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Complex Dynamics in Higher Dimension

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Dedicated to the memory of Michael Schneider

Abstract. We discuss a few new results in the area of complex dynamics in higher dimension. We investigate generic properties of orbits of biholomorphic symplectomorphisms of $\mathbb{C}^n$. In particular we show (Corollary 3.4) that for a dense G_δ set of maps, the set of points with bounded orbit has empty interior while the set of points with recurrent orbits nevertheless has full measure. We also investigate the space of real symplectomorphisms of $\mathbb{R}^n$ which extend to $\mathbb{C}^n$. For this space we show (Theorem 3.10) that for a dense G_δ set of maps, the set of points with bounded orbit is an F_σ with empty interior.

Contents

1. Introduction

In this paper we discuss low-dimensional dynamical systems described by complex numbers. There is a parallel theory for real numbers. The real numbers have the advantage of being more directly tuned to describing real-life systems. However, complex numbers offer additional regularity and besides, real systems usually complexify in a way that makes phenomena more clear: for example, periodic points disappear under parameter changes in the real case, but remain in the complex case.

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In the case of the solar system and other complicated systems, one has to resign oneself to studying the time evolution of a small number of variables, since if one wants to precisely predict long-term evolution one runs into unsurmountable computer problems. One cannot forget unavoidable errors that are just necessary limits of knowledge. And some knowledge is hence limited to a phenomenological type.

Here we give a brief overview of some of the open questions in the area of complex dynamics in dimension 2 or more. We also discuss some new results by the authors about symplectic geometry and Hamiltonian mechanics, belonging to higher-dimensional complex dynamics.

2. Questions in Higher-Dimensional Complex Dynamics

Complex dynamics in one complex dimension arose in the end of the last century as an outgrowth of studies of Newton's method and the three body problem in celestial mechanics. See [Alexander 1994] for a historical treatment.

2.1. Local Theory. In the local theory one studies the behavior near a fixed point, $f(x) = x$. This was the beginning of the theory in one complex variable: see [Schröder 1871]. Schröder discussed the case when the derivative $f'(x)$ of the map at the fixed point had absolute value less than one. He asked whether after a change of coordinates — that is, a conjugation — the map could be made linear in a small neighborhood (if the derivative was non-zero). This gave rise to the Schröder equation, which was later solved by Farkas [1884]. The case when the derivative was 1 was discussed by Fatou [1919; 1920a; 1920b] and Julia [1918], who proved the so-called flower theorem, describing the shape of the set of points whose orbit converges to the fixed point (the basin of attraction). The more general neutral case, i.e. when $|f'(x)| = 1$ is still not completely understood. The first result in this direction was proved by Siegel [1942]. He showed that f is conjugate to $f = e^{i\theta}z$ in case θ is sufficiently far from being rational. This was shown later to be valid for a larger class of angles by Brjuno [1965; 1971; 1972] and the question whether this was a necessary and sufficient condition was discussed by Yoccoz [1992].

The same problem arises for fixed points in higher dimension. In the case of a sufficiently irrational indifferent fixed point, Sternberg [1961] showed that the Theorem of Siegel is still valid. See [1987] for a more detailed history.

In general, let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a germ of a holomorphic map with $f(0) = 0$. The objective is to describe the local nature of the set of points converging to the fixed point. There is as yet no systematic study of this, and the work that has been done is more of a global nature. See the next section.

2.2. Global Theory. The case when $f'(x)$ has one eigenvalue 1 and the other is λ was studied by Ueda [1986], who showed that in the case of $|\lambda| < 1$, and when f is an automorphism of $\mathbb{C}^2$, the basin of attraction of the fixed point is a biholomorphic copy of $\mathbb{C}^2$ — what is called a Fatou–Bieberbach domain. A similar result when both eigenvalues are 1 and under suitable conditions on the higher order terms was proved recently by Weickert [1998]. A general result in any dimension was proved subsequently by Hakim [1998; 1997]. These two works are also of a local nature.

The global analogues of the distinction between attracting, repelling and indifferent behaviour at fixed points, are the distinction between Fatou sets, Julia sets and borderline cases, like Siegel domains. This started after the Montel Theorem was proved in one dimension, or with the equivalent notion of Kobayashi hyperbolicity in higher dimension.

In the theory of iteration of polynomials or rational functions in one variable, the Fatou sets are completely classified into 5 types [Sullivan 1985]. In higher dimension one has the same kinds of Fatou sets but there are others as well. There is as yet no complete classification of Fatou components for holomorphic maps on $\mathbb{P}^2$ say; see [Fornæss and Sibony 1995a]. For example, one knows that there are no wandering Fatou components Ω (that is, components Ω such that $f^n(\Omega) \cap \Omega = \varnothing$ for all n) in one dimension, but this is unknown in higher dimension. Another simple open problem in the case of polynomial automorphisms of $\mathbb{C}^2$ is whether a Fatou component can be biholomorphic to an annulus cross $\mathbb{C}$.

As far as the Julia set is concerned, one has a basic tool available, pluripotential theory. This is based on the fact that for example if one lifts a holomorphic map on $\mathbb{P}^n$ to a homogenous polynomial F on $\mathbb{C}^{n+1}$ of degree d, then the limit $G = \lim_{n\to\infty} d^{-n} \log \|F^n\|$ exists and the $(1,1)$ current $T = dd^c G$ has support precisely over the Julia set and therefore is an invariant object measuring the dynamics. This tool lies behind much of what is known about complex dynamics in higher dimension. See [Fornæss and Sibony 1994; Fornæss 1996].

The function G and the current T are naturally restricted to the case of iteration of polynomial and rational maps. In the case of entire maps in one and several variables, it seems one must get by without pluripotential theory. It is also appropriate to mention that one of the successful tools in iteration in one dimension, quasiconformal maps, have so far no higher-dimensional complex analogue. This is perhaps the reason why one hasn't so far been able to decide whether wandering Fatou components exist in higher dimension for polynomial automorphisms say. We should also say that although one doesn't have pluripotential theory in the study of entire maps, one has instead much more freedom to work with holomorphic functions, so in the case of holomorphic automorphisms of higher dimension and holomorphic endomorphisms in one dimension, one can show that wandering components exist [Fornaess and Sibony 1998a]. See [Bergweiler 1993] for the 1-dimensional case.

Several classes of maps on $\mathbb{C}^n$ have been studied. One can divide into two major classes, biholomorphic maps and endomorphisms.

The class that has been studied the most are the Hénon maps in $\mathbb{C}^2$, these are the polynomial automorphism with nontrivial dynamics. See [Bedford and Smillie 1991a; 1991b; 1992; 1998; $\geq$ 1999a; $\geq$ 1999b; Bedford et al. 1993; Hubbard and Oberste-Vorth 1994; 1995; Fornæss and Sibony 1992].

However, when it comes to the next step, extending the theory to $\mathbb{C}^3$ or higher, there are very few results at present (but see [Bedford and Pambuccian 1998]). A first step is to classify the polynomial automorphisms hopefully in a manner analogous to the Friedland–Milnor classification [1989] in $\mathbb{C}^2$. This has so far been done only for degree-2 maps in $\mathbb{C}^3$ [Fornæss and Wu 1998].

In the case of entire automorphisms, one can study for example the behaviour of orbits in general, asking whether they usually tend to infinity. This has been studied in [Fornæss and Sibony 1995b; [1996]; [1996]]. But we don't know for example, in the case of symplectomorphisms of $\mathbb{C}^{2n}$, whether there can exist a set of positive measure of bounded orbits which persist under all small perturbations. We discuss this type of question in the next section.

If we go to polynomial endomorphisms of $\mathbb{C}^2$ — that is, beyond the case of Hénon maps, which are invertible — the study is wide open. There is a classification [Alcarez and de Medrano $\geq$ 1999] of endomorphisms which are polynomial maps of degree 2 in $\mathbb{R}^2$ up to composition with linear automorphisms (not up to conjugation), but so far no systematic study exists of these classes.

2.3. Flows of Holomorphic Vector Fields. The iteration of maps refers to discrete dynamics. That is, one describes how a system changes in one unit of time. One can make an analogous study of continuous dynamics, where the maps are given by flows of holomorphic vector fields. Some work has been done on this [Fornæss and Sibony 1995b], but less than in the discrete case. Again one can ask questions about the local flow near a fixed point for the flow (a zero of the vector field) and about the global flow. Some work has been done in [Forstneric 1996] in the case of complete vector fields (those for which the flow is defined for all time), and in [Fornæss and Grellier 1996] on the question of the size of the set of points with exploding orbits (orbits which reach infinity in finite time).

2.4. Holomorphic Foliations and Laminations. Holomorphic vector fields foliate space by integral curves. In general, one can study foliations or more general, laminations. A compact set K is said to be laminated if there exist through every point $p \in K$, a complex manifold $M_p \subset K$ and these are either analytic continuations of each other or disjoint. One doesn't for example know if there can exist a lamination of some compact set in $\mathbb{P}^2$ so that no leaf is a compact complex manifold. See [Brunella 1994; 1996; Brunella and Ghys 1995; Camacho 1991; Camacho et al. 1992; Gómez-Mont 1988; 1987] for some work on foliations and further references.

In dealing with holomorphic endomorphisms of $\mathbb{P}^2$, one studies for example the unstable set of the saddle set of hyperbolic maps [Fornæss and Sibony 1998b]. This can locally be written as a union of graphs of local unstable manifolds. However, since these might not be pairwise disjoint in the endomorphism case (the unstable manifold through a point depends in general on the prehistory of the point), one gets to study a more general concept than laminations.

3. Symplectic Geometry and Hamiltonian Mechanics

We discuss here three new results. The first concerns the abundance of recurrent points for complex symplectomorphims of $\mathbb{C}^{2k}$. The second deals with real symplectomorphisms which can be complexified. Finally the third topic concerns the estimates of decompositions of Hamiltonians into sums of Hamiltonians whose associated Hamiltonian vector fields give arise to globally defined symplectomorphisms.

3.1. Recurrent points. In this section we discuss biholomorphic symplectomorphisms of $\mathbb{C}^{2k}$, that is, maps $f:\mathbb{C}^{2k}\to\mathbb{C}^{2k}$ which preserve the symplectic form $\omega := \sum_{j=1}^{k} dz_j \wedge dw_j$. We denote the class of all symplectomorphisms by $\mathcal{S}$. We put the topology of uniform convergence on compact sets on $\mathcal{S}$.

We are interested in the generic, long-term behavior of orbits. See [Fornæss and Sibony 1996].

Definition 3.1. Let $f\in\mathcal{S}$ and $p\in\mathbb{C}^{2k}$. We say that the point p is recurrent if for every neighborhood U of p there is a point $q\in U$ and an integer $n>1$ so that $f^n(q)\in U$.

Theorem 3.2. *The set of recurrent points R_f is of full measure for a G_δ dense set of symplectic maps f.*

This can be contrasted with a previous result from [Fornæss and Sibony 1996]:

Theorem 3.3. *There is a dense G_δ set $\mathcal{S}'\subset\mathcal{S}$ so that for each $f\in\mathcal{S}'$, the set $K_f\subset\mathbb{C}^{2k}$ of points whose orbit is bounded has empty interior.*

Combining these two results, one gets:

Corollary 3.4. *There is a dense G_δ set $\mathcal{S}''\subset\mathcal{S}$ so that for each $f\in\mathcal{S}''$, the set R_f has full measure, while K_f has no interior.*

Proof of Theorem 3.2. Let $f\in\mathcal{S}$. Let B_m denote the closed ball of center 0 and radius m, and set

$$U_m^f = \{x : x\in B_m \text{ and } |f^n(x)-x|<1/m \text{ for some } n>0\},$$
$$\mathcal{S}_m = \{f\in\mathcal{S} : |B_m\setminus U_m^f|<2^{-m}\}.$$

The set $\mathcal{S}_m$ is open for the compact open topology.

Claim 3.5. $\mathcal{S}_m$ is dense in $\mathcal{S}$.

Assuming the claim, the set $\mathcal{S}'=\cap\mathcal{S}_m$ is a G_δ dense set. Let $f\in\mathcal{S}'$. Define $R=\bigcup_N\bigl(\bigcap_{m>N}U_m^f\bigr)$. Then R is of full measure in any ball; indeed,

$$\bigl|\bigcup_{m>N}(B_m\setminus U_m^f)\bigr|<2^{-N}.$$

Every point x in R is recurrent.

We want now to prove the claim, thus completing the proof of the theorem. Let $f_0 \in \mathcal{S}$. We want to approximate f_0 on a given compact X by maps in $\mathcal{S}_m$. We can assume $X \subset B_m$. Set $\Omega_m := U_m^{f_0}$.

If Ω_m is nonempty we choose a compact set $K_m \subset \Omega_m$ such that $|\Omega_m \setminus K_m| < 2^{-2m}$. There is N_0 such that for $x \in K_m$ there is $n \le N_0$ with $|f^n(x) - x| < 1/m$. We want to enlarge the set of recurrent points (up to order $1/m$) by perturbing slightly f_0 on B_m.

We choose finitely many disjoint compact rectangles $\widehat{S}^j_{m\,j \le L} \subset B_m \setminus K_m$ with $\operatorname{diam}(\widehat{S}^j_m) < 1/(3m)$, and such that

$$\bigl|B_m \setminus (K_m \cup \textstyle\bigcup_j \widehat{S}^j_m)\bigr| < 2^{-(m+2)}.$$

Set $S^j_m := \widehat{S}^j_m \setminus \Omega_m$. Since every S^j_m is disjoint from Ω_m we have for every $l \ge 1$

$$f_0^l(S^j_m) \cap S^j_m = \varnothing$$

and consequently for $r, l \in \mathbb{Z}$ with $r \ne l$ we have

$$f_0^l(S^j_m) \cap f_0^r(S^j_m) = \varnothing. \tag{3–1}$$

Fix k such that $\bigcup_{|n| \le N_0} f_0^n(B_m) \subset \mathring{B}_k$. Since f_0 is volume preserving, condition (3–1) implies the existence of an $l_0 \ge N_0$ such that

$$\bigl\{x \in \textstyle\bigcup_j S^j_m : \text{there is } l \text{ satisfying } |l| \ge l_0 \text{ and } f^l(x) \in B_k\bigr\}$$

is of measure less than $2^{-(m+1)}$. Let B_r, for $r > k$, be a ball containing $\bigcup_{|n| \le l_0} f^n(B_m)$. Similarly there is an $l_1 \in \mathbb{N}$ such that $l_1 \ge l_0$ and

$$\bigl\{x \in \textstyle\bigcup_j S^j_m : \text{there is } l \text{ satisfying } |l| \ge l_1 \text{ and } f^l(x) \in B_r\bigr\}$$

is of measure less than $2^{-(m+1)}$. Shrinking the sets S^j_m we can assume that there are finitely many disjoint compact sets $\bigl(\widetilde{S}^j_m\bigr)_{j \le L'}$ in $B_m \setminus \Omega_m$ such that

$$|B_m \setminus (K_m \cup \widetilde{S}^j_m)| < 2^{-m}$$

and for $|l| \ge l_1$, the set $f^l(\widetilde{S}^j_m) \cap B_r$ is empty.

For any $x \in \widetilde{S}^j_m$ consider the complete orbit $\mathcal{O}(x) = \{f^n(x)\}_{n \in \mathbb{Z}}$. Let $n(x)$ be the first exit time of the orbit from B_r, and $-n'(x)$ the last entry time of the orbit into B_r. More precisely, $f^{n(x)}(x) \notin B_r$ but $f^{n(x)-p}(x) \in B_r$ for $0 < p < n(x)$; similarly $f^{-n'(x)}(x) \notin B_r$ but $f^{-n'(x)+p}(x) \in B_r$, if $n'(x) > p > 0$. Define

$$\Omega^+ = \bigl\{f^{n(x)}(x) : x \in \textstyle\bigcup_j \widetilde{S}^j_m\bigr\},$$
$$\Omega^- = \bigl\{f^{-n'(x)}(x) : x \in \textstyle\bigcup_j \widetilde{S}^j_m\bigr\}.$$

We can remove from $\bigcup_j \widetilde{S}^j_m$ a set of arbitrarily small measure such that the maps $x \to n(x)$ and $x \to n'(x)$ are locally constant.

The map $\phi : \Omega^+ \to \Omega^-$ defined by $\phi(f^{n(x)}(x)) = f^{-n'(x)}(x)$ is a bijection.

We can assume that we can cover $\bigcup_j \widetilde{S}_m^j$ by a finite union of small pairwise disjoint rectangles K^+, with $|\Omega^+ \setminus K^+|$ and $|K^+ \setminus \Omega^+| \ll 1$, and such that ϕ is locally given by $y \to f^{-s}(y)$ with $s \in \mathbb{N}$ fixed in a neighborhood.

We want to approximate $\phi_{|K^+}$ on most of K^+ by a global symplectomorphism σ such that σ is close to the identity on B_r. Then the map $f := f_0 \circ \sigma$ will be close to f_0 on X, and the orbit of most $f^{n(x)}(x) \in K^+$ will pass near x. Observe that we control the orbit of x under f in B_r, it stays close to the orbit of x under f_0. Hence $B_m \setminus U_f$ will be of arbitrarily small measure. We first need a lemma.

Lemma 3.6. *Let K be a compact set disjoint from the closed ball $\overline{B}$. Assume that K is a finite union of disjoint polynomially convex sets. Then for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset K$, such that $\overline{B} \cup K_\varepsilon$ is polynomially convex and $|K \setminus K_\varepsilon| < \varepsilon$.*

Proof of the Lemma. We first consider the case when $B = \varnothing$. Let l be a complex linear form. Fix a real number α. For any $0 < \delta \ll 1$ define

$$K_\delta^- = \{x \in K : \mathrm{Re}\ l(x) \le \alpha - \delta\},$$
$$K_\delta^+ = \{x \in K : \mathrm{Re}\ l(x) \ge \alpha + \delta\}.$$

For δ small enough the measure of $K \setminus (K_\delta^+ \cup K_\delta^-)$ is arbitrarily small. It is easy to verify that $\widehat{K_\delta^- \cup K_\delta^+} = \widehat{K}_\delta^- \cup \widehat{K}_\delta^+$. Repeating the process with finitely many hyperplanes one gets easily the set K_ε such that $\overline{B} \cup K_\varepsilon$ is polynomially convex.

Next consider the general case. Let l be a again complex linear form. Assume the real hyperplane $H = \{x : \mathrm{Re}\, l(x) = \alpha\}$ does not intersect $\overline{B}$, but possibly intersects K. For any $0 < \delta \ll 1$ define

$$Y = \overline{B} \cup K,$$
$$Y_\delta^- = \{x \in Y : \mathrm{Re}\ l(x) \le \alpha - \delta\},$$
$$Y_\delta^+ = \{x \in Y : \mathrm{Re}\ l(x) \ge \alpha + \delta\}.$$

For δ small enough the measure of $Y \setminus (Y_\delta^+ \cup Y_\delta^-)$ is arbitrarily small. It is easy to verify that $\widehat{Y_\delta^- \cup Y_\delta^+} = \widehat{Y}_\delta^- \cup \widehat{Y}_\delta^+$. Repeating the process with finitely many hyperplanes $l_j = \alpha_j$ one can choose things so that $\overline{B} \subset \bigcap_j \{l_j < \alpha_j\}$ while $K \subset \bigcup_j \{l_j > \alpha_j\}$, and one gets easily a set K_ε such that $\overline{B} \cup K_\varepsilon$ is polynomially convex. □

The second result we need is due to Forstneric.

Proposition 3.7 [Forstneric 1996]. *Let U be a simply connected Runge domain in $\mathbb{C}^{2k}$ such that $H^1(U, \mathbb{C}) = 0$. Let Φ_t be a biholomorphic map from U into $\mathbb{C}^{2k}$ of class $\mathcal{C}^2$ in $(t, z) \in [0, 1] \times U$. Assume each domain $U_t = \Phi_t(U)$ is Runge, and that every Φ_t is a symplectomorphism and Φ_1 can be approximated on U by global symplectomorphisms. Then Φ_0 can be approximated on U by global symplectomorphisms. A similar result holds for volume preserving maps in $\mathbb{C}^k$ (if one assumes that $H^{k-1}(U, \mathbb{C}) = 0$).*

We now finish the proof of the claim. Let K_ε be a compact obtained from K^+ by applying Lemma 3.6 to $\overline{B}_k \cup K^+$. We have to construct the family Φ_t such that $\phi = \Phi_1$ on a small neighborhood of $\overline{B} \cup K_\varepsilon$, which is topologically trivial. We are going to use the real hyperplanes as in the proof of the lemma. Suppose that the first hyperplane is just $H = \{(z, w) : \operatorname{Re} z_1 = \alpha\}$ and that $\overline{B} \subset \{\operatorname{Re} z_1 < \alpha\}$. Let $\chi(s)$ be a smooth approximation to $(s - \alpha)^+$. Define

$$\tau_s^t(z, w) = (z_1 + t\chi(s)w_1,\, w_1,\, z_2,\, \dots,\, w_k)$$

for $(z, w) \in \{\operatorname{Re} z_1 > \alpha\}$; here t is a large constant. Let $\psi_s := \tau_s^1 \circ \phi$. Then we have left the part in $\{\operatorname{Re} z_1 < \alpha\}$ unchanged and the part of K^+ in $\{\operatorname{Re} z_1 > \alpha + \varepsilon\}$, for $0 < \varepsilon \ll 1$, has slid far away.

Using sliding away from finitely many hyperplanes, removing possibly a set of very small measure from K_ε, we can construct $\Phi_s := \phi \circ \tau_s^l \circ \dots \circ \tau_s^1 \circ \phi$ such that $\Phi_0 = \phi$ and Φ_1 is defined on the ball and on finitely many rectangles.

The image of $\tilde{B} \cup K^+$ under ϕ_1 is contained in a union of balls $(B_j)_{j \le L}$ which are very far apart, in particular all their projections on the coordinate axes are disjoint. We can choose balls $(\tilde{B})_j$ with $B_j \Subset \tilde{B}_j$ and still the balls $\{\tilde{B}_j\}$ are very far apart. We connect ϕ_1^{-1} to the identity. Composing ϕ_1^{-1} with a finite number of shears (s_j) of type $(z_1 + h(w_1), w_1, z_2, \dots, w_k)$ with h entire we can achieve that the $\Theta_1 := s_p \circ \dots \circ s_1 \circ \phi_1^{-1}$ satisfies $\Theta_1(B_j) \Subset \tilde{B}_j$ and Θ_1 has a fixed point in each B_j, we can then write a homotopy to the identity. □

Remarks 3.8. 1. The authors proved in [Fornæss and Sibony 1996] the existence of a G_δ dense set $\mathcal{S}' \subset \mathcal{S}$ such that for any $f \in \mathcal{S}'$ the set of recurrent points R_f is a G_δ dense set. It follows from the previous theorem that generically, in the Baire sense, R_f is a G_δ dense set of full measure.

2. The same results hold for the group $\mathcal{V}$ of volume preserving biholomorphisms in $\mathbb{C}^k$. We can just start with a neighborhood U of K_ε which satisfies $H^{k-1}(U, \mathbb{C}) = 0$.

3.2. Real Symplectomorphisms. Let $\mathcal{S}_\mathbb{R} := \{f \in \mathcal{S} : f : \mathbb{C}^{2k} \to \mathbb{C}^{2k}$ such that $f(\mathbb{R}^{2k}) = \mathbb{R}^{2k}\}$. More precisely, let (z_j, w_j) be complex coordinates in $\mathbb{C}^{2k}$. Assume that $z_j = p_j + ip'_j, w_j = q_j + iq'_j$, the coordinates on $\mathbb{R}^{2k}$ are (p_j, q_j). The restriction of the form $\omega = \sum dz_j \wedge dw_j$ to $\mathbb{R}^{2k}$ is the standard symplectic form $\omega_0 = \sum dp_j \wedge dq_j$.

Proposition 3.9. *The group $\mathcal{S}_\mathbb{R}$ consists of diffeomorphisms f_0 of $\mathbb{R}^{2k}$ such that $f_0^* \omega_0 = \omega_0$, which extend biholomorphically to $\mathbb{C}^{2k}$.*

Proof. Left to the reader. □

A family $(f_i)_{i \in I}$ in $\mathcal{S}_\mathbb{R}$ converges to $f \in \mathcal{S}_\mathbb{R}$ if and only if (f_i) converge to f uniformly on compact sets of $\mathbb{C}^{2k}$ and the restriction of f_i to $\mathbb{R}^{2k}$ converges to $f_{|\mathbb{R}^{2k}}$ in the fine topology in $\mathbb{R}^{2k}$, which means that given any continuous function $\eta > 0$ on $\mathbb{R}^{2k}$ and given any n, $\sup_{|\alpha| \le n} |D^\alpha f_i - D^\alpha f|(p, q) < \eta(p, q)$ for i in a cofinal set.

It is easy to verify that $\mathcal{S}_{\mathbb{R}}$ with this topology is a Baire space.

THEOREM 3.10. *Let $k = 1, 2$. There is a G_δ dense set $\mathcal{S}'_{\mathbb{R}} \subset \mathcal{S}_{\mathbb{R}}$ such that for $f \in \mathcal{S}'_{\mathbb{R}}$ the set $K_f := \{(z, w) \in \mathbb{C}^{2k} : f^n(z, w)$ is bounded$\}$ is an F_σ of empty interior.*

PROOF. Let $H = \mathbb{C}^{2k} \times \mathcal{S}_{\mathbb{R}}$ with the product topology. Set

$$K := \{(z, w, f) \text{ with bounded forward orbit}\}.$$

Let Δ_n be a basis for the topology of $\mathbb{C}^{2k}$. For each n, define

$$\mathcal{S}_n := \{f : |f^m(z, w)| < n \text{ for every } m \text{ and every}(z, w) \in \overline{\Delta}_n\}.$$

If K has interior then some $\mathcal{S}_n$ has nonempty interior. Assume $n = R$. Let U_R be the interior of $\{(z, w, f) : |f^m(z, w)| \leq R$ for every $m\}$. Let U be the projection of U_R in $\mathcal{S}_R$.

For $(z, w, f) \in U_R$ let V_f be the slice of U_R for fixed f. The open set V_f is Runge for every f. Moreover it is clearly invariant under the map $(z, w) \to (\bar{z}, \bar{w})$. From the Schwarz Lemma, given ε_0 there is $\alpha > 0$ so that

$$|x - x'| < \alpha \Rightarrow |f^m(x) - f^m(x')| < \varepsilon_0 \tag{3–2}$$

for every $f \in U$, close enough to a given $f_0 \in U$.

Since the maps f are volume preserving on each slice, V_f is also backward invariant and each connected component of V_f is periodic. It follows from the Cartan Theorem and from the fact that the maps are volume preserving that for every such f and any component U_f of V_f the closure of the subgroup generated by f restricted to $\bigcup_n f^n(U_f)$ is a compact Abelian Lie group G_f. Consequently $G_f = T^l \times A$, where T is the unit circle, $l \in \mathbb{N}$, and A is a finite group. For $a \in U$, $a = (z, w)$, let $\bar{a} = (\bar{z}, \bar{w})$. Let $X_a = G_f(a)$, $Y_a = X_a \cup X_{\bar{a}}$.

LEMMA 3.11. *Let V be a Runge, bounded open set in $\mathbb{C}^{2k}$, for $k = 1, 2$, stable under $(z, w) \to (\bar{z}, \bar{w})$. Assume that V is invariant under a symplectic map $f \in \mathcal{S}_{\mathbb{R}}$. Let $G = \overline{(f^n_{|V})_n}$ and assume G is not discrete. There is a point a such that Y_a is polynomially convex and in every neighborhood of Y_a we can find $Y_{a'}$ such that $Y_a \cup Y_{a'}$ is polynomially convex and $Y_a \cap Y_{a'} = \varnothing$.*

PROOF. For any $x \in V$, the set Y_x is a union of disjoint tori (possibly points). The polynomially convex hull $\hat{Y}_x$ of Y_x is stable under f. As in [Fornæss and Sibony 1996, Lemma 4.3], we can find an a such that Y_a is polynomially convex. If Y_a is a finite set hence a union of periodic orbits the map f is then linearizable near each periodic orbit. Since the map is symplectic (holomorphic) it follows that the map is conjugate to a matrix with blocks

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix}.$$

One then computes easily that *most* orbits near the periodic points are polynomially convex. And even the union of finitely many of them is normally polynomially convex. More precisely: if, for example,

$$f(z_1, z_2, w_1, w_2) = (e^{i\theta_1} z_1,\, e^{-i\theta_1} w_1,\, e^{i\theta_2} z_2,\, e^{-i\theta_2} w_2),$$

the orbits that avoid the axes are polynomially convex.

Assume Y_a is a finite union of tori. Let $\tilde{Y}_a$ be the local complexification. If G acts effectively on Y_a, the dimension of Y_a equals that of G; let this dimension be l. Generically the orbits are of dimension l. Orbits close to Y_a are also polynomially convex. If $Y_{a'}$ is polynomially convex and the complexifications are disjoint then $Y_a \cup Y_{a'}$ is also polynomially convex. This finishes the lemma if $k = 1$.

We next consider the case where there is a non discrete isotropy group for Y_a and Y_a is not a finite union of periodic orbits. It suffices to consider only the identity component of G.

Let G_0 be the isotropy group. Then Y_a and G_0 are both a finite union of tori. Then G_0 is generated by a holomorphic Hamiltonian vector field with Hamiltonian H. Necessarily $\nabla H \equiv 0$ on Y_a, so H is constant on Y_a. Let V_a be a Runge neighborhood of Y_a, stable under f.

Next consider a point p close to a where $H(p)$ is nonzero. Let Y be the H orbit of p. Changing p a little, we may assume that Y is polynomially convex and still lies in the same level set of H. Let G' in G be a T^1 subgroup transverse to G_0. Then T^1 is generated by a holomorphic Hamiltonian vector field with Hamiltonian K.

Consider the K orbit Z of p. There is a projection of Z to Y_a given by mapping p to a and following the vectorfield of K. This can be extended to a holomorphic projection of a neighborhood of the full G orbit of p by letting the projection be constant on H orbits. Here we use the fact that the group G acts real analytically. Then it follows that Y_a together with the orbit of p is polynomially convex: Indeed Y_a is a totally real torus [Fornæss and Sibony 1996] and continuous functions on Y_a are uniform limits of polynomials [Wermer 1976]. To check polynomial convexity of a set X which projects on Y_a, it is enough by [Wermer 1976] to check the polynomial convexity of the fibers under π. □

We continue with the proof of Theorem 3.10. Let $Y_a, Y_{a'}$ be orbits under $G = G_{f_0}$ with $f_0 \in U$. Suppose $f_0^{n_0}$ is in the identity component of G. Assume $Y_a \cup Y_{a'}$ is polynomially convex as in the Lemma. Let f^t be a one parameter subgroup in G such that $f^1 = f_0^{n_0}$. Define $\xi = (df^t/dt)_{|t=0}$. It is proved in [Fornæss and Sibony 1996] Lemma 4.4 that ξ is a Hamiltonian vector field. We define $\tilde{\xi} = \xi$ on a neighborhood of Y_a and $\tilde{\xi} = -\xi$ in a neighborhood of $Y_{a'}$.

Let h be a Hamiltonian for $\tilde{\xi}$ defined on a Runge neighborhood of $Y_a \cup Y_{a'}$. We can assume that $h(\bar{z}, \bar{w}) = \bar{h}(z, w)$. We approximate h by a polynomial P, real on $\mathbb{R}^{2k}$, uniformly on a Runge neighborhood V_d containing a $d-$neighborhood of $Y_a \cup Y_{a'}$, we need the Hamiltonian vector field generated by P, has a small angle ε, with $\tilde{\xi}$ on $Y_a \cup Y_{a'}$.

We can write $P = \sum_{j=1}^{N} P_j$ where each P_j is a polynomial such that the associated Hamiltonian vector field is complete and the flow is a shear s_j. We can assume that the P_j are real on $\mathbb{R}^{2k}$. For $\delta > 0$, write $\delta P = \sum_{j=1}^{N} \delta P_j$, S_j^δ the shear associated to δP_j and $S^\delta = S_N^\delta \circ \cdots \circ S_1^\delta$.

Fix a large ball $B(0,r)$ in $\mathbb{C}^{2k}$. Assume $f(B(0,r)) \subset B(0,r')$. Choose $\delta \ll 1$ so that $|S^\delta - \mathrm{Id}|_{B(0,r')} \leq \varepsilon$.

We have $|S^\delta \circ f - f|_{B(0,r)} \leq \varepsilon$.

We will modify $S^\delta \circ f_0$ to bring it inside the open set U.

We show first that $(S^\delta \circ f_0)$ do not satisfy condition (3–2). Indeed, $(S^\delta \circ f_0)^{(m)}(x)$ is in V_d as soon as $m \leq d/(100\varepsilon\delta)$ for $x \in Y_a$, $x' \in Y_{a'}$. But S^δ push points in Y_a and $Y_{a'}$ in opposite directions so for $m \sim 1/\delta$ we have

$$\big|(S^\delta \circ f_0)^m(a) - (S^\delta \circ f_0)^m(a')\big| \sim 1 \geq \varepsilon_0,$$

contradicting (3–2).

However, we have to modify $S^\delta \circ f_0$ to keep the previous estimates and to put in the given neighborhood of f_0 in the fine topology.

Fix ε_j and $r_j \to \infty$ so that if $|f - f_0|_{B(r)} < \varepsilon_0$ and

$$|f - f_0|_{(B(r_{j+1}) \setminus B(r_j)) \cap \mathbb{R}^{2k}} < \varepsilon_j$$

then f is in the given neighborhood of f_0 in the fine topology where (3–2) is valid.

We need just to use inductively the following lemma.

Lemma 3.12. *Let $f \in \mathcal{S}_\mathbb{R}$ and fix $R_1 < R_2 < R_3$. Assume $f_{|B(0,R_2)\cap\mathbb{R}^{2k}}$ is close to the identity. Then there is $f_1 \in \mathcal{S}_\mathbb{R}$ such that f_1 is close to the identity on $B(0,R_1)$ and close to f on $(B(0,R_3) \setminus B(0,R_2)) \cap \mathbb{R}^{2k}$.*

Proof. We can write f as a time-1 map of a time-dependent Hamiltonian vector field $X(t)$, which is close to zero on $B(0,R_2)$. For every t the Hamiltonian is close to zero on $B(0,R_2)$. Multiply each Hamiltonian by a cut-off function equal to zero on $B(0,\, R_1 - \varepsilon)$ and 1 out of $B(0,R_1)$. We approximate each Hamiltonian on $B(0,R_1) \cup \big(B(0,R_3) \cap \mathbb{R}^{2k}\big)$ by entire functions real on reals. Then we consider the associated symplectomorphism and approximate by their composition, see [Fornæss and Sibony 1996, p. 316; Forstneric 1996]. □

This concludes the proof of the theorem. □

3.3. Decomposition of Homogeneous Polynomials. Let X denote a homogeneous polynomial in $\mathbb{C}^2$ of degree $d = m + 1$. In this section we will discuss how to decompose X into a finite sum of powers of linear functions. This problem arises in the study of symplectomorphisms when one wants to do computer calculations. More precisely, the problem is to truncate a power series for a symplectomorphism, and then to symplectify the truncation without “loosing too much". But that doesn't seem to be the case according to our computations. The problem arises also in approximation of symplectomorphisms with compositions of shears.

A step in this procedure is to consider symplectomorphisms of the form $F = \mathrm{Id} + (A_m, B_m) + O(\|z\|^{m+1})$.

Lemma 3.13. *Let $f = \mathrm{Id} + Q_m + O(|z|^{m+1})$ be a germ of holomorphic map in $\mathbb{C}^{2p}$, where Q_m is a polynomial mapping homogeneous of degree m. Then Q_m is a Hamiltonian vector field if $f^*(\omega) - \omega = O(|z|^m)$.*

Proof. To check that a holomorphic vector field X in $\mathbb{C}^{2p}$ is Hamiltonian we have to show that $X\rfloor\omega$ is a closed 1−form. Assume $X = (A_1, B_1, \ldots, A_p, B_p)$. Then

$$\omega(X, \cdot) = \sum_{j=1}^{p} (A_j dw_j - B_j dz_j),$$

and

$$d(X\rfloor\omega) = \sum (dA_j \wedge dw_j + dz_j \wedge dB_j).$$

On the other hand

$$\begin{aligned} f^*\omega &= \sum \big(dz_j + dA_j + dO(|z|^{m+1})\big) \wedge (dw_j + dB_j + \cdots) \\ &= \omega + d(X\rfloor\omega) + O(|z|^m). \end{aligned}$$

Hence $d(X\rfloor\omega) = 0$. □

Therefore one can write

$$(A_m, B_m) = \Big(-\frac{\partial X}{\partial w}, \frac{\partial X}{\partial z}\Big),$$

where X is a uniquely determined homogeneous polynomial of degree $m+1 = d$. It is easy to decompose $X = \sum_{j=0}^{d} c_j Q_j$ where the Q_j are powers of linear functions, forming a basis for the homogeneous polynomials.

Hence, letting $(C_j, D_j) = c_j(-\partial Q_j/\partial w, \partial Q_j/\partial z)$, we can write

$$\begin{aligned} F &= \tilde{F} + O(\|z\|^{m+1}), \\ \tilde{F} &= (\mathrm{Id} + (C_0, D_0)) \circ \cdots \circ (\mathrm{Id} + (C_d, D_d)). \end{aligned}$$

We are concerned here with the magnitude of the $c_j Q_j$. We will see below that the basis $\{Q_j\}$ can be chosen to be essentially as good as an orthonormal basis.

Note that if the (A_m, B_m) are less than some small ε, then we will show that the (C_j, D_j) are bounded by $c\varepsilon$ and hence the terms of $\tilde{F}$ of order at least $m+1$ are at most $c_m\varepsilon^2$. Hence, we will get that if we start with a symplectomorphism close to the identity, this process can be repeated a few times, to approximate the original map to higher and higher order and the resulting symplectomorphism remains close to the identity.

In the rest of this paper we will deal with the estimates on the $c_j Q_j$. We leave it for a later paper to carry this project further for arbitrary dimension and for the estimates on the $\tilde{F}$.

THEOREM 3.14. *Let $d \geq 2$ be an integer. There exist $d+1$ homogeneous polynomials P_d^j of degree d of the form*

$$P_d^j = \frac{\sqrt{(d+1)(d+2)}}{\pi}(\alpha_j z + \beta_j w)^d \qquad \text{for } j = 0, \ldots, d$$

such that $\|(\alpha_j, \beta_j)\| = \sqrt{|\alpha_j|^2 + |\beta_j|^2} = 1$, $\|P_d^j\|_{L^2(\mathbb{B})} = 1$, and the following properties are satisfied:

1. *The P_d^j is a basis for the space $\mathcal{P}_d$ of homogeneous holomorphic polynomials of degree d.*
2. *If $P \in \mathcal{P}_d$ is of the form $P = \sum_j c_j P_d^j$, then $|c_j| \leq C\sqrt{d}\|P\|_{L^2(B)}$, with C independent of P and d.*

REMARK 3.15. We can probably drop the coefficient $\sqrt{d}$ from the estimate in condition 2. See the end of the proof.

The main difficulty is that the powers of linear functions is only a $\mathbb{P}^1$ in the high-dimensional space $\mathbb{P}^d$ of all homogeneous polynomials (up to multiples). Hence it is somewhat remarkable that one can choose a basis practically as good as an orthonormal basis.

We are mainly interested for the moment in the asymptotic estimate when $d \to \infty$, rather than an optimal value for C. Hence we can restrict ourselves to large d.

Recall the following fact about Vandermonde determinants:

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+1} & x_{n+1}^2 & \ldots & x_{n+1}^n \end{vmatrix} = \textstyle\prod_{j>i}(x_j - x_i).$$

We also need a slightly more general classical formula:

$$\begin{vmatrix} 1 & x_1 & \cdots & x_1^{j-1} & x_1^{j+1} & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^{j-1} & x_2^{j+1} & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{j-1} & x_n^{j+1} & \cdots & x_{n+1}^n \end{vmatrix} = \textstyle\prod_{j>i}(x_j - x_i)S_{n-j},$$

where S_k is the sum of all distinct products of k of the x_i, $S_0 = 1$.

Our next step is to choose, for a given degree $d \geq 2$, a set of $d+1$ points in $\mathbb{P}^1$ that are evenly distributed in the spherical metric: say $p_{i,d} = [\alpha_{i,d} : \beta_{i,d}] = p_i = [\alpha_i : \beta_i]$, with $i = 0, \ldots d$. Moreover we assume that $|\alpha_i|^2 + |\beta_i^2| = 1$. For this, we will first describe the points p_i on a sphere and then project to the complex plane.

We assume that the points (a_i, b_i) are distributed on a 2-sphere of radius $\frac{1}{2}$ (so that the area is π, the area of $\mathbb{P}^1$ in the sperical metric) centered at $(0,0,0)$ in bands with angle from the negative z-axis between $(2j\sqrt{\pi/d})$ and $2(j+1)\sqrt{\pi/d}$, where $j = 0, \ldots, \lfloor\sqrt{d\pi}/2\rfloor - 1$. Set $\theta_j = (2j+1)\sqrt{\pi/d}$.

We want the points to be about $\sqrt{\pi/d}$ apart from each other. This is approximately what you can do of you put d points in a square lattice in $\mathbb{R}^2$ with width $\sqrt{\pi/d}$, covering an area π.

Each band has circumference $\pi\sin(\theta_j)$ and hence contains n_j evenly spread points, where n_j equals either $\lfloor \pi\sin(\theta_j)/\sqrt{\pi/d}\rfloor$ or one more than this value (the latter case is allowed if $(\sqrt{\pi d})/8 < 2j+1 < (7\sqrt{\pi d})/8$).

Next we move the sphere up to have center at $(0,0,1/2)$ and use stereographic projection to map the points to the $z-$plane. The points in the j-th band have $|z| = \tan(\theta_j/2) = (\sin\theta_j)/(1+\cos\theta_j)$ and are equally spaced. In projective coordinates they are

$$[\frac{\sin\theta_j}{1+\cos\theta_j} : 1] = [\frac{\sin\theta_j}{\sqrt{2(1+\cos\theta_j)}} : \sqrt{\frac{1+\cos\theta_j}{2}}] = [\alpha_j : \beta_j]$$

with $|\alpha_j|^2 + |\beta_j|^2 = 1$.

In fact, we let the n_j points have arguments ω^k in the first coordinate, where $k = 1, \ldots, n_j$ and ω a primitive n_j-th root of unity. So the points are of the form

$$\left[\frac{\omega^k \sin\theta_j}{\sqrt{1(1+\cos(\theta_j))}} : \sqrt{\frac{1+\cos(\theta_j)}{2}}\right].$$

Lemma 3.16.

$$d(p_i, p_k) \geq \sqrt{\pi/d} - O(1/d), \qquad \textit{for } i \neq k.$$

Proof. The distance can only be smaller than $\sqrt{\pi/d}$ if the two points are on the same circle for $\sqrt{\pi d}/8 < 2j+1 < 7\sqrt{\pi d}/8$. Hence we get

$$d(p_i, p_k) \geq \frac{\pi\sin((2j+1)\sqrt{\pi/d})}{\frac{\pi\sin((2j+1)\sqrt{\pi/d})}{\sqrt{\pi/d}} + 1},$$

$$d(p_i, p_k) \geq \sqrt{\frac{\pi}{d}}\frac{\pi\sin(\pi/8)}{\pi\sin(\pi/8) + \sqrt{\pi/d}},$$

$$d(p_i, p_k) \geq \sqrt{\frac{\pi}{d}} - \frac{1}{d\sin(\pi/8)}.$$

□

Lemma 3.17. *Let $i = 0, \ldots, d$. Then*

$$\int_{\mathbb{B}} |z|^{2i}|w|^{2d-2i} = \frac{\pi^2}{(d+2)(d+1)}\frac{1}{\binom{d}{i}}.$$

Proof. We calculate the integral in the w-direction and introduce polar coordinates to reduce the integral to calculation of

$$\frac{\pi^2}{d+1-i}\int_0^1 r^i(1-r)^{d+1-i}\,dr.$$

Now use induction to get the result. $\square$

Lemma 3.18. *The functions* $Q_{i,d} = Q_i := (1/\pi)\sqrt{(d+1)(d+2)}P_i$ *have* L^2 *norm* 1.

Proof. This follows from the previous lemma applied to the function z^d and using rotational invariance. $\square$

We will next estimate the deviation of the set Q_i from being an orthonormal basis. Observe first that the vectors

$$e_i := \frac{\sqrt{(d+1)(d+2)}}{\pi}\sqrt{\binom{d}{i}}z^i w^{d-i}$$

is an orthonormal basis by the lemma above.

We express first Q_i in terms of the e_j. The following formula is immediate. We are abusing notation from now on by writing a, b instead of α, β.

Lemma 3.19. $Q_i = \sum_{j=0}^{j=d} a_i^j b_i^{d-j}\sqrt{\binom{d}{j}}e_j =: \sum c_i^j e_j$.

The basic estimate we need is the determinant of the transition matrix from the e_j to the Q_i. The basic idea is that this determinant measures the failure of the $\{Q_i\}$ to be orthonormal.

Set $x_i = a_i/b_i$. The next lemma shows in particular that any $d+1$ distinct points in $\mathbb{P}^1$ gives rise to a basis.

Lemma 3.20.

$$\begin{aligned}
\begin{vmatrix} c_0^0 & c_0^1 & c_0^2 & \dots & c_0^d \\ c_1^0 & c_1^1 & c_1^2 & \dots & c_1^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_d^0 & c_d^1 & c_d^2 & \dots & c_d^d \end{vmatrix}
&= \prod_{i=0}^d \Bigl(\sqrt{\tbinom{d}{i}}b_i^d\Bigr)\begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^d \\ 1 & x_1 & x_1^2 & \dots & x_1^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_d & x_d^2 & \dots & x_d^d \end{vmatrix} \\
&= \Bigl(\prod_{i=0}^d \sqrt{\tbinom{d}{i}}b_i^d\Bigr)\prod_{j>i}\Bigl(\frac{a_j}{b_j}-\frac{a_i}{b_i}\Bigr) \\
&= \Bigl(\prod_{i=0}^d \sqrt{\tbinom{d}{i}}\Bigr)(\pm 1)\sqrt{\prod_{j\neq i} b_i b_j}\sqrt{\prod_{j\neq i}(a_j/b_j - a_i/b_i)} \\
&= \Bigl(\prod_{i=0}^d \sqrt{\tbinom{d}{i}}\Bigr)(\pm 1)\sqrt{\prod_{j\neq i}(a_j b_i - a_i b_j)}.
\end{aligned}$$

Proof. Immediate, using Vandermonde determinants. $\square$

Next, let $\Sigma_j := \mathrm{Span}(Q_0, \dots, \widehat{Q}_j, \dots, Q_d)$. Write

$$Q_{j+1} = Q'_{j+1} + Q''_{j+1},$$

where Q'_{j+1} is the component of Q_{j+1} perpendicular to Σ_j. Let $d_j := \|Q'_j\|_{L^2(B)}$.

Let $G_d := \det(\langle Q_i, Q_j\rangle)$ be the Gram determinant of the vectors $Q_0, \dots, Q_d$, and let $G_d^j := \det(\langle Q_i, Q_l\rangle)$ be the Gram determinant of $(Q_0, \dots, \widehat{Q}_j, \dots, Q_d)$. We will use the following classical fact:

Let $x_1, \dots, x_n$ be linearly independent vectors in a Hilbert space, spanning a subspace Σ. Suppose $y \notin \Sigma$ and let $Py = \sum_{i=1}^n a_i x_i$ be the orthogonal projection of y onto Σ, so that $(y - Py) \perp x_i$ for each i. Then

$$\|y - Py\|^2 = \|y\|^2 - \sum_{i=1}^n a_i \langle y, x_i\rangle = \frac{G^+}{G}, \tag{3–3}$$

where $G = \det(\langle x_i, x_j\rangle)$ is the Gram determinant of $(x_1, \dots, x_n)$ and G^+ is the Gram determinant of $(y, x_1, \dots, x_n)$. To see this, expand the determinant G^+ by its first row and use repeatedly the equalities

$$\langle y, x_j\rangle = \sum_{i=1}^n a_i \langle x_i, x_j\rangle.$$

Applying (3–3) to the situation at hand, we obtain

$$d_j = \frac{\sqrt{G_d}}{\sqrt{G_d^j}}.$$

With the notation of Lemma 4.5, set $A = (c_i^j)$. Then

$$G_d = \det(A^t \bar{A}) = \left(\textstyle\prod_{i=0}^d \binom{d}{i}\right) \left|\textstyle\prod_{j \neq i}(a_j b_i - a_i b_j)\right|.$$

Now consider G_d^j. For $0 \le j \le d$, let A_j be the matrix obtained from A by removing row j. Then

$$G_d^j = \det(A_j^t \overline{A}_j) = \sum_{0 \le l \le d} \left|\det(A_j^l)\right|^2,$$

where A_j^l is obtained from A_j by removing the l-th column. Hence

$$d_j^2 = \frac{G_d}{G_d^j} = \frac{|\det A^t \overline{A}|}{|\det A_j^t \overline{A}_j|} = \frac{|\det A|^2}{\sum_{l=0}^d |\det A_j^l|^2}.$$

It follows that

$$d_j^2 = \frac{\left(\prod_{i=0}^{d}\binom{d}{i}\right)\prod_{k\neq i}|a_k b_i - a_i b_k|}{\sum_{0\le l\le d}\left|\det\begin{bmatrix} c_0^0 & \dots & c_0^{l-1} & c_0^{l+1} & \dots & c_0^d \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{j-1}^0 & \dots & c_{j-1}^{l-1} & c_{j-1}^{l+1} & \dots & c_{j-1}^d \\ c_{j+1}^0 & \dots & c_{j+1}^{l-1} & c_{j+1}^{l+1} & \dots & c_{j+1}^d \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_d^0 & \dots & c_d^{l-1} & c_d^{l+1} & \dots & c_d^d \end{bmatrix}\right|^2}.$$

Therefore,

$$d_j^2 = \frac{\left(\prod_{i=0}^{d}\binom{d}{i}\right)\prod_{k\neq i}|a_k b_i - a_i b_k|}{\sum_{0\le l\le d}\left(\prod_{i\neq l}\binom{d}{i}\right)\left(\prod_{i\neq j}|b_i|^{2d}\right)\left|\det\begin{bmatrix} 1 & \dots & \left(\frac{a_0}{b_0}\right)^{d-1} \\ \vdots & \ddots & \vdots \\ 1 & \dots & \left(\frac{a_{j-1}}{b_{j-1}}\right)^{d-1} \\ 1 & \dots & \left(\frac{a_{j+1}}{b_{j+1}}\right)^{d-1} \\ \vdots & \ddots & \vdots \\ 1 & \dots & \left(\frac{a_d}{b_d}\right)^{d-1} \end{bmatrix}\right|^2 |S_{d-l}^j|^2}$$

where S_{d-l}^j is the sum of all distinct products of $d-l$ of the terms $a_0/b_0, \dots, a_{j-1}/b_{j-1}, a_{j+1}/b_{j+1}, \dots, a_d/b_d$ with $S_0^j = 1$.

We obtain

$$d_j^2 = \frac{\left(\prod_{i=0}^{d}\binom{d}{i}\right)\prod_i |b_i|^{2d}\prod_{k\neq i}\left|\frac{a_k}{b_k}-\frac{a_i}{b_i}\right|}{\sum_{0\le l\le d}\left(\prod_{i\neq l}\binom{d}{i}\right)\left(\prod_{i\neq j}|b_i|^{2d}\right)\left(\prod_{\substack{k\neq i\\ k,i\neq j}}\left|\frac{a_k}{b_k}-\frac{a_i}{b_i}\right|\right)|S_{d-l}^j|^2}.$$

Simplification leads to the following equality:

Lemma 3.21.

$$d_j^2 = \frac{|b_j|^{2d}\left(\prod_{k\neq j}\left|\frac{a_k}{b_k}-\frac{a_j}{b_j}\right|\right)^2}{\sum_{0\le l\le d}\frac{|S_{d-l}^j|^2}{\binom{d}{l}}}.$$

We can expand the numerator.

Lemma 3.22.

$$d_j^2 = \frac{|b_j|^{2d}\left(\left|\sum_{l=0}^{d}(-1)^{d-l}\left(\frac{a_j}{b_j}\right)^l S_{d-l}^j\right|\right)^2}{\sum_{0\le l\le d}\frac{|S_{d-l}^j|^2}{\binom{d}{l}}}.$$

We now try to write down a general formula. Let us number the circles C_k and suppose that the circle C_k contains n_k points $\frac{a_k}{b_k}\omega_k^i$. We choose our point $p = (a_j, b_j)$ on circle C_r. Fix any m. We want to find a formula for S_m^j. For any integer i such that $0 \le i < n_r$, consider any possible way of writing $i + \sum \delta_k n_k = m$ where each δ_k equals 0 or 1, and $\delta_r = 0$. So we are selecting those contributions coming from i points on C_r and all the points on the circles C_k, with $\delta_k = 1$. This procedure captures all non zero contributions to S_m^j. A general formula is $(a_j/b_j)S_m^j + S_{m+1}^j = S_{m+1}$, where S_m denotes all symmetric combinations of order m.

Lemma 3.23.

$$S_m^j = \sum_{i+\sum \delta_k n_k = m} \left(-\tfrac{a_j}{b_j}\right)^i \textstyle\prod_{\{\delta_k=1\}} \left(\tfrac{a_k}{b_k}\right)^{n_k}.$$

Lemma 3.24.

$$d_j^2 = \frac{|b_j|^{2d}\left(\left|\sum_{l=0}^{d}\sum_{i=0}^{n_r-1}(-1)^{d-l-i}\left(\frac{a_j}{b_j}\right)^{l+i}\sum_{\delta_k n_k = d-l-i}\prod\left(\frac{a_k}{b_k}\right)^{n_k}\right|\right)^2}{\sum_{0\le l\le d}\frac{|S_{d-l}^j|^2}{\binom{d}{l}}}.$$

Since the circle C_r is excluded from contributing to the last product, we have the restriction $d - l - i \le d - n_r$, so $l + i \ge n_r$.

Lemma 3.25.

$$d_j^2 = \frac{|b_j|^{2d}\left(\left|\sum_{s=n_r}^{d} n_r\left(-\frac{a_j}{b_j}\right)^{s}\sum_{\delta_k n_k = d-s}\prod\left(\frac{a_k}{b_k}\right)^{n_k}\right|\right)^2}{\sum_{0\le l\le d}\frac{|S_{d-l}^j|^2}{\binom{d}{l}}}.$$

The expression in the numerator has an obvious largest term. Namely, take $\delta_k = 1$ for those circles closer to the north pole than p. Set $\delta_k = 0$ otherwise. The term is

$$n_r\left(-\frac{a_j}{b_j}\right)^s \prod_{|b_k|<|b_j|}\left(\frac{a_k}{b_k}\right)^{n_k}, \qquad s = d - \sum \delta_k n_k.$$

This term is larger than the sum of all the others.

Before we proceed, we do some point counting. Let $t(r)$ denote the number of points on and the circle C_r and south of it. Put

$$S = \widetilde{S}_{t(r)}^j = \prod_{|b_k|<|b_r|}\left(\frac{a_k}{b_k}\right)^{n_k},$$

$$S = \textstyle\prod_{k>r}(\tan(\theta_k/2))^{\sqrt{\pi d}}\sin(\theta_k),$$

$$t(r) = \sum_{i=0}^{r} n_i = \sum_{i=0}^{r}\sqrt{\pi d}\,\sin\big((2i+1)\sqrt{\pi/d}\big).$$

Write $\theta_k = 2y$ to get $dk = \sqrt{d/\pi}\,dy$,

$$\begin{aligned}\log S &= \int_{r+1}^{\sqrt{d\pi}/2-1} \sqrt{\pi d}\sin(\theta_k)\log(\tan(\theta_k/2))\,dx\\ &\sim d\int_{r\sqrt{\pi/d}}^{\pi/2-\varepsilon} \sin(2y)\log(\tan(y))\,dy\\ &= -d\cos(2y)/2\log(\tan y)\big|_{r\sqrt{\pi/d}}^{\pi/2} + d\int_{r\sqrt{\pi/d}}^{\pi/2-\varepsilon}\frac{\cos(2y)}{\sin(2y)}dy\\ &= d/2\cos\theta_r\log(\tan(\theta_r/2)) - d/2\cos(\pi-2\varepsilon)\log(\tan(\pi/2-\varepsilon))\\ &\qquad + d/2\log(\sin(\pi-2\varepsilon)) - d/2\log(\sin\theta_r)\\ &\sim d/2\cos\theta_r\log(\tan(\theta_r)) - d/2\log(\sin\theta_r) + (d\log 2)/2.\end{aligned}$$

Hence we get:

Lemma 3.26.

$$\widetilde{S}^j_{t(r)} \sim 2^{d/2}\frac{(\tan(\theta_r/2))^{d\cos\theta_r/2}}{(\sin\theta_r)^{d/2}}.$$

Lemma 3.27. $t(r) \sim d/2 - d/2\cos\theta_r$.

Proof.

$$t(r) \sim \int_0^r \sqrt{\pi d}\sin((2x+1)\sqrt{\pi/d}\,)\,dx = d/2\cos(\sqrt{\pi/d}\,) - d/2\cos\theta_r. \qquad \square$$

So we get, for the numerator in d_j^2,

$$\left(\sqrt{\frac{1+\cos\theta_r}{2}}\right)^{2d}\left(\frac{\pi(\sin\theta_r)}{\sqrt{\pi/d}}\right)^2\left(\frac{\sin\theta_r}{1+\cos\theta_r}\right)^{d-d\cos\theta_r} 2^d\frac{(\tan(\theta_r/2))^{d\cos\theta_r}}{(\sin\theta_r)^d} = \pi d\sin^2\theta_r.$$

We have, more or less independently of which point is removed, the equality

$$S^j_{d-l} = \sqrt{\tbinom{d}{l}}.$$

But we want a lower bound for the numerator, hence cancellations are crucial.

We next estimate $S = S^j_r$.

Lemma 3.28.

$$(S^j_m)^2 \lesssim 2^d\frac{(\tan(\theta_r/2))^{d\cos\theta_r}}{(\sin\theta_r)^d}.$$

Proof. Recall that

$$S_j^m = \sum_{\{i+\sum\delta_k n_k = m\}}\left(-\frac{a_j}{b_j}\right)^i\prod_{\{\delta_k=1\}}\left(\frac{a_k}{b_k}\right)^{n_k}.$$

We should have

$$|S^j_m| \lesssim \prod\left|\frac{a_k}{b_k}\right|,$$

where the product is taken over the m points closest to the north pole.

Let C_r be the northernmost circle under those m points. By Lemma 4.14,

$$t(r) = \frac{d}{2} - \frac{d}{2}\cos\theta_r = d - m.$$

Let $\tilde{\jmath}$ be an index for a point on C_r. Then $|S^j_m| \lesssim \widetilde{S}^{\tilde{\jmath}}_{t(r)}$. Hence we have the estimate

$$|S^j_m| \lesssim 2^{d/2}\frac{(\tan\frac{\theta_r}{2})^{d\cos(\frac{\theta_r}{2})}}{(\sin\theta_r)^{d/2}}. \qquad \square$$

LEMMA 3.29.

$$|S^j_m|^2 \lesssim \binom{d}{m}.$$

PROOF. Since $m = d - t(r)$, we have $\binom{d}{m} = \binom{d}{t(r)}$. Then

$$\begin{aligned}
\frac{(S^j_m)^2}{\binom{d}{m}} &\lesssim \frac{2^d(\tan\frac{\theta_r}{2})^{d\cos\theta_r}}{(\sin\theta_r)^d}\frac{(\frac{d}{2}-\frac{d}{2}\cos\theta_r^{d/2-d/2\cos\theta_r})}{d^d}\bigl(d-\tfrac{d}{2}+\tfrac{d}{2}\cos\theta_r\bigr)^{d-d/2+d/2\cos\theta_r}\\
&= \frac{2^d(\tan\frac{\theta_r}{2})^{d\cos\theta_r}}{(\sin\theta_r)^d}\frac{(\frac{d}{2}-\frac{d}{2}\cos\theta_r^{d/2-d/2\cos\theta_r})}{d^d}\bigl(\tfrac{d}{2}+\tfrac{d}{2}\cos\theta_r\bigr)^{d/2+d/2\cos\theta_r}\\
&= \frac{2^d(\tan\frac{\theta_r}{2})^{d\cos\theta_r}}{(\sin\theta_r)^d}\frac{(1-\cos\theta_r^{d/2-d/2\cos\theta_r})}{2^d}(1+\cos\theta_r)^{d/2+d/2\cos\theta_r}\\
&= \frac{2^d(\tan\frac{\theta_r}{2})^{d\cos\theta_r}}{(\sin\theta_r)^d}\frac{(\sin\theta_r)^d}{2^d}\frac{(1+\cos\theta_r)^{d/2\cos\theta_r}}{(1-\cos\theta_r)^{d/2\cos\theta_r}}\\
&= \frac{(\tan\frac{\theta_r}{2})^{d\cos\theta_r}(1+\cos\theta_r)^{d/2\cos\theta_r}}{(1-\cos\theta_r)^{d/2\cos\theta_r}}\\
&= 1. \qquad \square
\end{aligned}$$

Hence:

LEMMA 3.30.

$$d_j^2 \gtrsim (\sin\theta_r)^2 \gtrsim \frac{1}{d}.$$

Remark that the estimate for d_j is most degenerate near the poles. Away from the poles, we have $d^j \sim 1$. However, we have not made optimal estimates and it is likely that the optimal lower bound is independent of whether or not we are close to the poles. So probably, we should have $d_j \sim 1$ everywhere.

Hence we get $d_j \sim 1/\sqrt{d}$.

Next, take any vector X of L^2 norm 1. We can write $X = X_j + c_jQ_j$ where $X_j \in \Sigma_j$. Then $X = (X_j + c_jQ''_j) + c_jQ'_j$. We get:

If X is homogenous of degree d, then $X = \sum c_j Q_j$, with $|c_j| \lesssim \|X\|_{L^2(\mathbb{B})}\sqrt{d}$.

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