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# Pseudoconvex-Concave Duality and Regularization of Currents

JEAN-PIERRE DEMAILLY

To the memory of Michael Schneider

Abstract. We investigate some basic properties of Finsler metrics on holomorphic vector bundles, in the perspective of obtaining geometric versions of the Serre duality theorem. We establish a duality framework under which pseudoconvexity and pseudoconcavity properties get exchanged — up to some technical restrictions. These duality properties are shown to be related to several geometric problems, such as the conjecture of Hartshorne and Schneider, asserting that the complement of a $q$-codimensional algebraic subvariety with ample normal bundle is $q$-convex. In full generality, a functorial construction of Finsler metrics on symmetric powers of a Finslerian vector bundle is obtained. The construction preserves positivity of curvature, as expected from the fact that tensor products of ample vector bundles are ample. From this, a new shorter and more geometric proof of a basic regularization theorem for closed $(1,1)$ currents is derived. The technique is based on the construction of a mollifier operator for plurisubharmonic functions, depending on the choice of a Finsler metric on the cotangent bundle and its symmetric powers.

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## Introduction

The goal of the present paper is to investigate some duality properties connecting pseudoconvexity and pseudoconcavity. Our ultimate perspective would be a geometric duality theory parallel to Serre duality, in the sense that Serre duality would be the underlying cohomological theory. Although similar ideas have already been used by several authors in various contexts — for example, for the study of direct images of sheaves [Ramis et al. 1971], or in connection with the study of Fantappie transforms and lineal convexity [Kiselman 1997], or in the study of Monge–Ampère equations [Lempert 1985] — we feel that the "convex-concave" duality theory still suffers from a severe lack of understanding.

Our main concern is about *Finsler metrics* on holomorphic vector bundles. As is well known, a holomorphic vector bundle $E$ on a compact complex manifold is ample in the sense of [Hartshorne 1966] if and only if its dual $E^\star$ admits a strictly pseudoconvex tubular neighborhood of 0, that is, if and only if $E^\star$ has a strictly plurisubharmonic smooth Finsler metric. In that case, we expect $E$ itself to have a tubular neighborhood of the zero section such that the Levi form of the boundary has everywhere signature $(r{-}1, n)$, where $r$ is the rank of $E$ and $n = \dim X$; in other words, $E$ has a Finsler metric whose Levi form has signature $(r, n)$. This is indeed the case if $E$ is positive in the sense of Griffiths, that is, if the above plurisuharmonic Finsler metric on $E^\star$ can be chosen to be *hermitian*; more generally, Sommese [1978; 1979; 1982] has observed that everything works well if the Finsler metric is fiberwise convex (in the ordinary sense). The Kodaira–Serre vanishing theorem tells us that strict pseudoconvexity of $E^\star$ implies that the cohomology of high symmetric powers $S^mE$ is concentrated in degree 0, while the Andreotti–Grauert vanishing theorem tells us that $(r, n)$ convexity-concavity of $E$ implies that the cohomology of $S^mE^\star$ is concentrated in degree $n$. Of course, both properties are connected from a cohomological view point by the Serre duality theorem, but the related geometric picture seems to be far more involved. A still deeper unsolved question is Griffiths' conjecture [1969] on the equivalence of ampleness and positivity of curvature for hermitian metrics.

One of the difficulties is that "linear" duality between $E$ and $E^\star$ is not sufficient to produce the expected biduality properties relating convexity on one side and concavity on the other side. What seems to be needed rather, is a duality between large symmetric powers $S^mE$ and $S^mE^\star$, asymptotically as $m$ goes to infinity ("polynomial duality"). Although we have not been able to find a completely satisfactory framework for such a theory, one of our results is that there is a functorial and natural construction which assigns Finsler metrics on all symmetric powers $S^mE$, whenever a Finsler metric on $E$ is given. The assignment has the desired property that the Finsler metrics on $S^mE$ are plurisubharmonic if the Finsler metric on $E$ was. The construction uses "polynomial duality" in an essential way, although it does not produce good metrics on the dual bundles $S^mE^\star$.

Several interesting questions depend on the solution to these problems. Robin Hartshorne [1970] raised the question whether the complement of an algebraic subvariety $Y$ with ample normal bundle $N_Y$ in a projective algebraic variety $X$ is $q$-convex in the sense of Andreotti–Grauert, with $q = \operatorname{codim} Y$. Michael Schneider [1973] proved the result in the case the normal bundle is positive is the sense of Griffiths, thus yielding strong support for Hartshorne's conjecture. As a consequence of Sommese's observation, Schneider's result extends the case if $N_Y^\star$ has a strictly pseudoconvex and fiberwise convex neighborhood of the zero section, which is the case for instance if $N_Y$ is ample and globally generated.

Other related questions which we treat in detail are the approximation of closed positive $(1,1)$-currents and the attenuation of their singularities. In general, a closed positive current $T$ cannot be approximated (even in the weak topology) by smooth closed positive currents. A cohomological obstruction lies in the fact that $T$ may have negative intersection numbergs $\{T\}^p \cdot Y$ with some subvarieties $Y \subset X$. This is the case for instance if $T = [E]$ is the current of integration on a the exceptional curve of a blown-up surface and $Y = E$. However, as we showed in [Demailly 1982; 1992; 1994], the approximation is possible if we allow the regularization $T_\varepsilon$ to have a small negative part. The main point is to control this negative part accurately, in term of the global geometry of the ambient geometry $X$. It turns out that more or less optimal bounds can be described in terms of the convexity of a Finsler metric on the tangent bundle $T_X$. Again, a relatively easy proof can be obtained for the case of a hermitian metric [Demailly 1982; 1994], but the general Finsler case, as solved in [Demailly 1992], still required very tricky analytic techniques. We give here an easier and more natural method based on the use of "symmetric products" of Finsler metrics.

Many of the ideas presented here have matured over a long period of time, for a large part through discussion and joint research with Thomas Peternell and Michael Schneider. Especially, the earlier results [Demailly 1992] concerning smoothing of currents were strongly motivated by techniques needed in our joint work [Demailly et al. 1994]. I would like here to express my deep memory of Michael Schneider, and my gratitude for his very beneficial mathematical influence.

## 1. Pseudoconvex Finsler Metrics and Ample Vector Bundles

Let $X$ be a complex manifold and $E$ a holomorphic vector bundle over $X$. We set $n = \dim_{\mathbb{C}} X$ and $r = \operatorname{rank} E$.

Definition 1.1 [Kobayashi 1975]. A (positive definite) Finsler metric on $E$ is a positive complex homogeneous function $\xi \mapsto \|\xi\|_x$ defined on each fiber $E_x$, that is, such that $\|\lambda\xi\|_x = |\lambda|\|\xi\|_x$ for each $\lambda \in \mathbb{C}$ and $\xi \in E_x$, and $\|\xi\|_x > 0$ for $\xi \neq 0$.

We will in general assume some regularity, e.g. continuity of the function $(x,\xi)\mapsto \|\xi\|_x$ on the total space $E$ of the bundle. We say that the metric is smooth if it is smooth on $E \smallsetminus \{0\}$. The logarithmic indicatrix of the Finsler metric is by definition the function

$$\chi(x,\xi) = \log \|\xi\|_x. \tag{1.2}$$

We will say in addition that the Finsler metric is *convex* if the function $\xi \mapsto \|\xi\|_x$ is convex on each fiber $E_x$ (viewed as a real vector space). A Finsler metric is convex if and only if it derives from a norm (hermitian norms are of course of a special interest in this respect); however, we will have to deal as well with non convex Finsler metrics.

The interest in Finsler metrics essentially arises from the following well-known characterization of ample vector bundles [Kodaira 1954; Grauert 1958; Kobayashi 1975].

THEOREM 1.3. *Let $E$ be a vector bundle on a compact complex manifold $X$. The following properties are equivalent.*

(1) *$E$ is ample in the sense of* [Hartshorne 1966].
(2) *$\mathcal{O}_{P(E)}(1)$ is an ample line bundle on the projectivized bundle $P(E)$ (of hyperplanes of $E$).*
(3) *$\mathcal{O}_{P(E)}(1)$ carries a smooth hermitian metric of positive Chern curvature form.*
(4) *$E^\star$ carries a smooth Finsler metric which is strictly plurisubharmonic on the total space $E^\star \smallsetminus \{0\}$.*
(5) *$E^\star$ admits a smoothly bounded strictly pseudoconvex tubular neighborhood $U$ of the zero section.*

Actually, the equivalence of (1), (2) is a purely algebraic fact, while the equivalence of (2) and (3) is a consequence of the Kodaira embedding theorem. The equivalence of (3) and (4) just comes from the observation that a Finsler metric on $E^\star$ can be viewed also as a *hermitian metric* $h^\star$ on the line bundle $\mathcal{O}_{P(E)}(-1)$ (as the total space of $\mathcal{O}_{P(E)}(-1)$ coincides with the blow-up of $E^\star$ along the zero section), and from the obvious identity

$$(\pi_{P(E)})^\star \Theta_{h^\star}(\mathcal{O}_{P(E)}(-1)) = -\frac{i}{2\pi}\partial\bar\partial\chi^\star,$$

where $\Theta_{h^\star}(\mathcal{O}_{P(E)}(-1))$ denotes the Chern curvature form of $h^\star = e^{\chi^\star}$, and $\pi_{P(E)} : E^\star \smallsetminus \{0\} \to P(E)$ the canonical projection. Finally, if we have a Finsler metric as in (4), then $U_\varepsilon = \{\xi^\star : \|\xi^\star\|^\star < \varepsilon\}$ is a fundamental system of strictly pseudoconvex neighborhood of the zero section of $E^\star$. Conversely, given such a neighborhood $U$, we can make it complex homogeneous by replacing $U$ with $U^\star = \bigcap_{|\lambda|\ge 1} \lambda U$. Then $U^\star$ is the unit ball bundle of a continuous strictly plurisubharmonic Finsler metric on $E^\star$ (which can further be made smooth thanks to Richberg's regularization theorem [1968], or by the much more precise results of [Demailly 1992], which will be reproved in a simpler way in Section 9).

REMARK 1.4. It is unknown whether the ampleness of $E$ implies the existence of a *convex* strictly plurisubharmonic Finsler metric on $E^\star$. Sommese [1978] observed that this is the case if $E$ is ample and generated by sections. In fact, if there are sections $\sigma_j \in H^0(X,E)$ generating $E$, then

$$h_0(\xi^\star) = \Big(\sum_j |\sigma_j(x)\cdot\xi^\star|^2\Big)^{1/2}$$

defines a weakly plurisubharmonic and strictly convex (actually hermitian) metric on $E^\star$. On the other hand, the ampleness implies the existence of a strictly plurisubharmonic Finsler metric $h_1$, thus $(1-\varepsilon)h_0+\varepsilon h_1$ is *strictly plurisubharmonic* and *strictly convex* for $\varepsilon$ small enough. Griffiths conjectured that ampleness of $E$ might even be equivalent to the existence of a *hermitian metric* with positive curvature, thus to the existence of a hermitian strictly plurisubharmonic metric on $E^\star$. Not much is known about this conjecture, except that it holds true if $r=\operatorname{rank}E=1$ (Kodaira) and $n=\dim X=1$ ([Umemura 1980]; see also [Campana and Flenner 1990]). Our feeling is that the general case should depend on deep facts of gauge theory (some sort of vector bundle version of the Calabi–Yau theorem would be needed).

## 2. Linearly Dual Finsler Metrics

Given a Finsler metric $\|\ \|$ on a holomorphic vector bundle $E$, one gets a dual (or rather *linearly dual*) Finsler metric $\|\ \|^\star$ on $E^\star$ by putting

$$\|\xi^\star\|^\star_x = \sup_{\xi\in E_x\smallsetminus\{0\}} \frac{|\xi\cdot\xi^\star|}{\|\xi\|_x}, \qquad \xi^\star\in E^\star_x. \tag{2.1}$$

Equivalently, in terms of the logarithmic indicatrix, we have

$$\chi^\star(x,\xi^\star) = \sup_{\xi\in E_x\smallsetminus\{0\}} \log|\xi\cdot\xi^\star| - \chi(x,\xi), \qquad \xi^\star\in E^\star_x. \tag{2.2}$$

It is clear that the linearly dual metric $\|\ \|^\star$ is always convex, and therefore the biduality formula $\|\ \|^{\star\star}=\|\ \|$ holds true if and only if $\|\ \|$ is convex.

A basic observation made in [Sommese 1978] is that the pseudoconvexity of a Finsler metric is related to some sort of pseudoconcavity of the dual metric, provided that the given metric is fiberwise convex. We will reprove it briefly in order to prepare the reader to the general case (which requires *polynomial* duality, and not only *linear* duality). We first need a definition.

DEFINITION 2.3. Let $E$ be equipped with a smooth Finsler metric of logarithmic indicatrix $\chi(x,\xi)=\log\|\xi\|_x$. We say that $\|\ \|$ has transversal Levi signature $(r,n)$ (where $r=\operatorname{rank}E$ and $n=\dim X$) if, at every point $(x,\xi)\in E\smallsetminus\{0\}$, the

Levi form $i\partial\overline{\partial}(e^{\chi})$ is positive definite along the fiber $E_x$ and negative definite on some $n$-dimensional subspace $W \subset T_{E,(x,\xi)}$ which is transversal to the fiber $E_x$.

This property can also be described geometrically as follows.

PROPOSITION 2.4. *The Finsler metric* $\|\ \|$ *on* $E$ *has transversal Levi signature* $(r, n)$ *if and only if it is fiberwise strictly pseudoconvex, and through every point* $(x_0, \xi_0)$ *of the unit sphere bundle* $\|\xi\| = 1$ *passes a germ of complex* $n$*-dimensional submanifold* $M_0$ *which is entirely contained in the unit ball bundle* $\{\|\xi\| \le 1\}$ *and has (strict) contact order* $2$ *at* $(x_0, \xi_0)$.

SKETCH OF PROOF. If the geometric property (2.4) is satisfied, we simply take $W = T_{M_0,(x_0,\xi_0)}$. Conversely, if $i\partial\overline{\partial}e^{\chi}$ has signature $(r, n)$ as in 2.3, then $i\partial\overline{\partial}\chi$ has signature $(r-1, n)$ (with one zero eigenvalue in the radial direction, since $\chi$ is log homogeneous). The Levi form of the hypersurface $\chi = 0$ thus has signature $(r-1, n)$ as well, and we can take the negative eigenspace $W \subset T_{E,(x_0,\xi_0)}$ to be tangent to that hypersurface. The germ $M_0$ is then taken to be the graph of a germ of holomorphic section $\sigma : (X, x_0) \to E$ tangent to $W$, with the second order jet of $\sigma$ adjusted in such a way that $\chi(x, \sigma(x)) \le -\varepsilon|x - x_0|^2$ $\big($as $\partial\chi(x_0) \ne 0$, one can eliminate the holomorphic and antiholomorphic parts in the second order jet of $\chi(x, \sigma(x))\big)$. □

The main part, (a), of the following basic result is due to A. Sommese [1978].

THEOREM 2.5. *Let* $E$ *be equipped with a smooth Finsler metric of logarithmic indicatrix* $\chi(x, \xi) = \log\|\xi\|_x$. *Assume that the metric is (fiberwise) strictly convex.*

(a) *If the metric* $\|\ \|$ *is strictly plurisubharmonic on* $E \smallsetminus \{0\}$, *then the dual metric* $\|\ \|^\star = e^{\chi^\star}$ *has transversal Levi signature* $(r, n)$ *on* $E^\star \smallsetminus \{0\}$.

(b) *In the opposite direction, if* $\|\ \|$ *has transversal Levi signature* $(r, n)$, *then* $\|\ \|^\star$ *is strictly plurisubharmonic on* $E^\star \smallsetminus \{0\}$.

REMARK 2.6. Theorem 2.5 still holds under the following more general, but more technical hypothesis, in place of the strict convexity hypothesis:

(H) *For every point* $(x, [\xi^\star]) \in P(E_x)$, *the supremum*

$$\chi^\star(x, \xi^\star) = \sup_{\xi \in E_x \smallsetminus \{0\}} \log|\xi \cdot \xi^\star| - \chi(x, \xi), \qquad \xi^\star \in E_x^\star$$

*is reached on a unique line* $[\xi] = f(x, [\xi^\star]) \in P(E_x)$, *where* $[\xi]$ *is a non critical maximum point along* $P(E_x)$.

Notice that the supremum is always reached in at least one element $[\xi] \in P(E_x)$, just by compactness. The assumption that there is a unique such point $[\xi] = f(x, [\xi^\star])$ which is non critical ensures that $f$ is smooth by the implicit function theorem, hence $\chi^\star$ will be also smooth.

The uniqueness assumption is indeed satisfied if the Finsler metric of $E$ is *strictly convex.* Indeed, if the maximum is reached for two non colinear vectors

$\xi_0$, $\xi_1$ and if we adjust $\xi_0$ and $\xi_1$ by taking multiples such that $\xi_0 \cdot \xi^\star = \xi_1 \cdot \xi^\star = 1$, then again $\xi_t \cdot \xi^\star = 1$ for all $\xi_t = (1-t)\xi_0 + t\xi_1 \in ]\xi_0, \xi_1[$, while the strict convexity implies $\chi(x, \xi_t) < \chi(x, \xi_0) = \chi(x, \xi_1)$, contradiction. We see as well that the maximum must be a non critical point, and that the Finsler metric $\| \ \|^\star$ is strictly convex. Thus, in this case, $\| \ \|$ is strictly plurisubharmonic if and only if $\| \ \|^\star$ has transversal Levi signature $(r, n)$.

Remark 2.7. In Theorem 2.5, the extra convexity assumption — or its weaker counterpart (H) — is certainly needed. In fact, if the conclusions were true without any further assumption, the linear bidual of a continuous plurisubharmonic Finsler metric would still be plurisubharmonic (since we can approximate locally such metrics by smooth strictly plurisubharmonic ones). This would imply that the convex hull of a pseudoconvex circled tubular neighborhood is pseudoconvex. However, if we equip the trivial rank two vector bundle $\mathbb{C} \times \mathbb{C}^2$ over $\mathbb{C}$ with the plurisubharmonic Finsler metric

$$\|\xi\|_x = \max\big(|\xi_1|,\ |\xi_2|,\ |x|\sqrt{|\xi_1|\,|\xi_2|}\big),$$

a trivial computation shows that the convex hull is associated with the metric

$$\|\xi\|'_x = \max\left(|\xi_1|,\ |\xi_2|,\ \frac{|x|^2}{1+|x|^2}(|\xi_1| + |\xi_2|)\right)$$

which is not plurisubharmonic in $x$.

Proof of Theorem 2.5. (a) First observe that $\exp(\chi^\star) = \| \ \|^\star$ is convex, and even strictly convex since the assumptions are not affected by small smooth $C^\infty$ or $C^2$ perturbations on $\chi$. Thus $i\partial\bar\partial \exp(\chi^\star)$ has at least $r$ positive eigenvalues eigenvalues along the vertical directions of $E^\star \to X$.

Let $f : P(E^\star) \to P(E)$ be defined as in condition 2.5 (H), and let $\widetilde f : E \smallsetminus \{0\}^\star \to E \smallsetminus \{0\}$ be a lifting of $f$. One can get such a global lifting $\widetilde f$ by setting e.g. $\widetilde f(x, \xi^\star) \cdot \xi^\star = 1$, so that $\widetilde f$ is uniquely defined. By definition of $\chi^\star$ and $f$, we have

$$\chi^\star(x, \xi^\star) = \log\big|\widetilde f(x, \xi^\star) \cdot \xi^\star\big| - \chi(x, \widetilde f(x, \xi^\star))$$

in a neighborhood of $(x_0, \xi_0^\star)$. Fix a local trivialization $E_{|U} \simeq U \times V$ xhere $V \simeq \mathbb{C}^r$ and view $\widetilde f$ as a map $\widetilde f : E^\star_{|U} \simeq U \times V^\star \to V$ defined in a neighborhood of $(x_0, \xi_0^\star)$. As $\dim E = n + r$ and $\dim V = r$, the kernel of the $\bar\partial$-differential

$$\bar\partial \widetilde f_{(x_0,\xi_0^\star)} : T_{E^\star,(x_0,\xi_0^\star)} \to V$$

is a complex subspace $W_0 \subset T_{E^\star,(x_0,\xi_0^\star)}$ of dimension $p \geq n$. By definition of $W_0$, there is a germ of $p$-dimensional submanifold $M \subset E^\star$ with $T_{M,(x_0,\xi_0^\star)} = W_0$, and a germ of holomorphic function $g : M \to V$ such that

$$\widetilde f(x, \xi^\star) = g(x, \xi^\star) + O(|x - x_0|^2 + |\xi^\star - \xi_0^\star|^2) \quad \text{on } M.$$

This implies

$$\chi^\star(x, \xi^\star) = \log|g(x, \xi^\star) \cdot \xi^\star| - \chi(x, g(x, \xi^\star)) + O(|x - x_0|^3 + |\xi^\star - \xi_0^\star|^3) \quad \text{on } M.$$

In fact, since $\xi_0 = \widetilde{f}(x_0, \xi_0^\star)$ is a stationary point for $\xi \mapsto \log|\xi \cdot \xi^\star| - \chi(x, \xi)$, the partial derivative in $\xi$ is $O(|x - x_0| + |\xi - \xi_0|)$, and a substitution of $\xi = \widetilde{f}(x, \xi^\star)$ by $\xi_1 = g(x, \xi^\star)$ introduces an error

$$O(|x - x_0| + |\xi - \xi_0| + |\xi_1 - \xi_0|)\,|\xi - \xi_1| = O(|x - x_0|^3 + |\xi^\star - \xi_0^\star|^3)$$

at most. Therefore

$$i\partial\bar{\partial}\chi^\star(x, \xi^\star) = -i\partial\bar{\partial}\chi(x, g(x, \xi^\star)) < 0 \quad \text{in restriction to } W_0 = T_{M,(x_0,\xi_0^\star)}.$$

This shows that $i\partial\bar{\partial}\chi^\star$ has at least $p \geq n$ negative eigenvalues. As there are already $r$ negative eigenvalues, the only possibility is that $p = n$.

(b) The assumption on $(E, \chi)$ means that for every $(x_0, \xi_0) \in E \smallsetminus \{0\}$, there is a germ of holomorphic section $\sigma : X \to E$ such that $-\chi(x, \sigma(x))$ is strictly plurisubharmonic and $\sigma(x_0) = \xi_0$. Fix $\xi_0^\star \in E_{x_0}^\star \smallsetminus \{0\}$ and take $\xi_0 \in E_{x_0} \smallsetminus \{0\}$ to be the unique point where the maximum defining $\chi^\star$ is reached. Then we infer that $\chi^\star(x, \xi^\star) \geq \log|\xi^\star \cdot \sigma(x)| - \chi(x, \sigma(x))$, with equality at $(x_0, \xi_0^\star)$. An obvious application of the mean value inequality then shows that $\chi^\star$ is plurisubharmonic and that $i\partial\bar{\partial}\chi^\star$ is strictly positive in all directions of $T_{E^\star}$, except the radial vertical direction. □

## 3. A Characterization of Signature $(r, n)$ Concavity

Let $E$ be a holomorphic vector bundle equipped with a smooth Finsler metric which satisfies the concavity properties exhibited by Theorem 2.5. We then have the following results about supremum of plurisubharmonic functions.

Theorem 3.1. *Assume that the Finsler metric $\|\ \|_E$ on $E$ has transversal Levi signature $(r, n)$. Then, for every plurisubharmonic function $(x, \xi) \mapsto u(x, \xi)$ on the total space $E$, the function*

$$M_u(x, t) = \sup_{\|\xi\|_E \leq |e^t|} u(x, \xi)$$

*is plurisubharmonic on $X \times \mathbb{C}$.*

Proof. First consider the restriction $x \mapsto M_u(x, 0)$, and pick a point $x_0$ in $X$. Let $\xi_0 \in E$, $\|\xi_0\|_E = 1$ be a point such that $M_u(x_0, 0) = u(x_0, \xi_0)$. By Proposition 2.4, there a germ of holomorphic section $\sigma : (X, x_0) \to E$ such that $\sigma(x_0) = \xi_0$, whose graph is contained in the unit ball bundle $\|\xi\|_E \leq 1$. Thus $M_u(x, 0) \geq u(x, \sigma(x))$ and $u(x_0, \sigma(x_0)) = M_u(x_0, 0)$. This implies that $M_u(x, 0)$ satisfies the mean value inequality at $x_0$. As $x_0$ is arbitrary, we conclude that $x \mapsto M_u(x, 0)$ is plurisubharmonic. The plurisubharmonicity in $(x, t)$ follows by considering the pull-back of $E^\star$ to $X \times \mathbb{C}$ by the projection $(x, t) \mapsto x$, equipped with the Finsler metric $|e^{-t}|\,\|\xi\|_E$ at point $(x, t)$. We again have osculating holomorphic sections contained in the unit ball bundle $\|\xi\|_E \leq |e^t|$, and the conclusion follows as before. □

We now turn ourselves to the "converse" result:

THEOREM 3.2. *Let $\| \ \|_E$ be a smooth Finsler metric on $E$ which is fiberwise strictly plurisubharmonic on all fibers $E_x$. Assume that $X$ is Stein and that*

$$M_u(x,t) = \sup_{\|\xi\|_E \le |e^t|} u(x,\xi)$$

*is plurisubharmonic on $X \times \mathbb{C}$ for every plurisubharmonic function $(x,\xi) \mapsto u(x,\xi)$ on the total space $E$. Then the Levi form of $\| \ \|_E$ has at least $n$ seminegative eigenvalues, in other words $\| \ \|_E$ is, locally over $X$, a limit of smooth Finsler metrics of transversal Levi signature $(r,n)$.*

PROOF. Once we know that there are at least $n$ seminegative eigenvalues, we can produce metrics of signature $(r,n)$ by considering

$$(x,\xi) \mapsto \|\xi\|_E \, e^{-\varepsilon|x|^2}, \qquad \varepsilon > 0$$

in any coordinate patch, whence the final assertion. Now, assume that the Levi form of $\| \ \|_E$ has at least $(r+1)$ positive eigenvalues at some point $(x_0,\xi_0) \in E$. Then the direct sum of positive eigenspaces in $T_{E,(x_0,\xi_0)}$ projects to a positive dimensional subspace in $T_{X,x_0}$. Consider a germ of smooth complex curve $\Gamma \subset X$ passing through $x_0$, such that its tangent at $x_0$ is contained in that subspace. Then (after shrinking $\Gamma$ if necessary) the restriction of the metric $\| \ \|_E$ to $E_{|\Gamma}$ is strictly plurisubharmonic. By the well-known properties of strictly pseudoconvex domains the unit ball bundle $\|\xi\|_E < 1$ admits a peak function $u$ at $(x_0,\xi_0)$, that is, there is a smooth strictly plurisuharmonic function $u$ on $E_{|\Gamma}$ which is equal to 0 at $(x_0,\xi_0)$ and strictly negative on the set $\{(x,\xi) \ne (x_0,\xi_0) : \|\xi\| \le 1\}$. As $u$ is smooth, we can extend it to $E_{|B}$, where $B = B(x_0,\delta)$ is a small ball centered at $x_0$. As $X$ is Stein, we can even extend it to $E$, possibly after shrinking $B$. Now $M_u(x,0)$ is equal to 0 at $x_0$ and strictly negative elsewhere on the curve $\Gamma$. This contradicts the maximum principle and shows that $M_u$ cannot be plurisubharmonic. Hence the assumption was absurd and the Levi form of $\| \ \|_E$ has at least $n$ seminegative eigenvalues. □

## 4. A Conjecture of Hartshorne and Schneider on Complements of Algebraic Subvarieties

Our study is closely connected to the following interesting (and unsolved) conjecture of R. Hartshorne, which was first partially confirmed by Michael Schneider [1973] in the case of a Griffiths positive normal bundle.

CONJECTURE 4.1. *If $X$ is a projective $n$-dimensional manifold and $Y \subset X$ is a complex submanifold of codimension $q$ with ample normal bundle $N_Y$, then $X \smallsetminus Y$ is $q$-convex in the sense of Andreotti–Grauert. In other words, $X \smallsetminus Y$ has a smooth exhaustion function whose Levi form has at least $n-q+1$ positive eigenvalues on a neighborhood of $Y$.*

Using Sommese's result 2.5(a), one can settle the following special case of the conjecture.

PROPOSITION 4.2 (Sommese). *In addition to the hypotheses in the conjecture, assume that $N_Y^\star$ has a* strictly convex *plurisubharmonic Finsler metric (this is the case for instance if $N_Y$ is generated by global sections). Then $X \smallsetminus Y$ is $q$-convex.*

PROOF. By adding $\varepsilon$ times a strictly plurisubharmonic Finsler metric on $N_Y^\star$ (which exists thanks to the assumption that $N_Y$ is ample), we can even assume that the metric on $N_Y^\star$ is strictly convex *and* strictly plurisubharmonic. Then the dual metric on $N_Y$ has a Levi form of signature $(q, n-q)$. Let $\widetilde{X} \to X$ be the blow-up of $X$ with center $Y$, and $\widetilde{Y} = P(N_Y^\star)$ the exceptional divisor. Then, by Theorem 2.5, the Finsler metric on $N_Y$ corresponds to a hermitian metric on

$$\mathcal{O}_{P(N_Y^\star)}(-1) \simeq N_{\widetilde{Y}} = \mathcal{O}_{\widetilde{X}}(\widetilde{Y})_{|\widetilde{Y}},$$

whose curvature form has signature $(q-1, n-q)$ on $\widetilde{Y}$. Take an arbitrary smooth extension of that metric to a metric of $\mathcal{O}_{\widetilde{X}}(\widetilde{Y})$ on $\widetilde{X}$. After multiplying the metric by a factor of the form $\exp(C\,d(z,\widetilde{Y})^2)$ in a neighborhood of $\widetilde{Y}$ (where $C \gg 0$ and $d(z,\widetilde{Y})$ is the riemannian distance to $\widetilde{Y}$ with respect to some metric), we can achieve that the curvature of $\mathcal{O}_{\widetilde{X}}(\widetilde{Y})$ acquires an additional negative eigenvalue in the normal direction to $\widetilde{Y}$. In this way, the curvature form of $\mathcal{O}_{\widetilde{X}}(\widetilde{Y})$ has signature $(q-1, n-q+1)$ in a neighborhood of $\widetilde{Y}$. We let $\sigma_{\widetilde{Y}} \in H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{Y}))$ be the canonical section of divisor $\widetilde{Y}$. An exhaustion of $X \smallsetminus Y = \widetilde{X} \smallsetminus \widetilde{Y}$ with the required properties is obtained by putting $\psi(z) = -\log \|\sigma_{\widetilde{Y}}(z)\|$. □

## 5. Symmetric and Tensor Products of Finsler Metrics

Let $E$ be a holomorphic vector bundle of rank $r$. In the sequel, we consider the $m$-th symmetric product $S^{m_1}E \times S^{m_2}E \to S^{m_1+m_2}E$ and the $m$-th symmetric power $E \to S^mE$, $\xi \mapsto \xi^m$, which we view as the result of taking products of polynomials on $E^\star$. We also use the duality pairing $S^mE^\star \times S^mE \to \mathbb{C}$, denoted by $(\theta_1, \theta_2^\star) \mapsto \theta_1 \cdot \theta_2^\star$. In multi-index notation, we have

$$(e)^\alpha \cdot (e^\star)^\beta = \delta_{\alpha\beta}\frac{\alpha!}{(|\alpha|)!},$$

where $(e_j)_{1\le j\le r}$ is a basis of $E$, $(e_j^\star)_{1\le j\le r}$ the dual basis in $E^\star$, $1 \le j \le r$, and $(e)^\alpha = e_1^{\alpha_1}\dots e_r^{\alpha_r}$ [Caution: this formula implies that $\theta_1^p \cdot \theta_2^{\star p} \ne (\theta_1 \cdot \theta_2^\star)^p$ for general elements $\theta_1 \in S^mE$, $\theta_2 \in S^mE^\star$, although this is true if $m = 1$.]

Whilst the linear dual $\|\ \|_{E^\star}$ of a Finsler metric $\|\ \|_E$ is not well behaved if $\|\ \|_E$ is not convex, we will see that (positive) symmetric powers and tensor powers can always be equipped with natural well behaved Finsler metrics. For an element $\theta^\star \in S^mE^\star$, viewed as a homogeneous polynomial of degree $m$ on $E$, we set

$$\|\theta^\star\|_{S^mE^\star,L^\infty_{1,m}} = \sup_{\xi\in E\smallsetminus\{0\}} \frac{|\theta^\star\cdot\xi^m|}{\|\xi\|_E^m} = \sup_{\|\xi\|_E\le 1} |\theta^\star\cdot\xi^m|. \tag{5.1}$$

[In the notation $L^\infty_{1,m}$, the upper index $\infty$ refers to the fact that we use sup norms, while the lower indices 1 refers to the fact that $\theta^\star$ appears with exponent 1, and $\xi^m$ with exponent $m$.] This definition just reduces to the definition of the dual metric in the case $m=1$, and thus need not be better behaved than the dual metric from the view point of curvature. On the other hand, for all $\theta_i^\star\in S^{m_i}E^\star$, $i=1,2$, it satisfies the submultiplicative law

$$\|\theta_1^\star\,\theta_2^\star\|_{S^{m_1+m_2}E^\star,L^\infty_{1,m_1+m_2}} \le \|\theta_1^\star\|_{S^{m_1}E^\star,L^\infty_{1,m_1}}\|\theta_2^\star\|_{S^{m_2}E^\star,L^\infty_{1,m_2}}.$$

On the "positive side", i.e. for $\tau\in S^mE$, we define a sequence of metrics $\|\ \|_{S^mE,L^\infty_{p,1}}$ on $S^mE$, $p\ge 1$, and their "limit" $\|\ \|_{S^mE,L^\infty_{\infty,1}}$ by putting

$$\begin{aligned}\|\tau\|_{S^mE,L^\infty_{p,1}} &= \sup_{\theta^\star\in S^{mp}E^\star\smallsetminus\{0\}}\Big(\frac{|\tau^p\cdot\theta^\star|}{\|\theta^\star\|_{S^{pm}E^\star,L^\infty_{1,pm}}}\Big)^{1/p}\\ &= \sup_{\|\theta^\star\|_{S^{mp}E^\star,L^\infty_{1,pm}}\le 1} |\tau^p\cdot\theta^\star|^{1/p},\end{aligned} \tag{5.2}$$

$$\|\tau\|_{S^mE,L^\infty_{\infty,1}} = \limsup_{p\to+\infty}\ \|\tau\|_{S^mE,L^\infty_{p,1}}. \tag{5.3}$$

In the case $m=1$, we have of course $S^1E=E$, but neither $\|\ \|_{S^1E,L^\infty_{p,1}}$ nor $\|\ \|_{S^1E,L^\infty_{\infty,1}}$ necessarily coincide with the original metric $\|\ \|_E$. In fact, by definition, it is easily seen that the unit ball bundle $\|\xi\|_{S^1E,L^\infty_{\infty,1}}\le 1$ is just the (fiberwise) *polynomial hull* of the ball bundle $\|\xi\|_E\le 1$. In particular, $\|\ \|_{S^1E,L^\infty_{\infty,1}}$ and $\|\ \|_E$ do coincide if and only if $\|\ \|_E$ is plurisubharmonic on all fibers $E_x$, which is certainly the case if $\|\ \|_E$ is globally plurisubharmonic on $E$. [By contrast, the unit ball bundle $\|\xi\|_{S^1E,L^\infty_{1,1}}\le 1$ is the convex hull of $\|\xi\|_E\le 1$, and need not be pseudoconvex even if the latter is; see Remark 2.7.] Our first observation is this:

PROPOSITION 5.4. *The $L^\infty$ metric $\|\ \|_{S^mE,L^\infty_{\infty,1}}$ is always well defined and non degenerate* (*in the sense that the* $\limsup$ *is finite and non zero for $\tau\ne 0$*), *and it defines a continuous Finsler metric on $S^mE$.*

PROOF. If in (5.2) we restrict $\theta^\star$ to be of the form $\theta^\star=(\xi^\star)^{mp}$, then

$$\tau^p\cdot\theta^\star=(\tau\cdot\xi^{\star m})^p,\qquad \|(\xi^\star)^{mp}\|_{S^{mp}E^\star,L^\infty_{1,mp}}=\|\xi^\star\|_{E^\star}^{mp},$$

where $\|\ \|_{E^\star}$ is the linear dual of $\|\ \|_E$. From this we infer

$$\|\tau\|_{S^mE,L^\infty_{p,1}}\ge \sup_{\|\xi^\star\|_{E^\star}\le 1}|\tau\cdot\xi^{\star m}|$$

for all $p=1,2,\ldots,\infty$, in particular $\|\tau\|_{S^mE,L^\infty_{\infty,1}}$ is non degenerate. In the other direction, we have to show that $\|\tau\|_{S^mE,L^\infty_{\infty,1}}$ is finite. We first make an explicit calculation when $\|\ \|_E$ is a hermitian norm. We may assume $E=\mathbb{C}^r$ with its standard hermitian norm. Then, writing $\theta^\star \cdot \xi^m = \sum_{|\alpha|=m} c_\alpha \xi^\alpha$ in multi-index notation, we get

$$\|\theta^\star\|^2_{S^m E^\star, L^\infty_{1,m}} = \sup_{\|\xi\|=1} \Big| \sum_{|\alpha|=m} c_\alpha \xi^\alpha \Big|^2 \ge \sup_{t_1+\cdots+t_r=1} \sum_{|\alpha|=m} |c_\alpha|^2 t^\alpha.$$

This is obtained by integrating over the $n$-torus $\xi_j = t_j^{1/2} e^{iu_j}$, $0 \le u_j < 2\pi$ (with $t = (t_j)$ fixed, $\sum t_j = 1$), and applying Parseval's formula. We can now replace the right hand supremum by the average over the $(n-1)$-simplex $\sum t_j = 1$. A short computation yields

$$\|\theta^\star\|^2_{S^m E^\star, L^\infty_{1,m}} \ge \frac{(r-1)!}{(m+1)(m+2)\ldots(m+r-1)} \sum_{|\alpha|=m} |c_\alpha|^2 \frac{\alpha!}{(|\alpha|)!}.$$

However $\sum_{|\alpha|=m} |c_\alpha|^2 \frac{\alpha!}{(|\alpha|)!}$ is just the hermitian norm on $S^m E^\star$ induced by the inclusion $S^m E^\star \subset (E^\star)^{\otimes m}$. The dual norm is the hermitian norm on $S^m E$. From this, we infer

$$\|\tau\|_{S^m E, L^\infty_{p,1}} \le \Big(\frac{(mp+1)(mp+2)\ldots(mp+r-1)}{(r-1)!}\Big)^{1/2p} \|\tau^p\|^{1/p}_{S^{mp} E,\mathrm{herm}},$$

$$\|\tau\|_{S^m E, L^\infty_{\infty,1}} \le \|\tau\|_{S^m E,\mathrm{herm}}$$

[using the obvious fact that hermitian norms are submultiplicative], whence the finiteness of $\|\tau\|_{S^m E, L^\infty_{\infty,1}}$. Finally, given any two Finsler metrics ${}_1\|\ \|_E$ and ${}_2\|\ \|_E$ such that ${}_1\|\ \|_E \le {}_2\|\ \|_E$, it is clear that ${}_1\|\tau\|_{S^m E, L^\infty_{\infty,1}} \le {}_2\|\tau\|_{S^m E, L^\infty_{\infty,1}}$. By comparing a given Finsler norm $\|\ \|_E = {}_1\|\ \|_E$ with a hermitian norm ${}_2\|\ \|_E$, we conclude that the metric $\|\ \|_{S^m E, L^\infty_{\infty,1}}$ must be finite. Moreover, comparing the metrics $\|\ \|_E$ at nearby points, we see that $\|\ \|_{S^m E, L^\infty_{\infty,1}}$ varies continuously (and that it depends continuously on $\|\ \|_E$). $\square$

Our next observation is that the $L^\infty$ metrics on the negative symmetric powers $S^m E^\star$ can be replaced by $L^2$ metrics without changing the final metric $\|\ \|_{S^m E, L^\infty_{\infty,1}}$ on $S^m E$. To see this, fix an arbitrary smooth positive volume form $dV$ on $P(E^\star_x)$, with (say) $\int_{P(E^\star_x)} dV = 1$. We can view any element $\theta^\star \in S^m E^\star$ as a section of $H^0(P(E^\star_x), \mathcal{O}_{P(E^\star_x)}(m))$. Let $\|\theta^\star\|^2_{\mathcal{O}(m)}$ be the pointwise norm on $\mathcal{O}_{P(E^\star)}(m)$ induced by $\|\ \|_E$, and let $d\sigma$ be the area measure on the unit sphere bundle $\Sigma(E)$ induced by $dV$. We then set

$$\|\theta^\star\|^2_{S^m E^\star, L^2_{1,m}} = \int_{P(E^\star_x)} \|\theta^\star\|^2_{\mathcal{O}(m)} dV = \int_{\xi\in\Sigma(E^\star_x)} |\theta^\star \cdot \xi^m|^2 d\sigma(\xi).$$

Clearly

$$\|\theta^\star\|_{S^m E^\star, L^2_{1,m}} \le \|\theta^\star\|_{S^m E^\star, L^\infty_{1,m}}.$$

On the other hand, there exists a constant $C$ such that

$$\|\theta^\star\|^2_{S^m E^\star, L^\infty_{1,m}} \le C\, m^{r-1} \|\theta^\star\|^2_{S^m E^\star, L^2_{1,m}}. \tag{5.5}$$

This is seen by applying the mean value inequality for subharmonic functions, on balls of radius $\sim 1/\sqrt{m}$ centered at arbitrary points in $P(E_x)$. In fact, in a suitable local trivialization of $\mathcal{O}_{P(E^\star)}$ near a point $[\xi_0] \in P(E_x^\star)$, we can write $\|\theta^\star\|^2_{\mathcal{O}(m)} = |\theta_0^\star|^2 e^{-m\psi}$ where $\theta_0^\star$ is the holomorphic function representing $\theta^\star$, and $\psi$ is the weight of the metric on $\mathcal{O}_{P(E^\star)}(1)$. We let $l$ be the holomorphic part in the first jet of $\psi$ at $[\xi_0]$, and apply the mean value inequality to

$$|\theta_0^\star e^{-ml}|^2 e^{-m(\psi - 2\,\mathrm{Re}\, l)}.$$

As $\psi - 2\,\mathrm{Re}\, l$ vanishes at second order at $[\xi_0]$, its maximum on a ball of radius $1/\sqrt{m}$ is $O(1/m)$. Hence, up to a constant independent of $m$, we can replace $|\theta_0^\star e^{-ml}|^2 e^{-m(\psi-2\,\mathrm{Re}\, l)}$ by the subharmonic function $|\theta_0^\star e^{-ml}|^2$. Inequality (5.5) then follows from the mean value inequality on the ball $B([\xi_0], 1/\sqrt{m})$ [noticing that the volume of this ball is $\sim 1/m^{r-1}$]. Now (5.5) shows that the replacement of $\|\theta^\star\|_{S^{pm}E^\star, L^\infty_{1,pm}}$ by $\|\theta^\star\|_{S^{pm}E^\star, L^2_{1,pm}}$ in (5.2) and (5.3) does not affect the limit as $p$ tends to $+\infty$.

If $\|\ \ \|_E$ is (globally) plurisubharmonic, we can even use more global $L^2$ metrics without changing the limit. Take a small Stein open subset $U \Subset X$ and fix a Kähler metric $\omega$ on $P(E^\star_{|U})$. To any section $\sigma \in H^0(\pi^{-1}(U), \mathcal{O}_{P(E^\star)}(m)) = H^0(U, S^m E^\star)$, we associate the $L^2$ norm

$$\|\sigma\|^2_{S^m E^\star, L^2_{1,m}(U)} = \int_{\pi^{-1}(U)} |\sigma|^2 dV_\omega,$$

where $\pi : P(E^\star) \to X$ is the canonical projection. In this way, we obtain a Hilbert space

$$\mathcal{H}_{E,m}(U) = \bigl\{\sigma : \|\sigma\|^2_{S^m E^\star, L^2_1(U)} < +\infty\bigr\} \subset H^0(\pi^{-1}(U), \mathcal{O}_{P(E^\star)}(m)),$$

and associated (non hermitian!) metrics

$$\begin{aligned}\|\tau\|_{S^m E_x, L^2_{p,1}(U)} &= \sup_{\sigma \in \mathcal{H}_{E,mp}(U),\, \|\sigma\| \le 1} |\sigma(x) \cdot \tau^p|^{1/p},\\ \|\tau\|_{S^m E_x, L^2_{\infty,1}(U)} &= \limsup_{p \to +\infty} \|\tau\|_{S^m E_x, L^2_{p,1}(U)}, \qquad \tau \in S^m E_x^\star.\end{aligned}$$

As these metrics are obtained by taking sups of plurisubharmonic functions ($(x,\tau) \mapsto \sigma(x)\cdot\tau^p$ is holomorphic on the total space of $S^m E$), it is clear that the corresponding metrics are plurisubharmonic on $S^m E$. Furthermore, an argument entirely similar to the one used for (5.5) shows that

$$\|\sigma(x)\|^2_{S^m E_x^\star, L^2_{1,m}} \le C\, m^n\, \|\sigma\|^2_{S^m E^\star, L^2_{1,m}(U)} \qquad \text{for all } x \in U' \Subset U.$$

In order to get this, we apply the mean value inequality on balls of radius $1/\sqrt{m}$ centered at points of the fiber $P(E_x^\star)$ and transversal to that fiber, in combination with Fubini's theorem. In the other direction, the Ohsawa–Takegoshi $L^2$ extension theorem [Ohsawa and Takegoshi 1987; Ohsawa 1988; Manivel 1993] shows that every element $\theta^\star \in S^m E^\star_x$, viewed as a section of $\mathcal{O}_{P(E^\star_x)}(m)$, can be extended to a section $\sigma \in H^0(\pi^{-1}(U), \mathcal{O}_{P(E^\star)}(m))$ such that

$$\|\sigma\|_{S^m E^\star, L^2_{1,m}(U)} \le C' \|\theta^\star\|_{S^m E^\star_x, L^2_{1,m}},$$

where $C'$ does not depend on $x \in U$. For this, we use the fundamental assumption that $\|\ \|_E$ is plurisubharmonic (and take profit of the fact that $\mathcal{O}_{P(E^\star)}(1)$ is relatively ample to get enough positivity in the curvature estimates: write e.g. $\mathcal{O}_{P(E^\star)}(m) = \mathcal{O}_{P(E^\star)}(m-m_0) \otimes \mathcal{O}_{P(E^\star)}(m_0)$, keep the original metric on the first factor $\mathcal{O}_{P(E^\star)}(m-m_0)$, and put a metric with uniformly positive curvature on the second factor). From this, we conclude that $\|\tau\|_{S^m E_x, L^2_{\infty,1}(U)}$ coincides with the metric defined in (5.3). Since this metric depends *in fine* only on $\|\ \|_E$, we will simply denote it by $\|\ \|_{S^m E}$. We have thus proven:

Theorem 5.6. *If $\|\ \|_E$ is (strictly) plurisubharmonic on $E$, then $\|\ \|_{S^m E}$ is (strictly) plurisubharmonic on $S^m E$.*

The case of strict plurisubharmonicity can be handled by more or less obvious perturbation arguments and will not be detailed here. As a consequence, we get the Finsler metric analogue of the fact that a direct sum or tensor product of ample vector bundles is ample.

Corollary 5.7. *If $E$, $F$ are holomorphic vector bundles, and $\|\ \|_E$, $\|\ \|_F$ are (strictly) plurisubharmonic Finsler metrics on $E$, $F$, there exist naturally defined (strictly) plurisubharmonic Finsler metrics $\|\ \|_{E\oplus F}$, $\|\ \|_{E\otimes F}$ on $E\oplus F$, $E\otimes F$ respectively.*

Proof. In the case of the direct sum, we simply set $\|\xi\oplus\eta\|_{E\oplus F} = \|\xi\|_E + \|\eta\|_F$. The logarithmic indicatrix is given by

$$\chi_{E\oplus F}(x,\xi,\eta) = \log\big(\exp(\chi_E(x,\xi)) + \exp(\chi_F(x,\eta))\big),$$

and it is clear from there that $\chi_{E\oplus F}$ is plurisubharmonic. Now, we observe that $S^2(E\oplus F) = S^2E \oplus S^2F \oplus (E\otimes F)$. Hence $E\otimes F$ can be viewed as a subbundle of $S^2(E\oplus F)$. To get the required Finsler metric on $E\otimes F$, we just apply Theorem 5.5 to $S^2(E\oplus F)$ and take the induced metric on $E\otimes F$. □

Remark 5.8. It would be interesting to know whether good Finsler metrics could be defined as well on the dual symmetric powers $S^m E^\star$. One natural candidate would be to use the already defined metrics $\|\ \|_{S^m E}$ and to set

$$\|\tau^\star\|_{S^m E^\star, L^\infty_{p,1}} = \sup_{\|\theta\|_{S^{pm}E} \le 1} |\tau^{\star p}\cdot\theta|^{1/p},$$

$$\|\tau^\star\|_{S^m E^\star, L^\infty_{\infty,1}} = \limsup_{p\to+\infty} \|\tau^\star\|_{S^m E^\star, L^\infty_{p,1}}.$$

However, we do not know how to handle these "bidually defined" Finsler metrics, and the natural question whether $\|\ \|_{S^m E^\star, L^\infty_{\infty,1}}$ has transversal signature $(\dim S^m E^\star, n)$ probably has a negative answer if $\|\ \|_E$ is not convex (although this might be "asymptotically true" as $m$ tends to $+\infty$).

5.9. Relation to cohomology vanishing and duality theorems. If $\| \ \|_{E^\star}$ is smooth and strictly plurisubharmonic, then $E$ is ample, thus its symmetric powers $S^m E$ have a lot of sections and the Kodaira–Serre vanishing theorem holds true, i.e.

$$H^q(X, S^m E \otimes \mathcal{F}) = 0, \qquad q \neq 0,$$

for every coherent sheaf $\mathcal{F}$ and $m \geq m_0(\mathcal{F})$ large enough. In a parallel way, if $\| \ \|_E$ has a metric of signature $(r, n)$, then the line bundle $\mathcal{O}_{P(E^\star)}(1)$ has a hermitian metric such that the curvature has signature $(r-1, n)$ over $P(E^\star)$. From this, by the standard Bochner technique, we conclude that

$$H^q(P(E^\star), \mathcal{O}_{P(E^\star)}(m) \otimes \mathcal{G}) = 0, \qquad q \neq n,$$

for every locally free sheaf $\mathcal{G}$ on $P(E^\star)$ and $m \geq m_0(\mathcal{G})$. The Leray spectral sequence shows that

$$H^q(X, S^m E^\star \otimes \mathcal{F}) = H^q(P(E^\star), \mathcal{O}_{P(E^\star)}(m) \otimes \pi^\star \mathcal{F}),$$

thus we have vanishing of this group as well is $\mathcal{F}$ is locally free and $q \neq n$, $m \geq m_0(\mathcal{F})$. The Serre duality theorem connects the two facts via an isomorphism

$$H^q(X, S^m E^\star \otimes \mathcal{F})^\star = H^{n-q}(X, S^m E \otimes \mathcal{F}^\star \otimes K_X).$$

What we are looking for, in some sense, is a "Finsler metric version" of the Serre duality theorem. Up to our knowledge, the duality works well only for convex Finsler metrics (and also asymptotically, for high symmetric powers $S^m E$ which carry positively curved hermitian metrics).

## 6. A Trick on Taylor Series

Let $\pi : E \to X$ be a holomorphic vector bundle, such that $E^\star$ is equipped with a continuous plurisubharmonic Finsler metric $\|\xi^\star\|_{E^\star} = \exp(\chi^\star(x, \xi^\star))$. Thanks to Section 5, we are able to define plurisubharmonic Finsler metrics $\| \ \|_{S^m E^\star}$ on all symmetric powers of $E^\star$. Our goal is to use these metrics in order to define plurisubharmonic sup functionals for holomorphic or plurisubharmonic functions. We first start with the simpler case when $\| \ \|_{E^\star}$ is convex.

Theorem 6.1. *Assume that $\| \ \|_{E^\star}$ is plurisubharmonic and convex, and let $\|\xi\|_E = \exp(\chi(x, \xi))$ be the (linearly) dual metric. Then, for every plurisubharmonic function $(x, \xi) \mapsto u(x, \xi)$ on the total space $E$, the function*

$$M_u^\chi(x, t) = \sup_{\|\xi\|_E \leq |e^t|} u(x, \xi)$$

*is plurisubharmonic on $X \times \mathbb{C}$.*

PROOF. This is a local result on $X$, so we can assume that $X$ is an open set $\Omega \subset \mathbb{C}^n$ and that $E = \Omega \times \mathbb{C}^r$ is trivial. By the standard approximation techniques, we can approximate $\| \ \|_{E^\star}$ by smooth strictly convex and strictly plurisubharmonic metrics ${}_\varepsilon\| \ \|_{E^\star} \geq \| \ \|_{E^\star}$ which decrease to $\| \ \|_{E^\star}$ as $\varepsilon$ decreases to 0. We then get a decreasing family $\lim\downarrow_{(\varepsilon\to 0)} {}_\varepsilon M_u(x,t) = M_u(x,t)$. It is thus enough to treat the case of smooth strictly convex and strictly plurisubharmonic metrics $\| \ \|_{E^\star}$. In that case, $\| \ \|_E$ has a Levi form of signature $(r,n)$ and we conclude by Theorem 3.1. □

Unfortunately, in the general case when $\| \ \|_{E^\star}$ is not convex, this simple approach does not work [in the sense that $M_u$ is not always plurisubharmonic]. We circumvent this difficulty by using instead the well-known trick of Taylor expansions, and replacing the sup with a more sophisticated evaluation of norms. If $f$ is a holomorphic function on the total space of $E$, the Taylor expansion of $f$ along the fibers of $E$ can be written as

$$f(x,\xi) = \sum_{m=0}^{+\infty} a_m(x)\cdot\xi^m, \qquad \xi \in E_x,$$

where $a_m$ is a section in $H^0(X, S^m E^\star)$. In that case, we set

$$\widehat{M}_f^\chi(x,t) = \sum_{m=0}^{+\infty} \|a_m(x)\|_{S^m E^\star} |e^{mt}|. \tag{6.2}$$

This is by definition a plurisubharmonic function on $X\times\mathbb{C}$. In fact, $\log \widehat{M}_f^\chi(x,t)$ is a plurisubharmonic function as well. As we will see in the following lemma, $\widehat{M}_f^\chi$ will play essentially the same role as $M_{|f|}$ could have played.

LEMMA 6.3. *Fix a hermitian metric* $\| \ \|_{E^\star,\mathrm{herm}} \geq \| \ \|_{E^\star}$, *and let* $\| \ \|_{E,\mathrm{herm}}$ *be the dual metric. Then there is an inequality*

$$\sup_{\|\xi\|_E \leq |e^t|} |f(x,\xi)| \leq \widehat{M}_f^\chi(x,t) \leq \Big(1+\frac{1}{\varepsilon}\Big)^r \sup_{\|\xi\|_{E,\mathrm{herm}} \leq (1+\varepsilon)|e^t|} |f(x,\xi)|.$$

PROOF. The left hand inequality is obtained by expanding

$$f(x,\xi) \leq \sum_{m=0}^{+\infty} |a_m(x)\cdot\xi^m| \leq \sum_{m=0}^{+\infty} \|a_m(x)\|_{S^m E^\star} \|\xi\|_E^m,$$

thanks to the fact that $a_m(x)^p \cdot \xi^{mp} = (a_m(x)\cdot\xi^m)^p$. In the other direction, we have

$$\begin{aligned}\|a_m(x)\|_{S^m E^\star} &\leq \|a_m(x)\|_{S^m E^\star,\mathrm{herm}} \\ &\leq \frac{(m+1)\dots(m+r-1)}{(r-1)!} \sup_{\|\xi\|_{E,\mathrm{herm}}\leq 1} |a_m(x)\cdot\xi^m|\end{aligned}$$

thanks to the inequalities obtained in the proof of Proposition 5.4. Now, the standard Cauchy inequalities imply

$$\sup_{\|\xi\|_{E,\mathrm{herm}}=1} |a_m(x)\cdot\xi^m| \le \frac{1}{R^m} \sup_{\|\xi\|_{E,\mathrm{herm}}=R} |f(x,\xi)|.$$

Combining all the above with $R>|e^t|$, we get

$$\begin{aligned}\widehat{M}_f^\chi(x,t) &\le \sup_{\|\xi\|_{E,\mathrm{herm}}\le R} |f(x,\xi)| \sum_{m=0}^{+\infty} \frac{|e^{mt}|}{R^m}\frac{(m+1)\ldots(m+r-1)}{(r-1)!}\\ &\le \frac{1}{\big(1-\frac{|e^t|}{R}\big)^r}\sup_{\|\xi\|_{E,\mathrm{herm}}\le R}|f(x,\xi)| \le \Big(1+\frac{1}{\varepsilon}\Big)^r \sup_{\|\xi\|_{E,\mathrm{herm}}\le(1+\varepsilon)|e^t|}|f(x,\xi)|.\end{aligned}$$

The lemma is proved. □

Remark 6.4. It is clear that the sup functional $M_{|f|}^\chi$ is submultiplicative, i.e.

$$M_{|fg|}^\chi(x,t) \le M_{|f|}^\chi(x,t)\, M_{|g|}^\chi(x,t).$$

However, the analogous property for $\widehat{M}_f^\chi$ would require to know whether

$$\|a\cdot b\|_{S^{m_1+m_2}E^\star} \le \|a\|_{S^{m_1}E^\star}\, \|b\|_{S^{m_2}E^\star},$$

(or a similar inequality with a constant $C$ independent of $m_1$, $m_2$). It is not clear whether such a property is true, since the precise asymptotic behaviour of the metrics $\|\ \ \|_{S^mE^\star}$ is hard to understand. In order to circumvent this problem, we select a non increasing sequence of real numbers $\rho_m\in\,]0,1]$ with $\rho_0=1$, such that

$$\rho_{m_1+m_2}\|a\cdot b\|_{S^{m_1+m_2}E^\star} \le \rho_{m_1}\rho_{m_2}\|a\|_{S^{m_1}E^\star}\|b\|_{S^{m_2}E^\star} \tag{6.5}$$

for all $m_1$, $m_2$. One can easily find such a sequence $\rho=(\rho_m)$ by induction on $m$, taking $\rho_m/\rho_{m-1}$ small enough. Then

$$\widehat{M}_f^{\chi,\rho}(x,t) := \sum_{m=0}^{+\infty} \rho_m\|a_m(x)\|_{S^mE^\star}|e^{mt}| \tag{6.6}$$

obviously satisfies the submultiplicative property. On the other hand, we lose the left hand inequality in Lemma 6.3. This unsatisfactory feature will create additional difficulties which we can only solve at the expense of using deeper analytic techniques.

We are mostly interested in the case when $E=T_X$ is the tangent bundle, and assume that a plurisubharmonic Finsler metric $\|\xi^\star\|_{T_X^\star}=\exp(\chi^\star(x,\xi^\star))$ is given. Locally, on a small coordinate open set $U\Subset U_0\subset X$ associated with a holomorphic chart

$$\tau: U_0\to\tau(U_0)\subset\mathbb{C}^n,$$

we have a corresponding trivialization $\tau': T_{X|U}\simeq\tau(U)\times\mathbb{C}^n$. Given a holomorphic function $f$ in a neighborhood of $\overline{U}$, we consider the holomorphic function such that $F(x,\xi) = f(\alpha(x,\xi))$ where $\alpha(x,\xi) = \tau^{-1}(\tau(x) + \tau'(x)\xi))$. It is defined on a sufficiently small ball bundle $B_\varepsilon(T_{X|U}) = \{(x,\xi) \in T_{X|U} : \|\xi\| < \varepsilon\}$, $\varepsilon > 0$. Thus

$$\widetilde{M}_f^{\chi,\rho}(x,t) := \widehat{M}_F^{\chi,\rho}(x,t) \tag{6.7}$$

makes sense for $|e^t| < c\varepsilon$, $c > 0$. Again, by construction, this is a plurisubharmonic function of $(x,t)$ on $U \times \{|e^t| < c\varepsilon\}$. This function will be used as a replacement of the sup of $f$ on the Finsler ball $\alpha(x, B(0,|e^t|)) \subset X$ (which we unfortunately know nothing about). However, the definition is not coordinate invariant, and we have to investigate the effect of coordinate changes.

Lemma 6.8. *Consider two holomorphic coordinate coordinate charts $\tau_j$ on a neighborhood of $\overline{U}$, for $j = 1, 2$, and the corresponding maps*

$$\alpha_j : B_\varepsilon(T_{X|U}) \to U_0, \qquad \alpha_j(x,\xi) = \tau_j^{-1}(\tau_j(x) + \tau_j'(x)\xi)).$$

*Let $F_j = f \circ \alpha_j$, $j = 1, 2$, and let $\delta > 0$ be fixed. Then there is a choice of a decreasing sequence $\rho = (\rho_m)$ such that*

$$\widehat{M}_{F_2}^{\chi,\rho}(x,t) \le (1+\delta)\widehat{M}_{F_1}^{\chi,\rho}(x,t),$$

*where $\rho$ depends on $U$, $\tau_1$, $\tau_2$, but not on $f$ (here $|e^t|$ is suppose to be chosen small enough so that both sides are defined). Any sequence $\rho$ with $\rho_m/\rho_{m-1}$ smaller that a given suitable sequence of small numbers works.*

In other words, if the sequence $(\rho_m)$ decays sufficiently fast, the functional $\widetilde{M}_f^{\chi,\rho}(x,t)$ defined above can be chosen to be "almost" coordinate invariant.

Proof. It is easy to check by the implicit function theorem that there exists a (uniquely) defined map $w : T_X \to T_X$, defined near the zero section and tangent to the identity at 0, such that

$$\alpha_2(x,\xi) = \alpha_1(x, w(x,\xi)), \qquad w(x,\xi) = \xi + O(\xi^2).$$

Hence, if we write

$$f \circ \alpha_j(x,\xi) = \sum_{m=0}^{+\infty} a_{m,j}(x) \cdot \xi^m, \qquad j = 1,\ 2,$$

the series corresponding to index $j = 2$ is obtained from the $j = 1$ series by substituting $\xi \mapsto w(x,\xi)$. It follows that

$$a_{m,2}(x) = a_{m,1}(x) + \sum_{\mu<m} L_{m,\mu}(x) \cdot a_{\mu,1}(x)$$

where $L_{m,\mu} : S^\mu T_X^\star \to S^m T_X^\star$ are certain holomorphic linear maps depending only on the chart mappings $\tau_1$, $\tau_2$. If $\rho_m/\rho_{m-1}$ is small enough, the contribution given by $\rho_m \sum_{\mu<m} L_{m,\mu}(x) \cdot a_{\mu,1}(x)$ is negligible compared to the $\rho_\mu \|a_{\mu,1}(x)\|_{S^\mu T_X^\star}$. The lemma follows. □

## 7. Approximation of Plurisubharmonic Functions by Logarithms of Holomorphic Functions

The next step is to extend the $\widetilde{M}_f^{\chi,\rho}$ functional to plurisubharmonic functions defined on a complex manifold, when the cotangent bundle $T_X^\star$ is equipped with a Finsler metric. The simplest way to do this is to approximate such functions by logarithms of holomorphic functions, by means of the Ohsawa–Takegoshi $L^2$ extension theorem [Ohsawa and Takegoshi 1987; Ohsawa 1988; Manivel 1993]. We reproduce here some of the techniques introduced in [Demailly 1992], but with substantial improvements. The procedure is still local and not completely canonical, so we will have later to apply a gluing procedure.

THEOREM 7.1. *Let $\varphi$ be a plurisubharmonic function on a bounded pseudoconvex open set $U \subset \mathbb{C}^n$. For every $p > 0$, let $\mathcal{H}_{p\varphi}(U)$ be the Hilbert space of holomorphic functions $f$ on $U$ such that $\int_U |f|^2 e^{-2p\varphi} d\lambda < +\infty$ and let*

$$\varphi_p = \frac{1}{2p} \log \sum |\sigma_l|^2,$$

*where $(\sigma_l)$ is an orthonormal basis of $\mathcal{H}_{p\varphi}(U)$. Then there are constants $C_1 > 0$ and $C_2 > 0$ independent of $p$ such that*

(i) $\varphi(z) - \dfrac{C_1}{p} \le \varphi_p(z) \le \sup\limits_{|\zeta - z| < r} \varphi(\zeta) + \dfrac{1}{p} \log \dfrac{C_2}{r^n}$

*for every $z \in U$ and $r < d(z, \partial U)$. In particular, $\varphi_p$ converges to $\varphi$ pointwise and in $L^1_{\mathrm{loc}}$ topology on $U$ when $p \to +\infty$ and*

(ii) $\nu(\varphi, z) - \dfrac{n}{p} \le \nu(\varphi_p, z) \le \nu(\varphi, z)$ *for every $z \in U$.*

PROOF. Note that $\sum |\sigma_l(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_{p\varphi}(U)$. As $\varphi$ is locally bounded above, the $L^2$ topology is actually stronger than the topology of uniform convergence on compact subsets of $U$. It follows that the series $\sum |\sigma_l|^2$ converges uniformly on $U$ and that its sum is real analytic. Moreover, we have

$$\varphi_p(z) = \sup_{f \in B_p(1)} \frac{1}{p} \log |f(z)| \tag{7.2}$$

where $B_p(1)$ is the unit ball of $\mathcal{H}_{p\varphi}(U)$. For $r < d(z, \partial U)$, the mean value inequality applied to the plurisubharmonic function $|f|^2$ implies

$$\begin{aligned} |f(z)|^2 &\le \frac{1}{\pi^n r^{2n}/n!} \int_{|\zeta - z| < r} |f(\zeta)|^2 d\lambda(\zeta) \\ &\le \frac{1}{\pi^n r^{2n}/n!} \exp\Big(2p \sup_{|\zeta - z| < r} \varphi(\zeta)\Big) \int_U |f|^2 e^{-2p\varphi} d\lambda. \end{aligned}$$

If we take the supremum over all $f \in B_p(1)$ we get

$$\varphi_p(z) \le \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{2p} \log \frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in (i) is proved. Conversely, the Ohsawa–Takegoshi extension theorem [Ohsawa and Takegoshi 1987; Ohsawa 1988; Manivel 1993] applied to the 0-dimensional subvariety $\{z\} \subset U$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function $f$ on $U$ such that $f(z) = a$ and

$$\int_U |f|^2 e^{-2p\varphi} d\lambda \leq C_3 |a|^2 e^{-2p\varphi(z)},$$

where $C_3$ only depends on $n$ and $\operatorname{diam} U$. We fix $a$ such that the right hand side is 1. This gives the other inequality

$$\varphi_p(z) \geq \frac{1}{p} \log |a| = \varphi(z) - \frac{\log C_3}{2p}.$$

The above inequality implies $\nu(\varphi_p, z) \leq \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x-z|<r} \varphi_p(x) \leq \sup_{|\zeta - z|<2r} \varphi(\zeta) + \frac{1}{p} \log \frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as $r$ tends to 0. The quotient by $\log r$ of the supremum of a plurisubharmonic function over $B(x, r)$ tends to the Lelong number at $x$. Thus we obtain

$$\nu(\varphi_p, x) \geq \nu(\varphi, x) - \frac{n}{p}. \qquad \square$$

Another important fact is that the approximations $\varphi_p$ do no depend much on the open set $U$, and they have a good dependence on $\varphi$ under small perturbations. In fact, let $U', U'' \subset U$ be Stein open subsets, and let $\varphi'$, $\varphi''$ be plurisubharmonic functions on $U'$, $U''$ such that $|\varphi' - \varphi''| \leq \varepsilon$ on $U' \cap U''$. If $f'$ is a function in the unit ball of $\mathcal{H}_{p\varphi}(U')$, then

$$\int_{U' \cap U''} |f'|^2 e^{-2p\varphi''} d\lambda \leq e^{2p\varepsilon}$$

by the hypothesis on $\varphi' - \varphi''$. For every $x_0 \in U'$, we can find a function $f'' \in \mathcal{H}_{p\varphi''}(U)$ such that $f''(x_0) = f'(x_0)$ and

$$\int_{U''} |f''|^2 e^{-2p\varphi''} d\lambda \leq \frac{C}{(d(x_0, \complement U))^{2n+2}} e^{2p\varepsilon} \int_{U'} |f'|^2 e^{-2p\varphi'} d\lambda.$$

This is done as usual, by solving the equation $\bar\partial g = \bar\partial(\theta f')$ with a cut-off function $\theta$ supported in the ball $B(x_0, \delta/2)$ and equal to 1 on $B(x_0, \delta/4)$, $\delta = d(x_0, \complement U)$, with the weight $2p\varphi(z) + 2n \log |z - x_0|$ ; the desired function is then $f'' = \theta f' - g$. By readjusting $f''$ by a constant so that $f''$ is in the unit sphere, and by taking the sup of $\log |f'(x_0)|$ and $\log |f''(x_0)|$ for all $f'$ and $f''$ in the unit ball of their respective Hilbert spaces, we conclude that

$$\varphi'_p(x) \leq \varphi''_p(x) + \varepsilon + \frac{1}{2p} \log \frac{C}{d(x, \complement(U' \cap U''))^{2n+2}} \qquad \text{on } U' \cap U'',$$

with some constant $C > 0$ depending only on the pair $(U', U'')$. By symmetry, we get

$$(7.3)\qquad |\varphi'_p(x) - \varphi''_p(x)| \le \varepsilon + \frac{1}{2p}\log\frac{C}{d(x, \complement(U'\cap U''))^{2n+2}} \qquad \text{on } U'\cap U''.$$

The next idea would be to take Taylor series much in the same way as we did in § 6, and look e.g. at

$$\Phi^{\chi,\rho,p}(x,t) = \sup_{f\in B_p(1)} \frac{1}{p}\log \widetilde{M}_f^{\chi,\rho}(x,t).$$

The main problem with this approach occurs when we want to check the effect of a change of coordinate patch. We then want to compare the jets with those of the functions $f$ obtained on another coordinate patch, say up to an order $Cp$ for $C \gg 0$ large. The comparison would be easy (by the usual Hörmander–Bombieri $\bar\partial$-technique, as we did for the 0-jets in (7.3)) for jets of small order in comparison to $p$, but going to such high orders introduces intolerable distortion in the required bounds. A solution to this problem is to introduce further approximations of $\varphi_p$ for which we have better control on the jets. This can be done by using Skoda's $L^2$-estimates for surjective bundle morphisms [Skoda 1972a; 1978]. This approach was already used in [Demailly 1992], but in a less effective fashion.

Let $K_p^U : \varphi \mapsto \varphi_p$ be the transformation defined above. This transformation has the effect of converting the singularities of $\varphi$, which are a priori arbitrary, into logarithmic analytic singularities (and, as a side effect, the multiplicities get discretized, with values in $\frac{1}{p}\mathbb{N}$). We simply iterate the process twice, and look at

$$(7.4)\qquad \varphi_{p,q} = K_{pq}^U(K_p^U(\varphi))$$

for large integers $q \gg p \gg 1$. In other words,

$$\varphi_p(z) = \frac{1}{2p}\log\sum|\sigma_l(z)|^2, \qquad \varphi_{p,q}(z) = \frac{1}{2pq}\log\sum|\tilde\sigma_l(z)|^2$$

where $\sigma = (\sigma_l)_{l\in\mathbb{N}}$ and $\tilde\sigma = (\tilde\sigma_l)_{l\in\mathbb{N}}$ are Hilbert bases of the $L^2$ spaces

$$\mathcal{H}_{p\varphi}(U) = \Big\{\int_U |f|^2 e^{-2p\varphi}d\lambda < +\infty\Big\}, \quad \mathcal{H}_{pq\varphi_p}(U) = \Big\{\int_U |f|^2|\sigma|^{-2q}d\lambda < +\infty\Big\}.$$

Theorem 7.1 shows that we still have essentially the same estimates for $\varphi_{p,q}$ as we had for $\varphi_p$, namely

$$(7.5\text{ i})\qquad \varphi(z) - \frac{C_1}{p} \le \varphi_{p,q}(z) \le \sup_{|\zeta - z|<r}\varphi(\zeta) + \Big(\frac{1}{p}+\frac{1}{pq}\Big)\log\frac{C_2}{r^n}$$

$$(7.5\text{ ii})\qquad \nu(\varphi,z) - n\Big(\frac{1}{p}+\frac{1}{pq}\Big) \le \nu(\varphi_{p,q},z) \le \nu(\varphi,z).$$

The major improvement is that we can now compare the jets when $U$ varies, even when we allow a small perturbation on $\varphi$ as well.

PROPOSITION 7.6. *Suppose that we have plurisubharmonic functions $\varphi'$, $\varphi''$ defined on bounded Stein open sets $U'$, $U'' \Subset \mathbb{C}^n$, with $|\varphi' - \varphi''| \le \varepsilon$ on $U' \cap U''$. Let $\sigma' = (\sigma'_l)_{l\in\mathbb{N}}$, $\sigma'' = (\sigma''_l)_{l\in\mathbb{N}}$ be the associated Hilbert bases of $\mathcal{H}_{p\varphi'}(U')$, $\mathcal{H}_{p\varphi''}(U'')$, and $\tilde\sigma' = (\tilde\sigma'_l)_{l\in\mathbb{N}}$, $\tilde\sigma'' = (\tilde\sigma''_l)_{l\in\mathbb{N}}$ the bases of $\mathcal{H}_{pq\varphi'_p}(U')$, $\mathcal{H}_{pq\varphi''_p}(U'')$. Fix a Stein open set $W \Subset U' \cap U''$ and a holomorphic function $f''$ on $U''$ such that*

$$\int_{U''} |f''|^2 |\sigma''|^{-2q} d\lambda \le 1, \qquad q > n+1.$$

(i) *One can write $f'' = \sum_{L\in\mathbb{N}^m} g_L (\sigma')^L$ on $W$ with $m = q-n-1$ and*

$$\int_W \sum_L |g_L|^2 |\sigma'|^{-2(n+1)} d\lambda \le C^{2q} e^{2pq\varepsilon}$$

*with a constant $C > 1$ depending only on $d(W, \complement(U' \cap U''))$.*

(ii) *There are holomorphic functions $h_l$ on $W$ such that $f'' = \sum h_l \hat\sigma'_l$ on $W$, and*

$$\sup_W \sum_l |h_l|^2 \le C_1(p) C^{2q} e^{2pq\varepsilon}$$

*where $C$ is as in* (i) *and $C_1(p)$ depends on $p$ (and $U', U'', W$ as well).*

PROOF. (i) Thanks to (7.3), we have $|\sigma'| \ge C^{-1} e^{-p\varepsilon} |\sigma''|$ on $W$ for some constant $C > 1$ depending only only on $d(W, \complement(U' \cap U''))$. Therefore

$$\int_W |f|^2 |\sigma'|^{-2q} d\lambda \le C^{2q} e^{2pq\varepsilon}.$$

We apply Skoda's $L^2$ division theorem (Corollary 10.6) with $r = n$, $m = q-n-1$, $\alpha = 1$, on the Stein open set $W$. Our assertion (i) follows, after absorbing the extra constant $(q-n)$ in $C^{2q}$.

(ii) We first apply (i) on a Stein open set $W_1$ such that $W \Subset W_1 \Subset U' \cap U''$, and write in this way $f'' = \sum_L g_L (\sigma')^L$ with the $L^2$ estimate as in (i). By [Nadel 1990] (see also [Demailly 1993]), the ideal sheaf $\mathcal{J}$ of holomorphic functions $v$ on $U'$ such that

$$\int_{U'} |v|^2 |\sigma'|^{-2(n+1)} d\lambda < +\infty$$

is coherent and locally generated by its global $L^2$ sections (of course, this ideal depends on the $\sigma'_l$, hence on $p$ and $\varphi'$). It follows that we can find finitely many holomorphic functions $v_1, \ldots, v_N$, $N = N(p)$, such that

$$\int_{U'} |v_j|^2 |\sigma'|^{-2(n+1)} d\lambda = 1$$

and $\mathcal{J}(W_1) = \sum v_j \mathcal{O}(W_1)$. As the topology given by the $L^2$ norm on the $L^2$ sections of $\mathcal{J}(W_1)$ is stronger than the Fréchet topology of uniform convergence on compact subsets, and as we have a Fréchet epimorphism $\mathcal{O}(W_1)^{\oplus N} \to \mathcal{J}(W_1)$,

$(a_1, \ldots, a_N) \mapsto \sum a_j v_j$, the open mapping theorem shows that we can write $g = \sum a_j v_j$ with

$$\sup_W \sum_{j=1}^N |a_j|^2 \leq A(p) \int_{W_1} |g|^2 |\sigma'|^{-2(n+1)} d\lambda$$

for every holomorphic function $g$ on $W_1$ for which the right hand side is finite [the constant $A(p)$ depends on $\mathcal{I}$, hence on $p$]. In particular, we can write $g_L = \sum_j a_{j,L} v_j$ with

$$\sup_W \sum_{j,L} |a_{j,L}|^2 \leq A(p) \int_{W_1} \sum_L |g_L|^2 |\sigma'|^{-2(n+1)} d\lambda \leq A(p) C^{2q} e^{2pq\varepsilon}.$$

We find

$$f'' = \sum_L g_L (\sigma')^L = \sum_{j,L} a_{j,L} g_j (\sigma')^L,$$

and as $L$ runs over all multiindices of length $m = q - n - 1$ we get

$$\int_{U'} \sum_{j,L} |g_j (\sigma')^L|^2 |\sigma'|^{-2q} d\lambda = \int_{U'} \sum_j |g_j|^2 |\sigma'|^{-2(n+1)} d\lambda = N = N(p).$$

We can therefore express the function $g_j(\sigma')^L$ in terms of the Hilbert basis $(\hat{\sigma}'_l)$

$$g_j(\sigma')^L = \sum_l b_{j,L,l} \hat{\sigma}'_l, \qquad b_{j,L,l} \in \mathbb{C}, \quad \sum_{j,L,l} |b_{j,L,l}|^2 = N(p).$$

Summing up everything, we obtain

$$f'' = \sum_{j,L,l} a_{j,L} b_{j,L,l} \hat{\sigma}'_l = \sum_l h_l \hat{\sigma}'_l, \qquad h_l = \sum_{j,L} a_{j,L} b_{j,L,l}.$$

The Cauchy–Schwarz inequality implies

$$\sup_W \sum_l |h_l|^2 \leq \sup_W \sum_{j,L} |a_{j,L}|^2 \sum_{j,L,l} |b_{j,L,l}|^2 \leq N(p) A(p) C^{2q} e^{2pq\varepsilon},$$

as desired. □

Now, assume that $X$ is a complex manifold such that $T_X^\star$ is equipped with a plurisubharmonic Finsler metric. As all constructions to be used are local, we may suppose that we are in a small coordinate open subset $U_0 \Subset X$ or, equivalently, in a Stein open set $U_0 \Subset \mathbb{C}^n$, with $\varphi$ being defined on $U_0$. We fix Stein open sets $U \Subset U_1 \Subset U_0$ and select a sequence $\rho = (\rho_m)$ satisfying property (6.5) on each fiber $T_{U,x}^\star$. Finally, for $(x,t) \in U \times \mathbb{C}$, we set

$$\Phi^{\chi,\rho,p,q}(x,t) := \sup_{f \in B_{p,q}(1)} \frac{1}{pq} \log \widehat{M}_F^{\chi,\rho}(x,t) + \frac{C_0}{p}, \qquad C_0 \gg 0, \tag{7.7}$$

where $f$ runs over the unit ball $B_{p,q}(1)$ of $\mathcal{H}_{pq\varphi_p}(U_1)$ and $F(x,\xi) = f(x+\xi)$. Then $\Phi^{\chi,\rho,p,q}(x,t)$ is well defined on $U \times \{\mathrm{Re}\, t < -A\}$ for $A \geq 0$ sufficiently large. Thanks to Lemma 6.8, the choice of coordinates on $U_0$ is essentially irrelevant when we compute $\widehat{M}_F^{\chi,\rho}(x,t)$, provided that $\rho$ decays fast enough. Moreover, a change of coordinate $\tau : U_0 \mapsto \tau(U_0)$ has the effect of replacing $\varphi$ by $\varphi^\tau = \varphi\circ\tau^{-1}$ and $\varphi_p$ by $\varphi_p^\tau = \varphi_p\circ\tau^{-1} + O(1/p)$, since the only change occurring in the definition of $\mathcal{H}_{p\varphi^\tau}(\tau(U_1))$ is the replacement of the Lebesgue volume form $d\lambda$ by $\tau^\star d\lambda$, which affects the $L^2$ norm by at most a constant. Similarly, the $L^2$ norm of $\mathcal{H}_{pq\varphi_p^\tau}(\tau(U_1))$ gets modified by an irrelevant multiplicative factor $\exp(O(q))$, inducing a negligible error term $O(1/p)$ in (7.7). If $C_0 \geq 0$ is large enough, (7.2) combined with 7.5 (i) implies that

$$\varphi(z) \leq \varphi_{p,q}(z) + \frac{C_0}{p} = \sup_f \frac{1}{pq}\log|f(x)| + \frac{C_0}{p} \leq \sup_{|\zeta-z|<r}\varphi(\zeta) + \Big(\frac{1}{p}+\frac{1}{pq}\Big)\log\frac{C}{r^n}$$

for some $C>0$, where $f$ runs over $B_{p,q}(1)\subset \mathcal{H}_{pq\varphi_p}(U_1)$. Lemma 6.3 applied with $r=|e^t|$ then gives

$$\text{(7.8 i)}\qquad \begin{aligned}\Phi^{\chi,\rho,p,q}(x,t) &\leq \sup_{|z-x|\leq C'|e^t|}\varphi_{p,q}(z) + \frac{C_0}{p}\\ &\leq \sup_{|z-x|\leq C|e^t|}\varphi(z) - n\Big(\frac{1}{p}+\frac{1}{pq}\Big)\operatorname{Re}t + \frac{C}{p},\end{aligned}$$

$$\text{(7.8 ii)}\qquad \begin{aligned}\Phi^{\chi,\rho,p,q}(x,t) &\geq \sup_{|z-x|\leq c_{\chi,\rho,p,q,\varphi,U}|e^t|}\varphi_{p,q}(z) + \frac{C_0}{p}\\ &\geq \sup_{|z-x|\leq c_{\chi,\rho,p,q,\varphi,U}|e^t|}\varphi(z) \geq \varphi(x)\end{aligned}$$

where $C$, $C'$ are universal constants, and $c_{\chi,\rho,p,q,\varphi,U}$ depends on all given data, but is independent of $x$. The last inequality is a simple consequence of the fact that the Taylor series $\widehat{M}_f^{\chi,\rho}(x,t) = \sum_{m\geq 0}\rho_m\|a_m(x)\||e^{mt}|$ are never identically zero, hence their behavior as $|e^t|\to 0$ is the same as for the series $\sum_{m\leq N}\rho_m\|a_m(x)\||e^{mt}|$, truncated at some rank $N=N_{p,q,\varphi,U}$. The constant $c_{\chi,\rho,p,q,\varphi,U}$ then essentially depends only on $\inf_{m\leq N}\rho_m$. The upper and lower bound provided by (7.8) imply in particular

$$\text{(7.9 i)}\qquad \lim_{\operatorname{Re}t\to-\infty}\frac{\Phi^{\chi,\rho,p,q}(x,t)}{\operatorname{Re}t} = \lim_{\operatorname{Re}t\to-\infty}\frac{\sup_{|z-x|\leq|e^t|}\varphi_{p,q}(z)}{\operatorname{Re}t} = \nu(\varphi_{p,q},x),$$

$$\text{(7.9 ii)}\qquad \lim_{\operatorname{Re}t\to-\infty}\left|\frac{\Phi^{\chi,\rho,p,q}(x,t)}{\operatorname{Re}t} - \frac{\sup_{|z-x|\leq|e^t|}\varphi_{p,q}(z)}{\operatorname{Re}t}\right| = 0,$$

where the second limit is uniform on $U$ [For this, we use the convexity of $\operatorname{Re}t\mapsto \sup_{|z-t|\leq|e^t|}\varphi(z)$ to check that the constants $C$ in $\sup_{|z-t|\leq C|e^t|}\varphi(z)$ are irrelevant.] For future reference, we also note

(7.10) The functions $\Phi^{\chi,\rho,p,q}(x,t)$ are continuous on $U\times\{\operatorname{Re}t<-A\}$.

This is an immediate consequence of the fact that the unit ball $B_{p,q}(1)$ of $\mathcal{H}_{pq\varphi_p}(U_1)$ is a normal family of holomorphic functions. We now investigate the effect of a perturbation on $\varphi$.

Proposition 7.11. *Let $U' \Subset U'_1 \Subset U_0$, $U'' \Subset U''_1 \Subset U_0$ and let $\varphi'$, $\varphi''$ be plurisubharmonic functions on $U'_1$, $U''_1$ such that $|\varphi'' - \varphi' - \operatorname{Re} g| \le \varepsilon$ on $U'_1 \cap U''_1$, for some holomorphic function $g \in \mathcal{O}(U'_1 \cap U''_1)$. There are constants $C_2(p)$ and $C_3$ (depending also on $\varphi'$, $\varphi''$, $g$, $U'$, $U''$) such that*

$$\left|\Phi''^{\chi,\rho,p,q}(x,t) - \Phi'^{\chi,\rho,p,q}(x,t) - \operatorname{Re} g(x)\right| \le 2\varepsilon + \frac{C_2(p)}{q} + \frac{C_3}{p}$$

*for all $x \in U' \cap U''$ and $|e^t| < r_0(\varepsilon)$ small enough.*

Proof. We first treat the simpler case when $g = 0$. By 7.6 (ii), every function $f'' \in B''_{p,q}(1) \subset \mathcal{H}_{pq\varphi_p}(U''_1)$ can be written

$$f'' = \sum_{l \in \mathbb{N}} h_l \hat{\sigma}'_l, \qquad \sup_W \sum_l |h_l|^2 \le C_1(p) C^{2q} e^{2pq\varepsilon}$$

on any relatively compact neighborhood $W$ of $\overline{U' \cap U''}$ in $U'_1 \cap U''_1$. Fix a small polydisk $\overline{D}(r) \subset \mathbb{C}^n$ such that $\overline{U' \cap U''} + \overline{D}(r) \subset U'_1 \cap U''_1$, and expand

$$h_l(x) = \sum_{\alpha \in \mathbb{N}^n} a_{l,\alpha} (x - x_0)^\alpha$$

as a power series at each point $x_0 \in U' \cap U''$. By integrating $\sum |h_l|^2$ over the polycircle $\prod \partial D(x_{0,j}, r_j)$, we find

$$\sum_{l \in \mathbb{N}, \alpha \in \mathbb{N}^n} |a_{l,\alpha}|^2 |r^\alpha|^2 \le C_1(p) C^{2q} e^{2pq\varepsilon}. \tag{7.12}$$

By substituting $h_l$ with its Taylor expansion in the definition of $f''$, we find

$$f''(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha(x)\, w_\alpha(x)$$

where

$$p_\alpha(x) = (x - x_0)^\alpha, \qquad w_\alpha(x) = \sum_{l \in \mathbb{N}} a_{l,\alpha} \hat{\sigma}'_l.$$

The $L^2$ norm of $w_\alpha$ in $\mathcal{H}_{pq\varphi'_p}(U'_1)$ is $\left(\sum_l |a_{l,\alpha}|^2\right)^{1/2}$, hence by definition

$$\frac{C_0}{p} + \frac{1}{pq} \log \frac{\widehat{M}^{\chi,\rho}_{w_\alpha}(x_0,t)}{\left(\sum_l |a_{l,\alpha}|^2\right)^{1/2}} \le \Phi'_{p,q}(x_0,t).$$

On the other hand, if $|e^t| \ll \|r\|$, Lemma 6.3 implies that

$$\widehat{M}^{\chi,\rho}_{p_\alpha}(x_0,t) \le \sup_{D(x_0,r/3)} |(x - x_0)^\alpha| \le 2^{-\alpha} r^\alpha$$

From this, we infer

$$\begin{aligned}\widehat{M}^{\chi,\rho}_{f''}(x_0,t) &\le \sum_{\alpha\in\mathbb{N}^n}\widehat{M}^{\chi,\rho}_{p_\alpha}(x_0,t)\widehat{M}^{\chi,\rho}_{w_\alpha}(x_0,t)\\ &\le \sum_{\alpha\in\mathbb{N}^n}2^{-\alpha}r^\alpha\Big(\sum_l|a_{l,\alpha}|^2\Big)^{1/2}\exp\big(pq(\Phi'_{p,q}(x_0,t)-C_0/p)\big)\\ &\le \sum_{\alpha\in\mathbb{N}^n}2^{-\alpha}\big(C_1(p)C^{2q}e^{2pq\varepsilon}\big)^{1/2}\exp\big(pq(\Phi'_{p,q}(x_0,t)-C_0/p)\big)\end{aligned}$$

thanks to (7.12). By taking $\frac{1}{pq}\log(\ldots)$ and passing to the sup over all $f''$, we get

$$\Phi''_{p,q}(x_0,t)\le\frac{1}{pq}\log\big(2^n\big(C_1(p)C^{2q}e^{2pq\varepsilon}\big)^{1/2}\big)+\Phi'_{p,q}(x_0,t).$$

Proposition 7.11 is thus proved for the case $g=0$, even with $\varepsilon$ instead of $2\varepsilon$ in the final estimate. In case $g$ is non zero, we observe that the replacement of $\varphi'$ by $\varphi'+\operatorname{Re}g$ yields isomorphisms of Hilbert spaces

$$\begin{aligned}\mathcal{H}_{p\varphi'}(U_1')&\to\mathcal{H}_{p(\varphi'+\operatorname{Re}g)}(U_1'), &\qquad f&\mapsto e^{pg}f,\\ \mathcal{H}_{pq\varphi'_p}(U_1')&\to\mathcal{H}_{pq(\varphi'_p+\operatorname{Re}g)}(U_1'), &\qquad f&\mapsto e^{pqg}f.\end{aligned}$$

The only difference occurring in the proof is that we get

$$f''=e^{pqg}\sum_{l\in\mathbb{N}}h_l\hat\sigma'_l$$

instead of $f''=\sum_{l\in\mathbb{N}}h_l\hat\sigma'_l$. In the upper bound for $\widehat{M}^{\chi,\rho}_{f''}(x_0,t)$, this introduces an extra term $\widehat{M}^{\chi,\rho}_{e^{pqg}}(x_0,t)$, which we evaluate as $\exp(pq(\operatorname{Re}g(x_0)+O(|e^t|)))$ thanks to Lemma 6.3. The general estimate follows, possibly with an additional $\varepsilon$ error when $|e^t|$ is small enough. ☐

The final step in the construction is to "glue" together the functions $\Phi^{\chi,\rho,p,q}(x,t)$, $(p,q)\in\mathbb{N}^2$. We choose a fast increasing sequence $p\mapsto q(p)$, in such a way that $C_2(p)/q(p)\le 1/p$, where $C_2(p)$ is the constant occurring in Proposition 7.11. We now define

$$\begin{aligned}\widetilde{M}^{\chi,\rho,s}_{\varphi}(x,t)&:=\Phi^{\chi,\rho,s}(x,t)\\ &:=\sup_{p\ge s}\Big(\Phi^{\chi,\rho,p,q(p)}(x,t-\log p)+\frac{\log p}{p}+n\Big(\frac1p+\frac{1}{p\,q(p)}\Big)\operatorname{Re}t\Big).\end{aligned}\tag{7.13}$$

[The terms in $\log p$ are there only for a minor technical reason, to make sure that $\widetilde{M}^{\chi,\rho,s}_{\varphi}(x,t)$ is a continuous function.] In this way, we achieve the expected goals, namely:

Proposition 7.14. *Let $\varphi$ be a plurisbharmonic function defined on a bounded Stein open set $U_0\Subset\mathbb{C}^n$ such that $T^\star_{U_0}$ is equipped with a plurisubharmonic smooth Finsler metric, and let $U\Subset U_0$. Then there is a functional $\widetilde{M}^{\chi,\rho,s}$ (associated with the choice of a sequence $q(p)$ which may have to be adjusted when $\varphi$ varies,* *but can be taken fixed if $\varphi$ remains in a bounded set of $L^1(U_0)$), such that the functions $\Phi^{\chi,\rho,s}(x,t) = \widetilde{M}_\varphi^{\chi,\rho,s}(x,t)$ satisfy the following properties*:

(i) *The functions $\Phi^{\chi,\rho,s}(x,t)$ are defined on $U \times \{\mathrm{Re}\, t < -A\}$ for $A > 0$ large, and are locally bounded continuous plurisubharmonic functions depending only on $\mathrm{Re}\, t$; moreover, $p \mapsto \Phi^{\chi,\rho,s}(x,t)$ is a decreasing family of functions.*

(ii) $\varphi(x) \le \Phi^{\chi,\rho,s}(x,t) \le \sup_{\|z-x\| \le Cs^{-1}|e^t|} \varphi(z) + C\,\dfrac{\log s}{s}$ *for some $C \gg 0$.*

(iii) $$\lim_{\mathrm{Re}\, t \to -\infty} \left| \frac{\Phi^{\chi,\rho,s}(x,t)}{\mathrm{Re}\, t} - \frac{\sup_{\|z-x\| \le |e^t|} \varphi(z)}{\mathrm{Re}\, t} \right| = 0$$

*uniformly on every compact subset of $U$, in particular*

$$\lim_{\mathrm{Re}\, t \to -\infty} \frac{\Phi^{\chi,\rho,s}(x,t)}{\mathrm{Re}\, t} = \nu(\varphi, x)$$

*for every $x \in U$.*

(iv) *For every holomorphic change of coordinates $\tau : U_0 \to \tau(U_0)$, the sequence $\rho = (\rho_m)$ can be chosen (depending only on $\tau$) such that for some constant $C > 0$ we have*

$$\left| \widetilde{M}^{\chi,\rho,s}_{\varphi\circ\tau^{-1}}(\tau(x),t) - \widetilde{M}^{\chi,\rho,s}_{\varphi}(x,t) \right| \le \frac{C}{s} \qquad \textit{for all } x \in U,$$

*when $T^\star_{\tau(U_0)}$ is equipped with the induced Finsler metric.*

(v) *Let $U' \Subset U'_1 \Subset U_0$, $U'' \Subset U''_1 \Subset U_0$ be Stein open subsets, and let $\varphi'$, $\varphi''$ be plurisubharmonic functions on $U'_1$, $U''_1$ such that $|\varphi' - \varphi| \le 1$ on $U'_1$, $|\varphi'' - \varphi| \le 1$ on $U''_1$ and $|\varphi'' - \varphi' - \mathrm{Re}\, g| \le \varepsilon$ on $U'_1 \cap U''_1$ for some holomorphic function $g \in \mathcal{O}(U'_1 \cap U''_1)$. Then*

$$\left| \widetilde{M}^{\chi,\rho,s}_{\varphi''}(x,t) - \widetilde{M}^{\chi,\rho,s}_{\varphi'}(x,t) \right| \le 2\varepsilon + \frac{C}{s} \qquad \textit{for all } x \in U' \cap U'' \textit{ and } |e^t| < r_0(\varepsilon),$$

*where $C = C(\varphi, U', U'')$.*

Proof. All properties are almost immediate consequences of the properties already obtained for $\Phi^{\chi,\rho,p,q}$, simply by taking the supremum. We check e.g. the continuity of $\Phi^{\chi,\rho,s}$, inequality (ii) and the second statement of (iii). In fact, (7.8 i,ii) imply

$$\begin{aligned}\varphi(x) + \frac{\log p}{p} + n\Big(\frac{1}{p} + \frac{1}{p\,q(p)}\Big)\,\mathrm{Re}\, t,\\ \le \Phi^{\chi,\rho,p,q(p)}(x, t - \log p) + \frac{\log p}{p} + n\Big(\frac{1}{p} + \frac{1}{p\,q(p)}\Big)\,\mathrm{Re}\, t\\ \le \sup_{\|z-x\| \le Cp^{-1}|e^t|} \varphi(z) + C\,\frac{\log p}{p}\end{aligned}$$

and (7.14 ii) follows from this. Moreover, the function $\Phi^{\chi,\rho,p,q(p)}(x, t - \log p) + \cdots$ converges to $\varphi(x)$ as $p \to +\infty$, while its terms get $> \varphi(x)$ for $p$ large, thanks to the lower bound. It follows that the sup in (7.13) is locally finite, therefore $\Phi^{\chi,\rho,s}$ is continuous. To prove (iii), we first observe that the right hand inequality in (i) gives

$$\lim_{\mathrm{Re}\,t\to-\infty}\frac{\Phi^{\chi,\rho,s}(x,t)}{\mathrm{Re}\,t}\ge\lim_{\mathrm{Re}\,t\to-\infty}\frac{\sup_{|z-x|\le|e^t|}\varphi(z)}{\mathrm{Re}\,t}=\nu(\varphi,x).$$

In the other direction, the definition of $\Phi^{\chi,\rho,s}(x,t)$ combined with (7.8 ii) implies

$$\begin{aligned}\Phi^{\chi,\rho,s}(x,t)&\ge\Phi^{\rho,p,q(p)}(x,t-\log p)+n\Big(\frac{1}{p}+\frac{1}{p\,q(p)}\Big)\,\mathrm{Re}\,t\\&\ge\sup_{|z-x|\le p^{-1}c_{\rho,p,q(p),\varphi,U}|e^t|}\varphi(z)+n\Big(\frac{1}{p}+\frac{1}{p\,q(p)}\Big)\,\mathrm{Re}\,t\end{aligned}$$

for all $p\ge s$, hence

$$\lim_{\mathrm{Re}\,t\to-\infty}\frac{\Phi^{\chi,\rho,s}(x,t)}{\mathrm{Re}\,t}\le\lim_{\mathrm{Re}\,t\to-\infty}\frac{\sup_{|z-x|\le|e^t|}\varphi(z)}{\mathrm{Re}\,t}+n\Big(\frac{1}{p}+\frac{1}{p\,q(p)}\Big).$$

We get the desired conclusion by letting $p\to+\infty$. □

## 8. A Variant of Kiselman's Legendre Transform

To begin with, let $\varphi$ be a plurisubharmonic function on a bounded pseudoconvex open set $U\Subset\mathbb{C}^n$. Consider the trivial vector bundle $T_U=U\times\mathbb{C}^n$, and assume that $T_U^\star$ is equipped with a smooth Finsler metric ${}_{\chi^\star}\|\xi^\star\|^\star=\exp(\chi^\star(z,\xi^\star))$ for $\xi^\star\in T^\star_{U,z}$. We assume that the curvature of the Finsler metric $\|\xi^\star\|^\star_z=e^{\chi^\star(z,\xi^\star)}$ on $T_U^\star$ satisfies

$$\frac{i}{\pi}\partial\bar\partial\chi^\star(z,\xi^\star)+\pi_U^\star u(z)\ge0 \tag{8.1}$$

for some nonnegative continuous $(1,1)$-form $u$ on $U$, where $\pi_U:T_U^\star\to U$ is the projection. If $\chi^\star=\log h^\star$ is a hermitian metric on $T_X^\star$, we let $h$ be the dual metric on $T_X$ and set

$$\Phi^h_\infty(z,w)=\sup_{{}_h\|\xi\|_z\le|e^w|}\varphi(z+\xi). \tag*{$(8.2)^h$}$$

By Theorem 6.1, this definition works equally well when $h^\star$ is a *fiberwise convex* Finsler metric. Clearly $\Phi^h_\infty(z,w)$ depends only on the real part $\mathrm{Re}\,w$ of $w$ and is defined on the open set $\Omega$ of points $(z,w)\in U\times\mathbb{C}$ such that $\mathrm{Re}\,w<\log d_z(z,\partial U)$, where $d_z$ denotes euclidean distance with respect to ${}_h\|\ \|_z$. Now, we would like to extend this to the case of a general Finsler metric, without any convexity assumption. As a replacement for the "sup formula" $(8.2)^h$, we set

$$\Phi^{\chi,\rho,s}_\infty(z,w)=\widetilde M^{\chi,\rho,s}_\varphi(z,w) \tag*{$(8.2)^\chi$}$$

where $\widetilde{M}^{\chi,\rho,s}$ denotes the functional associated with $\chi$, as in § 7. Here, however, $\chi(z,\xi^\star)$ need not be plurisubharmonic. This is not a real difficulty, since the definition of the $\widehat{M}^{\chi,\rho}$ functional in (6.6) shows that

$$\widetilde{M}_\varphi^{\chi,\rho,s}(z,w) = \widetilde{M}_\varphi^{\chi_v,\rho,s}(z,w-v(z))$$

for any smooth function $v$ on $U$ such that $\chi_v(z,\xi^\star) := \chi(z,\xi^\star)+v(z)$ is plurisubharmonic on $U$ (and such a function always exists by our assumption (8.1)). One of our main concern is to investigate singularities of $\varphi$ and these singularities are reflected in the way $\Phi_\infty^h(z,w)$ and $\Phi_\infty^{\chi,\rho,s}$ decay to $-\infty$ as $\mathrm{Re}\,w$ goes to $-\infty$. In this perspective, Proposition 7.14 (iii) shows that considering $\Phi_\infty^{\chi,\rho,s}(z,w)$ instead of $\Phi_\infty^h(z,w)$ does not make any difference. Moreover $\Phi_\infty^{\chi,\rho,s}(z,w)$ and $\Phi_\infty^h(z,w)$ are both convex increasing function of $\mathrm{Re}\,w$. For $(z,w)\in\Omega$ and $c>0$, we introduce the (generalized) *Legendre transform*

$$\Phi_c^h(z,w) = \inf_{t\le 0} \Phi_\infty^h(z,w+t) - ct, \tag{8.3$^h$}$$

$$\Phi_c^{\chi,\rho,s}(z,w) = \inf_{t\le 0} \Phi_\infty^{\chi,\rho,s}(z,w+t) - ct. \tag{8.3$^\chi$}$$

It is easy to see that these functions are increasing in $c$ and that

$$\lim_{c\to 0} \Phi_c^h(z,w) = \varphi(z), \qquad \lim_{c\to+\infty} \Phi_c^h = \Phi_\infty^h. \tag{8.4$^h$}$$

The analogue for $\Phi_c^{\chi,\rho,s}$ is

$$\begin{aligned} &\lim_{c\to 0} \Phi_c^{\chi,\rho,s}(z,w) = \lim_{\mathrm{Re}\,t\to-\infty} \Phi_\infty^{\chi,\rho,s}(z,\mathrm{Re}\,t) \in \big[\varphi(z),\ \varphi(z)+C\log s/s\big],\\ &\lim_{c\to+\infty} \Phi_c^{\chi,\rho,s} = \Phi_\infty^{\chi,\rho,s}. \end{aligned} \tag{8.4$^\chi$}$$

When ${}_h\|\xi\|_z$ is taken to be a constant metric, we know by [Kiselman 1978] that $\Phi_\infty^h$ and $\Phi_c^h$ are plurisubharmonic functions of the pair $(z,w)$, and that the Lelong numbers of $\Phi_c^h(\,\cdot\,,w)$ are given by

$$\nu\big(\Phi_c^h(\,\cdot\,,w),z\big) = \big(\nu(\varphi,z)-c\big)_+, \qquad \text{for all } (z,w)\in\Omega. \tag{8.5$^h$}$$

Since (8.5)$^h$ depends only on the maps $z\mapsto\Phi_\infty^h(z,w)$ with $w$ fixed, the equality is still valid when $h$ is a variable hermitian metric, and Proposition 7.14 (iii) even shows that the analogous property for $\Phi_c^{\chi,\rho,s}$ is true:

$$\nu\big(\Phi_c^{\chi,\rho,s}(\,\cdot\,,w),z\big) = \big(\nu(\varphi,z)-c\big)_+, \qquad \text{for all } (z,w)\in\Omega. \tag{8.5$^\chi$}$$

As usual we denote by

$$E_c(\varphi) = \big\{z\in U : \nu(\varphi,x)\ge c\big\} \tag{8.6}$$

the Lelong sublevel sets of $\varphi$. From now on, we omit the superscripts in the notation $\Phi_c^h$ or $\Phi_c^{\chi,\rho,s}$ since all properties are the same in both cases. In general, $\Phi_\infty$ is continuous on $\Omega$ and its right derivative $\partial\Phi_\infty(z,w)/\partial\,\mathrm{Re}\,w_+$ is upper semicontinuous; indeed, this partial derivative is the decreasing limit of $\big(\Phi_\infty(z,w+t)-\Phi_\infty(z,w)\big)/t$ as $t\downarrow 0_+$. It follows that $\Phi_c$ is continuous on

$\Omega \smallsetminus (E_c(\varphi) \times \mathbb{C})$: in fact, we have $\nu(\varphi, z) = \lim_{t \to -\infty} \partial\Phi_\infty(z,t)/\partial t_+ < c$ on every compact set $K \subset \Omega \smallsetminus (E_c(\varphi) \times \mathbb{C})$, so by the upper semicontinuity there is a constant $t_0$ such that $\partial\Phi_\infty(z, w+t)/\partial t_+ < c$ for $(z,w) \in K$ and $t < t_0$. Therefore

$$\Phi_c(z,w) = \inf_{t_0 \le t \le 0} \Phi_\infty(z, w+t) - ct \qquad \text{on } K,$$

and this infimum with compact range is continuous. Our next goal is to investigate the plurisubharmonicity of $\Phi_c$.

Proposition 8.7. *Assume the curvature of the Finsler metric $\|\xi^\star\|^\star_z = e^{\chi^\star(z,\xi^\star)}$ on $E^\star$ satisfies*

$$\frac{i}{\pi}\partial\bar\partial\chi^\star(z,\xi^\star) + \pi_X^\star u(z) \ge 0$$

*for some nonnegative continuous $(1,1)$-form $u$ on $X$, where $\pi_X : E^\star \to X$ is the projection. Then $\Phi_c = \Phi_c^{\chi,\rho,s}$ [and likewise $\Phi_c = \Phi_c^h$] enjoys the following properties.*

(i) *For all $\eta \ge 0$, we have*

$$\Phi_c(z, w-\eta) \ge \Phi_c(z,w) - \min\left\{\frac{\partial\Phi_\infty(z,w)}{\partial\operatorname{Re} w_-}\,,\, c\right\}\eta\,;$$

(ii) *For $(\zeta,\eta) \in T_U \times \mathbb{C}$ and $c \in \,]0,+\infty]$, the Hessian of $\Phi_c$ satisfies*

$$\frac{i}{\pi}\partial\bar\partial(\Phi_c)_{(z,w)}(\zeta,\eta) \ge -\min\left\{\frac{\partial\Phi_\infty(z,w)}{\partial\operatorname{Re} w_+}\,,\, c\right\} u_z(\zeta).$$

Proof. (i) For $\eta \ge 0$ and $t \le 0$, the convexity of $\Phi_\infty(z,w)$ in $\operatorname{Re} w$ implies

$$\Phi_\infty(z, w+t-\eta) \ge \Phi_\infty(z,w+t) - \eta\,\frac{\partial\Phi_\infty(z,w+t)}{\partial\operatorname{Re} w_-}$$

As $\partial\Phi_\infty(z,w)/\partial\operatorname{Re} w_-$ is increasing in $\operatorname{Re} w$, the infimum of both sides minus $ct$ gives

$$\Phi_c(z,w-\eta) \ge \Phi_c(z,w) - \eta\frac{\partial\Phi_\infty(z,w)}{\partial\operatorname{Re} w_-}.$$

On the other hand, the change of variables $t = t' + \eta$ yields

$$\Phi_c(z,w-\eta) \ge \inf_{t' \le -\eta} \Phi_\infty(z, w+t') - c(t'+\eta) \ge \Phi_c(z,w) - c\eta.$$

Property (i) follows.

(ii) Fix $(z_0, w_0) \in \Omega$ and a semipositive quadratic function $v(z)$ on $\mathbb{C}^n$ such that $\frac{i}{\pi}\partial\bar\partial v(0) > u_{z_0}$. Then the inequality $\frac{i}{\pi}\partial\bar\partial v(z - z_0) > u(z)$ still holds on a neighborhood $U_0$ of $z_0$, and the Finsler metric $\|\xi^\star\|^\star_z e^{v(z-z_0)}$ is plurisubharmonic on this neighborhood. From this, we conclude by Lemma 7.7 that the associated function

$$h(z,w) := \Phi_\infty(z, w + v(z-z_0))$$

is plurisubharmonic on $U_0$. Its Legendre transform

$$h_c(z,w) = \inf_{t\le 0} h(z,w-t) = \Phi_c(z, w+v(z-z_0))$$

is again plurisubharmonic. For small $(\zeta,\eta)\in T_U\times\mathbb{C}$, the mean value inequality yields

$$\begin{aligned}\int_0^{2\pi}\Phi_c(z_0+e^{i\theta}\zeta, w_0+e^{i\theta}\eta)\,\frac{d\theta}{2\pi} &= \int_0^{2\pi} h_c\big(z_0+e^{i\theta}\zeta, w_0+e^{i\theta}\eta - v(\zeta)\big)\,\frac{d\theta}{2\pi}\\ &\ge h_c\big(z_0, w_0-v(\zeta)\big) = \Phi_c\big(z_0,w_0-v(\zeta)\big)\\ &\ge \Phi_c(z_0,w_0) - \min\Big\{\frac{\partial\Phi_\infty(z_0,w_0)}{\partial\,\mathrm{Re}\,w_+}\,,\,c\Big\}v(\zeta)\end{aligned}$$

[the last inequality follows from (i)]. For $A>\partial\Phi_\infty(z_0,w_0)/\partial\mathrm{Re}\,w_+$, we still have $A>\partial\Phi_\infty(z,w)/\partial\mathrm{Re}\,w_+$ in a neighborhood of $(z_0,w_0)$ by the upper semicontinuity, and we conclude that the function $\Phi_c(z,w)+\min\{A,c\}v(z)$ satisfies the mean value inequality near $(z_0,w_0)$. Hence $\Phi_c(z,w)+\min\{A,c\}v(z)$ is plurisubharmonic near $(z_0,w_0)$. Since this is still true as $A$ tends to $\partial\Phi_\infty(z,w)/\partial\mathrm{Re}\,w_+$ and $\frac{i}{\pi}\partial\bar\partial v$ tends to $u_{z_0}$, the proof of (ii) is complete. $\square$

## 9. Regularization of Closed Positive $(1,1)$-Currents

The next step is to describe a gluing process for the construction of global regularizations of almost plurisubharmonic functions. We suppose that $T_X^\star$ is equipped with a Finsler metric $\|\xi^\star\|_x^\star = e^{\chi^\star(x,\xi^\star)}$ satisfying

$$\frac{i}{\pi}\partial\bar\partial\chi^\star(x,\xi^\star)+\pi_X^\star u(x)\ge 0,$$

where $u$ is a smooth semipositive $(1,1)$-form on $X$. Notice that $\frac{i}{\pi}\partial\bar\partial\chi^\star(z,\xi^\star)$ is just the Chern curvature of the induced hermitian metric on $\mathcal{O}_{TX}(1)$. An *almost positive* $(1,1)$-current is by definition a real $(1,1)$-current such that $T\ge\gamma$ for some real $(1,1)$-form $\gamma$ with locally bounded coefficients. An *almost psh function* is a function $\psi$ which can be written locally as $\psi=\varphi+w$ where $\varphi$ is plurisubharmonic and $w$ smooth. With these definitions, $\frac{i}{\pi}\partial\bar\partial\psi$ is almost positive if and only if $\psi$ is almost psh.

The following thereom was proved in [Demailly 1992] with a rather long and tricky proof. We present here a shorter and better approach using our modified Kiselman–Legendre transforms.

Theorem 9.1. *Let $T$ be a closed almost positive $(1,1)$-current and let $\alpha$ be a smooth real $(1,1)$-form in the same $\partial\bar\partial$-cohomology class as $T$, i.e. $T=\alpha+\frac{i}{\pi}\partial\bar\partial\psi$ where $\psi$ is an almost psh function. Let $\gamma$ be a continuous real $(1,1)$-form such that $T\ge\gamma$. Suppose that $\mathcal{O}_{TX}(1)$ is equipped with a smooth hermitian metric such that the Chern curvature form satisfies*

$$\Theta\big(\mathcal{O}_{TX}(1)\big)+\pi_X^\star u\ge 0$$

*with $\pi_X : P(T^\star X) \to X$ and with some nonnegative smooth $(1,1)$-form $u$ on $X$. Fix a hermitian metric $\omega$ on $X$. Then for every $c > 0$, there is a sequence of closed almost positive $(1,1)$-currents $T_{c,k} = \alpha + \frac{i}{\pi}\partial\bar{\partial}\psi_{c,k}$ such that $\psi_{c,k}$ is smooth on $X \smallsetminus E_c(T)$ and decreases to $\psi$ as $k$ tends to $+\infty$ (in particular, the current $T_{c,k}$ is smooth on $X \smallsetminus E_c(T)$ and converges weakly to $T$ on $X$), and such that*

(i) $T_{c,k} \geq \gamma - \min\{\lambda_k, c\}u - \varepsilon_k\omega$, *where*
(ii) $\lambda_k(x)$ *is a decreasing sequence of continuous functions on $X$ such that the limit* $\lim_{k\to+\infty}\lambda_k(x)$ *equals* $\nu(T,x)$ *at every point*,
(iii) $\varepsilon_k$ *is positive decreasing and* $\lim_{k\to+\infty}\varepsilon_k = 0$,
(iv) $\nu(T_{c,k},x) = \big(\nu(T,x) - c\big)_+$ *at every point* $x \in X$.

Proof. We first show that we indeed can write $T = \alpha + \frac{i}{\pi}\partial\bar{\partial}\psi$ with $\alpha$ smooth. Let $(U_j^0)$ be a finite covering of $X$ by coordinate balls and $(\theta_j)$ a partition of unity subordinate to $(U_j^0)$. If $T$ is written locally $T = \frac{i}{\pi}\partial\bar{\partial}\psi_j$ with $\psi_j$ defined on $U_j^0$, then $\psi = \sum\theta_j\psi_j$ has the property that $\alpha := T - \frac{i}{\pi}\partial\bar{\partial}\psi$ is smooth. This is an easy consequence of the fact that $\psi_k - \psi_j$ is plurisubharmonic, hence smooth, on $U_j^0 \cap U_k^0$, writing $T$ as $\frac{i}{\pi}\partial\bar{\partial}\psi_k$ over $U_k^0$. By replacing $T$ with $T - \alpha$ and $\gamma$ with $\gamma - \alpha$, we can assume that $\alpha = 0$ (in other words, Theorem 9.1 essentially deals only with the singular part of $T$).

We can therefore assume that $T = \frac{i}{\pi}\partial\bar{\partial}\psi$, where $\psi$ is an almost plurisubharmonic function on $X$ such that $T \geq \gamma$ for some continuous $(1,1)$-form $\gamma$. We select a finite covering $\mathcal{W} = (W_\nu)$ of $X$ by open coordinate charts. Given $\delta > 0$, we take in each $W_\nu$ a maximal family of points with (coordinate) distance to the boundary $\geq 3\delta$ and mutual distance $\geq \delta$. In this way, we get for $\delta > 0$ small a finite covering of $X$ by open balls $U_j$ of radius $\delta$, such that the concentric ball $U_j^0$ of radius $2\delta$ is relatively compact in the corresponding chart $W_\nu$. Let $\tau_j : U_j^0 \to B_j^0 := B(a_j, 2\delta)$ be the isomorphism given by the coordinates of $W_\nu$ and

$$
\begin{aligned}
&B_j \Subset B_j^1 \Subset B_j^0, && B_j = B(a_j,\delta), && B_j^1 = B(a_j,\sqrt{2}\,\delta), && B_j^0 = B(a_j,2\delta),\\
&U_j \Subset U_j^1 \Subset U_j^0, && U_j = \tau_j^{-1}(B_j), && U_j^1 = \tau_j^{-1}(B_j^1), && U_j^0 = \tau_j^{-1}(B_j^0).
\end{aligned}
$$

Let $\varepsilon(\delta)$ be a modulus of continuity for $\gamma$ on the sets $U_j^0$, such that $\lim_{\delta\to 0}\varepsilon(\delta) = 0$ and $\gamma_x - \gamma_{x'} \leq \frac{1}{2}\varepsilon(\delta)\,\omega_x$ for all $x, x' \in U_j^0$. We denote by $\gamma_j$ the $(1,1)$-form with constant coefficients on $B_j^0$ such that $\tau_j^\star\gamma_j$ coincides with $\gamma - \varepsilon(\delta)\,\omega$ at $\tau_j^{-1}(a_j)$. Then we have

$$
0 \leq \gamma - \tau_j^\star\gamma_j \leq 2\varepsilon(\delta)\,\omega \qquad \text{on } U_j \tag{9.2}
$$

for $\delta > 0$ small. We set $\psi_j = \psi \circ \tau_j^{-1}$ on $B_j^0$ and let $\tilde{\gamma}_j$ be the homogeneous quadratic function in $z - a_j$ such that $\frac{i}{\pi}\partial\bar{\partial}\tilde{\gamma}_j = \gamma_j$ on $B_j^0$. Finally, we set

$$
\varphi_j(z) = \psi_j(z) - \tilde{\gamma}_j(z) \qquad \text{on } B_j^0. \tag{9.3}
$$

It is clear that $\varphi_j$ is plurisubharmonic, since

$$\frac{i}{\pi}\partial\bar\partial(\varphi_j\circ\tau_j) = T - \tau_j^\star\gamma_j \geq \gamma - \tau_j^\star\gamma_j \geq 0.$$

We combine $(8.2)^\chi$ and $(8.3)^\chi$ to define "regularized" functions

$$(9.4)\quad \Phi_{j,c}^{\chi,\rho,s}(z,w) = \inf_{t\leq 0}\widetilde{M}_{\varphi_j}^{\chi,\rho,s}(z,w), \qquad z\in B_j^1,$$

$$(9.5)\quad \Psi_{j,c}^{\chi,\rho,s}(z,w) = \Phi_{j,c}^{\chi,\rho,s}(z,w) + \tilde\gamma_j(z) - \varepsilon(\delta)^{1/2}|z-a_j|^2, \qquad z\in B_j^1,$$

$$(9.6)\quad \Psi_{c}^{\chi,\rho,s}(x,w) = \sup_{U_j^1\ni x}\Psi_{j,c}^{\chi,\rho,s}(\tau_j(x),w), \qquad x\in X,$$

for $\mathrm{Re}\,w < -A$, with $A\gg 0$. We have to check that the gluing procedure used in the definition of $\Psi_c^{\chi,\rho,s}$ does not introduce discontinuities when $x$ passes through a boundary $\partial U_j^1$. For this, we must compare $\Psi_{j,c}^{\chi,\rho,s}(\tau_j(x),w)$ and $\Psi_{k,c}^{\chi,\rho,s}(\tau_k(x),w)$ on overlapping open sets $U_j^1$, $U_k^1$. The comparison involves two points:

- effect of replacing $\psi_j$ with $\psi_j - \tilde\gamma_j$, and
- effect of coordinate changes.

First assume for simplicity that $U_j^1$ and $U_k^1$ are contained in the same coordinate patch $W_\nu$ (in such a way that $\tau_j = \tau_k$ on $U_j^1\cap U_k^1$, therefore in this case, we do not have to worry about coordinate changes). Then $\psi_j = \psi_k$ on $B_j^1\cap B_k^1$, and therefore $\varphi_k - \varphi_j = \tilde\gamma_j - \tilde\gamma_k$ is a quadratic function whose Levi form is $O(\varepsilon(\delta))$, by the assumption on the modulus of continuity of $\gamma$. This quadratic function can be written as

$$\tilde\gamma_j(z) - \tilde\gamma_k(z) = \mathrm{Re}\,g_{jk}(z) + q_{jk}(z - z_{jk}^0),$$

the sum of an affine pluriharmonic part $\mathrm{Re}\,g_{jk}$ and a quadratic term $q_{jk}(z-z_{jk}^0)$ which takes $O(\varepsilon(\delta)\delta^2)$ values (since $\mathrm{diam}\,B_j^1\cap B_k^1\leq\delta$). Therefore we have

$$|\varphi_k - \varphi_j - \mathrm{Re}\,g_{jk}| \leq C\varepsilon(\delta)\delta^2.$$

By 7.14 (v), we conclude that

$$\big|\Phi_{k,c}^{\chi,\rho,s}(z,w) - \Phi_{j,c}^{\chi,\rho,s}(z,w) - \mathrm{Re}\,g_{jk}(z)\big| \leq 2C\varepsilon(\delta)\delta^2 + \frac{C'}{s}$$

for some constants $C$, $C'$, hence

$$\big|\big(\Phi_{k,c}^{\chi,\rho,s}(z,w) + \tilde\gamma_k(z)\big) - \big(\Phi_{j,c}^{\chi,\rho,s}(z,w) + \tilde\gamma_j(z)\big)\big| \leq 3C\varepsilon(\delta)\delta^2 + \frac{C'}{s}.$$

Now, in case $U_j^1$ and $U_k^1$ are not equipped with the same coordinates, 7.14 (iv) shows that an extra error term $C/s$ is introduced by the change of coordinates $\tau_{jk} = \tau_j\circ\tau_k^{-1}$, and also possibly a further $O(\delta^3)$ term due to the fact that

$\tilde{\gamma}_j \circ \tau_{jk}$ differs from a quadratic function by terms of order 3 or more in the $\tau_k$-coordinates. Combining everything together, we get

$$\begin{aligned}\big|\big(\Phi^{\chi,\rho,s}_{k,c}(\tau_k(z),w) + \tilde{\gamma}_k(\tau_k(z))\big) - \big(\Phi^{\chi,\rho,s}_{j,c}(\tau_j(z),w) + \tilde{\gamma}_j(\tau_j(z))\big)\big| \\ \le C''\Big(\varepsilon(\delta)\delta^2 + \delta^3 + \frac{1}{s}\Big) \le C'''\varepsilon(\delta)\delta^2\end{aligned}$$

if we choose $s \ge 1/(\varepsilon(\delta)\delta^2)$. We assume from now on that $s$ is chosen in this way. For $x \in \partial U_j^1 = \tau_j^{-1}(S(a_j, \sqrt{2}\delta))$, formula (9.5) yields

$$\Psi^{\chi,\rho,s}_{j,c}(\tau_j(z),w) = \Phi^{\chi,\rho,s}_{j,c}(\tau_j(z),w) + \tilde{\gamma}_j(\tau_j(z)) - \varepsilon(\delta)^{1/2}2\delta^2,$$

whereas there exists $k$ such that $x \in U_k = \tau_k^{-1}(B(a_k,\delta))$, hence

$$\Psi^{\chi,\rho,s}_{k,c}(\tau_k(z),w) \ge \Phi^{\chi,\rho,s}_{k,c}(\tau_j(z),w) + \tilde{\gamma}_k(\tau_k(z)) - \varepsilon(\delta)^{1/2}\delta^2.$$

We infer from this

$$\Psi^{\chi,\rho,s}_{k,c}(\tau_k(z),w) - \Psi^{\chi,\rho,s}_{j,c}(\tau_j(z),w) \ge \varepsilon(\delta)^{1/2}\delta^2 - C'''\varepsilon(\delta)\delta^2 > 0$$

for $\delta$ small enough. This shows that formula (9.6) makes sense for $\delta$ small. Formulas (9.2) and (9.5) show that

$$\frac{i}{\pi}\partial\bar{\partial}_z\Psi^{\chi,\rho,s}_{j,c}(\tau_j(z),w) \ge \frac{i}{\pi}\partial\bar{\partial}_z\Phi^{\chi,\rho,s}_{j,c}(\tau_j(z),w) + \gamma - C\varepsilon(\delta)^{1/2}\omega \tag{9.7}$$

for some constant $C > 0$. The sequence of approximations $\psi_{c,k}$ needed in the theorem is obtained by taking sequences $\delta_k \downarrow 0$, $s_k \ge 1/(\varepsilon(\delta_k)\delta_k^2)$ and $A_k \uparrow +\infty$, and putting

$$\tilde{\psi}_{c,k}(z) = \Psi^{\chi,\rho,s_k}_c(z,-A_k) + \frac{1}{k}$$

where $\Psi^{\chi,\rho,s_k}_c$ is constructed as above by means of an open covering $\mathcal{U}_k$ of $X$ with balls of radii $\sim \delta_k$. By (9.7) and Proposition 8.7 ii), we find

$$\frac{i}{\pi}\partial\bar{\partial}\tilde{\psi}_{c,k} \ge -\min\Big(\frac{\partial\Phi^{\chi,\rho,s_k}_\infty}{\partial\,\mathrm{Re}\,w_-}(z,-A_k), c\Big)u - C\varepsilon(\delta_k)^{1/2}\omega.$$

As $\displaystyle\lim_{\mathrm{Re}\,w\to-\infty}\frac{\partial\Phi^{\chi,\rho,s}}{\partial\,\mathrm{Re}\,w_-}(z,w) = \nu(\varphi,z) = \nu(\psi,z)$, a suitable choice of $A_k$ ensures that

$$\tilde{\lambda}_k(z) := \frac{\partial\Phi^{\chi,\rho,s_k}_\infty}{\partial\,\mathrm{Re}\,w_-}(z,-A_k) \to \nu(\psi,z) \qquad \text{as } k \to +\infty.$$

Furthermore, an appropriate choice of the sequences $\delta_k$, $s_k$, $A_k$ guarantees that the sequence $\tilde{\psi}_{c,k}$ is non increasing. [The only point we have to mind about is the effect of a change of the open covering, as the radius $\delta_k$ of the covering balls decreases to 0. However, Proposition 7.14 (iv, v) shows that the effect can be made negligible with respect to $\frac{1}{k} - \frac{1}{k+1}$, and then everything is ok.] We can ensure as well that $\lambda_k$ is decreasing, by replacing if necessary $\tilde{\lambda}_k$ with

$$\lambda_k(z) = \sup_{l\ge k}\tilde{\lambda}_l(z).$$

Finally, the functions $\tilde{\psi}_{c,k}$ that we got are (a priori) just known to be continuous on $X \smallsetminus E_c(T)$, thanks to Proposition 7.14 (i) and the discussion before Proposition 8.7. Again, Richberg's approximation theorem [1968] shows that we can replace $\tilde{\psi}_{c,k}$ with a smooth approximation $\psi_{c,k}$ on $X \smallsetminus E_c(T)$, with $|\tilde{\psi}_{c,k} - \psi_{c,k}|$ arbitrarily small in uniform norm, and at the expense of losing an extra error term $\varepsilon_k \omega$ in the lower bound for $\frac{i}{\pi}\partial\bar{\partial}\psi_{c,k}$. Theorem 9.1 is proved. □

## 10. Appendix: Basic Results on $L^2$ Estimates

We state here the basic $L^2$ existence theorems used in the above sections, concerning $\bar{\partial}$ equations or holomorphic functions. The first of these is the intrinsic manifold version of Hörmander's $L^2$ estimates [Hörmander 1965; 1966], based on the Bochner–Kodaira–Nakano technique. See also [Andreotti and Vesentini 1965].

Theorem 10.1. *Let $L$ be a holomorphic line bundle on a weakly pseudoconvex $n$-dimensional manifold $X$ equipped with a Kähler metric $\omega$. Suppose that $L$ has a smooth hermitian metric whose curvature form satisfies*

$$2\pi\,\Theta(L) + i\partial\bar{\partial}\varphi \geq A\,\omega$$

*where $\varphi$ is an almost psh function and $A$ a positive continuous function on $X$. Then for every form $v$ of type $(n,q)$, $q \geq 1$, with values in $L$, such that $\bar{\partial}v = 0$ and*

$$\int_X \frac{1}{A}|v|^2 e^{-\varphi} dV_\omega < +\infty,$$

*there exists a form $u$ of type $(n, q-1)$ with values in $L$ such that $\bar{\partial}u = v$ and*

$$\int_X |u|^2 e^{-\varphi} dV_\omega \leq \frac{1}{q}\int_X \frac{1}{A}|v|^2 e^{-\varphi} dV_\omega.$$

A *weakly pseudoconvex manifold* is by definition a complex manifold possessing a smooth weakly pseudoconvex exhaustion function (examples: Stein manifolds, compact manifolds, the total space of a Griffiths weakly negative vector bundle, and so on). Suppose that $\varphi$ has Lelong number $\nu(\varphi, x) = 0$ at a given point $x$. Then for every $m$ the weight $e^{-m\varphi}$ is integrable in a small neighborhood $V$ of $x$ [Skoda 1972b]. Let $\theta$ be a cut-off function equal to 1 near $x$, with support in $V$. Let $z$ be coordinates and let $e$ be a local frame of $L$ on $V$. For $\varepsilon$ small enough, the curvature form

$$2\pi\,\Theta(L) + i\partial\bar{\partial}\big(\varphi(z) + 2\varepsilon\theta(z)\log|z-x|\big)$$

is still positive definite. We apply A.1 to the bundle $L^m$ equipped with the corresponding weight $m(\varphi(z) + 2\varepsilon\theta(z)\log|z-x|)$, and solve the equation $\bar{\partial}u = v$ for the $(n,1)$-form $v = \bar{\partial}(\theta(z)P(z)dz_1 \wedge \cdots \wedge dz_n \otimes e^m)$ associated to an arbitrary polynomial $P$. The $L^2$ estimate shows that the solution $u$ has to vanish at order $\geq q+1$ at $x$ where $q = [m\varepsilon] - n$, hence

$$\theta(z)P(z)dz_1 \wedge \cdots \wedge dz_n \otimes e^m - u(z)$$

is a holomorphic section of $K_X \otimes L^m$ with prescribed jet of order $q$ at $x$.

COROLLARY 10.2. *Suppose that $2\pi\,\Theta(L) + i\partial\overline{\partial}\varphi \geq \delta\,\omega$ for some $\delta > 0$. Let $x \in X$ be such that $\nu(\varphi, x) = 0$. Then there exists $\varepsilon > 0$ such that the sections in $H^0(X, K_X \otimes L^m)$ generate all jets of order $\leq m\varepsilon$ at $x$ for $m$ large.* □

We now state the basic $L^2$ extension theorem which was needed in several occasions. A detailed proof can be found in [Ohsawa and Takegoshi 1987; Ohsawa 1988; Manivel 1993]; see also [Demailly 1996, Theorem 13.6]. Only the case $q = 0$ (dealing with holomorphic sections) does play a role in this work.

THEOREM 10.3 (OHSAWA–TAKEGOSHI). *Let $X$ be a weakly pseudoconvex $n$-dimensional complex manifold equipped with a Kähler metric $\omega$, let $L$ be a hermitian holomorphic line bundle, $E$ a hermitian holomorphic vector bundle of rank $r$ over $X$, and $s$ a global holomorphic section of $E$. Assume that $s$ is generically transverse to the zero section, and let*

$$Y = \{x \in X : s(x) = 0,\ \Lambda^r ds(x) \neq 0\}, \qquad p = \dim Y = n - r.$$

*Moreover, assume that the $(1,1)$-form $\mathrm{i}\Theta(L) + r\,\mathrm{i}\,\partial\overline{\partial}\log|s|^2$ is semipositive and that there is a continuous function $\alpha \geq 1$ such that the following two inequalities hold everywhere on $X$ :*

(a) $\mathrm{i}\Theta(L) + r\,\mathrm{i}\,\partial\overline{\partial}\log|s|^2 \geq \alpha^{-1}\dfrac{\{\mathrm{i}\Theta(E)s, s\}}{|s|^2}$,

(b) $|s| \leq e^{-\alpha}$.

*Then for every smooth $\overline{\partial}$-closed $(0,q)$-form $f$ over $Y$ with values in the line bundle $\Lambda^n T_X^\star \otimes L$ (restricted to $Y$), such that $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} dV_\omega < +\infty$, there exists a $\overline{\partial}$-closed $(0,q)$-form $F$ over $X$ with values in $\Lambda^n T_X^\star \otimes L$, such that $F$ is smooth over $X \smallsetminus \{s = \Lambda^r(ds) = 0\}$, satisfies $F_{\restriction Y} = f$ and*

$$\int_X \frac{|F|^2}{|s|^{2r}(-\log|s|)^2}\, dV_{X,\omega} \leq C_r \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega}\,,$$

*where $C_r$ is a numerical constant depending only on $r$.*

COROLLARY 10.4. *Let $Y$ be a pure dimensional closed complex submanifold of $\mathbb{C}^n$, let $\Omega$ be a bounded pseudoconvex open set and let $\varphi$ be a plurisubharmonic function on $\Omega$. Then for any holomorphic function $f$ on $Y \cap \Omega$ with*

$$\int_{Y\cap\Omega} |f|^2 e^{-\varphi} dV_Y < +\infty,$$

*there exists a holomorphic extension $F$ to $\Omega$ such that*

$$\int_{\Omega} |F|^2 e^{-\varphi} dV \leq A \int_{Y \cap \Omega} |f|^2 e^{-\varphi} dV_Y < +\infty.$$

*Here $A$ depends only on $Y$ and on the diameter of $\Omega$.*

Finally, a crucial application of Skoda's $L^2$ estimates [1972a; 1978] for ideals of holomorphic functions was made in Section 5:

THEOREM 10.5. *Let $\varphi$ be a plurisubharmonic function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ and let $\sigma_1, \ldots, \sigma_N$ be holomorphic functions on $\Omega$ (the sequence $\sigma_j$ can be infinite). Set $r = \min\{N-1, n\}$ and $|\sigma|^2 = \sum |\sigma_j|^2$. Then, for every holomorphic function $f$ on $\Omega$ such that*

$$\int_{\Omega} |f|^2 |\sigma|^{-2(r+1+\alpha)} e^{-\varphi} dV < +\infty, \ \alpha > 0,$$

*there exist holomorphic functions $g_1, \ldots, g_N$ on $\Omega$ such that $f = \sum_{1 \leq j \leq N} g_j \sigma_j$ and*

$$\int_{\Omega} |g|^2 |\sigma|^{-2(r+\alpha)} e^{-\varphi} dV \leq \frac{\alpha+1}{\alpha} \int_{\Omega} |f|^2 |\sigma|^{-2(r+1+\alpha)} e^{-\varphi} dV < +\infty.$$

COROLLARY 10.6. *With the same notations, suppose that*

$$\int_{\Omega} |f|^2 |\sigma|^{-2(r+m+\alpha)} e^{-\varphi} dV < +\infty$$

*for some $\alpha > 0$ and some integer $m \geq 1$. Then there exist holomorphic functions $g_L$ for all $L = (l_1, \ldots, l_m) \in \{1, \ldots, N\}^m$ such that*

$$f = \sum_L g_L \sigma^L \text{ with } \sigma^L = \sigma_{l_1} \sigma_{l_2} \ldots \sigma_{l_m},$$

$$\int_{\Omega} \sum_L |g_L|^2 |\sigma|^{-2(r+\alpha)} e^{-\varphi} dV \leq \frac{\alpha+m}{\alpha} \int_{\Omega} |f|^2 |\sigma|^{-2(r+m+\alpha)} e^{-\varphi} dV < +\infty.$$

PROOF. Use induction on $m$: if the result is true for $(m-1, \alpha+1)$ then $f = \sum_{\Lambda} g_{\Lambda} \sigma^{\Lambda}$ with $\Lambda$ of length $m-1$, and each function $g_{\Lambda}$ can be written $g_{\Lambda} = \sum_{l_m} g_L \sigma_{l_m}$ with $L = (\Lambda, l_m)$ and

$$\int_{\Omega} \sum_{l_m} |g_L|^2 |\sigma|^{-2(r+\alpha)} e^{-\varphi} dV \leq \frac{\alpha+1}{\alpha} \int_{\Omega} |G_{\Lambda}|^2 |\sigma|^{-2(r+1+\alpha)} e^{-\varphi} dV < +\infty,$$

$$\int_{\Omega} \sum_{\Lambda} |g_{\Lambda}|^2 |\sigma|^{-2(r+1+\alpha)} e^{-\varphi} dV \leq \frac{\alpha+m}{\alpha+1} \int_{\Omega} |f|^2 |\sigma|^{-2(r+m+\alpha)} e^{-\varphi} dV < +\infty. \ \square$$

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Jean-Pierre Demailly
Université de Grenoble I
Département de Mathématiques
Institut Fourier, UMR 5582 du CNRS
38402 Saint-Martin d'Hères Cedex
France
demailly@ujf-grenoble.fr