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# Attractors in $\mathbb{P}^2$

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Abstract. We investigate attractors for holomorphic maps from $\mathbb{P}^k \to \mathbb{P}^k$, emphasizing the case $k = 2$. The interest in attractors stems from the fact that when a map is subject to small random perturbations, the long-term dynamics of the resulting system live near the map's attractors. In the case $k = 1$, that is, the case of rational functions on the Riemann sphere, the attractors are either periodic orbits or the whole sphere. In higher dimensions, however, there are other possibilities, which we call nontrivial. In addition to giving some examples of nontrivial attractors, we prove some general results about such attractors in $\mathbb{P}^2$, among them that a given map can have at most one nontrivial attractor $K$, that $K$ is then connected, has pseudoconvex complement, and contains a nonconstant entire image of $\mathbb{C}$, and that an attractor for a map $f$ is also an attractor for any iterate $f^n$.

Contents



## 1. Introduction

We recall first some general notions from the theory of dynamical systems. See [Ruelle 1989] for background.

Let $(X, d)$ be a compact metric space and $f$ a continuous map from $X$ to $X$. The sequence $(x_j)_{1\le j\le n}$ is an $\varepsilon$-pseudo-orbit if $d(f(x_j), x_{j+1}) < \varepsilon$ for $j = 1, \dots, n-1$. For $a, b \in X$, we write $a \succ b$ if for every $\varepsilon > 0$ there is an $\varepsilon$-pseudo-orbit from $a$ to $b$. We also write $a \succ a$. We write $a \sim b$ if $a \succ b$ and $b \succ a$, and

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denote by $[a]$ the equivalence class of $a$ under this relation. Define an *attractor* to be a minimal equivalence class for $\sim$. The following proposition is an easy consequence of Zorn's lemma.

PROPOSITION 1.1. *Let $f : X \to X$ be a continuous map on a compact metric space $X$. Then given any $x \in X$, there is an attractor $[a]$ such that $x \succ a$.*

It is also easy to show that an attractor $K$ is compact and satisfies $f(K) = K$. See [Ruelle 1989].

We have also the notion of an *attracting set.* A nonempty compact subset $K \subset X$ is an attracting set if it satisfies these conditions:

(i) There exists an open neighborhood $U \supset K$ such that $f(U) \Subset U$.
(ii) $K = \bigcap f^n(U)$.

LEMMA 1.2. *Suppose $\varnothing \neq U \subset X$ is an open set such that $f(U) \Subset U$. Then $U$ contains an attracting set $\bigcap f^n(U)$.*

PROOF. See [Ruelle 1989, Proposition 8.2]. □

LEMMA 1.3. *Let $K$ be an attractor. Then $K$ is a decreasing limit of countably many attracting sets.*

PROOF. Let $U$ be any open neighborhood of $K$. Then there exists $\rho > 0$ such that no $\rho$-pseudo-orbit from $K$ leaves $U$. For $\varepsilon < \rho$, let $V$ be the set of points which can be reached by an $\varepsilon$-pseudo-orbit starting at $K$. Then $V$ is an open subset of $U$, and, for each $x \in V$, we have $d(f(x), \partial V) \geq \varepsilon$; otherwise points in $V^c$ could be reached from $K$ by an $\varepsilon$-pseudo-orbit, contradicting the definition of $V$. Thus $f(V) \Subset V$, and

$$K' := \bigcap f^n(V)$$

is an attracting set, by Lemma 1.2. Since $f(K) = K$, we have $K' \supset K$. Since $U$ was arbitrary, we are done. □

## 2. Size of Attractors

THEOREM 2.1. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of degree at least two. Suppose that $K$ is an attractor for $f$. Then either $K$ is an attracting periodic orbit for $f$, or $K$ contains a nonconstant, entire image of $\mathbb{C}$.*

PROOF. By the previous lemma, $K$ is a decreasing limit of attracting sets. So we can put $K = \bigcap_{i=1}^{\infty} K_i$ where the $K_i$ are attracting sets, $K_{i+1} \subset K_i$. We can also find open sets $U_i$, $U_{i+1} \Subset U_i$ and $f(U_i) \Subset U_i$, $K_i = \bigcap f^n(U_i)$, $K = \bigcap U_i$. Fix $i$. Let $\mathcal{A}_i$ denote the affine automorphisms of $\mathbb{P}^k$ close enough to Id. More precisely we want all $A \in \mathcal{A}_i$ to have the property that $A \circ f(U_i) \Subset U_i$. Let $\delta > 0$ be so small that if $\mathrm{dist}(p, q) < \delta$ then there exists an $A_{p,q} \in \mathcal{A}_i$ so that $A_{p,q}(p) = q$.

Let $t = t_i < 1$ be fixed so that if $p \in \mathbb{P}^k$, there exists an $A_p \in \mathcal{A}_i$ which fixes $p$ and for which the derivative at $p$ is scaling by the factor $1/t$. Suppose next that $p \in K_i$ and that $q := f^n(p)$ is closer to $p$ than $\delta$. (In fact, we can take any point $w \in K_i$ and let $p := f^m(w)$ for large $m$.) Let

$$B := A_{q,p} \circ f \circ A_{f^{n-1}(p)} \circ f \circ \cdots \circ A_{f(p)} \circ f.$$

Then $B(U_i) \Subset U_i$ and $B(p) = p$.

There are two cases:

Suppose first that for each $i$ we can always find at least one such $B$ with some eigenvalue of $B'(p)$ strictly larger than 1 in modulus. In that case, let $\xi$ be a corresponding eigenvector. Let $\phi : \Delta \to U$ be a holomorphic map with $\phi(0) = p$ and $\phi'(p)$ a nonzero multiple of $\xi$. Using the sequence $\phi_n := B^n \circ \phi_1$ we get a map $\psi_i$ from the unit disc into $U_i$ with $\psi_i(0) = p$ and $|\psi_i'(0)| > i$. By Brody's theorem there must be a nonconstant entire image $X$ of $\mathbb{C}$ in $\bigcap U_i = K$.

The second case is that for some $i$ one never can have some eigenvalue of some such $B'(p)$ larger than one. In that case, it follows that

$$A_{f^n(p),p} \circ f^n(p)$$

has derivative bounded by $t^{n-1}$ whenever $\operatorname{dist}(f^n(p), p) < \delta$. We cover $K_i$ by a finite number $\{\Delta_j\}_{j=1}^k$ of discs of radius $\delta$. Consider any finite orbit $\{f^n(p)\}_{j=1}^N$, where $p \in K$. We can always break the orbit up in at most $k$ blocks. The first and last point of each block of consecutive iterates are in the same disc. To define the first block, take all the iterates up to and including the last one in the same disc as the first. To get the second block, take the first iterate after the block and take all interates up to and including the last one in that disc, etc. It follows from the above estimates that for some $C > 0$, we have $\|f^n(p)'\| \leq Ct^n$ for any $p \in K$ and any $n \geq 1$. Hence $f^M|_K$ is contracting for large $M$. Since $f(K) = K$, it follows that the attractor is just an attracting periodic orbit. $\square$

Corollary 2.2. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of degree at least two, and let $K$ be an attractor for $f$. Then either $K$ is an attracting periodic orbit for $f$, or $K \cap J \neq \varnothing$, where $J$ is the Julia set for $f$.*

Proof. It is a result of Ueda [1994] that the Fatou set for $f$ is Kobayashi hyperbolic. By the theorem, if $K$ is not an attracting periodic orbit, it contains an entire nonconstant image of $\mathbb{C}$, in which case the hyperbolicity of the Fatou set implies that $K \cap J \neq \varnothing$. $\square$

Example 2.3. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map which restricts to a polynomial self map of $\mathbb{C}^2$ and preserves the line $L$ at infinity. Then the line at infinity is an attracting set. The map $f : L \to L$ can be chosen to have a Siegel disc or a parabolic basin and no other Fatou components (except preimages). In that case $L$ is an attractor which lies partly in the Fatou set and partly in the Julia set.

Definition 2.4. We say that an attractor is trivial if it consists of a finite periodic attracting orbit or the whole space. Otherwise we say that the attractor is nontrivial.

We want to analyze nontrivial attractors. From the proof of the above theorem, we get in particular:

Theorem 2.5. *Suppose that $K$ is a nontrivial attractor and that $U$ is an open set containing $K$. Then there exist an open set $W$ with $K \subset W \subset U$, a positive number $t < 1$ such that if $A_p$ is the linear map expanding by a factor $1/t$ at $p$, then $A_p \circ f(W) \Subset W$, and a $\delta > 0$ so that if $p, q \in K$ with $\operatorname{dist}(p, q) < \delta$, then there exists a linear map $A_{p,q}$ close to $\mathrm{Id}$ so that $A_{p,q}(p) = q$ and $A \circ f(W) \Subset W$. Moreover, there exists a point $p \in K$ and an integer $n$ such that $\operatorname{dist}(f^n(p), p) < \delta$ and $B := A_{f^n(p),p} \circ f \circ A_{f^{n-1}(p)} \circ \cdots \circ A_{f(p)} \circ f$ satisfies $B(p) = p$ and for some nonzero tangent vector $\xi$ we have $B'(p)(\xi) = \lambda\xi$, with $|\lambda| > 1$.*

Corollary 2.6. *Let $q \in K$, a nontrivial attractor and $W, U$ as above. Then there exists a map $g : \mathbb{P}^2 \to \mathbb{P}^2$ with $g(W) \Subset W$ and $g(q) = q$, and also some vector $\xi \neq 0$ such that $g'(q)\xi = \lambda\xi$ with $|\lambda| > 1$.*

Proof. Let $f_1, \dots f_m, \tilde{f}_1, \dots, \tilde{f}_k$ be small perturbations of $f$ mapping $W$ relatively compact to $W$, with $f_m \circ \cdots f_1(q) = p$ and $\tilde{f}_k(p) \circ \cdots \circ \tilde{f}_1(p) = q$. Wiggling a little more, we may assume that $q$ and $p$ are not critical points for $f_n \circ \cdots \circ f_1$ and $\tilde{f}_k \circ \cdots \circ \tilde{f}_1$, respectively. Then the composition $\tilde{f}_k \circ \cdots \circ \tilde{f}_1 \circ B^N \circ f_n \circ \cdots \circ f_1$ works for large $N$. □

Corollary 2.7. *Let $U$ be any neighborhood of a nontrivial attractor $K$. Then for every point $p \in K$, and any $R > 0$ there exists a holomorphic map $\Phi : \Delta \to U$ with $\Phi(0) = p$, $\|\Phi'(0)\| = R$.*

Theorem 2.8. *A nontrivial attractor $K$ is connected.*

Proof. Suppose not. Then there exists two open sets $U, V$ with $K \subset U \cup V$, $\overline{U} \cap \overline{V} = \varnothing$. Define $K_1 := K \cap U \neq \varnothing$ and $K_2 := K \cap V \neq \varnothing$. By the above construction, there exist entire images $\Phi_i(\mathbb{C}) \subset K_i$. The theorem follows then from the following two results. □

## 3. Pseudoconvexity of the Complement of an Attractor

Lemma 3.1. $\mathbb{P}^2 \setminus \overline{\Phi_i(\mathbb{C})}$ *is pseudoconvex.*

Proof. If not, there is a Hartogs figure $H$ in $\mathbb{P}^2 \setminus \overline{\Phi_i(\mathbb{C})}$, so that part of $\Phi_i(\mathbb{C})$ is in $\tilde{H} \setminus H$. But then one can find a bounded subharmonic non-constant function on $\Phi_i(\mathbb{C})$, hence on $\mathbb{C}$, which is impossible. □

Proposition 3.2. *A pseudoconvex set in $\mathbb{P}^2$ has connected complement.*

Corollary 3.3. *$f$ can have at most one nontrivial attractor.*

COROLLARY 3.4. *A nontrivial attractor $A$ for $f$ is also an attractor for any iterate $f^n$.*

PROOF. Since $A$ is a countable decreasing intersection of attracting sets for $f$, $A$ is also a countable intersection of attracting sets for $f^n$. Hence $A$ contains an attractor $B$ for $f^n$. Since $B$ is an attractor for $f^n$, we have $f^n(B) = B$. For any $x \in A$, we have $x \succ [a]$ under $f^n$ for some attractor $[a]$. But since any pseudo-orbit for $f^n$ is a pseudo-orbit for $f$, we must have $[a] \subset A$. Since $A$ contains no attracting periodic orbits, $[a]$ must be nontrivial, and thus by Corollary 3.3 $[a] = B$. For $i < n$, let $O$ be an $\varepsilon$-pseudo-orbit for $f^n$ from $f^i(B)$ to $B$. Then $f^{n-i}(O)$ is an $L^{n-i}\varepsilon$- pseudo-orbit from $f^n(B) = B$ to $f^{n-i}(B)$, where $L$ is a Lipschitz constant for $f^n$. Since $\varepsilon$ was arbitrary, we have $B \succ f^{n-i}(B)$. Since $B$ is an attractor for $f^n$, we must have $f^{n-i}(B) \subset B$. This holds for each $i < n$. Applying $f^i$ to this inclusion, we obtain $B \subset f^i(B)$ for each $i < n$. Thus $B = f^i(B)$ for each $i < n$. In particular, $f(B) = B$.

Now let $a \in A$ and $b \in B$. Given $\varepsilon > 0$, there is an $\varepsilon$-pseudo-orbit for $f$ from $b$ to $a$. We may write

$$a = \tau_k \circ f \circ \ldots \circ \tau_1 \circ f \circ \tau_0(b),$$

where each $\tau_i$ is a translation by a vector in $B(0, \varepsilon)$. Let $j = k \bmod n$. Write

$$\tau_j \circ f \circ \ldots \circ \tau_1 \circ f \circ \tau_0 = \sigma_0 \circ f^j$$

if $j \geq 1$, where $\sigma_0$ is a translation by a vector in $B(0, \varepsilon')$, and where $\varepsilon' \to 0$ as $\varepsilon \to 0$. If $j = 0$, just take $\sigma_1 = \tau_0$. Similarly, write

$$\tau_{i+n-1} \circ f \circ \ldots \circ \tau_i \circ f = \sigma_{(i-j-1+n)/n} \circ f^n$$

for $i \in \mathbb{N}$ with $i = j + 1 \bmod n$, where again each $\sigma$ is translation by a vector of modulus $\varepsilon''$, where $\varepsilon'' \to 0$ as $\varepsilon \to 0$. We may assume that $\varepsilon'' > \varepsilon' > \varepsilon$.

We have constructed an $\varepsilon''$-pseudo-orbit for $f^n$ from $f^j(b)$ to $a$. But since $f^j(b) \in B$, we may also find an $\varepsilon''$-pseudo-orbit for $f^n$ from $b$ to $f^j(b)$. Putting them together, we have a $2\varepsilon''$-pseudo-orbit from $b$ to $a$. Since we may make $\varepsilon''$ as small as we like, we have $b \succ a$ for $f^n$. But then $a \in B$ by the definition of $B$. □

LEMMA 3.5. $C \cap A \neq \varnothing$.

PROOF. Obvious since the complement of the critical set is pseudoconvex. □

In fact, we get for the same reason:

LEMMA 3.6. *Let $X$ be any algebraic curve. Then $X \cap A \neq \varnothing$.*

PROPOSITION 3.7. *There is an open neighborhood $U \supset A$ so that if $p \in A$, then there exists a map $g : U \to U$ such that $g(p) = p$ and $p$ is a saddle point.*

PROOF. First, there is a $g$ with $g(p) = p$ and at least one eigenvalue is expanding. If the other is not attracting, we insert a detour from $p$ close to $C \cap A$ to make the other eigenvalue small. □

COROLLARY 3.8. *The attracting eigenvalue of $g$ at $p$ might be taken to be* 0.

This is obvious from the previous proof.

THEOREM 3.9. *The complement of an attractor is pseudoconvex.*

PROOF. Suppose not. Pick a point $p \in K$ with a Hartogs figure $H$, with $H \cap K = \varnothing$ and $p \in \tilde{H}$. We may also assume that there exists an open set $U \supset K$ such that $U \cap K = \varnothing$. Then the unstable manifold for $p$ for $g$ as in the previous proposition is parametrized by $\mathbf{C}$ and hence the proof of Lemma 3 applies. □

DEFINITION 3.10. A compact set $L \subset A$ is minimal if the orbit of any point of $L$ is dense in $L$.

REMARK 3.11. By Zorn's Lemma, the closure of the forward orbit of any point in $A$ contains a minimal $L$.

LEMMA 3.12. *An attracting set $S \supset A$ contains $A$ and a possibly infinite collection of attracting periodic orbits.*

This is clear.

## 4. Description of Fatou Components That Intersect $A$

To fix notation, let $U_n$ be a sequence of neighborhoods of $A$,

$$V_{n+1} := f(U_{n+1}) \Subset U_{n+1} \Subset U_n,$$

such that $A = \bigcap U_n$.

THEOREM 4.1. *Let $\Omega$ be a Fatou component, $\overline{\Omega} \cap A \neq \varnothing$. Then $f^n_{|\Omega} \to A$ u.c.c.*

PROOF. It suffices to prove that if $\gamma : [0,1] \to \Omega$ is a continuous curve and $\gamma(0) \in U_n$, then $f^m(\gamma([0,1]) \subset U_n$ for all large enough $m$. Let $[0, r_m]$ be the largest interval for which $f^m(\gamma([0,1]) \subset \overline{U}_n$. Notice then that $r_m$ is an increasing sequence. However, since $f^{m+1}(\gamma(r_m)) \in \overline{V}_{n+1}$, it follows by uniform continuity that there is a fixed $\varepsilon > 0$ such that $r_{m+1} \geq \min\{1, r_m + \varepsilon\}$. Hence we are done. □

The proof shows, a little more generally:

COROLLARY 4.2. *If some Fatou component $\Omega$ intersects $U_n$, then $f^n_{|\Omega} \to U_n$ u.c.c.*

## 5. Simple Cases of Attractors

In this section we try to gain insight into attractors by working our way through examples which are gradually more complicated.

THEOREM 5.1. *Suppose that* $f : \mathbb{P}^2 \to \mathbb{P}^2$ *is a holomorphic map which restricts to a polynomial on* $\mathbb{C}^2$. *If* $A$ *has a nontrivial attractor*, *then this is the line at infinity and* $f$ *there is a rational map without attracting basins*. *The converse is also true*.

This is clear.

THEOREM 5.2. *Let* $A$ *be a totally invariant attractor for* $f : \mathbb{P}^2 \to \mathbb{P}^2$. *Then either* $A = \mathbb{P}^2$ *or* $A$ *is contained in a pluripolar set*, *and* $\mathbb{P}^2 \setminus A$ *is not hyperbolic. If in addition* $A$ *is algebraic*, *then* $A$ *is a hyperplane or a nonsingular quadratic curve*.

PROOF. By [Fornæss and Sibony 1994, Theorem 4.5] and the following discussion, if a proper subvariety $V$ of $\mathbb{P}^2$ satisfies $f^{-1}(V) = V$, then either $V$ is a nonsingular quadratic curve and $f$ must then have odd degree, or $V$ is a union of hyperplanes, and $f$ has one of the forms

$$
\begin{aligned}
[z:w:t] &\mapsto \bigl[f_0([z:w:t]) : f_1([z:w:t]) : t^d\bigr]\\
[z:w:t] &\mapsto \bigl[f_0([z:w:t]) : w^d : t^d\bigr]\\
[z:w:t] &\mapsto \bigl[z^d : w^d : t^d\bigr],
\end{aligned}
$$

depending on whether $V$ consists of one, two, or three hyperplanes. It is easy to verify directly that in the last two cases $V$ is not an attractor. Thus the only possibility for a totally invariant algebraic attractor is the first case, where the attractor is a hyperplane and $f$ is a suspension of a holomorphic map on $\mathbb{P}^1$ with empty Fatou set, or a nonsingular quadratic curve. This proves the second statement.

To prove the first, we note that by a special case of a result of Russakovskii and Shiffman [1997], given a holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$, there exists a pluripolar set $\mathcal{E}$ such that, for any probability measure $\nu$ which gives no mass to $\mathcal{E}$,

$$((f^n)^*\nu)/d^{nk} \to \mu.$$

Taking $\nu$ to be the mass of a single point, we see that given any $p \notin \mathcal{E}$, the successive inverse images of $p$ cluster all over supp $\mu$. Since the complement of $\mathcal{E}$ is dense in $\mathbb{P}^k$, for any such $p$ and any $q \in \operatorname{supp}\mu$, we have $q \succ p$. Thus if an attractor $A$ contains any point of supp $\mu$, then $A = \mathbb{P}^k$. If $A \not\subset \mathcal{E}$, then there exists $p \in A$ whose inverse images cluster all over supp $\mu$. Since $A$ is totally invariant and closed, supp $\mu \subset A$. Thus $A = \mathbb{P}^k$.

To prove that $\Omega := \mathbb{P}^2 \setminus A$ is not hyperbolic, since supp $\mu \subset \Omega$, by a result of Briend [1996] there exists a repelling periodic point $p \in \Omega$. Since $f(\Omega) = \Omega$, there exist arbitrarily large analytic disks in $\Omega$ through $p$. □

Next we turn to attractors which are not totally invariant.

EXAMPLE 5.3. The map $[(z-2w)^2 : t^2+z^2 : zt/2]$ has the line at infinity as an attractor whose preimage also containts the line $(z=0)$.

Details: The line at infinity is forward invariant and the map restricts to a critically finite preperiodic map on the line at infinity. We need to show that $(t=0)$ is attracting. We cover the line at infinity with two sets, $U_1 = \{|z| < |w|\}$ and $U_2 = \{|w| < |z|\}$. We introduce a metric equal the Euclidean metric in each of the two coordinates Then we show that the normal derivative if the map is at most $\frac{1}{2}$ at any point. Let $Z = (z-2w)^2$, $W = z^2+t^2$, $T = zt/2$.

On $U_1$ we have $|Z|/|W| = (1-2|w|/|z|)^2 > 1$. Hence $U_1$ is mapped into $U_2$. Hence the map takes the form $(z:1:t) \to (1:W:T)$, or $(z,t) \to (z^2/(z-2)^2, zt/(2(z-2)^2))$. Hence the $t$ derivative of the second coordinate is when $t=0$, $|z|/(|2(z-2)|^2) < \frac{1}{4}$ since $|z|<1$.

On $U_2$ there are two cases. If $|Z| < |W|$, the map takes the form

$$(w,t) \to ((1-2w)^2, t/2)$$

and the normal derivative is $\frac{1}{2}$.

If $|W| < |Z|$, the map takes the form

$$(w,t) \to \big(1/(1-2w)^2,\ t/(2(1-2w))^2\big);$$

hence the normal derivative is $1/|2(1-2w)|^2$. But in this set $|1/(1-2w)^2| < 1$, so we are done again.

A general technique for generating examples of this kind for maps of degree $d \geq 3$ : Take any rational map $[P(z,w) : Q(z,w)]$ of degree $d \geq 3$ without attracting basin. Then we can define $[P+t^d : Q : zt^{d-1}]$ or if necessary put the $t^d$ on the second term.

Next we give an example of an attractor which is a smooth rational curve, but not a line. The attractor is the set $V = (zw = t^2)$. Consider, for small $\delta \neq 0$, the map

$$F_\delta = [X:Y:Z] = \big[(z+4w-4t)^2 : z^2 : z(z+4w-4t) + \delta(t^2 - zw)\big].$$

First consider $F_0 : T^2 - XY \equiv 0$ and the point of indeterminacy is $[0:1:1] \notin V$. Hence $V$ is mapped holomorphically into itself and the map is holomorphic in a neighborhood of $V$. Also $F$ can be calculated on $V$, parametrized by $\tau \to (\tau, 1/\tau, 1)$, which is mapped to $[(\tau-2)^4/\tau^2 : \tau^2 : (\tau-2)^2] = [(\tau-2)^2/\tau^2 : \tau^2/(\tau-2)^2 : 1]$; hence the map reduces to $x \to (x-2)^2/x^2$ which is a critically finite maps whose Julia set is all of $\mathbb{P}^1$. For small $\delta \neq 0$, it follows that $V$ is an attractor.

Another example, similar to the previous one: Use the map $z \to \lambda(1-2/z)^3$, where $\lambda \in \mathbb{C}$ is chosen to make the map critically finite, with Julia set equal to

$\mathbb{P}^1$, again realized as $zw = t^2$.

$$f : \mathbb{P}^2 \to \mathbb{P}^2,$$
$$[z : w : t] \mapsto \big[\lambda(z+4w-4t)^3 : z^3/\lambda : z(z-2t)(z+4w-4t)+2(z-2t)(zw-t^2)\big].$$

In this case one calculates that $ZW-T^2 = 4(zw-t^2)^2\big(3(z-2t)^2+16(zw-t^2)\big)$ and hence the variety $zw = t^2$ is contained in the critical set. Hence it is an attractor.

Given $f : \mathbb{P}^2 \to \mathbb{P}^2$, let $C_1$ denote the critical locus of $f$. Let

$$D_1 = \bigcup_{n\geq 1} f^n(C_1) \quad \text{and} \quad E_1 = \bigcap_{n\geq 0} f^n(D_1).$$

Using the terminology of Jonsson [1998], we call $f$ *1-critically finite* if $D_1$ is algebraic (or, equivalently, if the union defining it is finite) and if $E_1$ and $C_1$ have no common irreducible component. Note that $E_1$ is algebraic if $D_1$ is. If $f$ is 1-critically finite, define

$$C_2 = C_1 \cap E_1, \quad D_2 = \bigcup_{n\geq 1} f^n(C_2), \quad E_2 = \bigcap_{n\geq 0} f^n(D_2).$$

Ueda [1998] has proved that these are finite sets. Call $f$ *2-critically finite* if $C_2 \cap E_2 = \varnothing$. It has been proved by Fornaess and Sibony [1992] and by Ueda [1998] that if $f$ is 2-critically finite its Fatou set is empty. Further work of Jonsson [1998] and Briend [1996] has shown that for such $f$, $\operatorname{supp}\mu = \mathbb{P}^2$. For 2-critically finite maps, therefore, $\mathbb{P}^2$ is an attractor. We wish to study maps which are 1-critically finite, but not necessarily 2-critically finite.

THEOREM 5.4. *Suppose that $f : \mathbb{P}^2 \to \mathbb{P}^2$ is* 1*-critically finite. Let $A$ be a nontrivial attractor for $f$. Then either $A = \mathbb{P}^2$, or $A$ contains a periodic cycle whose multiplier has one zero eigenvalue.*

PROOF. From Lemma 3.6, we have $C_1 \cap A \neq \varnothing$ and $E_1 \cap A \neq \varnothing$. By assumption, $E_1$ is algebraic, and by its definition its irreducible components are periodic. We may assume that $f$ is not 2-critically finite; otherwise $A = \mathbb{P}^2$. Thus $E_2$ contains a critical point $p$. Since all the points in $E_2$ are periodic, $p$ is periodic.

Let $V$ be an irreducible component of $E_1$ containing $p$. Since by Corollary 3.4 an attractor for $f$ is also an attractor for $f^n$, we may replace $f$ by an iterate without loss of generality. Thus we may assume that $V$ is invariant. If $V \subset A$, we are done. Otherwise, let $U$ be an open set containing $A$ with $f(U) \Subset U$. Assume that $U$ was chosen small enough that $V \not\subset U$, and let $U' = V \cap U$. We can assume that there are only finitely many irreducible components of $V \cap U$, and these are mapped to each other. Replacing $f$ by an iterate if necessary, we may assume that there is a component $V'$ which is mapped into itself and which intersects $A$. Let $\pi : \tilde{V} \to V$ be a normalization of $V$, and let $\tilde{f}$ be a lift of $f|_V$ to $\tilde{V}$. Let $\tilde{V}' = \pi^{-1}(V')$. Since $\tilde{f}(\tilde{V}') \Subset \tilde{V}'$, there is a fixed point $\tilde{q}$ for $\tilde{f}$ in $\tilde{V}'$. If $\tilde{V}$ is hyperbolic, $\tilde{V}$ has only finitely many nonconstant holomorphic self maps, so this is impossible. If $\tilde{V} = \mathbb{P}^1$, then $\tilde{f}$ is a rational function with an attracting fixed point at $\tilde{q}$. If $\tilde{q}$ is not critical, then there is a critical point in its basin with infinite forward orbit. Then the image under $\pi$ of this point is a critical point of $f$ in $E_1$ with infinite forward orbit. This is impossible, since $D_1$ is a finite set. Thus again $\tilde{q}$ is critical. The final possibility is that $\tilde{V}$ is a torus. But then, since $\tilde{f}$ is not injective, every periodic point of $\tilde{f}$ is repelling, contradicting the existence of $\tilde{q}$. Thus $\tilde{q}$ is a fixed critical point for $\tilde{f}$, and $q := \pi(\tilde{q})$ is a fixed critical point for $f$. Since $V'$ intersects $A$, and $f^n(z) \to q$ for all $z \in V'$, we must have $q \in A$. Since we have replaced $f$, possibly, with higher iterates, we conclude that the map we started with had a critical periodic orbit in $A$. □

Example 5.5. Consider the map $[z : w : t] \mapsto [(z-2w)^2 : z^2 : t^2]$. The line $(t = 0)$ is an attractor, and $[1 : 1 : 0]$ is a fixed critical point in the attractor.

Proposition 5.6. *If an attractor $A$ contains a repelling periodic point $p$ or a Siegel domain, then any path connecting $p$ to the complement of $A$ must intersect $\overline{D}_1$.*

Proof. In the case of a Siegel domain $\Omega$, we have already that $\partial\Omega \subset D_1$. In the case of a repelling periodic point $p$, which we may assume to be fixed, take a neighborhood of $p$ on which branches of inverses of $f^n$. are defined and converge to the constant map $p$. Let $q$ outside $A$ be connected to $p$ by a path which doesn't intersect $\overline{D}_1$. We may extend all the branches of inverses of $f^n$ previously defined along that path, and they form a normal family in the resulting open set, by a result of Ueda. Any convergent subsequence must converge to the constant map $p$. Thus inverse images of $p$ cluster on $q$. Thus $p \succ q$. But this is a contradiction. □

Corollary 5.7. *If $D_1$ is algebraic, there are no repelling periodic points in $A$ unless $A = \mathbb{P}^2$. There are no Siegel domains anywhere.*

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