of*

$$(\mathbb{P}_2, \mathcal{O}(-1)), \quad (\mathbb{P}_2, \mathcal{O}(-2)), \quad (\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-1,-1)), \quad (Q_0, \mathcal{O}(-1)),$$

   where Q_0 is the quadric cone.
   (2c) *Y is smooth and φ is the blow-up of Y along a smooth curve.*

φ is called an extremal contraction.

Y is (a possibly singular) Kähler space in all cases except possibly (2c). Moreover in all cases but possibly (2c), φ is the contraction of an extremal ray in the cone of curves $\overline{NE}(X)$.

A normal complex space is **Kähler** if there is a Kähler metric h on the regular part of X with the following property. Every singular point has a neighborhood U and a closed embedding $U \subset V$ where V is an open subset of some $\mathbb{C}^n$ such that there is Kähler metric h' on V with $h'|U \setminus \mathrm{Sing}(X) = h$.

REMARK. Theorem 3.18 has been proved for all smooth compact Kähler threefolds X unless X is simple with $\kappa(X) = -\infty$; see [Peternell 1998c]. The same paper also proves abundance for minimal Kähler threefolds which are not both simple and non-Kummer; see 3.20 below.

ABOUT THE PROOF OF THEOREM 3.18. In order to apply Theorem 3.17 one needs a non-splitting family (C_t) of rational curves with $-4 \le K_X \cdot C_t < 0$. For this it is sufficient to have one rational curve C with $-4 \le K_X \cdot C < 0$. Then one can apply deformation theory to obtain a family; if this family splits, take an irreducible part C' of a splitting member with $K_X \cdot C' < 0$ and deform again. This procedure must terminate since X is Kähler.

In case (I) one shows that $K_{X_{t_\nu}}$ is not nef in terms of the algebraic approximation (this is of course a major step) and then apply Mori theory in the algebraic case to obtain a rational $C_{t_\nu} \subset X_{t_\nu}$ for a fixed t_ν with $K_{X_{t_\nu}} \cdot C_{t_\nu} < 0$. This curve can then be deformed into X_0 to obtain a rational curve C_0 with $K_{X_0} \cdot C_0 < 0$.

In case (II) we consider the linear system $|mK_X|$ defining a meromorphic map $f : X \rightharpoonup Y$ to a projective surface. Then we choose a general element $D_0 \in |mK_X|$. Now our linear system must have fixed components A_i and has a movable part B. Examining carefully the structure of B and A_i we first obtain some curve C with $K_X \cdot C < 0$ and then in a second step a rational one. A similar thing can be done if $\kappa(X) = 1$ to find at least some curve C with $K_X \cdot C < 0$. $\square$

3.19. Let X be a compact Kähler threefold with K_X not nef. We have seen that at least with some additional assumptions we can construct a rational curve C with $K_X \cdot C < 0$. Hence we can construct a map $\phi : X \to Y$ as described in Theorem 3.18. In order to continue the process in case $\dim Y = 3$ it is now very important that ϕ can be chosen in such a way that Y is again Kähler. Let E denote that exceptional locus of ϕ. If $\dim \phi(E) = 0$ then it turns out that Y is *always* Kähler. But if $\dim \phi(E) = 1$, that is, if ϕ is the blow-up of a smooth curve in the manifold Y, this is not necessarily the case, even in the projective case (Y could be Moishezon). Instead one has — in the projective case — to choose ϕ carefully: it has to be the contraction of an extremal ray in $\overline{NE}(X)$. In the Kähler case we can introduce the dual cone $\overline{NA}(X)$ to the Kähler cone in $H^{2,2}(X)$ and can prove that Y is Kähler if and only if the ray $R = \mathbb{R}_+[l]$ is extremal in $\overline{NA}(X)$, where l is a fiber of ϕ.

Problems. (1) Is Y Kähler if and only if R is extremal in $\overline{NE}(X)$?
(2) How can one find extremal rays in $\overline{NA}(X)$ or in $\overline{NE}(X)$? Is there a "Cone Theorem"?

3.20. Even if one has shown the existence of contraction $\phi : X \to Y$ for a compact Kähler threefold X with K_X not nef such that Y is Kähler, it is still necessary to do the same also for normal projective $\mathbb{Q}$-factorial Kähler threefolds X with at most terminal singularities in order to be able to repeat the process. Of course then one will run into the same trouble as in the algebraic case, namely that sometimes a small contraction will appear so that Y has bad singularities and we have to flip. However the existence and termination of flips are basically analytically local and have been settled in [Kawamata 1988; Mori 1988].

We next indicate how the expected answer to the problems (1) and (2) in Section 3.10 would give a new insight into the structure of non-algebraic Kähler threefolds far away from the "usual" algebraic applications of Mori theory.

A compact Kähler manifold X is **simple** if there is no covering family of positive dimensional subvarieties (hence through a very general point of X there is no positive dimensional compact subvariety). Note that using cycle space methods, the classification of compact Kähler manifolds can be reduced to a large extent to the classification of the simple manifolds; see [Grauert et al. 1994] and the references given there. In dimension three, simple compact Kähler threefolds are conjectured to be "Kummer" in the following sense. X is called Kummer if X is bimeromorphic to a variety T/G, with T a torus and G a finite group acting on T.

Observe that the set of points of T having non-trivial isotropy is finite in this situation.

THEOREM 3.21. *If* (3.10(1)) *and* (3.10(2)) *have positive answers in dimension three*, *every simple smooth compact Kähler threefold is simple*.

INDICATION OF PROOF. If X is simple, it cannot be uniruled, hence it has a minimal model by 3.10(1). By 3.10(2), $mK_{X'}$ is generated by global sections for $m \gg 0$. Again by the simplicity it follows $\kappa(X') = 0$, hence $mK_{X'} = \mathcal{O}_{X'}$. Assume X' Gorenstein and $m = 1$ for the sake of simplicity (in general one has to pass to a covering $\tilde{X} \to X'$ which is étale over the smooth part of X'). Then one can apply Riemann–Roch and obtain

$$\chi(X', \mathcal{O}_{X'}) = 0.$$

Since $\dim H^3(X', \mathcal{O}_{X'}) = 1$ by Serre duality, and since

$$H^2(X', \mathcal{O}_{X'}) \neq 0$$

(pass to a desingularisation, apply Theorem 3.7 and come back to X' using the rationality of the singularities of X') we deduce

$$H^1(X', \mathcal{O}_{X'}) \neq 0.$$

Therefore we have an Albanese map $X' \to \mathrm{Alb}(X')$. Now the structure of X' allows one to prove that the Albanese map is an isomorphism. □

We close the section with the following recent structure theorem from [Campana and Peternell 1998]:

THEOREM 3.22. *Let X be a smooth compact Kähler threefold which is not both simple and non-Kummer*. *Then*

(i) *If $\kappa(X) = -\infty$, X is uniruled*.
(ii) *If $\kappa(X) = 0$ and if X carries a holomorphic* 2-*form* (*for example*, *if X is not projective*), *then X is bimeromorphic to some threefold X'* (*possibly with quotient singularities*) *which has a finite cover $\tilde{X}'$ étale in codimension* 1 *such that $\tilde{X}'$ is either a torus or a product of an elliptic curve and a* K3-*surface*.

4. Topological Classification

In this section we mainly discuss the following question: given a compact complex manifolds X, can one describe all complex structures on the underlying topological (differentiable) manifold, if X has some nice properties (Fano etc.). In other words, we consider a topological manifold and ask for all complex structures if there is any. A typical question: if X is "nice" and Y homeomorphic to X, is $X \simeq Y$ biholomorphically? And: what are the analytically defined topological invariants?

4.1. In dimension 1 everything is clear: there is one (in fact analytically defined) topological invariant, the genus, and $X \simeq Y$ if and only if $g(X) = g(Y)$. Moreover every compact topological 2-dimensional real manifold carries a complex structure. The structure is unique if and only if $g = 0$, i.e., $X \simeq \mathbb{P}_1$. This already gives a hint that we should look for in higher dimension to those manifolds which are "natural" generalisations of $\mathbb{P}_1$. If $g \geq 2$, or, in higher dimensions, if X is of general type, then the task is to describe moduli spaces. This is a completely different topic and therefore systematically omitted.

4.2. In dimension 2 there has been spectacular progress in the last fifteen years due to the work of Freedman, the Donaldson theory and the Seiberg–Witten invariants. It is now known that the Kodaira dimension is a $\mathcal{C}^\infty$-invariant of compact Kähler surfaces but not a topological invariant. We will completely ignore this vast area and refer to [Donaldson and Kronheimer 1990; Okonek and Van de Ven 1990; Friedman and Morgan 1994; Okonek and Teleman 1999]. In the topological case there are still open problems, for example whether there is a surface of general type homeomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ (although the answer is known to be negative for $\mathbb{P}_2$.

The surface results imply that the Kodaira dimension is not a differentiable invariant of compact Kähler threefolds: let S be the Barlow surface, a minimal surface of general type homeomorphic to $\mathbb{P}_2$ blown up in 8 points. Then take an elliptic curve C and let $X_1 = C \times S$ and $X_2 = C \times \mathbb{P}_2(x_1, \dots, x_8)$. Then $\kappa(X_1) = 2$ whereas $\kappa(X_2) = -\infty$. Note that X_1 and X_2 are even diffeomorphic since topological and differentiable equivalences are the same here. If we take C to have genus ≥ 2, then we even find a threefold with K_X ample diffeomorphic to a threefolds with negative Kodaira dimension.

We will now go to higher dimensions and will see that only few things are known. The most basic question is certainly the following:

QUESTION 4.3. What are the complex structures on the complex projective space $\mathbb{P}_n$?

A first answer was given by Hirzebruch and Kodaira [1957]:

THEOREM 4.4. *Let X be a compact Kähler manifold homeomorphic to $\mathbb{P}_n$. Then $X \simeq \mathbb{P}_n$ biholomorphically unless n is even and K_X is ample.*

The proof makes essential use of the fact that the Pontrjagin classes $p_i(X) \in H^{4i}(X, \mathbb{R})$ are topological invariants. Actually in 1957 it was only known that the $p_i(X)$ were differentiable invariants, so Hirzebruch and Kodaira could formulate only a differentiable version of Theorem 4.4, but afterwards Novikov [1965] proved that the Pontrjagin classes are actually topological invariants. Hirzebruch and Kodaira could determine only the sign of $c_1(X)^n$, so that in even dimension the case K_X ample (and divisible by 4) remained open until Yau proved the Calabi conjectures. Using the latter one can rule out the case of K_X ample as follows. By calculating invariants one finds the following Chern class equality

$$nc_1(X)^n = 20 = 2(n+1)c_2(X)c_1(X)^{n-2}.$$

This is just the borderline for the Yau inequality and by the existence of a Kähler–Einstein metric, a classical differential-geometric argument shows that the universal cover of X is the unit ball in $\mathbb{C}^n$. On the other hand X is simply connected, contradiction. This is the only known argument to rule out the existence of complex structures of general type on projective space.

THEOREM 4.5. *Let X be a compact Kähler manifold homeomorphic to $\mathbb{P}_n$ Then $X \simeq \mathbb{P}_n$ analytically.*

One can ask the same question for, e.g., the n-dimensional quadric Q_n, $n \geq 3$. If $n = 2$, one has to admit the Hirzebruch surfaces $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2n))$; this case is special because $b_2 = 2$. In this context Brieskorn [1964] proved a result analogous to Theorem 4.4, with the same exception, namely that there could be a projective manifold of even dimension n with K_X ample homeomorphic to Q_n. Since there are, for example, surfaces with $c_1^2 = c_2$ which are simply connected, one would need here a completely different argument from Yau's.

In Theorem 4.4 the Kähler assumption, which is obviously equivalent to projectivity, is important. On one hand it allows to compute $H^q(X, \mathcal{O}_X)$ by Hodge decomposition, on the other hand one can use the Kodaira vanishing theorem to calculate $\chi(X, \mathcal{O}_X(k))$ for the ample generator $\mathcal{O}_X(1)$. If no Kähler assumption is made, then the problem gets very complicated and is essentially unsolved. Here is a possibly tractable subproblem:

PROBLEM 4.6. Let X be a compact manifold homeomorphic to $\mathbb{P}_n$. Assume that $\dim X \geq 3$ and $a(X) > 0$. Is $X \simeq \mathbb{P}_n$?

If $n \geq 4$ nothing is known in this regard except for a result of Nakamura [1992] for $n = 4$, which gives a positive answer if $a(X) = 4$ and X not of general type. If $n = 3$ and $a(X) = 3$, the problem is completely solved [Kollár 1991b; Nakamura 1987; Peternell 1986; 1998a]:

THEOREM 4.7. *Every Moishezon threefold homeomorphic to $\mathbb{P}_3$ is $\mathbb{P}_3$.*

The same holds for the quadric and some Fano threefolds (V_5 and the cubic); see [Kollár 1991b; Nakamura 1988; 1996].

If $a(X) < \dim X$, virtually nothing is known. There is an interesting relation to the existence problem on complex structures on 6-spheres, we will come to this in Section 4.18. If $a(X) = 0$ the problem seems hopeless at the moment, but at least for threefolds Theorem 4.6 seems not to be unsolvable.

4.8. It is conjectured that in the situation of Theorem 4.4 it is not actually necessary to assume that X and $\mathbb{P}_n$ are homeomorphic. It should be sufficient to assume that the cohomology rings $H^*(X, \mathbb{Z})$ and $H^*(\mathbb{P}_n, \mathbb{Z})$ are isomorphic (as graded rings). This is proven by Van de Ven and Fujita up to dimension

6 [van de Ven 1962; Fujita 1980]. It should be mentioned that Mumford has constructed surfaces of general type with $b_1 = 0$ and $b_2 = 1$ so that in Theorem 4.4 it is not sufficient to assume equality of the Betti numbers.

There is another weakening of the problem of complex structures on projective space: one considers only complex structures near to the standard one. This has been solved by Siu [1989] (see also [Hwang 1996]):

THEOREM 4.9. *Let $\mathcal{X} = (X_t)_{t\in\Delta}$ be a family of compact complex manifolds* (Definition 3.8), *parametrised by the unit disc $\Delta \subset \mathbb{C}$. Assume that $X_t \simeq \mathbb{P}_n$ for all $t \neq 0$. Then $X_0 \simeq \mathbb{P}_n$.*

In other words, $\mathbb{P}_n$ is stable under *global* deformations. Note that automatically all X_t are Moishezon and X_0 has a lot of vector fields. The analogous problem for the quadric was solved by Hwang [1995] and for hermitian symmetric manifolds with $b_2 = 1$ by Hwang and Mok [1998]. It should also be true for rational-homogeneous manifolds. More generally, one can ask:

PROBLEM 4.10. Let X_0 be a rational-homogeneous manifold with $b_2 = 1$. Let X be a compact manifold homeomorphic to X_0. Does it follow that $X \simeq X_0$?

PROBLEM 4.11. Let X_0 be a Fano manifold with $b_2(X_0) = 1$ and X a projective manifold homeomorphic to X_0. What is the structure of X? Is $\kappa(X) = -\infty$? What happens for $b_2(X_0) \geq 2$?

We shall restrict the discussion now to $\dim X = 3$. First we discuss the case $b_2 = 1$. We should expect that X is again Fano. This, however, is unknown even in very simple cases. For example:

PROBLEM 4.12. Let $X_0 \subset \mathbb{P}_4$ be a cubic hypersurface. Is there a projective threefold X with K_X ample such that X_0 and X are homeomorphic?

The difficulty is the lack of topological invariants, compared to surfaces we do not know any new topological invariant; however it might be possible to solve Problem 4.12 by carefully examining the linear system $|L|$ or $|2L|$, where L is the ample generator of $\mathrm{Pic}(X) = \mathbb{Z}$. Maybe it is now time to loose some words on topological invariants.

4.13. Here are the known topological invariants — by which we mean analytic invariants which a posteriori turn out to be topological invariants. First we have the Chern class $c_n(X)$, which by Hopf's theorem is nothing that the topological Euler characteristic $\chi_{\mathrm{top}}(X)$. By Hodge decomposition $q(X) = h^1(X, \mathcal{O}_X)$ is a topological invariant. Next we have the second Stiefel–Whitney class

$$w_2(X) = c_1(X)/ \mod 2 \in H^2(X, \mathbb{Z}_2).$$

Finally there are the Pontrjagin classes

$$p_i(X) \in H^{4i}(X, \mathbb{R}).$$

We have $p_1(X) = 2c_2 - c_1^2$ and $p_2(X) = 2c_4 - 2c_1c_3 + c_2^2$.

PROBLEM 4.14. Are $h^q(X, \mathcal{O}_X)$ topological invariants of compact Kähler manifolds? Is at least $\chi(X, \mathcal{O}_X)$ a topological invariant of compact Kähler threefolds?

4.15. We now look at Fano threefolds with higher b_2. So let X_0 be a Fano threefold with $b_2 = 2$. Let X be a projective threefold homeomorphic to X_0. By Hodge decomposition we have

$$H^2(X_0, \mathcal{O}) \simeq H^2(X, \mathcal{O}).$$

In order to make progress we need to assume that $b_3 = 0$. Then we conclude that $H^3(X_0, \mathcal{O}) \simeq H^3(X, \mathcal{O})$. Thus $\chi(X, \mathcal{O}_X) = 1$. Now a fundamental theorem of Miyoka [1987] says that a threefold X with K_X nef has

$$\chi(X, \mathcal{O}_X) \leq 0.$$

This is a consequence of his inequality $c_1^2 \leq 3c_2$ for minimal threefolds. Hence K_X cannot be nef and therefore by Mori theory there must be an extremal contraction $\phi : X \to Y$. This gives us a tool to investigate the structure of X and one can prove:

THEOREM 4.16. *Let X_0 be a Fano threefold with $b_2 = 2$ and $b_3 = 0$. Let X be a projective threefold homeomorphic to X. Then $X_0 \simeq X$ or there is an explicit description for X.*

An example for "an explicit" description as mentioned in the theorem is the following. Let $X_0 = \mathbb{P}(T_{\mathbb{P}_2})$ and take for X a vector bundle E on $\mathbb{P}_2$ with the same Chern classes and let $X = \mathbb{P}(E)$.

The theorem is proved in [Campana and Peternell 1994] in the case that X_0 is not the blow-up of another Fano threefold along a smooth curve and in [Freitag 1994] in the remaining cases. One also might ask whether one can release the projectivity assumption. In this context the paper [Summerer 1997] proves that the flag manifold $\mathbb{P}(T_{\mathbb{P}_2})$ is rigid under global deformation and that $\mathbb{P}_1 \times \mathbb{P}_2$ has only "the obvious" (projective) deformations.

We saw in (4.2) that the statement "K_X is not nef" is not topologically invariant. However, if we start with a threefold X_0 such that $\chi(X_0, \mathcal{O}_{X_0}) > 0$, any projective threefold X homeomorphic to X_0 is not minimal, that is, carries an extremal contraction, *once* we know that $\chi(X_0, \mathcal{O}_{X_0}) = \chi(X, \mathcal{O}_X)$. This means that the problem of projective complex structures for threefolds with $b_2 > 1$ is most tractable in the case of positive holomorphic Euler characteristic, for example Fano threefolds.

4.17. A somehow related result of [Campana and Peternell 1994] is the following. Some non-projective Moishezon twistor space X_0 is constructed with the property that there is not projective threefold homeomorphic to X_0.

4.18. We next mention the fundamental problem asking which topological manifolds admit a complex structure. We concentrate on simply connected manifolds of dimension 6. The topological 6-manifolds which have torsion-free homology are classified by the work of Wall [1966] and Jupp [1973]. They can be completely described by a system of invariants: $H^2(X, \mathbb{Z})$, the Betti number $b_3(X)$, the cup product on $H^2(X, \mathbb{Z})$, the Pontrjagin class $p_1(X)$, the Stiefel–Whitney class $w_2(X)$ and the triangulation class $\tau(X) \in H^4(X, \mathbb{Z})$ with a certain relation. Now several fundamental questions arise:

(a) Which complex cubics can be realised as cup form of a compact complex threefold (up to equivalence)?
(b) Which systems of invariants can be realised by almost complex manifold?
(c) Which systems of invariants can be realised by complex manifolds (by Kähler manifolds)?

Instead of describing results we refer to the papers [Okonek and Van de Ven 1995; Schmitt 1995; 1997; 1996].

4.19. One of the most natural questions in the context of Section 4.18 is certainly the problem of complex structures on spheres. The situation is as follows.

(a) The only complex structure on S^2 is of course the complex structure $\mathbb{P}^1$.
(b) The spheres S^{2n} do not admit almost complex structures for $n \geq 2, n \neq 3$. If S^{2n} is equipped with the standard differentiable structure, this is due to Kirchhoff [1948], in general to Borel and Serre [1953].
(c) There remains the question of complex structures on S^6. Here almost complex structures do exist; one induced by the Cayley numbers: see [Steenrod 1951]. Hence there is the question of integrability. This is still unsolved. It is clear that a complex structure on S^6 is far from being Kähler.
(d) The only result is the following [Campana et al. 1998a]: If X is a compact manifold homeomorphic to S^6, then X does not admit a non-constant meromorphic function. More generally one can show:

Theorem 4.20. *Let X be a smooth compact threefold with $b_2(X) = 0$. If $a(X) \geq 1$, then either $b_1(X) = 1$ and $b_3(X) = 0$ or $b_1(X) = 0$ and $b_3(X) = 2$.*

The first alternative is realised by Hopf manifolds, and the second by Calabi–Eckmann manifolds, which are complex structures on $S^3 \times S^3$. Note finally the relation between complex structures on S^6 and $\mathbb{P}_3$: let X be a complex structure on S^6 and $\hat{X} \to X$ the blow-up of a point $p \in X$. Then $\hat{X}$ is a complex structure on $\mathbb{P}_3$. In [Huckleberry et al. 1999] it is shown that there is no complex Lie group acting on X with an open orbit, in particular X can have at most two independent vector fields. A consequence: If S^6 has a complex structure, then $\mathbb{P}_3$ has a 1-dimensional family of exotic complex structures.

5. The Fundamental Group

A very interesting topological invariant is the fundamental group of a compact Kähler or projective manifold. We survey here some results concerning the following questions:

(a) Which groups are Kähler (that is, of the form $\pi_1(X)$, for some adequate compact Kähler manifold X)? Many restrictions are known.
(b) How does $\kappa(X)$ influence $\pi_1(X)$ or $\widetilde{X}$, the universal cover of X (any compact Kähler manifold)?
(c) Do the classes of groups of the form $\pi_1(X)$ for X compact Kähler and X projective differ?

We shall concentrate on question (b), and to some extend on (c), which is closely related to (b).

5.1. Restrictions on Kähler groups. We shall only give here some very brief indications, refering to [Amorós et al. 1996] for more details, where the known obstructions and examples are systematically surveyed.

There are three main types of known restrictions:

5.1.1. Restrictions on the lower central series of $\pi_1(X)$. These are deduced from classical Hodge theory (the $\partial\bar\partial$-Lemma). The basic two restrictions are that (up to torsion) this lower central series is determined by its first 2 terms (that is, by the natural map $\bigwedge^2 H^1(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})$). This was shown with $\mathbb{R}$-coefficients in [Deligne et al. 1975], and was later related to the Albanese map in [Campana 1995b].

Notice, however, that even for nilpotent groups, it is not known which ones are Kähler. Only recently were non-trivial examples given [Campana 1995b; Sommese and Van de Ven 1986]). Only for the very special case of Heisenberg groups is the situation more or less understood [Campana 1995b; Carlson and Toledo 1995]. But no example is known of torsion-free nilpotent Kähler groups of nilpotency class 3 or more, although no obstruction to their existence is known.

5.1.2. The second type of restriction is that $\overline{H}^1\big(\pi_1(X)\big), \ell^2\big(\pi_1(X)\big) \neq 0$ implies that $\pi_1(X)$ is commensurable to a surface group (proved by M. Gromov, using L^2-methods). These methods show that a Kähler group has at most one end [Arapura et al. 1992]. See [Amorós et al. 1996, Sections 1 and 4] for more details.

5.1.3. Obstructions for lattices in semisimple Lie groups to be Kähler. These are derived from the theory of harmonic maps to negatively curved manifolds. Its extension to the case of Bruhat–Tits buildings and negatively curved metric spaces, which appears in papers by Gromov and Schoen and by Korevaar and Schoen, seems to be a very promising new tool. See [Amorós et al. 1996, Sections 5, 6, 7].

5.1.4. Remark. The Kähler assumption seems very essential (and minimal) to obtain restrictions on $\pi_1(X)$. Indeed: any finitely presented group is the fundamental group of a compact complex 3-fold, which can be choosen symplectic, and a twistor space on some appropriate self-dual Riemannian 4-fold after a deep result of C. Taubes. Notice that a twistor space which is Kähler (Hitchin) or even bimeromorphic to Kähler [Campana 1991] is simply-connected, so that the twistor construction does not produce any non-trivial fundamental group in the Kähler case.

5.2. Kodaira dimension and fundamental group. We denote by X a compact Kähler manifold. In Riemannian or Kähler geometry, positivity assumptions on the Ricci curvature imply restrictions on the fundamental group (compare Section 6). For example:

5.2.1. If $\mathrm{Ricci}(X) > 0$, then $\pi_1(X)$ is finite (denoted: $|\pi_1(X)| < +\infty$).

5.2.2. If $\mathrm{Ricci}(X) \geq 0$, then $\pi_1(X)$ is almost abelian (that is, has a finite index subgroup which is abelian).

The analogous numerical assumptions read:

1. $c_1(X) > 0$ and
2. $c_1(X) \geq 0$ respectively.

In fact the analogous statements turn out to be true (due to the existence of Kähler–Einstein metrics in the case of 5.2.2′):

5.2.1′. If X is Fano, then $\pi_1(X) = 1$.

5.2.2′. If $c_1(X) = 0$, then $\pi_1(X)$ is almost abelian. (We assume that X is Kähler!)

As in Section 2, however, one expects this kind of result to be true under weaker assumptions, since π_1 is a birational invariant, and the results above should remain true for "minimal models". The questions then become:

QUESTION 5.2.1″. Assume $\kappa^+(X) = -\infty$. Is then $\pi_1(X) = 1$? (We shall see below that this is true).

Observe that here, the condition $\kappa^+(X) = -\infty$ a priori is weaker than assuming that X is birational to some Fano manifold. So that a vanishing theorem for π_1 in that case is the best one can expect by using a hypothesis on Kodaira dimensions.

QUESTION 5.2.2″. Assume $\kappa(X) = 0$. Is then $\pi_1(X)$ almost abelian?

This is unknown, but is conjectured to be true. We shall see an important special case below. It holds for surfaces and also for projective threefolds by a result of Y. Namikawa and J. Steenbrink [1995].

In fact a relative version of 5.2.2″ can reasonably be expected, too:

QUESTION 5.2.3. Let X be a projective manifold with $\kappa(X) \geq 0$; let $\Phi : X \to Y$ be its Iitaka fibration, and $\Phi_* : \pi_1(X) \to \pi_1(Y)$ the induced map (it is well defined if we assume, as we can, that X and Y are smooth -moreover it is surjective since Φ is connected). Let $K := \operatorname{Ker}\Phi_*$. Is then K almost abelian — at least after replacing X by some suitable finite étale cover $\overline{X} \to X$?

Observe that the generic fiber X_y of Φ has $\kappa = 0$. (However, the natural map $\pi_1(X_y) \mapsto K$ is not surjective in general, so that Question 5.2.3 does not reduce to 5.2.2″.)

Notice that this question is empty when X is of general type. Thus the only really new fundamental groups are to be found in this class — for X a surface, by Lefschetz theorem in the projective case.

A similar question can be asked for the algebraic reduction.

QUESTION 5.2.4. Let X be a compact Kähler manifold; let $r : X \to A$ be its algebraic reduction, and $r_* : \pi_1(X) \to \pi_1(A)$ be the induced map. (As in Question 5.2.3 this is well-defined and onto). Let $R := \operatorname{Ker} r_*$. Is then R almost-abelian? Here the generic fiber X_a of r has $\kappa(X_a) \le 0$.

Some special cases are known, which shall be discussed below.

5.3. Γ-reduction

THEOREM 5.3.1 [Campana 1994]. *Let X be a compact Kähler manifold. There exists a quasi-fibration $\gamma_X : X \to \Gamma(X)$ such that for a general in X, the fiber X_a of γ_X passing through a is the largest among the* ***connected*** *compact analytic subsets A of X containing a such that the natural map : $i_* : \pi_1(\widehat{A}) \to \pi_1(X)$ has finite image, where i_* is induced by the inclusion of A in X composed with the normalisation map $\nu : \widehat{A} \to A$.*

The map γ_X, called the ***Γ-reduction*** *of X, is bimeromorphically invariant; its generic fiber is smooth. We denote by $\gamma d(X) := \dim \Gamma(X)$ its* ***γ-dimension****.*

The special case where X is projective has been shown independently by J. Kollár [1993], who named the map above the **Shafarevich map** of X.

The result above has been shown in [Campana 1994] with another (trivially equivalent) formulation for the universal cover $\widetilde{X}$ (or any Galois cover) of X. See also [Campana 1994] or [Kollár 1993] for the relation ship of Theorem 5.3.1 with Shafarevich's conjecture. (The Shafarevich conjecture implies in particular that γ_X is regular, and that $a \in X$ can be any point.)

Note that $\gamma d(X) = 0$ is equivalent to $A = X$ and also to $|\pi_1(X)| < +\infty$; on the other extreme: $\gamma d(X) = \dim X$ means that for any positive dimensional A through general a, the map $\pi_1(\widehat{A}) \to \pi_1(X)$ has infinite image. Obviously, X is not uniruled in that case (in fact: if $\widehat{A} = \mathbb{P}_1$, the map above has trivial image). This remark will be generalized below.

EXAMPLES. (a) Curves: $\gamma d(X) = 0$ if $g(X) = 0$; and $\gamma d(X) = 1$ if $g(X) \ge 1$.
(b) Surfaces: If $\kappa(X) < 2$, then $\gamma d(X) = q'(X) + \chi'(\mathcal{O}_X)$, where

$$q'(X) = \inf\big(q(X), 1\big) \quad \text{and} \quad \chi'(\mathcal{O}_X) = \begin{cases} 0 & \text{if } \chi(\mathcal{O}_X) \neq 0, \\ 1 & \text{if } \chi(\mathcal{O}_X) = 0. \end{cases}$$

This formula can be checked directly from the classification of Enriques–Kodaira if $\kappa(X) \leq 0$; in the elliptic case with $\kappa(X) = 1$, it can be shown that $\mathrm{Ker}\big(\pi_1(X) \to \pi_1(B)\big)$ is infinite precisely when the singular fibers are all multiple elliptic, that is, when $\chi(\mathcal{O}_X) = 0$. (See [Gurjar and Shastri 1985], for example). For surfaces of general type, there does not seem to be any simple relationship between c_1^2 and c_2 and $\gamma d(X)$. The values attained by $\gamma d(X)$ are all possible (0, 1 or 2).

(c) Tori: We have $\gamma d(X) = \dim X$ if X is a complex torus, since its universal cover is Stein. Notice that we also have $\chi(\mathcal{O}_X) = 0$.

Theorem 5.3.2 [Campana 1994; Kollár et al. 1992a]. *If X is rationally connected*, $\pi_1(X) = \{1\}$. *In particular*, *if X is Fano*, $\pi_1(X) = \{1\}$.

Proof. This is an easy consequence of Theorem 5.3.1. Let a, b be general in X. They can be joined by a connected rational chain A. Then $\pi_1(\widehat{A})$ maps trivially to $\pi_1(X)$. So a and b are in the some fiber of γ_X, which is thus constant. □

This proof is the one given in [Campana 1994] (except that one works on $\widetilde{X}$ there). The proof given in [Kollár et al. 1992a] is more difficult, since it uses first that rational connectedness implies strong rational connectedness, and then uses this stronger property to conclude. (In the case of strong rational connectedness a simple argument does exist; see [Campana 1991].)

Actually, 5.3.2 holds in the relative version as well:

Theorem 5.3.3 [Kollár 1993]. *Let $f : X \to Y$ be a dominant rational map with X, Y smooth. Assume the generic fiber of f is rationally connected. Then $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.*

The proof rests on Theorem 5.3.2 and an analysis of the π_1 of fibers of f in codimension 1 on Y, to show they are simply connected.

Notice that the result above is no longer valid if one only assumes that the generic (smooth) fibers of X are simply connected:

Example 5.3.4. Let S be an Enriques surface; $u : \widetilde{S} \mapsto S$ its universal cover (a K3 surface) and E be an elliptic curve. Let $\mathbb{Z}_2$ act on $\overline{X} := E \times \widetilde{S}$ by $i(z, \tilde{s}) = \big(-z; j(\tilde{s})\big)$, where i is a generator of $\mathbb{Z}_2$ and $j : \widetilde{S} \to \widetilde{S}$ the "Enriques Involution" $\big(\widetilde{S}/(j) = S\big)$. Let $\pi : X := \overline{X}/(i) \to E/\pm \simeq \mathbb{P}_1$ be induced by the first projection of $\overline{X}$. Then $\pi_* : \pi_1(X) \to \pi_1(\mathbb{P}_1) = \{1\}$ has infinite kernel (observe that the singular fibers are not simply connected, and that $\chi(\mathcal{O}_{\widetilde{S}}) = 2 > 1$).

As a consequence of Theorem 5.3.3, we have:

Corollary 5.3.5. *If X is rationally generated*, *then* $\pi_1(X) = \{1\}$.

Proof. We just have to iterate the MRC fibrations to eventually arrive at a point. □

This corollary will be strengthened below (with rational generatedness replaced by $\kappa^+(X) = -\infty$).

The drawback in Theorem 5.3.1 is that $a \in X$ has to be choosen to be general, so that it does not solve the following problem:

QUESTION 5.3.6 (M. Nori). Let X be a smooth projective surface. Assume that X contains a rational curve C (singular possibly), such that $C^2 > 0$. Is then $\pi_1(X)$ finite?

If $a \in X$ could be choosen to be any point, the answer would be yes. In particular, if the Shafarevich conjecture holds this is the case (as observed first by Gurjar). It is easy to see, using the Albanese map, that $q(X) = 0$.

The best results obtained are that linear representations of $\pi_1(X)$ have finite image. The results below also show easily that $\kappa(X) = 2$ if $\pi_1(X)$ is infinite (this was first shown by Gurjar–Shastri using classification of surfaces and showing that the Shafarevich's conjecture holds for surfaces with $\kappa \leq 1$).

5.4. The comparison theorem

THEOREM 5.4.1 [Campana 1995a]. *Let X be a compact Kähler manifold with $\chi(\mathcal{O}_X) \neq 0$. Then either $\kappa^+(X) \geq \gamma d(X)$ or $\kappa^+(X) = -\infty$ and $\pi_1(X) = \{1\}$.*

REMARKS 5.4.2. (1) The condition $\chi(\mathcal{O}_X) \neq 0$ cannot be dropped: tori X present the maximum failure to the inequality ($\kappa^+ = 0, \gamma d = n$). They might be characterized (birationally up to finite etale covers) by that property. See below.

(2) Theorem 5.4.1 extends an earlier result of M. Gromov [1991]: "If the universal cover $\widetilde{X}$ of X does not contain any positive dimensional compact subvariety and $\chi(\mathcal{O}_X) \neq 0$, then X is projective". The generalisation of this result lead to the introduction of the invariants γd, κ^+ and the construction of the Γ-reduction $\gamma_X : X \to \Gamma(X)$.

(3) The proof of Gromov (and of Theorem 5.4.1) rests on L_2-methods, and especially the Atiyah's L_2-index theorem.

When $\kappa^+(X) \leq 0$, Theorem 5.4.1 gives finiteness criteria for $\pi_1(X)$, as follows:

COROLLARY 5.4.3. *Let $\kappa^+(X) = -\infty$. Then $\pi_1(X) = \{1\}$.*

PROOF. It is easy to show that $h^0(X, \Omega^p_X) = 0$ ($p > 0$) if $\kappa^+(X) = -\infty$. Thus $\chi(\mathcal{O}_X) = 1 \neq 0$, and Theorem 5.4.1 applies. □

Notice that $\kappa^+(X) = -\infty$ if X is rationally generated. So we get Corollary 5.3.5 again by a different method. Conjecturally if X is rationally generated, then $\kappa^+(X) = -\infty$; if it is false, Corollary 5.4.3 is strictly stronger than 5.3.5.

COROLLARY 5.4.4. *Let $\kappa^+(X) = 0$, and let $\chi(\mathcal{O}_X) \neq 0$. Then $|\pi_1(X)| \leq 2^{n-1}/|\chi(\mathcal{O}_X)|$, where $n = \dim X$ and $|\pi_1(X)|$ is the cardinality of $\pi_1(X)$.*

PROOF. By Theorem 5.4.1, only the inequality has to be shown (finiteness results from 5.4.1). So we are reduced to bounding π_1^{alg}, the algebraic fundamental group instead of π_1. This follows from the usual covering trick, plus the following easy inequality:

LEMMA 5.4.5. *Assume* $\kappa^+(X) = 0$. *Then*

$$h^0(X, \Omega_X^p) \leq \binom{n}{p} \quad \text{and} \quad |\chi(\mathcal{O}_X)| \leq 2^{n-1}. \qquad \square$$

Conjecturally, Corollary 5.4.4 should hold with $\kappa^+(X) = 0$ replaced by $\kappa(X) = 0$. If $\kappa(X) = 0$, Lemma 5.4.5 is a conjecture of K. Ueno (proved by Y. Kawamata if $p = 1$).

COROLLARY 5.4.6. *Let* $\chi(\mathcal{O}_X) \neq 0$ *and assume* $c_1(X) = 0$. *Then* $|\pi_1(X)| \leq 2^{n-1}$ *if* X *is projective.*

Indeed: $\kappa^+(X) = 0$ in that case. Of course, by the existence of Ricci-flat metrics, this is known if X is Kähler. But the proof given here is more elementary.

A special case is:

COROLLARY 5.4.7. *Let* X *be a* $K3$ *surface* (*so that* $q(X) = 0$ *and* $K_X = \mathcal{O}_X$). *Then* $\pi_1(X) = \{1\}$.

The proof of 5.4.1 shows this (even without assuming X to be Kähler). This is for sure the simplest proof of this result, not requiring any knowledge of either deformation theory or Ricci-flat metrics.

In a similar vein:

COROLLARY 5.4.8. *Let* X *be a compact Kähler manifold with* $a(X) = 0$ (*that is*, X *has no non-constant meromorphic function*). *Asume that* $\chi(\mathcal{O}_X) \neq 0$, *too. Then* $|\pi_1(X)| \leq 2^{n-1}$.

Indeed, $a(X) \geq \kappa^+(X)$.

Notice that general tori (with $a(X) = 0$) again show that the assumption $\chi(\mathcal{O}_X) \neq 0$ cannot be dropped.

When $n = 3$, the assumption $\chi(\mathcal{O}_X) \neq 0$ can be weakened and the bound improved:

COROLLARY 5.4.9. *Let* X *a compact Kähler* 3*-fold with* $a(X) = q(X) = 0$. *Then* $|\pi_1(X)| \leq 3$ (*hence* $\pi_1(X) = \{1\}$, $\mathbb{Z}_2$ *or* $\mathbb{Z}_3$; *the last two possibilities are probably impossible*).

PROOF. We only need to show that $0 \neq \chi(\mathcal{O}_X)$ $(= 1 - q + n^{2,0} - h^{3,0} \geq h^{2,0} > 0)$; the first inequality holds because $h^{3,0} \leq 1$, the second because $h^{2,0} = 0$ implies X is projective (Kodaira). $\square$

The only known compact Kähler 3-folds with $a(X) = q(X) = 0$ are bimeromorphically ruled fibrations $\pi : X \to S$ where S is a K3-surface with $a(S) = 0$. Conjecturally, these are the only ones. If one assumes the Kähler version of the minimal model program and aboundance conjecture in dimension 3, this conjecture is true (see [Peternell 1998b] and Section 3; the main point is to give a meaning to the statement "K_X is nef" in this situation).

The method of proof of Theorem 5.4.1 gives also a part of the relative versions of Questions 5.2.3 and 5.2.4:

THEOREM 5.4.10 [Campana 1994]. *Let $\Psi : X \to Y$ be either the Iitaka fibration or the algebraic reduction of the compact Kähler manifold X. Let $L := \operatorname{Ker}\big(\psi_* : \pi_1(X) \to \pi_1(Y)\big)$ be the kernel of the induced map. Let X_y be a smooth fiber of ψ, and $j_* : \pi_1(X_y) \to K$ be the morphism induced by the natural inclusion $j : X_y \hookrightarrow X$. Then the image of j_* is finite if $\chi(\mathcal{O}_X) \neq 0$* (*and if moreover $\kappa^+(X_y) = \kappa(X_y) = 0$ in case ψ is the Iitaka fibration*).

This motivates the following question:

QUESTION 5.4.11. Let X be compact Kähler with $\chi(\mathcal{O}_X) \neq 0$; let $r : X \to A$ be its algebraic reduction. Is then $\operatorname{Ker}\big(r_* : \pi_1(\overline{X}) \to \pi_1(\overline{A})\big)$ a finite group for some suitable finite étale cover $\overline{X}$ of X?

For the general case of Questions 5.2.3 and 5.2.4, there is another partial positive answer. In order to state it, we introduce some notation. For any group Γ, set $\Gamma^{\mathrm{nilp}} = \Gamma/\Gamma'_\infty$, where $\Gamma'_\infty := \bigcap_{n\geq 2} \Gamma'_n$ with

$$\Gamma'_n := \operatorname{Ker}\big(\Gamma \to \Gamma/\Gamma_n \to (\Gamma/\Gamma_n)/\operatorname{Torsion}\big).$$

THEOREM 5.4.12 [Campana 1995b]. *Let $\Psi : X \to Y$ be either the algebraic reduction, or the Iitaka fibration of the compact Kähler manifold X. Let $\psi_*^{\mathrm{nilp}} : \pi_1(X)^{\mathrm{nilp}} \to \pi_1(Y)^{\mathrm{nilp}}$ be the natural morphism* (see [Campana 1995b] for the precise definition). *Then $K := \operatorname{Ker}(\psi_*^{\mathrm{nilp}}) \cong \mathbb{Z}^{\oplus 2s}$, where $s = q(X) - q(Y)$, and the exact sequence*

$$1 \to K \to \pi_1(X)^{\mathrm{nilp}} \to \pi_1(Y)^{\mathrm{nilp}} \to 1$$

splits (*non-canonically*).

The proof is given in [Campana 1995b] only for the algebraic reduction, but the sume proof applies for the Iitaka fibration.

We conclude this section with some conjectures concerning n-dimensional compact Kähler manifolds X:

CONJECTURES 5.4.13. (1) Let X be such that $\kappa^+(X) = 0$ (or $\kappa(X) = 0$), $\gamma d(X) = n$. Then X is bimeromorphic to some X_0 which is covered by a torus.

(2) More generally: assume that $\kappa^+(X) = 0$ (or $\kappa(X) = 0$), and that $\gamma d(X) = d$. Then: some finite étale cover $\overline{X}$ of X is bimeromorphic to a product $\overline{Y} \times T$, where T is a torus and $\kappa^+(\overline{Y}) = \kappa(\overline{Y}) = 0$ with $\pi_1(\overline{Y}) = \{1\}$.

Conjecture 5.4.12(2) should also have a relative version (for the Iitaka fibration).

Finally, we refer to [Kollár 1993] to see some other aspects of the application of Theorem 5.4.1 in the projective setting.

6. Biregular Classification

By "biregular classification" we mean a more or less explicit description of varieties of a certain type. Of course this is only possible under very restrictive circumstances. In differential geometry one classifies roughly in terms of curvature conditions: positive, negative and zero curvature. A curvature condition is suitable for biregular classification rather than birational classification because the sign of the curvature makes the variety more or less rigid in the birational category: blow-ups destroy the curvature condition. In the context of complex geometry a slightly more general notion than the sign of curvature will be useful as we shall see in this section. However we can still, cum grano salis, say that the aim of this section is to understand projective or Kähler manifolds with semipositive (bisectional or Ricci) curvature. The class of negatively curved manifolds is much larger and it is hopeless to get a biregular classification. We begin with a very short review of the situation in dimension 1.

6.0. Let X be a compact Riemann surface. If $-K_X$ is ample, then $X \simeq \mathbb{P}_1$, if $K_X = \mathcal{O}_X$, then X is a torus and if K_X is ample, then X has genus ≥ 2. The same classification holds in terms of positive, zero and negative curvature. In this case of course the holomorphic bisectional and Ricci curvatures are equivalent. In higher dimensions the tangent bundle T_X and the anticanonical bundle $-K_X$ are no longer the same, that is, we have to distinguish between bisectional and Ricci curvature; we will first look at the tangent bundle.

The classification theory in higher dimensions starts with this result:

THEOREM 6.1 (Mori). *Let X be a compact manifold with ample tangent bundle. Then $X \simeq \mathbb{P}_n$.*

X is automatically projective and Mori's proof is to rediscover the lines (through a given point). A priori however it is not at all clear whether there is any rational curve; these are constructed by Mori's reduction to characteristic p. Given a projective manifold X with $K_X \cdot C < 0$ for some curve C, Mori constructs a rational curve with the same property. The point is that in characteristic p, the inequality $K_X \cdot C < 0$ allow one to deform C, at least after having applied a suitable Frobenius. The rational curve appears since a certain rational map is not a morphism.

For a proof of Theorem 6.1 not using characteristic p, see [Peternell 1996].

In the same year (1979) Siu and Yau proved Theorem 6.1 with a weaker assumption, namely that X has a Kähler metric with positive holomorphic bisectional curvature.

In the spirit of Siu and Yau, but using characteristic p, Mok [1988] proved the following result:

THEOREM 6.2. *Let X be a compact Kähler manifold of semi-positive holomorphic bisectional curvature. Then, after taking a finite étale cover, X is of the* *form*

$$X \simeq T \times \prod Y_j,$$

where T is a torus and the Y_j are hermitian symmetric manifolds with $b_2 = 1$.

In Mori's theorem no assumption on curvature is made; ampleness is "just" an algebraic property. To check it, it is not necessary to construct a metric. We are looking for an equivalent result in the semipositive case. Note by the way that in Mok's theorem one needs a **Kähler** metric of semipositive curvature which is much stronger than just assuming the existence of some hermitian metric with the same curvature. In the case of line bundles on projective manifolds it is clear how to get rid of curvature conditions: one assumes L to be nef, that is, $L\cdot C \geq 0$ for all curve $C \subset X$. In the Kähler case however this definition clearly fails. The substitute is Definition 3.11. In the vector bundle case we define:

DEFINITION 6.3. Let X be a compact manifold. A vector bundle E on X is nef, if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is *nef* on $\mathbb{P}(E)$.

Now the problem is: *Determine the structure of compact Kähler manifolds X such that T_X is nef, or, alternatively, $-K_X$ nef.*

For T_X nef we have a structure theorem, proved in [Demailly et al. 1994]. In order to state it we introduce the following "irregularity":

$$\tilde{q}(X) = \sup\{q(\tilde{X}) \mid \tilde{X} \to X \text{ is finite étale }\}.$$

THEOREM 6.4. *Let X be a compact Kähler manifold with T_X nef.*

(1) *The Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a surjective submersion with nef relative tangent bundle.*
(2) *If $\tilde{q}(X) = q(X)$, then the fibers of α are Fano manifolds.*
(3) *X is Fano if and only if $c_1(X)^n \neq 0$.*
(4) *$\pi_1(X)$ is almost abelian, that is, an extension of $\mathbb{Z}^m$ by a finite group.*

Note that the structure of α is not arbitrary: it has a flat nature in the sense that the bundles $\alpha_*(-mK_X)$ are numerically flat (nef and with nef dual). An important step in the proof of Theorem 6.4 is the study of numerically flat vector bundles:

THEOREM 6.5. *Let X be a compact Kähler manifold. Let E be a numerically flat vector bundle on X. Then E admits a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_p = E$$

by subbundles such that the quotients E_i/E_{i+1} are hermitian flat, that is, defined by a representation $\pi_1(X) \to U(r)$.

For proofs see [Demailly et al. 1994]. For ideas and background relevant to all of this section see also [Peternell 1996].

Theorem 6.5 is used in the proof of 6.4 to show the existence of a 1-form after finite étale cover, if there is a p-form for p odd.

Theorem 6.4 reduces the structure problem for manifolds with nef tangent bundles to that of Fano manifolds with nef tangent bundles. If X is Fano with T_X nef, then consider the contraction of an extremal ray, say $\varphi : X \to Y$. One can prove that φ is a surjective submersion with nef relative tangent bundle, so that the main difficulty is provided by Fano manifolds with $b_2(X) = 1$. Here is the main conjecture about these varieties.

Conjecture 6.6. Let X be a Fano manifold with T_X nef. Then X is rational homogeneous.

As already said, the main difficulty arises when $b_2(X) = 1$. If this case is settled, then one has to study Mori fibrations over rational homogeneous manifolds whose fibers are rational homogeneous and need to lift vector fields. The evidence for Conjecture 6.6 is the validity in dimensions 2 and 3 and that X behaves as if it is homogeneous: every effective divisor is nef, the deformations of a rational curve fill up all of X etc. The classification in dimension 3 uses however classification theory and therefore does not shed any light on the higher-dimensional case. One is tempted to prove the existence at least of some vector fields by proving

$$\chi(X, T_X) > 0 \tag{$*$}$$

which together with the vanishing $H^q(X, T_X) = 0, q \geq 2$ would give us some vector field. The nefness of T_X yields inequalities for the Chern classes of X. Unfortunately these inequalities are not strong enough to give $(*)$ via Riemann–Roch. Instead one should study the family of rational curves of minimal degree in X. They already cover X and experience shows that they dictate the geometry of X. For more comments see [Peternell 1996].

We now turn to compact Kähler manifolds X with $-K_X$ nef. The building blocks of these varieties are Fano manifolds, that is, $-K_X$ is ample, and manifolds with $K_X \equiv 0$, i.e. tori, Calabi–Yau manifolds and symplectic manifolds up to finite étale cover. We want to see how manifolds with $-K_X$ nef are constructed from these "prototypes". The starting point is to separate the torus part by considering the Albanese map.

Theorem 6.7. *Let X be a compact Kähler manifold with $-K_X$ hermitian semi-positive* (*that is*, *there is a metric on $-K_X$ with semi-positive curvature*). *Then the Albanese map is a surjective submersion.*

The proof goes by constructing for every holomorphic 1-form ω a differentiable vector field v such that the contraction gives $\|v\|^2$. Now the curvature condition implies that v is holomorphic, since ω is holomorphic and therefore $\|v\|$ is a constant. Hence ω has no zeroes which proves the claim. See [Demailly et al. 1993] for details. If $-K_X$ is merely nef, the proof apparently does not work. At least the surjectivity was proved by Qi Zhang [1996] in the algebraic case:

Theorem 6.8. *Let X be a projective manifold with $-K_X$ nef. Then the Albanese map is surjective.*

PROOF. If not, X would admit a map onto a variety Y of general type, which can be ruled by cutting down to a curve in Y and applying the results of [Miyaoka 1993]. □

This last paper relies on characteristic p, so the Kähler case remains unsettled in general. However in [Campana et al. 1998b] it is shown that a compact Kähler n-fold with $-K_X$ nef cannot have a map onto a variety of general type of dimension 1 (this case is proved in [Demailly et al. 1993]), $n-2$ or $n-1$. This settles in particular Theorem 6.8 in the Kähler case up to dimension 4.

Concerning smoothness, the following theorem settles the threefold case:

THEOREM 6.9 [Peternell and Serrano 1998]. *Let X be a smooth projective threefold with $-K_X$ nef. Then the Albanese is smooth.*

The proof relies on a careful analysis of the Mori contractions on X. The Kähler case is settled in [Demailly et al. 1998].

In the hermitian semi-positive case one can prove much more [Demailly et al. 1996]:

THEOREM 6.10. *Let X be a compact Kähler manifold with $-K_X$ hermitian semi-positive. Then*:

(1) *The universal cover $\tilde{X}$ admits a holomorphic and isometric splitting*
$$\tilde{X} \simeq \mathbb{C}^q \times \prod X_i,$$
where the X_i are Calabi–Yau manifolds or symplectic manifolds or manifolds having the property that
$$H^0(X_i, \Omega_{X_i}^{\otimes m}) = 0$$
for all $m > 0$.

(2) *There exists a finite étale Galois cover $\hat{X} \to X$ such that the Albanese map is a locally trivial fiber bundle to the q-dimensional torus A whose fibers are all simply connected and of types descirbed in* (1).

(3) *We have $\pi_1(\hat{X}) \simeq \mathbb{Z}^{2q}$.*

In the nef case it is at least known that $\pi_1(X)$ has subexponential growth. In [Campana 1995a] (see also Definiton 1.9) a refined version of Kodaira dimension is defined:

DEFINITION 6.11. Let X be a compact manifold. Then
$$\kappa^+(X) = \max\{\kappa(\det \mathcal{F}) \mid \mathcal{F} \subset \Omega_X^p \text{ for some } p > 0\}.$$

Replacing Ω_X^p by $\Omega_X^{\otimes m}$ we obtain an invariant $\kappa^{++}(X)$. In these terms the varieties X in Theorem 6.10 which are neither Calabi–Yau nor symplectic satisfy $\kappa^{++}(X) = -\infty$. Moreover we have $\kappa_+(X) = \kappa^{++}(X)$ if $-K_X$ is hermitian semi-positive.

To conclude we collect problems on varieties with nef anticanonical bundles as well as problems on the new Kodaira type invariants.

PROBLEMS 6.12. Let X be compact Kähler.

(1) What is the relation between $\kappa^+(X)$ and $\kappa^{++}(X)$? Of course $\kappa^+(X) \leq \kappa^{++}(X)$.
(2) Suppose $\kappa^+(X) = -\infty$. Is X rationally generated or even rationally connected, at least if $-K_X$ is nef?
(3) Is the structure theorem 6.10 true for compact Kähler manifolds with $-K_X$ nef?
(4) Assume $-K_X$ nef. Is $\kappa^+(X) \leq 0$?

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