Several Complex Variables
MSRI Publications
Volume **37**, 1999

Remarks on Global Irregularity in the $\bar{\partial}$–Neumann Problem

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Abstract. The Bergman projection on a general bounded, smooth pseudoconvex domain in two complex variables need not be globally regular, that is, need not preserve the class of all functions that are smooth up to the boundary. In this article the construction of the worm domains is reviewed, with emphasis on those features relevant to their role as counterexamples to global regularity. Prior results, and related issues such as the commutation method and compactness estimates, are discussed. A model in two real variables for global irregularity is discussed in detail. Related work on real analytic regularity, both local and global, is summarized. Several open questions are posed.

Contents

Research supported by NSF grant DMS 96-23007.

1. Introduction

Let $n > 1$, and let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^∞ boundary. The $\bar\partial$–Neumann problem for $(0,1)$-forms on Ω is a boundary value problem

$$\Box u = f \qquad \text{on } \Omega, \tag{1–1}$$
$$u \,\lrcorner\, \bar\partial\rho = 0 \qquad \text{on } \partial\Omega, \tag{1–2}$$
$$\bar\partial u \,\lrcorner\, \bar\partial\rho = 0 \qquad \text{on } \partial\Omega. \tag{1–3}$$

where u, f are $(0,1)$-forms, ρ is any defining function for Ω, $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and $\lrcorner$ denotes the interior product of forms. $\mathbb{C}^n$ is regarded as being equipped with its canonical Hermitian metric, and $\bar\partial^*$ denotes the formal adjoint of $\bar\partial$ with respect to that metric.

The boundary conditions may be reformulated so as to apply to functions that are not very regular at the boundary: $u \in \text{Domain}(\bar\partial^*)$ and $\bar\partial u \in \text{Domain}(\bar\partial^*)$ [Folland and Kohn 1972]. In the L^2 setting there is then a satisfactory global theory [Folland and Kohn 1972; Catlin 1984]; if Ω is pseudoconvex, then for every $f \in L^2(\Omega)$ there exists a unique solution $u \in L^2(\Omega)$. Moreover, if $\bar\partial f = 0$, then $\bar\partial u = f$, and u is the solution with smallest L^2 norm. The Neumann operator N is the bounded linear operator on $L^2(\Omega)$ that maps datum f to solution u.

The $\bar\partial$–Neumann problem is useful as a tool for solving the primary equation $\bar\partial u = f$ because it often leads to a solution having good regularity properties at the boundary. For large classes of domains, in particular for all strictly pseudoconvex domains, it is a hypoelliptic boundary value problem, that is, the solution u is smooth[1] in any relatively open subset of $\overline{\Omega}$ in which the datum f is smooth. Whereas the main goal of the theory is regularity in spaces and norms such as C^∞, C^k, Sobolev or Hölder, basic estimates and existence and uniqueness theory are most naturally expressed in L^2.

For some time it remained an open question whether the global C^∞ theory was as satisfactory as the L^2 theory.

THEOREM 1.1 [Christ 1996b]. *There exist a smoothly bounded, pseudoconvex domain in $\mathbb{C}^2$ and a datum $f \in C^\infty(\overline{\Omega})$ such that the unique solution $u \in L^2(\Omega)$ of the $\bar\partial$–Neumann problem does not belong to $C^\infty(\overline{\Omega})$.*

There were antecedents. Barrett [1984] gave an example of a smoothly bounded, nonpseudoconvex domain for which the Bergman projection B fails to preserve $C^\infty(\overline{\Omega})$. Kiselman [1991] showed that B fails to preserve $C^\infty(\overline{\Omega})$ for certain bounded but nonsmooth pseudoconvex Hartogs domains. Barrett [1992] added a fundamental insight and deduced that for the so-called worm domains, which are smoothly bounded and pseudoconvex, B fails to map the Sobolev space H^s to itself, for large s. Finally Christ [1996b] proved an *a priori* H^s estimate

[1] "Smooth" and "C^∞" are synonymous throughout this article.

for smooth solutions on worm domains, and observed that this estimate would contradict Barrett's result if global C^∞ regularity were valid.

This article discusses background, related results, the proof of global irregularity, and open questions. It is an expanded version of lectures given at MSRI in the Fall of 1995. A brief report on analytic hypoellipticity is also included.

I am indebted to Emil Straube for useful comments on a preliminary draft.

2. Background

The equation $\Box u = f$ is a linear system of n equations. The operator $\Box$ is simply a constant multiple of the Euclidean Laplacian, acting diagonally with respect to the standard basis $\{d\bar z_j\}$, so is elliptic. However, the boundary conditions are not coercive; that is, the *a priori* inequality

$$\sum_{|\alpha|\le 2} \|\partial^\alpha u\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$$

for all $u \in C^\infty(\overline\Omega)$ satisfying the boundary conditions (1–1), (1–1) is not valid for any nonempty Ω. For strictly pseudoconvex domains one has a weaker *a priori* inequality: the H^1 norm of u is bounded by a constant multiple of the $L^2 = H^0$ norm of f, provided that $u \in C^\infty(\overline\Omega)$ satisfies the boundary conditions [Kohn 1963; 1964]. Even this inequality breaks down for domains that are pseudoconvex but not strictly pseudoconvex; the regularity of solutions is governed by geometric properties of the boundary.

There are two different fundamental notions of regularity in the C^∞ category, hypoellipticity and global regularity.[2] Hypoellipticity means that for every open set $V \subset \mathbb{C}^n$ and every $f \in L^2(\Omega)\cap C^\infty(V\cap\overline\Omega)$, the $\bar\partial$–Neumann solution u belongs to $C^\infty(V\cap\overline\Omega)$. Global regularity in C^∞ means that for every $f \in C^\infty(\overline\Omega)$, the $\bar\partial$–Neumann solution u also belongs to $C^\infty(\overline\Omega)$. Hypoellipticity thus implies global regularity.

Consider any linear partial differential operator L, with C^∞ coefficients, defined on a smooth compact manifold M without boundary. Such an operator is said to be hypoelliptic if for any open set $V \subset M$ and any $u \in \mathcal{D}'(V)$ such that $Lu \in C^\infty(V)$, necessarily $u \in C^\infty(V)$. It is globally regular in C^∞ if for all $u \in \mathcal{D}'(M)$ such that $Lu \in C^\infty(M)$, necessarily $u \in C^\infty(M)$. The definitions given for the $\bar\partial$–Neumann problem in the preceding paragraph are the natural analogues of these notions for boundary value problems, with minor modifications.

In general, global C^∞ regularity is a far weaker property than hypoellipticity. As a first example, consider the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, equipped with coordinates (x_1, x_2). Let $L = \partial_{x_1} + \alpha\partial_{x_2}$, where α is a real constant. The vector field L defines a foliation of $\mathbb{T}^2$, and any function u defined in some

[2]The latter is sometimes called global hypoellipticity.

open subset and locally constant along each leaf is annihilated by L. From the relationship $\widehat{Lu}(k) = (2\pi i)(k_1 + \alpha k_2)\hat{u}(k)$ it follows that L is globally regular in C^∞ if and only if α satisfies a Diophantine inequality $|k_1 + \alpha k_2| \geq |k|^{-N}$ for some $N < \infty$ as $|k| \to \infty$. Thus global regularity holds for almost every α. No such Diophantine behavior has been encountered for the $\bar{\partial}$–Neumann problem for domains in $\mathbb{C}^n$; irregularity for that problem has a rather different source.[3]

As a second example, consider any torus $\mathbb{T}^n$ and distribution $K \in \mathcal{D}'(\mathbb{T}^n)$. Denote by 0 the identity element of the group $\mathbb{T}^n$. The convolution operator $Tf = f * K$ then preserves $C^\infty(\mathbb{T}^n)$. On the other hand, T is pseudolocal[4] if and only if $K \in C^\infty(\mathbb{T}^n \backslash \{0\})$.

The principal results known, in the positive direction, concerning hypoellipticity in the $\bar{\partial}$–Neumann problem for smoothly bounded pseudoconvex domains in $\mathbb{C}^n$ are as follows. For all strictly pseudoconvex domains, the $\bar{\partial}$–Neumann problem is hypoelliptic [Kohn 1963; 1964]. For all $s \geq 0$, the solution belongs to the Sobolev class H^{s+1} in every relatively compact subset of any relatively open subset of $\overline{\Omega}$ in which the datum belongs to H^s. Precise results describe the gain in regularity in various function spaces and the singularities of objects such as the Bergman kernel.[5]

Hypoellipticity holds more generally, for all domains of finite type in the sense of [D'Angelo 1982]. (The defining property of such domains is that at any $p \in \partial\Omega$, no complex subvariety of $\mathbb{C}^n$ has infinite order of contact with $\partial\Omega$.) The $\bar{\partial}$–Neumann problem satisfies subelliptic estimates up to the boundary: there exists $\varepsilon > 0$ such that for every $s \geq 0$, every relatively open subset U of $\overline{\Omega}$ and every datum $f \in L^2(\Omega) \cap C^\infty(U)$, the $\bar{\partial}$–Neumann solution u belongs to $H^{s+\varepsilon}$ on every relatively compact subset of U [Catlin 1987]. Conversely, subellipticity implies finite type. No characterization of the optimal ε is known in general.

The case of domains of finite type in $\mathbb{C}^2$ is far simpler than that in higher dimensions, and is well understood. Finite type in $\mathbb{C}^2$ is characterized by Lie brackets of vector fields in $T^{1,0} \oplus T^{0,1}(\partial\Omega)$, and the optimal exponent ε equals $2/m$ where m is the type as defined by Lie brackets or by the maximal order of contact of complex submanifolds with $\partial\Omega$.[6] Closely related to the $\bar{\partial}$–Neumann problem for domains of finite type in $\mathbb{C}^2$ is the theory of sums of squares of smooth real vector fields satisfying the bracket condition of Hörmander [1967].

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[3]Somewhat artificial examples of operators with variable coefficients exhibiting similar behavior are analyzed in [Himonas 1995].

[4]An operator T is said to be pseudolocal if it preserves $\mathcal{D}'(\mathbb{T}^n) \cap C^\infty(V)$ for every open subset V of $\mathbb{T}^n$.

[5]There is likewise a gain of one derivative in the Hölder and L^p-Sobolev scales [Greiner and Stein 1977]. Moreover, there is a gain of two derivatives in the so-called "good" directions; for any smooth vector fields V_1, V_2 defined on $\overline{\Omega}$ such that V_i and JV_i are tangent to $\partial\Omega$, $V_1V_2u \in H^s$ wherever $f \in H^s$ [Greiner and Stein 1977].

[6]There is still a gain of two derivatives in good directions, and a gain of $2/m$ derivatives in the Hölder and L^p-Sobolev scales, for the $\bar{\partial}$–Neumann problem as well as for a related equation on $\partial\Omega$. See [Chang et al. 1992; Christ 1991a; 1991b] and the many references cited there.

So far as this author is aware, little has been established in the positive direction concerning hypoellipticity for domains of infinite type. There are however several interesting theorems guaranteeing global C^∞ regularity, or a closely related property, for classes of domains for which hypoellipticity need not hold. The first result of this type [Kohn 1973] concerned the weighted $\bar\partial$–Neumann problem, associated to any plurisubharmonic function $\varphi \in C^\infty(\bar\Omega)$. In this problem $\Box$ is replaced by $\Box_\varphi = \bar\partial\bar\partial^*_\varphi + \bar\partial^*_\varphi\bar\partial$, where $\bar\partial^*_\varphi$ is the formal adjoint of $\bar\partial$ in the Hilbert space $L^2(\Omega, e^{-\varphi}dzd\bar z)$, and the boundary conditions are that $u, \bar\partial u$ should belong to the domain of $\bar\partial^*_\varphi$ (on forms of degrees one and two, respectively). Kohn showed that given any Ω and any exponent $s \geq 0$, there exists φ such that for every $f \in H^s(\Omega)$, the solution u of the $\bar\partial$–Neumann problem with weight $\exp(-\varphi)$ also belongs to $H^s(\Omega)$. Work of Bell and Ligocka [1980], however, demonstrated that the problem for $\varphi \equiv 0$ has a special significance.

Consider the quadratic form

$$Q(u,u) = \|\bar\partial u\|^2_{H^0(\Omega)} + \|\bar\partial^* u\|^2_{H^0(\Omega)}.$$

Compactness of the Neumann operator is equivalent to an inequality

$$\|u\|^2_{H^0} \leq \varepsilon Q(u,u) + C_\varepsilon \|u\|^2_{H^{-1}} \tag{2–1}$$

for all $u \in C^1(\bar\Omega)$ satisfying the first boundary condition (1–1), for all $\varepsilon > 0$. Subellipticity implies compactness, which in turn implies [Kohn and Nirenberg 1965] global regularity. See [Catlin 1984; Sibony 1987] for compactness criteria in terms of auxiliary plurisubharmonic functions having suitable growth properties.

A second type of result asserts that global C^∞ regularity holds for all domains enjoying suitable symmetries, in particular, for any Reinhardt domain, or more generally, for any circular or Hartogs domain for which the orbit of the symmetry group is transverse to the complex tangent space to Ω at every boundary point.[7] Such results are essentially special cases of a general principle to the effect that global regularity always holds in the presence of suitable global symmetries, one version of which is formulated in the real analytic category in [Christ 1994a].

More general results in the positive direction have been obtained by Boas and Straube [Boas and Straube 1991a; 1991b; 1993], after earlier work of Bonami and Charpentier [1988]. Denote by $W_\infty \subset \partial\Omega$ the set of points at which the boundary has infinite type. A sufficient condition for global C^∞ regularity is that there exist a smooth real vector field V defined on some neighborhood of W_∞ in $\partial\Omega$ and transverse to $[T^{1,0} \oplus T^{0,1}](\partial\Omega)$ at every point of W_∞, such that

$$[V, T^{1,0} \oplus T^{0,1}] \subset T^{1,0} \oplus T^{0,1}. \tag{2–2}$$

In fact, it suffices that for each $\varepsilon > 0$ there exist V_ε, defined on some neighborhood $U = U_\varepsilon$ of W_∞ in $\partial\Omega$ and transverse to $T^{1,0} \oplus T^{0,1}$ at every point of W_∞,

[7] Results of this genre have been obtained by numerous authors including So-Chin Chen, Cordaro and Himonas [Cordaro and Himonas 1994], Derridj [1997], Barrett, and Straube.

such that[8]

$$[V_\varepsilon, T^{1,0} \oplus T^{0,1}] \subset T^{1,0} \oplus T^{0,1} \text{ modulo } O(\varepsilon) \text{ on } U. \tag{2–3}$$

For Hartogs or circular domains having transverse symmetries, the action of the symmetry group S^1 gives rise to a single vector field V having the stronger commutation property $[V, \bar\partial] = 0,\ [V, \bar\partial^*] = 0$.

One corollary of the theorem of Boas and Straube is $\bar\partial$–Neumann global regularity for all *convex* domains [Boas and Straube 1991b; Chen 1991]. To formulate a second special case, consider any Ω for which the set $W_\infty \subset \partial\Omega$ of all boundary points of infinite type consists of a smoothly bounded, compact complex submanifold $\mathcal{V}$ of $\mathbb{C}^n$ with boundary, of positive dimension. A second corollary is global C^∞ regularity for Ω whenever $\mathcal{V}$ is simply connected. A third case where the required vector field exists is when there exists a defining function $\rho \in C^\infty(\overline{\Omega})$ that is plurisubharmonic at the boundary[9] [Boas and Straube 1991b].[10]

3. Exact Regularity and Positivity

Consider any smoothly bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Denote by $L^2(\Omega)$ the space of square integrable $(0,1)$-forms defined on Ω, and for each $s \geq 0$ denote by $H^s = H^s(\Omega)$ the space of $(0,1)$-form valued functions Ω possessing s derivatives in L^2 in the usual sense of Sobolev theory.

The Neumann operator N (for $(0,1)$-forms) is the unique bounded linear operator on $L^2(\Omega)$ that maps any f to the unique solution u of the $\bar\partial$–Neumann problem with datum f. Existence and uniqueness stem from the fundamental inequality

$$\|u\|^2_{L^2(\Omega)} \leq C\|\bar\partial u\|^2_{L^2(\Omega)} + C\|\bar\partial^* u\|^2_{L^2(\Omega)}, \tag{3–1}$$

valid for all $u \in C^1(\overline{\Omega})$ satisfying the first boundary condition (1–1). A proof may be found in [Catlin 1984].

Definition. For each $s \geq 0$, the $\bar\partial$–Neumann problem for Ω is exactly regular in H^s if the Neumann operator N maps $H^s(\Omega)$ into $H^s(\Omega)$.

Corresponding notions may be defined for an operator L on a compact manifold without boundary. By virtue of the Sobolev embedding theorem, exact regularity implies global C^∞ regularity in either setting. There is of course no converse in general, as illustrated by the operators $\partial_{x_1} + \alpha\partial_{x_2}$ on $\mathbb{T}^2$. If $|k_1 + \alpha k_2| \geq c|k|^{-N}$ as $|k| \to \infty$, then L^{-1} exists modulo a finite dimensional kernel and cokernel,

[8] Fix finitely many coordinate patches $O_\alpha \subset \partial\Omega$ whose union contains W_∞ and fix, for each α, a basis of sections $X_{\alpha,j}$ of $T^{1,0} \oplus T^{0,1}(O_\alpha)$. It is required that for each ε and each $N < \infty$ there exist V_ε such that for all α and all j, $[V_\varepsilon, X_{\alpha,j}]$ may be decomposed in $U_\varepsilon \cap O_\alpha$ as a section of $T^{1,0} \oplus T^{0,1}(O_\alpha)$ plus a vector field whose C^N norm is at most ε.

[9] The complex Hessian of ρ is required to be positive semidefinite at each point of $\partial\Omega$.

[10] This result has been reproved and refined by Kohn [$\geq$ 1999].

and maps $H^s(\mathbb{T}^2)$ to H^{s-N} for all s, but since the limit infimum of $|k_1 + \alpha k_2|$ always equals zero, L^{-1} cannot preserve any class H^s.[11]

Why is exact regularity of such importance? The theory begins with an H^0 estimate, $\|u\|_{H^0} \le C\|\Box u\|_{H^0}$. For the very degenerate boundary conditions arising at boundary points of infinite type, there is no hope of any parametrix formula that will express u in terms of $\Box u$, modulo a smoothing term. Attempts to exploit the H^0 inequality to majorize derivatives of u lead to error terms, for instance from the commutation of $\Box$ with partial derivatives, which appear on the right hand side of an inequality. One arrives at an estimate of the form

$$\|u\|_{H^t} \le C\|\Box u\|_{H^s} + C'\|u\|_{H^s}. \tag{3–2}$$

Such an inequality is useful only if (i) $t > s$, (ii) both $t = s$ and $C' < 1$, or (iii) $C' = 0$ because all commutator terms vanish identically.

For general pseudoconvex domains whose boundaries contain points of infinite type, there is no smoothing effect to make $t > s$. Estimates with $t \le s$ are highly unstable, potentially being destroyed by perturbations by operators of order zero. In practice, (ii) requires that C' be made arbitrarily small, to ensure that it is < 1. Commutator terms can be expected to vanish identically only for domains with symmetries.

For any smoothly bounded, pseudoconvex domain Ω there exists $\delta > 0$ such that the $\bar\partial$–Neumann problem is exactly regular in H^s for all $0 \le s < \delta$. This holds essentially because $C' = O(s)$ in (3–2) for small $s \ge 0$.

All proofs of exact regularity have relied on $Q(u,u)$ being sufficiently large relative to commutator terms. Consider first the compact case. The H^0 inequality (2–1) leads for each s and each $\varepsilon > 0$ to an inequality

$$\varepsilon^{-1}\|u\|_{H^s} \le \left[\|\Box u\|_{H^s} + C\|u\|_{H^s}\right] + C'_{\varepsilon,s}\|u\|_{H^0},$$

where C depends only on s. The factor ε^{-1} on the left hand side permits absorption of the term $C\|u\|_{H^s}$, whence the H^s norm of u is majorized by the H^s norm of $\Box u$.

Consider next the weighted theory. Fix Ω and a strictly plurisubharmonic function $\varphi \in C^\infty(\overline{\Omega})$. Denote by $\bar\partial^*_\lambda$ the adjoint of $\bar\partial$ in $\mathcal{H}_\lambda = L^2(\Omega, \exp(-\lambda\varphi))$, and set $Q_\lambda(u,u) = \|\bar\partial u\|^2_{\mathcal{H}_\lambda} + \|\bar\partial^*_\lambda u\|^2_{\mathcal{H}_\lambda}$. Then for all $u \in C^1(\overline{\Omega})$ satisfying the first boundary condition (1–1), $\|u\|^2_{\mathcal{H}_\lambda} \le C\lambda^{-1}Q_\lambda(u,u)$. This inequality is intermediate between the basic unweighted majorization $\|u\|^2_{L^2} \le CQ(u,u)$ and the compactness inequality (2–1). The norms of $\mathcal{H}_\lambda$ and L^2 are equivalent, though not uniformly in λ, so the weighted inequality implies [Kohn 1973] that for all sufficiently large λ, for all $s \le c\lambda^{1/2}$ and all $u \in C^\infty(\overline{\Omega})$, $\|u\|_{H^s} \le C_\lambda\|\Box_\lambda u\|_{H^s}$. It is possible to pass from this *a priori* majorization to the conclusion that the

[11]No analogous example is known to this author for the $\bar\partial$–Neumann problem for domains in $\mathbb{C}^n$; global C^∞ regularity has always been been proved via exact regularity. Kohn has asked whether they are in fact equivalent.

$\bar\partial$–Neumann problem for Ω with weight $\exp(-\lambda\varphi)$ is exactly regular in H^s, in the range $s \le c\lambda^{1/2}$.

Finally, in the results of Boas and Straube, one begins with a weaker inequality $\|u\|^2 \le CQ(u,u)$ for a fixed constant C. Outside any neighborhood of the set W_∞ of boundary points of infinite type, this is supplemented by a compactness estimate. By exploiting the special vector field V it can be arranged that for each s, the commutator terms leading to the potentially harmful term $C'\|u\|_{H^s}$ on the right hand side of (3–2) are of three types. Those of the first type are supported outside a neighborhood of W_∞, hence are harmless by virtue of the compactness inequality. Those of the second type are majorized by arbitrarily small multiples of $\|u\|_{H^s}$. Those of the third type, arising from the $T^{1,0}\oplus T^{0,1}(\partial\Omega)$ components of commutators of V with sections of $T^{1,0}\oplus T^{0,1}$, are majorized by lower order Sobolev norms of u.

The common theme is that a successful analysis is possible because the basic L^2 inequality is stronger than the harmful commutator terms. In the first situation, the L^2 inequality is arbitrarily strong; in the third, the error terms are arbitrarily weak near W_∞, and in the second, the weight $\exp(-\lambda\varphi)$ is chosen so as to make the L^2 inequality sufficiently strong relative to the error terms.

4. Worm Domains

The worm domains, invented by Diederich and Fornæss [1977], are examples of pseudoconvex domains whose closures have no Stein neighborhood bases. This means that there exists $\delta > 0$ such that there exists no pseudoconvex domain containing $\overline{\Omega}$, and contained in $\{z : \text{distance}(z,\Omega) < \delta\}$.

Definition. A worm domain in $\mathbb{C}^2$ is a bounded open set of the form

$$\mathcal{W} = \bigl\{z : |z_1 + e^{i\log|z_2|^2}|^2 < 1 - \phi(\log|z_2|^2)\bigr\} \tag{4–1}$$

having the following properties:

(i) $\mathcal{W}$ has smooth boundary and is pseudoconvex.
(ii) $\phi \in C^\infty$ takes values in $[0,1]$, vanishes identically on $[-r,r]$ for some $r>0$, and vanishes nowhere else.
(iii) $\mathcal{W}$ is strictly pseudoconvex at every boundary point where $\bigl|\log|z_2|^2\bigr| > r$.

We will sometimes write $\mathcal{W}_r = \mathcal{W}$.

Diederich and Fornæss proved that ϕ can be chosen so that these properties hold; beyond this the choice of ϕ is of no consequence. Important properties of worm domains include:

(i) $\partial\mathcal{W}_r$ contains the annular complex manifold with boundary

$$\mathcal{A}_r = \bigl\{z : z_1 = 0 \text{ and } \bigl|\log|z_2|^2\bigr| \le r\bigr\}. \tag{4–2}$$

(ii) $\mathcal{W}$ is strictly pseudoconvex at every boundary point not in $\mathcal{A}_r$.

If $r \geq \pi$ then $\partial \mathcal{W}_r$ contains the annulus $\mathcal{A}_\pi$ as well as the two circles

$$\{z : |z_1 + e^{i\pi}| = 1 \text{ and } \log|z_2|^2 = \pm\pi\}.$$

Applying the standard extension argument, one finds that any function holomorphic in any neighborhood of the union of $\mathcal{A}_\pi$ and the two circles must extend holomorphically to a fixed such neighborhood, which thus is contained in every pseudoconvex neighborhood of $\overline{\mathcal{W}_r}$. But if $r < \pi$ then $\overline{\mathcal{W}_r}$ does have a basis of pseudoconvex neighborhoods [Fornæss and Stensønes 1987; Bedford and Fornæss 1978].

The worm domains had long been regarded as important test cases for global regularity when Barrett [1992] achieved a breakthrough.

THEOREM 4.1. *For each $r > 0$ there exists $t \in \mathbb{R}^+$ such that for any worm domain $\mathcal{W}_r$ and any $s \geq t$, the $\bar\partial$–Neumann problem fails to be exactly regular in H^s. Moreover $t \to 0$ as $r \to \infty$.*

The proof focused on the Bergman projection B rather than on the Neumann operator. B is the orthogonal projection mapping scalar valued functions in $L^2(\Omega)$ onto the closed subspace of all holomorphic square integrable functions. It is related to the $\bar\partial$–Neumann problem via the formula [Kohn 1963; 1964].

$$B = I - \bar\partial^* N \bar\partial, \tag{4–3}$$

where I denotes the identity operator. In $\mathbb{C}^2$, for any exponent s, B preserves (scalar valued) H^s if and only if N preserves ($(0,1)$-form valued) H^s; B preserves $C^\infty(\overline{\Omega})$ if and only if N does so [Boas and Straube 1990][12].

The proof had two parts, of which the first was an elegant direct analysis of the nonsmooth domains

$$\mathcal{W}'_r = \{z : \big|z_1 + e^{i\log|z_2|^2}\big| < 1 \text{ and } -r < \log|z_2|^2 < r\}.$$

B not only fails to preserve H^t, but even fails to map $C^\infty(\overline{\mathcal{W}'_r})$ to H^t.

This step has much in common with the contemporaneous proof by Christ and Geller [1992] that the Szegő projection for certain real analytic domains of finite type fails to be analytic pseudolocal. In both analyses, separation of variables leads to a synthesis of the projection operator in terms of explicitly realizable projections onto one dimensional subspaces.[13] The expression for such a rank one projection carries a factor of the reciprocal of the norm squared of a basis element. Analytic continuation of this reciprocal with respect to a natural Fourier parameter leads to poles off of the real axis, which are the source of irregularity.

The second part was a proof by contradiction. It was shown that if the Bergman projection for $\mathcal{W}_r$ preserves some H^s, then the Bergman projection for

[12] There exists a generalization valid for all dimensions [Boas and Straube 1990].

[13] This decomposition and synthesis in [Christ and Geller 1992] was taken from work of Nagel [1986].

$\mathcal{W}'_r$ must also preserve H^s. The reasoning relied on a scaling argument, in which it was essential that the norms H^s on the left and right hand sides of the *a priori* inclusion inequality have identical scaling properties. Consequently this indirect method did not exclude the possibility that B might map H^s to $H^{s-\varepsilon}$, for all $\varepsilon > 0$, for all s.

5. A Cohomology Class

The worm domains have another property of vital importance for any discussion of global regularity, whose significance in this context was recognized by Boas and Straube [1993]. Consider any smoothly bounded domain Ω for which the set W_∞ of all boundary points of infinite type forms a smooth, compact complex submanifold R, with boundary. The worm domains are examples.

The embedding of R into the Cauchy–Riemann manifold $\partial\Omega$ induces an element of the de Rham cohomology group $H^1(R)$, defined as follows. Fix any purely real, nowhere vanishing one-form η, defined in a neighborhood in $\partial\Omega$ of R, that annihilates $T^{0,1} \oplus T^{1,0}(\partial\Omega)$. Fix likewise a smooth real vector field V, transverse to $T^{0,1} \oplus T^{1,0}$, satisfying $\eta(V) \equiv 1$. Consider the one-form $\alpha = -\mathcal{L}_V \eta\big|_R$, the Lie derivative of $-\eta$ with respect to V, restricted to R.[14] Moreover, if Ω is pseudoconvex, then α is a closed form [Boas and Straube 1993], hence represents an element $[\alpha]$ of the cohomology group $H^1(R)$. This element is independent of the choices of η and of V.

Definition. The winding class $w(R, \partial\Omega)$ of $\partial\Omega$ about R is the cohomology class $[\alpha] \in H^1(R)$.

This class is determined by the first-order jet of the CR structure of $\partial\Omega$ along R. A fundamental property of worm domains is that

$$\text{For every worm domain, } w(\mathcal{A}_r, \partial\mathcal{W}_r) \neq 0. \tag{5–1}$$

A theorem of Boas and Straube [1993] asserts that if $w(R, \partial\Omega) = 0$, then there exist vector fields V satisfying the approximate commutation relation (2–3). Consequently the $\bar\partial$–Neumann problem is globally regular in C^∞.

To understand $w(R, \partial\Omega)$ in concrete terms [15], suppose that $\Omega \subset \mathbb{C}^2$ and R is a smooth Riemann surface with boundary, embeddable in $\mathbb{C}^1$. Choose coordinates $(x + iy,\, t) \in \mathbb{C} \times \mathbb{R}$ in a neighborhood of R in $\partial\Omega$ such that $R \subset \{t = 0\}$; identify R with $\{x + iy : (x + iy, 0) \in R\}$. A Cauchy–Riemann operator has the form $\bar\partial_b = X + iY$ where X, Y are real vector fields of the form $X = \partial_x + a\partial_t,\ Y = \partial_y + b\partial_t$, where a, b are smooth real valued functions of

[14] α is a section over R of the tangent bundle TR, not of $T\partial\Omega$.

[15] Bedford and Fornæss [1978] gave a geometric interpretation of $w(R, \partial\Omega)$, and had shown that whenever it is smaller than a certain threshold value in a natural norm on cohomology, $\overline{\Omega}$ has a pseudoconvex neighborhood basis. α had appeared earlier in work of D'Angelo [1979; 1987].

(x, y, t) and $a(x+iy,\, 0) \equiv 0 \equiv b(x+iy,\, 0)$. The Levi form may be identified with the function $\lambda(x+iy,\, t) = (b_x + ab_t) - (a_y + ba_t)$, where the subscripts indicate partial differentiation. By hypothesis, $R = \{(x+iy,\, t) : \lambda(x+iy,\, t) = 0\}$.

By choosing $\eta = dt - a\,dx - b\,dy$ and $V = \partial_t$, we obtain $-\mathcal{L}_V\eta = a_t\,dx + b_t\,dy$ and hence $\alpha(x+iy) = a_t(x, y, 0)\,dx + b_t(x, y, 0)\,dy$, for $x+iy \in R$. Note that

$$d\alpha = (a_{t,y} - b_{t,x})(x+iy,\, 0)\,dx\,dy = (\partial_t\lambda)(x+iy,\, 0)\,dx\,dy.$$

Pseudoconvexity of $\partial\Omega$ means that $\lambda(x+iy,\, t) \geq 0$ everywhere, which forces $\partial_t\lambda(x+iy,\, 0) \equiv 0$ since $\lambda(x+iy,\, 0) \equiv 0$. Thus α is indeed closed.

To what extent does the CR structure of $\partial\Omega$ coincide with the Levi flat CR structure $R \times \mathbb{R}$ near R? More precisely, do there exist coordinates $(x+iy,\, t)$ in which $R \subset \{t = 0\}$ and $\bar\partial_b$ takes the form $(\partial_x + \tilde a\partial_t) + i(\partial_y + \tilde b\partial_t)$ with $\tilde a(x+iy,\, t), \tilde b(x+iy,\, t) = O(t^2)$ as $t \to 0$ for every $x+iy \in R$? By an elementary analysis, the answer is affirmative if and only if $w(R, \partial\Omega) = 0$. Thus the theorem of Boas and Straube asserts rather paradoxically that global regularity holds whenever the CR structure near R is sufficiently degenerate.

In the absence of any pseudoconvexity hypothesis, α need not be closed, but exactness of the form α remains the criterion for existence of the desired coordinate system. There exists a hierarchy of invariants $w_k(R, \partial\Omega)$, with $w(R, \partial\Omega) = w_1$. Each w_k is defined if $w_{k-1} = 0$, and represents the obstruction to existence of coordinates in which $\tilde a, \tilde b = O(t^{k+1})$. Each w_k is an equivalence class of forms modulo exact forms; in the pseudoconvex case, w_k is represented by a closed form for even k. These invariants have no relevance to C^∞ regularity, but we believe that they may play a role in the theory of global regularity in Gevrey classes, partially but not completely analogous to the role of w_1 in the C^∞ case.

6. Special Vector Fields and Commutation

The use of auxiliary vector fields V satisfying the commutation equations

$$[V, X_j] \in \operatorname{span}\{X_i\} \qquad \text{for all } j \tag{6–1}$$

together with a transversality condition, for sums of squares operators $L = \sum_i X_i^2$, and of analogous commutation equations in related situations such as the $\bar\partial$–Neumann problem, has not been restricted to the question of global C^∞ regularity. Real analytic vector fields satisfying (6–1) globally on a compact manifold have been used by Derridj and Tartakoff [1976], Komatsu [1975; 1976] and later authors to prove global regularity in C^ω [Tartakoff 1996]. This work has depended also on what are known as maximal estimates and their generalizations.[16]

[16]Maximal estimates and their connection with representations of nilpotent Lie groups are the subject of a deep theory initiated by Rothschild and Stein [1976] and developed by Helffer and Nourrigat in a series of works including [Helffer and Nourrigat 1985] and leading up to [Nourrigat 1990] and related work of Nourrigat.

For sums of squares operators, maximal estimates take the form

$$\sum_{i,j} \|X_i X_j u\|_{L^2} \leq C\|Lu\|_{L^2} + C\|u\|_{L^2}.$$

They are used to absorb certain error terms that arise from commutators $[V, X_j]$ in a bootstrapping argument in which successively higher derivatives are estimated. Chen [1988; 1989], Cordaro and Himonas [1994], Derridj [1997] and Christ [1994a] have obtained cruder results based on the existence of vector fields for which the commutators vanish identically. Such results require far weaker bounds than maximal estimates.

Auxiliary vector fields with this commutation property have also been used to establish analytic hypoellipticity in certain cases. In the method of Tartakoff [1980], this requires the modification of V by cutoff functions having appropriate regularity properties, to take into account the possible lack of global regularity or even global definition of the data. Sjöstrand [1982; 1983] has developed a microlocal analogue, in which a vector field corresponds to a one parameter deformation of the operator being studied.

The use of auxiliary vector fields having this commutation property should be regarded not as a special trick, but rather as the most natural approach to exact regularity. The remainder of this section is devoted to a justification of this assertion. For simplicity we restrict the discussion to any sum of squares operator $L = \sum_j X_j^2$, on a compact manifold M without boundary.[17] We assume $\|u\|_{L^2} \leq C\|Lu\|_{L^2}$, for all $u \in C^2$.

Consider any first order elliptic, self adjoint, strictly positive pseudodifferential operator Λ on M. Then the powers Λ^s are well defined for all $s \in \mathbb{C}$, and Λ^s maps $H^r(M)$ bijectively to H^{r-s} for all $s, r \in \mathbb{R}$. Define $L_s = \Lambda^s \circ L \circ \Lambda^{-s}$. Then for each $0 \leq s \in \mathbb{R}$, L is exactly regular in H^s if and only if for all $u \in H^{-s}(M)$,

$$L_s u \in H^0 \text{ implies } u \in H^0. \tag{6–2}$$

Thus one seeks an *a priori* inequality for all $u \in C^\infty(M)$ of the form[18]

$$\|u\|_{L^2} \leq C_s\|L_s u\|_{L^2} + C_s\|u\|_{H^{-1}}. \tag{6–3}$$

Since such an inequality holds for $L_s = L$, it is natural to ask whether L_s may be analyzed as a perturbation of L. Now $L_s = \sum_j (\Lambda^s X_j \Lambda^{-s})^2$. Moreover,

$$\Lambda^s[X_j, \Lambda^{-s}] = -s\Lambda^{-1}[X_j, \Lambda]$$

[17]The same analysis applies equally well to the $\bar\partial$–Neumann problem on any pseudoconvex domain in $\mathbb{C}^2$, by the method of reduction to the boundary as explained in § 9.

[18]From an inequality of this type for all $s \geq 0$, with C_s bounded uniformly on compact sets, it is possible to deduce (6–2) for all $s \in \mathbb{R}$ by a continuity argument, using approximations to the identity and pseudodifferential calculus.

modulo a pseudodifferential operator of order ≤ -1; the contribution of any such operator can be shown always to be negligible for our discussion, by exploiting the L^2 inequality

$$\|X_j u\| \le C\|Lu\| + C\|u\|. \tag{6–4}$$

Therefore modulo harmless error terms,

$$L_s \approx L - s\sum_j (X_j B_j + B_j X_j) + s^2 \sum_j B_j^2, \tag{6–5}$$

where $B_j = \Lambda^{-1}[X_j, \Lambda]$ has order ≤ 0. Since each factor $\Lambda^{-1}[X_j, \Lambda]$ has order ≤ 0, (6–4) implies

$$\|(L_s - L)u\| \le C(|s| + s^2)(\|Lu\| + \|u\|) + C\|u\|_{H^{-1}} + C\|Lu\|_{H^{-1}}.$$

Thus (6–3) holds, and L is exactly regular in H^s, for all sufficiently small s.

Moreover, for any pseudodifferential operator E of strictly negative order, any perturbation term of the form EX_iX_j is harmless, even if multiplied by an arbitrarily large coefficient, since it ultimately leads to an estimate in terms of some negative order Sobolev norm of u after exploiting (6–4) in evaluating the quadratic form $\langle L_s u,\, u\rangle$. Thus in order to establish (6–3), it would suffice for there to exist Λ such that each commutator $[X_j, \Lambda]$ can be expressed as $\sum_i B_{i,j} X_i$ modulo an operator of order 0, where each $B_{i,j}$ is some pseudodifferential operator of order 0. This is a property of the principal symbol of Λ alone. Moreover, by virtue of standard microlocal regularity estimates, it suffices to have such a commutation relation microlocally in a conic neighborhood of the characteristic variety $\Sigma \subset T^*M$ defined by the vanishing of the principal symbol of L.

Let us now specialize the discussion to the case where at every point of M, $\{X_j\}$ are linearly independent and span a subspace of the tangent space having codimension one. Then Σ is a line bundle. We suppose this bundle to be orientable. Thus Σ splits as the union of two half line bundles, and there exists a globally defined vector field T transverse at every point to $\operatorname{span}\{X_j\}$.

In a conic neighborhood of either half, Λ may be expressed as a smooth real vector field V, plus a perturbation expressible as a finite sum of terms $E_{i,j}X_iX_j$ where $E_{i,j}$ has order ≤ -1, plus a negligible term of order 0. V is transverse to $\operatorname{span}\{X_j\}$, because Λ is elliptic. The commutator of any X_j with any of these perturbation terms has already the desired form. Thus if there exists Λ for which each commutator $[X_j, \Lambda]$ takes the desired form, then there must exist V satisfying $[V, X_j] \in \operatorname{span}\{X_i\}$ for all j.

7. A Model

Global C^∞ irregularity for the worm domains was discovered by analyzing the simplest instance of a more general problem. Consider a finite collection of smooth real vector fields X_j on a compact manifold M without boundary, and an operator $L = -\sum_j X_j^2 + \sum_j b_j X_j + a$ where $a, b_j \in C^\infty$. Under what circumstances is L globally regular in C^∞?

Denote by $\|\cdot\|$ the norm in $L^2(M)$, with respect to some smooth measure. A Lipschitz path $\gamma : [0,1] \mapsto M$ is said to be admissible if $\frac{d}{ds}\gamma(s) \in \operatorname{span}\{X_j(\gamma(s))\}$ for almost every s. A collection of vector fields X_j is said to satisfy the bracket hypothesis if the Lie algebra generated by them spans the tangent space to M.

We impose three hypotheses in order to preclude various pathologies and to mimic features present in the $\bar\partial$–Neumann problem for arbitrary smoothly bounded, pseudoconvex domains in $\mathbb{C}^2$.

- There exists $C < \infty$ such that for all $u \in C^2(M)$, $\|u\| \le C\|Lu\|$.
- For every $x, y \in M$ there exists an admissible path γ satisfying[19] $\gamma(0) = x$ and $\gamma(1) = y$.
- $\{X_j\}$ satisfies the bracket hypothesis on some nonempty subset $U \subset M$.

Under these hypotheses, must L be globally regular in C^∞?

This is not a true generalization of the $\bar\partial$–Neumann problem. But as will be explained in § 9, the latter may be reduced (in $\mathbb{C}^2$) to a very similar situation, where the vector fields are the real and imaginary parts of $\bar\partial_b$ on $\partial\Omega$.

The first hypothesis mimics the existence of an L^2 estimate for the $\bar\partial$–Neumann problem. The second and third mimic respectively the absence of compact complex submanifolds without boundary in boundaries of domains in $\mathbb{C}^n$, and the presence of strictly pseudoconvex points in boundaries of all such domains, respectively. Each hypothesis excludes the constant coefficient examples on $\mathbb{T}^2$ discussed in § 2. The first may be achieved, for any collection of vector fields and coefficients b_j, by adding a sufficiently large positive constant to a. These assumptions complement one another. The third builds in a certain smoothing effect, while the second provides a mechanism for that effect to propagate to all of M.

Global C^∞ regularity does not necessarily hold in this situation. As an example,[20] let $M = \mathbb{T}^2$ and fix a coordinate patch $V_0 \subset M$ along with an identification of V_0 with $\{(x,t) \in (-2,2) \times (-2\delta, 2\delta)\} \subset \mathbb{R}^2$. Set $J = [-1,1] \times \{0\}$. Let X, Y be any two smooth, real vector fields defined on M satisfying the following hypotheses.

(i) $X, Y, [X,Y]$ span the tangent space to M at every point of $M \backslash J$.
(ii) In V_0, $X \equiv \partial_x$ and $Y \equiv b(x,t)\partial_t$.
(iii) For all $|x| \le 1$ and $|t| \le \delta$, $b(x,t) = \alpha(x)t + O(t^2)$, where $\alpha(x)$ vanishes nowhere.

The collection of vector fields $\{X, Y\}$ then satisfies the second and third hypotheses imposed above.

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[19]This property is called reachability by some authors [Sussmann 1973].

[20]The global structure of M is of no importance in this example.

The role of the Riemann surface R in the discussion in § 5 is taken here by J, even though $H^1(J) = 0$. Although there appears to be no direct analogue of the one-forms η, α of that discussion, there exists no vector field V transverse to $\operatorname{span}\{X, Y\}$ such that $[V, X]$ and $[V, Y]$ belong to $\operatorname{span}\{X, Y\}$; nor does a family of such vector fields exist with the slightly weaker approximate commutation property (2–3).

THEOREM 7.1 [Christ 1995a]. *Let X, Y, M be as above. Let L be any operator on M of the form $L = -X^2 - Y^2 + a$, such that $a \in C^\infty$ and $\|u\|^2 \le C\langle Lu, u\rangle$ for all $u \in C^2(M)$. Then L is not globally regular in C^∞.*

The close analogy between this result and the $\bar\partial$–Neumann problem for worm domains will be explained in § 9. A variant of Theorem 7.1 is actually proved in [Christ 1995a], but the same proof applies.

Before discussing the proof, we will formulate more precise conclusions giving some insight into the nature of the problem and the singularities of solutions. For $|x| \le 1$, write $a(x,t) = \beta(x) + O(t)$. Consider the one parameter family of ordinary differential operators

$$\mathcal{H}_\sigma = -\partial_x^2 + \sigma\alpha(x)^2 + \beta(x).$$

Define Σ_0 to be the set of all $\sigma \in \mathbb{C}$ for which the Dirichlet problem

$$\begin{cases} \mathcal{H}_\sigma f = 0 & \text{on } [-1,1], \\ f(\pm 1) = 0 \end{cases}$$

has a nonzero solution. Then Σ_0 consists of a discrete sequence of real numbers $\lambda_0 < \lambda_1 < \dots$ tending to $+\infty$. Define

$$\Sigma = \left\{ s \in [0,\infty) : (s - 1/2)^2 \in \Sigma_0 \right\} .$$

Write $\Sigma = \{s_0 < s_1 < \dots\}$. It can be shown [Christ 1995a] that $s_0 > 0$.

Under our hypotheses, L^{-1} is a well defined bounded linear operator on $L^2(M)$.

THEOREM 7.2. *L has the following global regularity properties.*

- *For every $s < s_0$, L^{-1} preserves $H^s(M)$.*
- *For each $s > s_0$, L^{-1} fails to map $C^\infty(M)$ to H^s.*
- *Suppose that $0 \le s < r < s_0$, or $s_j < s < r < s_{j+1}$ for some $j \ge 0$. Then any $u \in H^s(M)$ satisfying $Lu \in H^r(M)$ must belong to H^r.*
- *For each $s \notin \Sigma$ an a priori inequality is valid: There exists $C < \infty$ such that for every $u \in H^s(M)$ such that $Lu \in H^s$,*

$$\|u\|_{H^s} \le C\|Lu\|_{H^s}. \tag{7–1}$$

- *For each $s \notin \Sigma$, $\{f \in H^s(M) : L^{-1}f \in H^s\}$ is a closed subspace of H^s with finite codimension.*

To guess the nature of the singularities of solutions, consider the following simpler problem. Define

$$\mathcal{L} = -\partial_x^2 - \alpha^2(x)(t\partial_t)^2 + \beta(x). \tag{7–2}$$

Consider the Dirichlet problem

$$\begin{cases} \mathcal{L}u = g & \text{on } [-1,1]\times\mathbb{R}, \\ u(x,t) \equiv 0 & \text{on } \{\pm 1\}\times\mathbb{R}. \end{cases} \tag{7–3}$$

To construct a singular solution for this Dirichlet problem, fix $s \in \Sigma$, set $\sigma = (s-1/2)(s+1/2)$, and fix a nonzero solution of $\mathcal{H}_\sigma f = 0$ with $f(\pm 1) = 0$. Fix $\eta \in C_0^\infty(\mathbb{R})$, identically equal to one in some neighborhood of 0. Set $u(x,t) = \eta(t)f(x)t^{s-1/2}$ for $t > 0$, and $u \equiv 0$ for $t < 0$. Then $u \in L^2([-1,1]\times\mathbb{R})$ is a solution of (7–3) for a certain $g \in C_0^\infty([-1,1]\times\mathbb{R})$. Thus the Dirichlet problem (7–3) for $\mathcal{L}$ on the strip is globally irregular.

The proof of Theorem 7.1 consists in reducing the global analysis of L on M to the Dirichlet problem for $\mathcal{L}$. Unfortunately, we know of no direct construction of nonsmooth solutions for L on M that uses the singular solution of the preceding paragraph as an Ansatz.

Instead, the proof[21] consists in two parts [Christ 1995a]. First, the *a priori* inequality (7–1) is established. Second, emulating Barrett [1992], we prove that for any $s \ge s_0$, L cannot be exactly regular in H^s.

With these two facts in hand, suppose that L were globally regular in C^∞. Fix any $s_0 < s \notin \Sigma$. Given any $f \in H^s$, fix a sequence $\{f_j\} \subset C^\infty$ converging to f in H^s. Then $\{L^{-1}f_j\}$ is Cauchy in H^s, by the *a priori* inequality, since $L^{-1}f_j \in C^\infty$ by hypothesis. On the other hand, since L^{-1} is bounded on L^2, $L^{-1}f_j \to L^{-1}f$ in L^2 norm. Consequently $L^{-1}f \in H^s$. This contradicts the result that L fails to be exactly regular in H^s.

8. A Tale of Three Regions

The main part of the analysis is the proof of the *a priori* estimate (7–1) for $0 < s \notin \Sigma$. The main difficulty is as follows.

Associated to the operator L is a sub-Riemannian structure on the manifold M. Define a metric $\rho(x,y)$ to equal the minimal length of any Lipschitz path γ joining x to y, such that the tangent vector to γ is almost everywhere of the form $s_1X + s_2Y$ with $s_1^2 + s_2^2 \le 1$. Points having coordinates (x,ε) with $|x| \le 1/2$ are at distance $> 1/2$ from J in this degenerate metric, no matter how small $\varepsilon > 0$ may be; paths approaching J "from above" have infinite length, but paths such as $s \mapsto (s,0)$ approaching J "from the side" have finite length.

For the purpose of analyzing L, M is divided naturally into three regions. Region I is $M\backslash J$; L satisfies the bracket hypothesis on any compact subset of

[21] We have subsequently found a reformulation of the proof that eliminates the second part of the argument and has a less paradoxical structure. But this reformulation involves essentially the same ingredients, and is no simpler.

region I, so a very satisfactory regularity theory is known: L is hypoelliptic and gains at least one derivative. Region II is an infinitesimal tubular neighborhood $\{|x| \le 1, 0 \ne t \sim 0\}$. Here L is an elliptic polynomial in $\partial_x, t\partial_t$, so a natural tool for its analysis is the partial Mellin transform in the variable t. The subregions $t > 0$, $t < 0$ are locally decoupled where $|x| < 1$; the relationship between $u(x, 0^+)$ and $u(x, 0^-)$ is determined by global considerations. Region III is another infinitesimal region, lying to both sides of J, where $t \sim 0$ and $1 < |x| \sim 1$. In this transitional region, if Y is expanded as a linear combination $c(x)\partial_t + O(t)\partial_t$, the coefficient $c(x)$ vanishes to infinite order as $|x| \to 1^+$. In such a situation no parametrix construction can be hoped for. The only tool available appears to be *a priori* L^2 estimation stemming from integration by parts.

One needs not only an analysis for each region, but three compatible analyses. No attack by decomposing M into three parts by a partition of unity has succeeded; error terms resulting from commutation of L with the partition functions are too severe to be absorbed.

The proof of the *a priori* estimate proceeds in several steps. For simplicity we assume $u \in C^\infty$. The following discussion is occasionally imprecise; correct statements may be found in [Christ 1995a].

First step. For any $\varepsilon > 0$, u may be assumed to be supported where $|x| < 1+\varepsilon$ and $|t| < \varepsilon$. Indeed, since $X, Y, [X, Y]$ span the tangent space outside J, the H^{s+1} norm of u is controlled on any compact subset of $M \backslash J$ by $\|Lu\|_{H^s} + \|u\|_{H^0}$.

Second step. Fix a globally defined, self adjoint, strictly positive elliptic first order pseudodifferential operator Λ on M, and set $L_s = \Lambda^s \circ L \circ \Lambda^{-s}$. Then L satisfies an *a priori* exact regularity estimate in H^s if and only if there exist ε, C such that

$$\|u\| \le C\|L_s u\| + C\|u\|_{H^{-1}}$$

for all $u \in C^\infty$ supported where $|x| < 1+\varepsilon$ and $|t| < \varepsilon$, where all norms without subscripts are L^2 norms. In particular, we may work henceforth on $\mathbb{R}^2$ rather than on M.

Denote by $\Gamma \subset T^*M$ the line bundle $\{(x, t; \xi, \tau) : (x, t) \in J \text{ and } \xi = 0\}$. Microlocally on the complement of Γ, the H^1 norm of u is controlled by the H^0 norm of $L_s u$ plus the H^{-1} norm of u, for every s.

Third step.

$$L_s = -\partial_x^2 - (Y_s + A_1)(Y_s + A_2) + \beta(x) + A_3,$$

where $\beta(x) = a(x, 0)$ and Y_s is a real vector field which, where $|x| \le 1$, takes the form

$$Y_s = \alpha(x)(t\partial_t + s) + O(t^2)\partial_t.$$

The principal symbol σ_0 of each $A_j \in S^0_{1,0}$ vanishes identically on Γ.

Fourth step. Integration by parts yields

$$\|\partial_x u\| \le C\|L_s u\| + C\|u\| \qquad \text{for all } u \in C^2. \tag{8–1}$$

By itself this inequality is of limited value, since $\|u\|$ appears on the right hand side rather than on the left.

Fifth step. The fundamental theorem of calculus together with the vanishing of $u(x,t)$ for all $|x| > 1+\varepsilon$ yield

$$\|u\|_{L^2(\{|x|>1\})} + \|u\|_{L^2(\{-1,1\}\times\mathbb{R})} \le C\varepsilon^{1/2}\|\partial_x u\| \le C\varepsilon^{1/2}\left[\|L_s u\| + C\|u\|\right].$$

Combining this with (8–1) and absorbing certain terms into the left hand side gives the best information attainable without a close analysis of the degenerate region II.

$$\|u\| + \|\partial_x u\| + \varepsilon^{-1/2}\|u\|_{L^2(\{-1,1\}\times\mathbb{R})} \le C\|L_s u\| + C\|u\|_{L^2([-1,1]\times\mathbb{R})}. \qquad (8\text{–}2)$$

By choosing ε to be sufficiently small, we may absorb the last term on the right into the left hand side of the inequality.

It remains to control the L^2 norm of u in $[-1,1]\times\mathbb{R}$. In the next step we prepare the machinery that will be used to achieve this control in step seven.

Sixth step. Define $\mathcal{L}_s = -\partial_x^2 - \alpha(x)^2(t\partial_t + s)^2 + \beta(x)$; note that $\mathcal{L}_s$ is an elliptic polynomial in $\partial_x, t\partial_t$. Conjugation with the Mellin transform[22] in the variable t reduces the analysis of $\mathcal{L}_s$ on $L^2([-1,1]\times\mathbb{R})$ to that of the one parameter family[23] of ordinary differential operators

$$\mathcal{H}_{(s+i\tau-\frac{1}{2})^2} = -\partial_x^2 - \alpha(x)^2(s+i\tau-\tfrac{1}{2})^2 + \beta(x), \qquad \tau\in\mathbb{R}.$$

The assumption that $s\notin\Sigma$ is equivalent to the assertion that for each $\tau\in\mathbb{R}$, the nullspace of $\mathcal{H}_{(s+i\tau-\frac{1}{2})^2}$ on $L^2([-1,1])$ with Dirichlet boundary conditions is $\{0\}$. Thus $\mathcal{H}_{(s+i\tau-\frac{1}{2})^2}g = f$ may be solved in $L^2([-1,1])$, with arbitrarily prescribed boundary values, and the solution is unique. On the other hand, because $\mathcal{H}_{(s+i\tau-\frac{1}{2})^2}$ is an elliptic polynomial in ∂_x and $i\tau$, the same holds automatically for all sufficiently large $|\tau|$. Quantifying all this and invoking the Plancherel and inversion properties of the Mellin transform, one deduces that the Dirichlet problem for $\mathcal{L}_s$ is uniquely solvable in $L^2([-1,1]\times\mathbb{R})$. Moreover, if $u\in C^2([-1,1]\times\mathbb{R})$ has compact support and $\mathcal{L}_s u = f_1 + t\partial_t f_2 + (t\partial_t)^2 f_3$ in $[-1,1]\times\mathbb{R}$, then[24]

$$\|u\|_{L^2([-1,1]\times\mathbb{R})} + \|\partial_x u\|_{L^2([-1,1]\times\mathbb{R})} \le C\sum_j \|f_j\|_{L^2([-1,1]\times\mathbb{R})} + C\|u\|_{L^2(\{\pm1\}\times\mathbb{R})}. \qquad (8\text{–}3)$$

[22] This applies for $t>0$; the region $t<0$ is handled by substituting $t\mapsto -t$ and repeating the same analysis.

[23] s is shifted to $s-\frac{1}{2}$ in order to take into account the difference between the measures dt and $t^{-1}\,dt$; the latter appears in the usual Plancherel formula for the Mellin transform.

[24] Up to two factors of $t\partial_t$ are permitted on the right hand side of the equation for $\mathcal{L}_s u$, because $\mathcal{H}_{(s+i\tau)^2}$ is an elliptic polynomial of degree two in $\partial_x, i\tau$ for each s.

Seventh step. On $[-1,1]\times\mathbb{R}$, $\mathfrak{L}_s u = L_s u + (\mathfrak{L}_s - L_s)u$. The remainder term $(\mathfrak{L}_s - L_s)u$ may be expressed as $(t\partial_t)^2 A_1 u + t\partial_t A_2 u + A_3 u$, where $\sigma_0(A_j)\equiv 0$ on Γ. Thus by (8–3),

$$\|u\|_{L^2([-1,1]\times\mathbb{R})} \le C\|u\|_{L^2(\{\pm1\}\times\mathbb{R})} + C\sum_j \|A_j u\|.$$

Since the H^0 norm of u is controlled microlocally by the H^{-1} norms of $L_s u$ and of u on the complement of Γ, and since $\sigma_0(A_j)\equiv 0$ on Γ,

$$\|A_j u\| \le C_\eta \|L_s u\|_{H^{-1}} + C_\eta\|u\|_{H^{-1}} + \eta\|u\|_{H^0}$$

for every $\eta > 0$. By inserting this into the preceding inequality and combining the result with the conclusion of the fifth step, we arrive at the desired *a priori* inequality majorizing $\|u\|$ by $\|L_s u\| + \|u\|_{H^{-1}}$. □

The simpler half of the proof is the demonstration that L is not exactly regular in H^{s_0}. The operator $\mathfrak{L} = -\partial_x^2 - (\alpha(x)t\partial_t)^2 + \beta(x)$ is obtained from L, in the region $|x|\le 1$, by substituting $t=\varepsilon\tilde t$, and letting $\varepsilon\to 0$. At typical points where $|x|>1$, the coefficient of ∂_t^2 in $\mathfrak{L}$ will be nonzero, and this scaling will lead to $\varepsilon^{-1}\partial_{\tilde t}$, hence in the limit to an infinite coefficient.

First step. There exists $f\in C_0^\infty((-1,1)\times\mathbb{R})$ for which the unique solution $u\in L^2([-1,1]\times\mathbb{R})$ of $L_s u = f$ with boundary condition $u(\pm1,t)\equiv 0$ is singular, in the sense that $|\partial_t|^{s_0}u\notin L^2([-1,1]\times\mathbb{R})$. This follows from a Mellin transform analysis, in the spirit of the sixth step above.

The remainder of the proof consists in showing that for any s, if L is exactly regular in $H^s(M)$, then there exists $C<\infty$ such that for every $f\in C_0^\infty((-1,1)\times\mathbb{R})$, there exists a solution $u\in L^2([-1,1]\times\mathbb{R})$ satisfying $\mathfrak{L}u=f$ and the boundary condition $u\equiv 0$ on $\{\pm1\}\times\mathbb{R}$, such that $|\partial_t|^s u\in L^2([-1,1]\times\mathbb{R})$ and

$$\||\partial_t|^s u\|_{L^2} \le C\|f\|_{H^s}. \tag{8–4}$$

Second step. Fix $s>0$, and suppose L to be exactly regular in $H^s(M)$. Fix $f\in C_0^\infty((-1,1)\times\mathbb{R})$. To produce the desired solution u, recall that L^{-1} is a well defined bounded operator on $L^2(M)$. For each small $\varepsilon>0$, for (x,t) in a fixed small open neighborhood in M of J, set

$$u_\varepsilon(x,t) = (L^{-1}f_\varepsilon)(x,\varepsilon t) \qquad \text{where } f_\varepsilon(x,t) = f(x,\varepsilon^{-1}t).$$

f_ε is supported where $|x|<1-\eta$ and $|t|<C\varepsilon$ for some $C,\eta\in\mathbb{R}^+$; we extend it to be identically zero outside this set, so that it is globally defined on M. The hypothesis that L^{-1} is bounded on $H^s(M)$ implies that in a neighborhood of J, u_ε and $\partial_x u_\varepsilon$ satisfy (8–4); the essential point is that the highest order derivative with respect to t on both sides of (8–4) is $|\partial_t|^s$, hence both sides scale in the same way under dilation with respect to t, as $\varepsilon\to 0$.

Since L^{-1} is bounded on H^0, the same reasoning leads to the conclusion that $u_\varepsilon, \partial_x u_\varepsilon$ are uniformly bounded in $L^2(M)$.

Third step. Define u to be a weak $*$ limit of some weakly convergent sequence u_{ε_j}. Then $u, \partial_x u, |\partial_t|^s u \in L^2([-1,1]\times\mathbb{R})$, with norms bounded by $\|f\|_{H^s}$. Passing to the limit in the equation defining u_ε, and exploiting the *a priori* bounds, we obtain $\mathfrak{L}u = f$ in $[-1,1]\times\mathbb{R}$.

The scaling and limiting procedure of steps two and three is due to Barrett [1992], who carried it out for the Bergman projection, rather than for a differential equation.[25]

Fourth step. It remains to show that $u(\pm 1, t) = 0$ for almost every $t \in \mathbb{R}$. Because $\partial_x u \in L^2$ and $u \in L^2$, $u(\pm 1, t)$ is well defined as a function in $L^2(\mathbb{R})$.

For $|x| > 1$, the differential operator obtained from this limiting procedure has infinite coefficients, and no equation for u is obtained. Instead, recall that for any neighborhood U of J, L^{-1} maps $H^0(M)$ boundedly to $H^1(M\backslash U)$. Now the space H^1 scales differently from H^0. From this it can be deduced that $u_\varepsilon \to 0$ in L^2 norm in $M\backslash U$. Coupling this with the uniform bound on $\partial_x u_\varepsilon$ in L^2, it follows that $u_\varepsilon(\pm 1, t) \to 0$ in $L^2(\mathbb{R})$. Therefore u satisfies the Dirichlet boundary condition. □

Paradoxically, then, the Dirichlet boundary condition arises from the failure for $|x| > 1$ of the same scaling procedure that gives rise to the differential operator $\mathfrak{L}$ for $|x| < 1$. Global singularities arise from the interaction between the degenerate region J and the nondegenerate region $|x| > 1$ that borders it.

This analysis is objectionable on several grounds. First, it is indirect. Second, it yields little information concerning the nature of singularities, despite strong heuristic indications that for $|x| < 1$ and $t > 0$, singular solutions behave like $g(x)t^{s_j - \frac{1}{2}}$ modulo higher powers of t. Third, it relies on the ellipticity of $\mathfrak{L}$ with respect to $t\partial_t$ in order to absorb terms that are $O(t^2\partial_t)$. No such ellipticity is present in analogues on three dimensional CR manifolds, such as the boundary of the worm domain.

In § 5 we pointed out another paradox: the regularity theorem of Boas and Straube guarantees global regularity whenever the CR structure near a Riemann surface R embedded in $\partial\Omega$ is sufficiently degenerate. It is interesting to reexamine this paradox from the point of view of the preceding analysis. Consider the Dirichlet problem on $[-1,1]\times\mathbb{R}$ for the operator

$$\mathfrak{L} = -\partial_x^2 - \alpha^2(x)(t^m\partial_t)^2 + \beta(x).$$

The case $m = 1$ has already been analyzed; exponents $m > 1$ give rise to more degenerate situations. When $m > 1$, separation of variables leads to solutions

$$f_\lambda(x,t) = g_\lambda(x)e^{-\lambda t^{1-m}}\,\chi_{t>0}$$

[25]The Dirichlet boundary condition was not discussed in [Barrett 1992]. Instead, the limiting operator was identified as a Bergman projection by examining its actions on the space of square integrable holomorphic functions and on its orthocomplement.

where λ is a nonlinear eigenvalue parameter, χ is the characteristic function of $\mathbb{R}^+$, and g_λ satisfies the ordinary differential equation

$$-g'' - \alpha^2(x)(m-1)^2\lambda^2 g + \beta(x)g = 0$$

on $[-1,1]$ with boundary conditions $g(\pm 1) = 0$. When $\lambda > 0$, these solutions are C^∞ at $t = 0$. The larger m becomes, the more rapidly f vanishes at $t = 0$, and hence the milder is its singularity (in the sense of Gevrey classes, for instance).

9. More on Worm Domains

We next explain how analysis of the $\bar\partial$–Neumann problem on worm domains may be reduced to a variant of the two dimensional model discussed in the preceding section. Assume $\Omega \Subset \mathbb{C}^2$ to have smooth defining function ρ.

The $\bar\partial$–Neumann problem is a boundary value problem for an elliptic partial differential equation, and as such is amenable to treatment by the method of reduction to a pseudodifferential equation on the boundary.[26] This reduction is achieved by solving instead the elliptic boundary value problem

$$\begin{cases} \square u = f & \text{on } \Omega, \\ \quad u = v & \text{on } \partial\Omega, \end{cases} \tag{9–1}$$

where v is a section of a certain complex line bundle $\mathcal{B}^{0,1}$ on $\partial\Omega$. The section v depends on f and is to be chosen so that the unique solution u satisfies the $\bar\partial$–Neumann boundary conditions; The problem (9–1) is explicitly solvable via pseudodifferential operator calculus, modulo a smoothing term, and there is a precise connection between the regularity of the solution and of the data.

The section v has in principle two components, but the first $\bar\partial$–Neumann boundary condition says that one component vanishes identically. The second boundary condition may be expressed as an equation $\square^+ v = g$ on $\partial\Omega$, where $\square^+$ is a certain pseudodifferential operator of order 1, and $g = (\bar\partial G f \lrcorner \bar\partial\rho)$ restricted to $\partial\Omega$, where Gf is the unique solution of the elliptic boundary value problem $\square(Gf) = f$ on Ω and $Gf \equiv 0$ on $\partial\Omega$.

On $\partial\Omega$ a Cauchy–Riemann operator is the complex vector field

$$\bar\partial_b = (\partial_{\bar z_1}\rho)\partial_{\bar z_2} - (\partial_{\bar z_2}\rho)\partial_{\bar z_1}.$$

Define $\bar L = \bar\partial_b$, $L = \bar\partial_b^*$. The principal symbol of $\square^+$ vanishes only on a line bundle Σ^+ that is one half of the characteristic variety defined by the vanishing of the principal symbol of $\bar\partial_b$. After composing $\square^+$ with an elliptic pseudodifferential operator of order $+1$, $\square^+$ takes the form

$$\mathcal{L} = \bar L L + B_1 \bar L + B_2 L + B_3 \tag{9–2}$$

microlocally in a conic neighborhood of Σ^+, where each B_j is a pseudodifferential operator of order less than or equal to 0. For each $s > 0$, if $t = s - 1/2$ then

[26] A detailed presentation is in [Chang et al. 1992].

the Neumann operator preserves $H^s(\Omega)$ if and only if whenever $v \in H^{-1/2}(\partial\Omega)$ and $\mathcal{L}v \in H^t(\partial\Omega)$, necessarily $v \in H^t(\partial\Omega)$. Since $\Box^+$ is elliptic on the complement of Σ^+, all the analysis may henceforth be microlocalized to a small conic neighborhood of Σ^+.

For worm domains, the circle group acts as a group of automorphisms by $z \mapsto R_\theta z = (z_1, e^{i\theta} z_2)$, inducing corresponding actions on functions and forms. The Hilbert space of square integrable $(0,k)$-forms decomposes as the orthogonal direct sum $\bigoplus_{j\in\mathbf{Z}} \mathcal{H}_j^k$ where $\mathcal{H}_j^k$ is the set of all $(0,k)$-forms f satisfying $R_\theta f \equiv e^{ij\theta} f$. The Bergman projection and Neumann operator preserve $\mathcal{H}_j^0$ and $\mathcal{H}_j^1$, respectively.

Proposition 9.1. *For each worm domain there exists a discrete subset $S \subset \mathbb{R}^+$ such that for each $s \notin S$ and each $j \in \mathbf{Z}$ there exists $C_{s,j} < \infty$ such that for every $(0,1)$-form $u \in \mathcal{H}_j^1 \cap C^\infty(\overline{\mathcal{W}})$ such that $Nu \in C^\infty(\overline{\mathcal{W}})$,*

$$\|Nu\|_{H^s(\mathcal{W})} \le C_{s,j} \|u\|_{H^s(\mathcal{W})}.$$

We do not know whether $C_{s,j}$ may be taken to be independent of j. The proof does imply that it is bounded by $C_s(1+|j|)^N$, for some exponent N independent of s. Thus our *a priori* inequalities can be formulated for all $u \in C^\infty$, rather than for each $\mathcal{H}_j$, but in such a formulation the norm on the right hand side should be changed to H^{s+N}.[27]

The Hilbert space $L^2(\partial\mathcal{W})$ decomposes into an orthogonal direct sum of subspaces $\mathcal{H}_j$, consisting of functions automorphic of degree j with respect to the action of the rotation group S^1 in the variable z_2. $\mathcal{H}_j$ may be identified with $L^2(\partial\mathcal{W}/S^1)$. The operators $\mathcal{L}, \bar{L}, L, B_j$ in (9–2) may be constructed so as to commute with the action of S^1, hence to preserve each $\mathcal{H}_j$. Thus for each j, the action of $\mathcal{L}$ on $\mathcal{H}_j(\partial\mathcal{W})$ may be identified with the action of an operator $\mathcal{L}_j$ on $L^2(\partial\mathcal{W}/S^1)$.

The quotient $\partial\mathcal{W}/S^1$ is a two dimensional real manifold. Coordinatizing $\partial\mathcal{W}$ by (x,θ,t) in such a way that $z_2 = \exp(x+i\theta)$ and $z_1 = \exp(i2x)(e^{it}-1)$ where $\left|\log|z_2|^2\right| \le r$, $\mathcal{L}_j$ takes the form $\bar{L}L + B_1\bar{L} + B_2 L + B_3$ where $\bar{L}$ is a complex vector field which takes the form $\bar{L} = \partial_x + it\alpha(t)\partial_t$ where $|x| \le r/2$, $\alpha(0) \ne 0$, and each B_k is a classical pseudodifferential operator of order ≤ 0, which depends on the parameter j in a nonuniform manner.

Setting $J = \{(x,t) : |x| \le r/2 \text{ and } t = 0\}$, and writing $\bar{L} = X + iY$, the vector fields $X, Y, [X,Y]$ span the tangent space to $\partial\mathcal{W}/S^1$ at every point in the complement of J, and are tangent to J at each of its points. Thus the operator $\mathcal{L}_j$ on $\partial\mathcal{W}/S^1$ is quite similar to the two dimensional model discussed in § 7, with two added complications: There are pseudodifferential factors, and the reduction of the $\bar\partial$–Neumann problem to $\mathcal{L}$, and thence to $\mathcal{L}_j$, requires only a microlocal

[27]The extra N derivatives are tangent to the Riemann surface $R = \mathcal{A}$ in $\partial\mathcal{W}$ along $\mathcal{A}$, and hence are essentially invariant under scaling in the direction orthogonal to $\mathcal{A}$, just as was $t\partial_t$ in the discussion in § 7.

a priori estimate for $\mathcal{L}_j$ in a certain conic subset of phase space. The proof of Theorem 7.1 can be adapted to this situation.

The lower order terms $B_1\bar{L}, B_2L, B_3$ are not negligible in this analysis; indeed, they determine the values of the exceptional Sobolev exponents $s \in \Sigma$, but the analysis carries through for any such lower order terms. The set Σ turns out to be independent of j.

At the end of § 8 we remarked that the two dimensional analysis relies on a certain ellipticity absent in three dimensions. For the worm domain, the global rotation symmetry makes possible a reduction to two dimensions; the lack of ellipticity results in a lack of uniformity of estimates with respect to j, but has no effect on the analysis for fixed j.

10. Analytic Regularity

This section is a brief report on recent progress on analytic hypoellipticity and global analytic regularity not only for the $\bar{\partial}$–Neumann problem, but also for related operators such as sums of squares of vector fields, emphasizing the author's contributions. More information, including references, can be found in the expository articles [Christ 1995b; 1996c]. Throughout the discussion, all domains and all coefficients of operators are assumed to be C^ω.

It has been known since about 1978, through the fundamental work of Tartakoff [1978; 1980] and Treves [1978], that the $\bar{\partial}$–Neumann problem is analytic hypoelliptic (that is, the solution is real analytic up to the boundary wherever the datum is) for all strictly pseudoconvex domains. Other results and methods in this direction have subsequently been introduced by Geller, Métivier and Sjöstrand.

On the other hand, Baouendi and Goulaouic discovered that

$$\partial_x^2 + \partial_y^2 + x^2\partial_t^2$$

is not analytic hypoelliptic, despite satisfying the bracket hypothesis. Métivier generalized this by showing that for sums of squares of d linearly independent real vector fields in $\mathbb{R}^{d+1}$, analytic hypoellipticity fails to hold if an associated quadratic form, analogous to the Levi form, is degenerate at every point of an open set. Nondegeneracy of this form is equivalent to the characteristic variety defined by the vanishing of the principal symbol being a symplectic submanifold of $T^*\mathbb{R}^{d+1}$.

There remained the intermediate case, which arises in the study of the $\bar{\partial}$–Neumann problem for bounded, pseudoconvex, real analytic domains in $\mathbb{C}^n$. Subsequent investigations have fallen into three categories.

(i) Analytic hypoellipticity has been proved in certain weakly pseudoconvex and nonsymplectic cases, by extending the methods known for the strictly pseudoconvex and symplectic case. Much work in this direction has been done, in particular, by Derridj and Tartakoff [1988; 1991; 1993; 1995]; perhaps the furthest advance is [Grigis and Sjöstrand 1985]. All this work has required that the degeneration from strict pseudoconvexity to weak pseudoconvexity have a very special algebraic form; the methods seem to be decidedly limited in scope.

(ii) Global C^ω regularity has been proved for certain very special domains and operators possessing global symmetries [Chen 1988; Christ 1994a; Derridj 1997; Cordaro and Himonas 1994].

(iii) Various counterexamples and negative results have been devised. Some of these will be described below. Despite progress, there still exist few theorems of much generality; one of those few is in [Christ 1994b].

At present a wide gap separates the positive results from the known negative results. However, through the development of these negative results it has become increasingly evident that analytic hypoellipticity, and even global regularity in C^ω, are valid only rarely in the weakly pseudoconvex/nonsymplectic setting. While analytic hypoellipticity remains an open question for most weakly pseudoconvex domains, we believe that it fails to hold in the vast majority of cases.[28] Thus any method for proving analytic hypoellipticity must necessarily be very limited in scope.

An interesting conjecture has recently been formulated by Treves [1999], concerning the relationship between analytic hypoellipticity of a sum of squares operator, and the symplectic geometry of certain strata of the characteristic variety defined by the vanishing of its principal symbol.

Another proposed connection between hypoellipticity, in the real analytic, Gevrey, and C^∞ categories, and symplectic geometry is explored in [Christ 1998].

10.1. Global Counterexamples. It had been hoped that in both the C^∞ and the C^ω categories, at least global regularity would hold in great generality.

THEOREM 10.1. *There exist a bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^2$ with C^ω boundary and a function $f \in C^\omega(\partial\Omega)$, whose Szegő projection does not belong to $C^\omega(\partial\Omega)$.*

The analysis [Christ 1996d] is related in certain broad aspects to the proof of global C^∞ irregularity for worm domains. Symmetry permits a reduction in dimension; more sophisticated analysis permits a reduction to one real dimension modulo certain error terms; existence of nonlinear eigenvalues for certain associated operators is at the core of the analysis; a deformation is introduced to evade the nonlinear eigenvalues; *a priori* estimates are proved for certain deformations; coupling these with singularities at the nonlinear eigenvalue parameters leads to a contradiction.

[28]This is another context in which second order equations are less well behaved than are those of first order. For operators of principal type, such as $\bar\partial_b$, there is a very satisfactory theory, and many such operators are analytic hypoelliptic, microlocally in appropriate regions.

This example has been refined by Tolli [1998]: there exists a convex domain having the same property, which is weakly pseudoconvex at only a single boundary point.

10.2. Victory in $\mathbb{R}^2$. For a relatively simple test class of operators with no artificial symmetry assumptions, analytic hypoellipticity has essentially been characterized. Consider any two real, C^ω vector fields X, Y, satisfying the bracket condition in an open subset of $\mathbb{R}^2$.

THEOREM 10.2 [Christ 1995a]. *For generic*[29] *pairs of vector fields*, $L = X^2 + Y^2$ *is analytic hypoelliptic at a point* $p \in \mathbb{R}^2$ *if and only if there exist an exponent* $m \ge 1$ *and coordinates with origin at* p *in which*

$$\operatorname{span}\{X, Y\} = \operatorname{span}\{\partial_x, x^{m-1}\partial_t\}. \tag{10–1}$$

Equality of these spans is to be understood in the sense of C^ω modules, not pointwise.

Sufficiency of the condition stated was proved long ago by Grušin; what is new is the necessity. The principal corollary is that analytic hypoellipticity holds quite rarely indeed. We believe that the same happens in higher dimensions and for other operators.

The main step is to show that L is analytic hypoelliptic if and only if a certain nonlinear eigenvalue problem has no solution. This problem takes the following form. To L is associated a one parameter family of ordinary differential operators $\mathcal{L}_z = -\partial_x^2 + Q(x, z)^2$, with parameter $z \in \mathbb{C}^1$, where Q is a homogeneous polynomial in $(x, z) \in \mathbb{R} \times \mathbb{C}$ that is monic with respect to x, and has degree $m - 1$ where m is the "type" at p; that is, the bracket hypothesis holds to order exactly m at p. The polynomial Q, modulo a simple equivalence relation, and a numerical quantity $q \in \mathbb{Q}^+$ used to define it, are apparently new geometric invariants of a pair of vector fields, satisfying the bracket condition, in $\mathbb{R}^2$. These invariants are not defined in terms of Lie brackets; q is related to a sort of directed order of contact at p between different branches of the complexified variety in $\mathbb{C}^2$ defined by the vanishing of the determinant of X, Y. The analytic hypoelliptic case arises precisely when this variety is nonsingular at p, that is, has only one branch. The pair $\{X, Y\}$ satisfies (10–1) if and only if $Q(x, z) \equiv x^{m-1}$ (modulo the equivalence relation).

A parameter z is said to be a nonlinear eigenvalue if $\mathcal{L}_z$ has nonzero nullspace in $L^2(\mathbb{R})$.

THEOREM 10.3 [Christ 1996a]. *If there exists at least one nonlinear eigenvalue for* $\{\mathcal{L}_z\}$, *then* L *fails to be analytic hypoelliptic in any neighborhood of* p.

[29] The meaning of "generic" will not be explained here; the set of all nongeneric pairs has been proved to be small, and may conceivably be empty. There is a corresponding microlocal theorem for $(X + iY) \circ (X - iY)$, a model for $\bar\partial_b^*\bar\partial_b$, under a pseudoconvexity hypothesis, in which no assumption of genericity is needed.

For generic[30] *polynomials* Q, *there exist infinitely many nonlinear eigenvalues.*

The restriction to $\mathbb{R}^2$ is essential to the analysis. However, the restriction to two vector fields is inessential and has been made only for the sake of simplicity.

For operators $(X + iY) \circ (X - iY)$ under a suitable "pseudoconvexity" hypothesis, there is an analogous but complete theory [Christ 1996a]: analytic hypoellipticity microlocally in the appropriate conic subset of $T^*\mathbb{R}^2$, the geometric condition (10–1), and nonexistence of nonlinear eigenvalues for the associated family of ordinary differential operators are all equivalent. Moreover, nonlinear eigenvalues fail to exist if and only if Q is equivalent to x^{m-1}.

For analyses of two classes of nonlinear eigenvalue problems for ordinary differential operators see [Christ 1993; 1996a].

10.3. Gevrey Hypoellipticity. Consider any sum of squares operator L in any dimension. Assume that the bracket hypothesis holds to order exactly m at a point p. Then, by [Derridj and Zuily 1973], L is hypoelliptic in the Gevrey class G^s for all $s \ge m$. Until about 1994, for every example known to this author, either L was analytic hypoelliptic, or it was Gevrey hypoelliptic for no $s < m$. The proof of Theorem 10.2 led to detailed information on Gevrey regularity, and in particular to the discovery of a whole range of intermediate behavior.

A simplified analysis applies to the following examples. They are of limited interest in themselves, but serve to demonstrate the intricacy of the Gevrey theory, and the fact that subtler geometric invariants than m come into play. Let $1 \le p \le q \in \mathbb{N}$, let (x, t) be coordinates in $\mathbb{R} \times \mathbb{R}^2$, and define

$$L = \partial_x^2 + x^{2(p-1)}\partial_{t_1}^2 + x^{2(q-1)}\partial_{t_2}^2.$$

Through work of Grušin, Oleĭnik and Radkevič, these are known to be analytic hypoelliptic if and only if $p = q$. The bracket condition is satisfied to order $m = q$ at 0.

THEOREM 10.4 [Christ 1997b]. *L is G^s hypoelliptic in some neighborhood of* 0 *if and only if $s \ge q/p$.*

This result has been reproved from another point of view by Bove and Tartakoff [1997], who obtained a still more refined result in terms of certain nonisotropic Gevrey classes.

An example in the opposite direction has been developed by Yu [1998]. In $\mathbb{R}^5$ with coordinates $(x, y, t) \in \mathbb{R}^{2+2+1}$ consider the examples

$$L_m = \partial_{x_1}^2 + (\partial_{y_1} + x_1^{m-1}\partial_t)^2 + \partial_{x_2}^2 + (\partial_{y_2} + x_2\partial_t)^2\ .$$

[30]The set of nongeneric polynomials has Hausdorff codimension at least two, in a natural parameter space. We do not know whether it is empty; this question is analogous to one raised by Barrett [1995].

L_m is analytic hypoelliptic when $m = 2$. For $m > 2$ it is Gevrey hypoelliptic of all orders $s \geq 2$ [Derridj and Zuily 1973]. Clearly it becomes more degenerate as m increases; brackets of length m in ∂_{x_1} and $\partial_{y_1} + x_1^{m-1}\partial_t$ are required to span the direction ∂_t. What is less clear is that increasing degeneracy should have no effect on Gevrey hypoellipticity.

THEOREM 10.5 [Yu 1998]. *For any even $m \geq 4$, L_m fails to be analytic hypoelliptic. More precisely, L_m is G^s hypoelliptic only if $s \geq 2$.*

The proof relies on the asymptotic behavior of nonlinear eigenvalues ζ_j as $j \to \infty$, not merely on the existence of one eigenvalue. It is quite a bit more intricate than the treatment of examples like $\partial_{x_1}^2 + (\partial_{y_1} + x_1^{m-1}\partial_t)^2$ in $\mathbb{R}^3$.

10.4. Speculation.

PREDICTION. In nonsymplectic and weakly pseudoconvex situations, analytic hypoellipticity holds very rarely, and only for special types of degeneracies. The algebraic structure of a degeneracy is decisive.

One instance in which this deliberately vague principle can be made precise is the theory for operators $X^2 + Y^2$ in $\mathbb{R}^2$. According to Theorem 10.2, for generic vector fields, analytic hypoellipticity holds at $p \in \mathbb{R}^2$ if and only if the complex variety $W \subset \mathbb{C}^2$ defined by the vanishing of $\det(X, Y)$ has a single branch at p.

For operators $X^2 + Y^2$ in $\mathbb{R}^3$, and for the $\bar\partial$–Neumann problem for weakly pseudoconvex, real analytic domains in $\mathbb{C}^2$, we believe that the following examples are the key to understanding what condition might characterize analytic hypoellipticity. With coordinates $(x, y, t) \in \mathbb{R}^3$ consider vector fields $X = \partial_x$, $Y = \partial_y + a(x, y)\partial_t$, which correspond to so called "rigid" CR structures. The fundamental invariant is the Levi form $\lambda(x, y, t) = \lambda(x, y) = \partial a(x, y)/\partial x$.

Let $(x, y, t; \xi, \eta, \tau)$ be coordinates in $T^*\mathbb{R}^3$. Consider examples

$$\lambda_1(x, y) = x^{2p} + y^{2p}$$
$$\lambda_2(x, y) = x^p y^p + x^{2q} + y^{2q}$$

where $0 < p < q$ and p is even. In each case, the variety in $T^*\mathbb{R}^3$ defined by the vanishing of the principal symbols of X, Y and $[X, Y]$ is the symplectic submanifold $V = \{\xi = \eta = x = y = 0\}$. The Poisson stratifications conjectured by Treves [1999] to govern analytic hypoellipticity do not distinguish between λ_1 and λ_2. Operators with $\partial a/\partial x = \lambda_1$ are known to be analytic hypoelliptic [Grigis and Sjöstrand 1985]. There is an algebraic obstruction to the application of existing methods to Levi forms λ_2, and analytic hypoellipticity remains an open question in this case.

QUESTION 10.1. Are operators $X^2 + Y^2$ in $\mathbb{R}^3$ with Levi forms $[X, Y] = \lambda_2(x, y)\partial_t$ analytic hypoelliptic?

Further remarks explaining the difference between λ_2 and λ_1 can be found in [Christ 1998].

11. Questions

We conclude with speculations and possible directions for further investigation. Many of the questions posed here have been raised by earlier authors and are of long standing. Throughout the discussion we assume that $\Omega \Subset \mathbb{C}^n$ is smoothly bounded and pseudoconvex. Denote by Λ^δ and G^s the usual Hölder and Gevrey classes, respectively.

Question 11.1. Let Ω be a domain of finite type in $\mathbb{C}^n$, $n \geq 3$. Does the Neumann operator map L^∞ to $\Lambda^\delta(\overline{\Omega})$ for some $\delta > 0$?

Convex domains behave better than general pseudoconvex domains, in several respects: (i) Global C^∞ regularity always holds.[31] (ii) The Bergman and Szegő projections and associated kernels for smoothly bounded convex domains of finite type are reasonably well understood in the C^∞, L^p Sobolev and Hölder categories, through work of McNeal [1994] and McNeal and Stein [1994], whereas much less is known for general pseudoconvex domains of finite type in $\mathbb{C}^n$, $n > 2$. (iii) For any convex domain in $\mathbb{C}^2$, the equation $\bar\partial u = f$ has an L^p solution for any L^p datum, for all $1 < p < \infty$ [Polking 1991].

Question 11.2. Is the equation $\bar\partial u = f$ solvable in L^p and Hölder classes, for all smoothly bounded convex domains in $\mathbb{C}^n$, for all n?

A basic example of a nonconvex, pseudoconvex domain of finite type is the cross of iron in $\mathbb{C}^3$:

$$\Omega^\dagger : y_0 > |z_1|^6 + |z_1 z_2|^2 + |z_2|^6 ,$$

where $z_j = x_j + iy_j$. Separation of variables leads to formulae for the Bergman and Szegő kernels, analogous to but more complicated than the formula of Nagel [1986] for certain domains in $\mathbb{C}^2$. So far as this author is aware, all questions beyond the existence of subelliptic estimates are open, including pointwise bounds for the Szegő and Bergman kernels, L^p and Hölder class mapping properties, analyticity, and analytic pseudolocality.[32] It might be possible to extract some information from the kernel formulae.

For further information concerning Hölder, supremum and L^p norm estimates, see [Sibony 1980/81; 1993; Fornæss and Sibony 1991; 1993]. A survey concerning weakly pseudoconvex domains is [Sibony 1991].

Problem 11.3. Analyze $\Omega^\dagger$.

Work of Morimoto [1987b] and of Bell and Mohammed [1995] suggests the following conjecture concerning hypoellipticity (in C^∞) for domains of infinite type. Denote by $\lambda(z)$ the smallest eigenvalue of the Levi form at a point $z \in \partial\Omega$, and by W_∞ the set of all boundary points at which Ω is not of finite type.

[31] On the other hand, Tolli [1998] has proved that for a certain convex real analytic domain in $\mathbb{C}^2$ having only a single weakly pseudoconvex boundary point, the Szegő projection fails to preserve the class of functions globally real analytic on the boundary.

[32] I am indebted to J. McNeal for useful conversations concerning $\Omega^\dagger$.

CONJECTURE 11.4. Suppose that W_∞ is contained in a smooth real hypersurface M of $\partial\Omega$. Suppose that there exist $c > 0$ and $0 < \delta < 1$ such that for all $z \in \partial\Omega$,

$$\lambda(z) \geq c \exp(-\mathrm{distance}(z, M)^{-\delta}).$$

Then the $\bar{\partial}$–Neumann problem for Ω is hypoelliptic.

Conversely, there exist domains for which $\lambda(z) \geq c \exp(-C\, \mathrm{distance}(z, M))$, yet the $\bar{\partial}$–Neumann problem is not hypoelliptic.

The hypothesis that W_∞ is contained in a smooth hypersurface is unnatural; if this conjecture can be proved then a further generalization more in the spirit of Kohn's work [1979] on subellipticity and finite ideal type should be sought.

Now that global C^∞ regularity is known not to hold in general, it is natural to seek sufficient conditions. Compactness of the $\bar{\partial}$–Neumann problem is a more robust property that may prove more amenable to a satisfactory analysis. It is a purely local property; Diophantine inequalities and related pathology should not intervene in discussions of compactness.

PROBLEM 11.5. Characterize compactness of the Neumann operator N for pseudoconvex domains in $\mathbb{C}^2$.

At the least, this should be feasible for restricted classes of domains. Compactness is equivalent to the absence of complex discs in the boundary for Reinhardt domains, and presence of complex discs precludes compactness for arbitrary domains at least in $\mathbb{C}^2$, but the equivalence breaks down for Hartogs domains [Matheos 1998]. This problem and the next question appear to be related to the existence of nowhere dense compact subsets of $\mathbb{C}^1$ with positive logarithmic capacity.

A satisfactory characterization of global C^∞ regularity appears not to be a reasonable goal, but at least two natural questions beckon.

QUESTION 11.6. Does there exist a smoothly bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^2$ whose boundary contains no analytic discs, yet the $\bar{\partial}$–Neumann problem for Ω is not globally regular in C^∞?

QUESTION 11.7. For the $\bar{\partial}$–Neumann problem on smoothly bounded pseudoconvex domains, does global C^∞ regularity always imply exact regularity in H^s for all s?

I suspect the answer to be negative. Barrett [1995; 1998] has studied exact regularity for domains in $\mathbb{C}^2$ for which W_∞ is a smoothly bounded Riemann surface, and shown that (i) exact regularity is violated whenever a certain nonlinear eigenvalue problem on the Riemann surface has a positive solution, and (ii) for generic Riemann surfaces, the nonlinear eigenvalue problem has indeed a positive solution. If there exist exceptional Riemann surfaces without nonlinear eigenvalues, some of those would be candidates for examples of global regularity without exact regularity. Whether such domains exist remains an open question.

It is also conceivable that the very instability of estimates with loss of derivatives could be exploited to show that for some one parameter family of domains Ω_t, global regularity holds for generic t even though exact regularity does not, in the same way that the Diophantine condition $|k_1 + \alpha k_2| \ge c|k|^{-N}$ holds for generic α, without establishing global regularity for any particular value of t.[33]

QUESTION 11.8. Does global regularity fail to hold for every domain in $\mathbb{C}^2$ for which W_∞ is a smoothly bounded Riemann surface R, satisfying $w(R, \partial\Omega) \neq 0$?

There is some interesting intuition for global C^∞ irregularity for the operators described in § 7, based on the connection between degenerate elliptic second order operators with real coefficients and stochastic processes. For information on this connection see [Bell 1995]. This intuition, together with conditional expectation arguments applied to random paths, predicts global irregularity for the models discussed in § 7, and explains in more geometric terms the seemingly paradoxical regularity for the more degenerate cases $m > 1$ discussed at the end of that section.

PROBLEM 11.9. Understand global C^∞ irregularity, for second order degenerate elliptic operators with real coefficients, from the point of view of Malliavin calculus and related stochastic techniques.

The next two problems and next question concern gaps in our understanding of global C^∞ irregularity for worm domains, and are of lesser importance.

PROBLEM 11.10. Prove global C^∞ irregularity for worm domains by working directly on the domain, rather than by reducing to the boundary.

PROBLEM 11.11. Generalize the analysis of the worm domains to higher dimensional analogues.

QUESTION 11.12. For worm domains, for Sobolev exponents s not belonging to the discrete exceptional set, is there an *a priori* estimate for the Neumann operator in H^s, with no loss of derivatives?

This amounts to asking whether bounds are uniform in the parameter j.

Much of the interest in global regularity for the $\bar\partial$–Neumann problem stems from a theorem of Bell and Ligocka [1980]: If $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ are bounded, pseudoconvex domains with C^∞ boundaries, if $f : \Omega_1 \mapsto \Omega_2$ is a biholomorphism, and if the Bergman projection for each domain preserves $C^\infty(\bar\Omega_j)$, then f extends to a C^∞ diffeomorphism of their closures. For worm domains, the Bergman projection fails to preserve smoothness up to the boundary; this property is equivalent to global regularity for the $\bar\partial$–Neumann problem [Boas and Straube 1990]. But the proof leads to no counterexample for the mapping problem. Chen [1993] has shown that every automorphism of any worm domain is a rotation $(z_1, z_2) \mapsto (z_1, e^{i\theta} z_2)$, and hence certainly extends smoothly to the boundary.

[33] For a slightly related problem in which estimates with loss of derivatives have been established by exploiting such instability see [Christ and Karadzhov $\ge$ 1999; Christ et al. 1996].

QUESTION 11.13. Does every biholomorphic mapping between two smoothly bounded, pseudoconvex domains extend to a diffeomorphism of their closures?

Our next set of questions concerns the real analytic theory.

QUESTION 11.14. Suppose that $\Omega \Subset \mathbb{C}^2$ is pseudoconvex and has a real analytic boundary. For which Ω is the $\bar\partial$–Neumann problem analytic hypoelliptic? For which is it globally regular in C^ω?

Relatively recent examples [Christ 1997a; 1997b; Bove and Tartakoff 1997; Yu 1998] have demonstrated that for analytic nonhypoelliptic operators, determination of the optimal exponent for Gevrey hypoellipticity is a subtle matter. It is not at all apparent how geometric properties of the domain determine this exponent.[34]

QUESTION 11.15. If Ω has a real analytic boundary but the $\bar\partial$–Neumann problem is not analytic hypoelliptic, for which exponents s is it hypoelliptic in the Gevrey class G^s? For which exponents is it globally regular in G^s?

There appear to exist wormlike domains whose defining functions belong to every Gevrey class G^s with $s > 1$, and for which the higher invariants w_k are nonzero. Both the examples discussed at the end of § 7 and formal analysis of commutators suggest that for $m > 1$, nonvanishing of w_m may be related to global irregularity in Gevrey classes G^s for $s < m/(m-1)$.

QUESTION 11.16. Do the higher invariants w_k introduced in § 5 play a role in the theory of global Gevrey class hypoellipticity?

Another fundamental issue pertaining to singularities is their propagation. Consider the operator $\bar\partial_b^* \circ \bar\partial_b$, on the boundary of any real analytic, pseudoconvex domain $\Omega \Subset \mathbb{C}^2$. Suppose there exists a smooth, nonconstant curve $\gamma \subset \partial\Omega$ whose tangent vector lies everywhere in the span of the real and imaginary parts of $\bar\partial_b$, and which is contained in the set of all weakly pseudoconvex points of $\partial\Omega$.[35] Consider only functions u whose analytic wave front sets are contained in the subset of phase space in which $\bar\partial_b^*\bar\partial_b$ is C^∞ hypoelliptic.

QUESTION 11.17. If γ intersects the analytic singular support of u, must γ be contained in its analytic singular support?

The same may be asked for operators $X^2 + Y^2$, where X, Y are equal to, or analogous to, the real and imaginary parts of $\bar\partial_b$. For these, the question has been answered affirmatively by Grigis and Sjöstrand [1985] in the special case where the type is 3 at every "weakly pseudoconvex" point.

The $\bar\partial$–Neumann problem is a method for reducing the overdetermined first order system $\bar\partial u = f$ to a determined second order equation, analogous to Hodge

[34] A conjecture in this direction has been formulated by Bove and Tartakoff [1997].

[35] It has been shown [Christ 1994b] that whenever such a curve exists, $\bar\partial_b^*\bar\partial_b$ fails to be analytic hypoelliptic, microlocally in the region of phase space where it is C^∞ hypoelliptic.

theory. At the root of the counterexamples discovered in the last few years for global C^∞ regularity, for analytic hypoellipticity, and for global C^ω regularity are certain nonlinear eigenvalue problems that are associated to second order equations, but seem to have no counterparts for first order equations. Other methods for solving $\bar\partial u = f$ do exist, including solution by integral operators, and generalization of the $\bar\partial$–Neumann method to a twisted $\bar\partial$ complex [Ohsawa and Takegoshi 1987; McNeal 1996; Siu 1996]. Another method, which solves the $\bar\partial$ and $\bar\partial_b$ equations globally in the C^ω category, has been described by Christ and Li [1997].

QUESTION 11.18. Do these counterexamples represent limitations inherent in the nature of second order equations, or can the method of reduction of the $\bar\partial$ system to a determined second order equation be modified so as to avoid them? Do they have analogues for the $\bar\partial$ system itself?

Addenda. After this paper was written the first part of Conjecture 11.4 was proved for $\mathbb{C}^2$ by the author.

(Added in proof.) Since this paper was written, the author has learned of additional references concerning hypoellipticity of infinite type sums of squares of vector fields. They include [Kajitani and Wakabayashi 1991; Morimoto 1987a; Morimoto and Morioka 1997]. Further speculation may be found in [Christ 1998].

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