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Analytic Hilbert Quotients

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Abstract. We give a systematic treatment of the quotient theory for a holomorphic action of a reductive group $G = K^{\mathbb{C}}$ on a not necessarily compact Kählerian space X. This is carried out via the complex geometry of Hamiltonian actions and in particular uses strong exhaustion properties of K-invariant plurisubharmonic potential functions.

The open subset $X(\mu)$ of momentum semistable points is covered by analytic Luna slice neighborhoods which are constructed along the Kempf–Ness set $\mu^{-1}\{0\}$. The analytic Hilbert quotient $X(\mu) \to X(\mu)/\!/G$ is defined on these Stein neighborhoods by complex analytic invariant theory. If X is projective algebraic, then these quotients are those given by geometric invariant theory.

The main results here appear in various contexts in the literature. However, a number of proofs are new and we hope that the systematic treatment will provide the nonspecialist with basic background information as well as details of recent developments.

Contents

1. Introduction

As the title indicates, we focus here on a certain quotient construction for group actions on complex spaces. Our attention is primarily devoted to actions of (linear) reductive complex Lie groups, i.e., complex matrix groups which are complexifications $G = K^{\mathbb{C}}$ of their maximal compact subgroups.

The building blocks for these groups are $\mathbb{C}^* = (S^1)^{\mathbb{C}}$ and the simple complex Lie groups, e.g., $\mathrm{SL}_n(\mathbb{C}) = (\mathrm{SU}_n)^{\mathbb{C}}$. In fact, after a finite central extension, a connected reductive Lie group is just a product of such groups.

A holomorphic action of a complex Lie group G on a complex space X is a holomorphic map $G \times X \to X$, $(g, x) \mapsto g(x)$, which is defined by a homomorphism $G \to Aut_{\mathcal{O}}X$. Although the reductive groups themselves are easily listed, understanding their actions is an entirely different matter.

As a first step one often considers invariants of a given action. At the geometric level this can mean the orbit space construction: Two points $x_1, x_2 \in X$ are deemed equivalent if there exists $g \in G$ with $g(x_1) = x_2$.

For a compact group a quotient $\pi : X \to X/\!\sim\, =: X/K$ of this type is quite reasonable. For example, the base is Hausdorff, has a stratified manifold structure and can be dealt with methods from semi-algebraic geometry. Furthermore, this quotient has a natural invariant theory interpretation: $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$ for every invariant continuous function $f \in \mathcal{C}(X)^K$.

On the other hand, one leaves the complex analytic category, e.g., the quotient of $\mathbb{C}$ by the standard S^1-action in $\mathbb{R}^{\geq 0}$.

From a complex analytic viewpoint an orbit space construction involving the complex Lie group G would be preferable. However, not to even mention the difficulties of constructing a complex quotient structure, due to the non-compactness of G, the orbit space X/G is often not even Hausdorff.

At least in situations where there are plenty of functions, the appropriate first steps in an analytic theory would seem to be of an invariant theoretic nature.

For example, in a case where algebraic methods are of great use, if $G \times V \to V$ is a linear action of a reductive group on a complex vector space, then the ring $R = \mathbb{C}[V]^G$ of invariant polynomials in finitely generated [Kraft 1984].

If $f_1, \ldots, f_m \in R$ is a choice for such generators, then the image of the map $F = (f_1, \ldots, f_m)$ is a good model for the quotient. Due to Hilbert's original impact on the finite generatedness of R, these are quite often referred to as Hilbert quotients. The base is the affine variety $\mathrm{Spec}(R)$ and the notation $\pi : V \to \mathrm{Spec}(R) =: V/\!/G$ serves as a reminder that this is not necessarily an orbit space construction.

For example, for the action of $G = \mathbb{C}^*$ on $\mathbb{C}^2$ by $\lambda(z, w) = (\lambda z, \lambda^{-1} w)$, the ring R is generated by $f := zw$ and the fiber of $\pi : \mathbb{C}^2 \to \mathbb{C}^2/\!/G = \mathbb{C}$ over $0 \in \mathbb{C}$ consists of three orbits.

Using arguments of affine algebraic geometry along with the identity principle $\mathbb{C}[V]^G = \mathbb{C}[V]^K$ one shows that the Hilbert quotient is indeed of a geometric nature. The quotient π is surjective, and $x_1 \sim x_2$ if and only if $\overline{Gx} \cap \overline{Gy} \neq \varnothing$ (see Section 3). For more general spaces and actions this may not be an equivalence relation.

Since polynomials are dense in $\mathcal{O}(V)^G$, the Hilbert quotient $\pi : V \to V/\!/G$ for a linear action serves as the appropriate quotient if one considers $G \times V \to V$ as a holomorphic action.

Now, using the existence of a closed, equivariant embedding $X \hookrightarrow V$ in a representation space, the algebraic quotient theory is immediately extended to the category of affine varieties. For Stein spaces, even in the smooth case, there do not in general exist such equivariant embeddings [Heinzner 1988]. Nevertheless, the algebraic invariant theory can be applied, but in a certain sense only locally.

The invariant theoretic quotient $\pi : X \to X/\!/G$ for a holomorphic action of a reductive group on a normal Stein space was constructed by D. Snow [1982]. Later the normality condition was removed and the quotient was also constructed for action of compact groups on Stein spaces [Heinzner 1989; 1991].

The key to Snow's construction is his adaptation of Luna's slice theorem (see Section 4) to the complex analytic setting. From our point of view the controlling tool for the well-definedness of the quotient is a K-invariant, strictly plurisubharmonic exhaustion function.

In the development of the complex analytic quotient theory since that time, the convexity and exhaustion properties of invariant plurisubharmonic functions have played a decisive role.

The Stein context and its rich function theory is however a bit misleading. The key geometric information is provided by K-invariant Kählerian structures and the availability of invariant plurisubharmonic potential functions which turn out to be exhaustions along the fibers of the quotients that are to be constructed.

The conceptual underlying factor is that of symplectic reduction. In order to introduce the notation we now recall this construction.

If (M, ω) is a symplectic manifold equipped with a smooth action $K \times M \to M$ of a Lie group of symplectic diffeomorphisms, a vector field $\xi_M \in \mathrm{Vect}_\omega(M)$ coming from the action of a 1-parameter group given by $\xi \in \mathrm{Lie}\, K^*$ is said to be Hamiltonian if it is associated to a function $\mu_\xi \in C^\infty(M)$ by $d\mu_\xi = i_{\xi_M}\omega$.

If these functions can be bundled together to an equivariant map $\mu : M \to \mathrm{Lie}(K)^*$ with coordinates $\xi \circ \mu = \mu_\xi$, then μ is said to be an equivariant moment map.

Vector fields $V_H \in \mathrm{Ham}(M)$ associated to K-invariant Hamiltonians $H \in C^\infty(M)$ by the rule $dH = i_{V_H}\omega$ have the full moment map as a constant of motion. Thus one is led to study the μ-fibers and their induced geometry.

It is most often sufficient to analyze a fiber of the type $M_0 := \mu^{-1}(0)$ and its embedding $i : M_0 \hookrightarrow M$. Of course, unless something is known about the group action so that, e.g., normal form theorems are applicable, M_0 may be quite singular, and in any case the form $\omega_0 := i^*\omega$ will be degenerate in the orbit directions.

If K is acting properly, e.g., for K compact, $M_{\text{red}} := M_0/K$ has only mild singularities and ω_0 can be pushed down to a stratified symplectic structure. See [Sjamaar and Lerman 1991], and also [Heinzner et al. 1994] for an embedding in the Stein space setting.

The technique of symplectic reduction has been used in numerous situations to better understand existing complex group quotients; see, for example, [Mumford et al. 1994; Kirwan 1984; Neeman 1985; Guillemin and Sternberg 1984]. Here we construct quotients in the Kählerian setting by using this principle in combination with methods involving plurisubharmonic potential functions. This approach is implicit in considerations of [Heinzner et al. 1994], and is carried out in general in [Heinzner and Loose 1994]. Kählerian quotients are also constructed in [Sjamaar 1995], in the presence of an appropriate, proper Morse function.

Although we carry out proofs in the singular case, for introductory purposes it is sufficient to consider a Kähler manifold (X, ω) equipped with a holomorphic action $G \times X \to X$ of a reductive group. For a choice of a maximal compact subgroup K of G it is assumed that ω is K-invariant and that there exists a K-moment map $\mu : X \to \operatorname{Lie}^* (K)$.

The 0-fiber $X_0 := \mu^{-1}(0)$, referred to as the Kempf–Ness set, provides a foundation for the entire study.

First, the set for which a quotient can be constructed, the set of semi-stable points, is defined by

$$X(\mu) := \{x \in X : \overline{Gx} \cap X_0 \neq \varnothing\}.$$

Using exhaustion properties of plurisubharmonic potential functions ρ with $\omega = dd^c\rho$ it is shown that $X(\mu)$ is open, that the Hausdorff quotient $\pi : X(\mu) \to Q$ exists and that the inclusion $X_0 \hookrightarrow X(\mu)$ induces a homeomorphism $X_0/K = Q$.

The complex structure on the quotient, which is given by the invariant part $U \mapsto \pi_* \mathcal{O}(U)^G$ of the direct image sheaf, is understood via an analytic version of Luna's slice theorem. In this way, using plurisubharmonic potential functions as controlling devices, we return to invariant theory. Hence the classical notation $X(\mu)/\!/G := Q$ is used. Since $\pi : X(\mu) \to X(\mu)/\!/G$ is a Stein map [Heinzner et al. 1998] and is locally an invariant theoretic quotient, this is an example of an analytic Hilbert quotient.

Although it is almost always the case that $X \neq X(\mu)$, there are certain important situations where $X = X(\mu)$. For example, if X is Stein, $\rho : X \to \mathbb{R}$ is a K-invariant, proper exhaustion and $\omega := dd^c\rho$, then $X = X(\mu)$ and $X(\mu) \to X(\mu)/\!/G$ is the invariant theory quotient.

One of the main goals of this paper is to make the basic methods for actions of reductive groups accessible to non-specialists. In particular, discussing from the point of view of complex geometry, in Sections 2 and 3 we attempt to systematically build a foundation for the developments in the last 10 years in complex analytic quotient theory. The existence of the quotient and its essential properties are proved in Section 4.

2. Stein Homogeneous Spaces of Reductive Groups

In the invariant theoretic as well as the Hamiltonian approach, Stein homogeneous spaces play a central role in the construction and understanding of analytic Hilbert quotients. Here we describe the basic properties of these spaces via an analysis of invariant plurisubharmonic functions and Kählerian reduction at the group level.

2.1. Decomposition Theorems. Let G be a reductive complex Lie group. A maximal compact subgroup K determines a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{p} := i\mathfrak{k}$. Define $P := \exp \mathfrak{p}$.

THE KP-DECOMPOSITION. *The map* $\exp : \mathfrak{p} \to P$ *is a diffeomorphism onto a closed submanifold* P *of* G. *Group multiplication*, $(k.p) \mapsto k{\cdot}p$, *then defines a diffeomorphism* $K \times P \to G$.

EXAMPLE (THE KP-DECOMPOSITION OF $\mathrm{GL}_n(\mathbb{C})$). Let $G := \mathrm{GL}_n(\mathbb{C})$ and $\sigma : G \to G$ be the anti-holomorphic involution defined by $\sigma(A) = {}^t\bar{A}^{-1}$. The real form $K := U_n = \mathrm{Fix}(\sigma)$ is a maximal compact subgroup of G.

Recall that $\mathfrak{g} = \mathrm{Mat}(n \times n, \mathbb{C})$ and note that at the Lie algebra level $\sigma_* : \mathfrak{g} \to \mathfrak{g}$ is given by $A \mapsto -{}^t\bar{A}$. Thus $\mathfrak{k} = \{A \in \mathrm{Mat}(n \times n, \mathbb{C}) : A + {}^t\bar{A} = 0\}$ and $\mathfrak{p} = i\mathfrak{k}$ is the set of Hermitian matrices.

The exponential map $\exp : \mathfrak{g} \to G$, $A \mapsto e^A = \sum_{n=0}^{\infty} A^n/n!$, maps $\mathfrak{p}$ into the closed submanifold $H^{>0}$ of Hermitian positive-definite matrices. For $h \in H^{>0}$ there exists $k \in K$ such that khk^{-1} is a diagonal matrix.

Since $\mathfrak{p}$ is invariant under the $\mathrm{Int}(K)$-action, i.e., $p \mapsto kpk^{-1}$, and the diagonal elements of $H^{>0}$ are clearly in $\exp(\mathfrak{p})$, it follows that $P = H^{>0}$.

That $\exp : \mathfrak{p} \to P$ and $KP \to G$ are diffeomorphisms follows from concrete calculations with matrices; see [Chevalley 1946].

The theorem on the KP-decomposition for an arbitrary reductive group is proved via this example. For this, first embed K in U_n by a faithful unitary representation $\tau : K \to U_n$. It can be shown that τ can be uniquely extended to a holomorphic representation $\tau^{\mathbb{C}} : G \to \mathrm{GL}_n(\mathbb{C})$ which is in fact biholomorphic onto its image; see [Hochschild 1965]. The $U_n{\cdot}H^{>0}$-decomposition of $\mathrm{GL}_n(\mathbb{C})$ restricts to G to give its KP-decomposition, as desired.

For notational convenience assume that G is connected and let T be a maximal torus in K. Let K act on $\mathfrak{k}$ by the adjoint representation. It follows that $K{\cdot}\mathfrak{t} = \mathfrak{k}$. Since $\mathfrak{p} = i\mathfrak{k}$, this can be interpreted in the context of the KP-decomposition. Before doing so, we recall several basic facts concerning the Weyl group. As a basic reference, see [Wallach 1973], for instance.

If for $k \in K$ there exists $\xi \in \mathfrak{t}$ with $k(\xi) \in \mathfrak{t}$, then it in fact follows that $k(\mathfrak{t}) = \mathfrak{t}$ and k is in the normalizer $N_K(T)$. Of course T acts trivially on $\mathfrak{t}$. So the $N_K(T)$ action factors through the action of the Weyl group $W = W_K(T) = N_K(T)/T$.

Since the tori do not possess continuous families of Lie group automorphisms and T is maximal, it follows that $W = W_K(T)$ is finite.

In fact W is generated in a natural way by reflections and therefore has a closed fundamental region $\mathfrak{t}^+$ which is an intersection of finitely many closed half-spaces. Consequently the Weyl-chamber $\mathfrak{t}^+$ serves as a fundamental region for the $\mathrm{Ad}(K)$-representation: The map $K \times \mathfrak{t}^+ \to \mathfrak{k}$, $(k, \xi) \mapsto \mathrm{Ad}(k)(\xi)$, is bijective.

The KAK-Decompositon. *Let G be a connected reductive group, K a maximal compact subgroup, $T < K$ a maximal torus and let $A := \exp(i\mathfrak{t}) \subset P$. Let $K \times K$ act on G by multiplication, $g \mapsto k_1 g k_2^{-1}$. Then:*

(i) $(K \times K)A = G$.
(ii) *For $A^+ := \exp(i\mathfrak{t}^+)$ it follows that the map $K \times K \times A^+ \to G$, $(k_1, k_2, a) \mapsto k_1 a k_2^{-1}$, is bijective.*
(iii) *Restriction defines an isomorphism $\mathcal{E}(G)^{K\times K} \cong \mathcal{E}(A)^W$.*

Proof. Since $P = \exp(i\mathfrak{k})$, K acts on $P \cong \mathfrak{k}$ by its adjoint representation. Thus A^+ is a fundamental region for its action and $(K \times K)\cdot A^+ = K\cdot \mathrm{Ad}(K)\cdot A^+ = K\cdot P = G$. It follows from the KP-decomposition and the fact that A^+ is a fundamental region that the map $K \times A^+ \times K \to G$ is injective. This proves (i) and (ii).

If $f \in \mathcal{E}(G)^{K\times K}$, then its restriction is clearly W-invariant. Since $G = KAK$, the restriction map is injective. For the surjectivity, given $f \in \mathcal{E}(A)^W$, let h be any smooth extension to G and define $F \in \mathcal{E}(G)^{K\times K}$ by averaging:

$$F(x) = \int h(k_1 x k_2^{-1})\, dV,$$

where dV is an invariant probability measure on $K\times K$. It follows that $F|A = f$. □

Remark. Since $G = K\cdot G^\circ$, the assumption of connectivity in the KAK-decomposition is of no essential relevance for applications.

2.2. Invariant Plurisubharmonic Functions.

a. Critical points. Throughout this section G is a reductive group, H a closed complex subgroup and $X = G/H$ the associated complex homogeneous manifold. Fix a maximal compact subgroup $K < G$. Note that, since $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$, it follows that the real dimension $\dim_{\mathbb{R}} Kx$ of an arbitrary K-orbit in X is at least the complex dimension $\dim_{\mathbb{C}} X$.

Lemma 2.2.1. *Suppose that H has finitely many components. Then it is reductive if and only if there exists $x \in X$ with $\dim_{\mathbb{R}} Kx = \dim_{\mathbb{C}} X$.*

Proof. If H is reductive, then it has a maximal compact subgroup L so that $H = L^{\mathbb{C}}$. After replacing H by a conjugate, we may assume that $L < K$. Thus the associated orbit of the neutral point $x \in X$ is $Kx = K/L$ and

$$\dim_{\mathbb{R}} Kx = \dim_{\mathbb{R}} K - \dim_{\mathbb{R}} L = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} H = \dim_{\mathbb{C}} X.$$

Conversely, if $\dim_{\mathbb{R}} Kx = \dim_{\mathbb{C}} X$, then, defining $L := H \cap K$, the preceding string of equalities shows that $\dim_{\mathbb{C}} L^{\mathbb{C}} = \dim_{\mathbb{C}} H$. (For this we also use the fact that $\mathfrak{l}$ is totally real.) Thus H^0 is reductive and, since H has at most finitely many components, it follows that H is reductive. □

Numerous arguments of the present work involve critical points of K-invariant strictly plurisubharmonic functions $\rho : X \to \mathbb{R}$. It is important in this regard to underline the role of the moment map.

As above let $X = G/H$ be a complex homogeneous space of a reductive group $G = K^{\mathbb{C}}$. Let D be a K-invariant open set in X equipped with a smooth, K-invariant Kähler form ω. Assume that there exits an equivariant moment map $\mu : D \to (\text{Lie } K)^*$ with $\mu^{-1}(0) \neq \varnothing$.

Given $x_0 \in \mu^{-1}(0)$ let N be a convex neighborhood of $0 \in \mathfrak{p}$ so that the open set $U = K\exp(iN)x_0$ is contained in D.

Lemma 2.2.2. $\mu^{-1}(0) \cap U = Kx_0$.

Proof. If for $k \in K$ and $\xi \in N$ the point $x = k\exp(i\xi)x_0$ is in $\mu^{-1}(0) \cap U$, then the same is true of $x_1 := \exp(i\xi)x_0$.

Now let $x_t := \exp(i\xi t)x_0$. From the defining property $d\mu_\xi = i_{\xi_D}\omega$ of the moment function μ_ξ it follows that $J\xi_D$ is the gradient field of μ_ξ with respect to the Riemannian metric induced by ω and the complex structure J. Thus, either $t \mapsto \mu_\xi(x_t)$ is strictly increasing or $x_t = x_0$ is the constant curve. Since $\mu_\xi(x_0) = \mu_\xi(x_1) = 0$, we are in the latter situation and therefore $x_1 = k(x_0) \in Kx_0$ □

By definition, the K-orbits in $\mu^{-1}(0)$ are isotropic, i.e., the pull-back of ω to such an orbit vanishes identically.

Isotropic submanifolds of maximal dimension, i.e., of half the dimension of the ambient symplectic manifold, are called Lagrangian. For Kählerian symplectic structures, $\omega(v, Jv) > 0$ for all tangent vectors v. Thus Lagrangian submanifolds are totally real.

Corollary 2.2.3. *The orbit $Kx_0 = \mu^{-1}(0) \cap U$ is Lagrangian and therefore totally real. Furthermore, $G^0_{x_0} = (K^0_{x_0})^{\mathbb{C}}$.*

Proof. It remains to prove the last statement. However, it follows immediately from the fact that $\dim_{\mathbb{R}} Kx_0 = \dim_{\mathbb{C}} D$. □

If ρ is a K-invariant, strictly plurisubharmonic function on D, then we may apply the preceding observations to the Kähler form $\omega := dd^c\rho$ and the equivariant moment map defined by $\mu_\xi := -J\xi_D(\rho)$. In this case $x_0 \in \mu^{-1}(0)$ if and only if $d\rho(x_0) = 0$.

A direct translation then yields the following result.

Corollary 2.2.4. *If D is a K-invariant domain in the homogeneous space $X = G/H$ of a complex reductive group G and ρ is a K-invariant, strictly plurisubharmonic function on D with $d\rho(x_0) = 0$, then, in a K-invariant neighborhood U of x_0 in D, the critical set $\{x \in U : d\rho(x) = 0\}$ consists of exactly the totally real orbit Kx_0 where ρ has its minimum, i.e.,* $\min\{\rho(x) : x \in U\} = \rho(x_0)$. *Furthermore, if H has finitely many components, then it is reductive.*

Proof. The statement concerning the minimum of ρ follows immediately from the positive definiteness of the complex Hessian. The other statements are translations of results above. □

Remark. If D is a K-invariant Stein domain, then it possesses a K-invariant exhaustion $\rho : D \to \mathbb{R}^{\geq 0}$. It follows that at least H^0 is reductive. Of course H itself may not be reductive. For example, if H is an infinite discrete group, every K-orbit has a K-invariant Stein neighborhood.

At the group level the convexity argument above is in fact global.

Proposition 2.2.5 (Exhaustion theorem for reductive groups). *Let G be a reductive group, K a maximal compact subgroup and $\rho : G \to \mathbb{R}$ a K-invariant strictly plurisubharmonic function. Then ρ is a proper exhaustion function if and only if $\{d\rho = 0\} \neq \varnothing$. In this case, if $d\rho(x_0) = 0$ and $\rho(x_0) =: c_0$, then $\rho : X \to [c_0, \infty)$ and $\{d\rho = 0\} = Kx_0$.*

Proof. In this case we have $U = X$. Thus, either $\{d\rho = 0\} = \varnothing$, in which case ρ is clearly not a proper exhaustion, or $\{d\rho = 0\} = Kx_0$ is a K-orbit where ρ takes on its minimum.

Using the KP-decomposition we may then regard $\rho : \mathfrak{p} \to \mathbb{R}$ as a function on the vector space $\mathfrak{p}$ which has a minimum at $0 \in \mathfrak{p}$.

Since for all $\xi \in \mathfrak{k}$ the function $z \mapsto \rho(\exp(\xi z)x_0)$ is a strictly subharmonic $\mathbb{R}$-invariant function on the complex plane, it follows that, regarded as a function on $\mathfrak{p}$, ρ is strictly convex along lines through the origin. Consequently, if m denotes the minimum of the normal derivatives of ρ along the unit sphere in $\mathfrak{p}$, it follows that $\rho(v) \geq m\,\|v\|$. □

Remark. The above very useful convexity argument was brought to our attention by Azad and Loeb; see [Azad and Loeb 1993]. As in that reference, we will also apply it to the case of homogeneous spaces.

b. The Theorem of Matsushima–Onitshick. For homogeneous spaces $X = G/H$ of reductive groups there is a close connection between holomorphic and algebraic phenomena. For this the key ingredient is the density of G-finite holomorphic functions.

Definition. Let G be a group acting linearly on a $\mathbb{C}$-vector space V. A vector $v \in V$ is called G-finite if the linear span $\langle g(v) : g \in G\rangle_{\mathbb{C}}$ is finite-dimensional.

The next result is a consequence of the theorem of Peter and Weyl; see [Akhiezer 1995], for example.

Proposition 2.2.6. *Let G be a reductive group acting holomorphically on a complex space X. Then the G-finite holomorphic functions are dense in $\mathcal{O}(X)$.*

Another fundamental property of complex reductive groups is the algebraic nature of their representations; see [Chevalley 1946], for example.

Proposition 2.2.7. *If G is a reductive group realized as a locally closed complex subgroup of some $\mathrm{GL}_n(\mathbb{C})$, then it is an affine subvariety. This is the unique affine structure on G which is compatible with the group and complex manifold structure. A holomorphic representation $\tau : G \to \mathrm{GL}(V)$ is automatically algebraic.*

As a consequence, the existence of sufficiently many independent holomorphic functions implies the algebraicity of X.

Proposition 2.2.8. *Let $X = G/H$ be a complex homogeneous space of the reductive group G. Let $n := \dim_{\mathbb{C}} X$ and suppose that there exist $f_1, \dots, f_n \in \mathcal{O}(X)$ such that $df_1 \wedge \cdots \wedge df_n \not\equiv 0$. Then H is an algebraic subgroup of G.*

Proof. By Proposition 2.2.6 we may assume that the f_j are G-finite. Let $V = \langle Gf_1, \dots, Gf_n \rangle_{\mathbb{C}}$ be the vector space spanned by the G-orbits and define $F : X \to V^*$ by $F(x)(f) = f(x)$. It follows that F is G-equivariant and generically of maximal rank. The image $F(x)$ is therefore a G-orbit $Gv = G/H_1$ with H_1/H discrete. But the G-representation on V is algebraic and consequently H_1 is an algebraic subgroup. Thus, being a subgroup of finite index in a $\mathbb{C}$-algebraic group, H is likewise $\mathbb{C}$-algebraic. □

Remark. Using a theorem of Grauert and Remmert or a version of Zariski's main theorem [1984], the finite cover $X = G/H_1 = Gv$ can be extended to a finite G-equivariant ramified covering $Z \to \overline{Gv} \subset V$ of an affine closure Z of X. In particular X is quasi-affine.

Here is another important tool for the analysis of holomorphic functions in the presence of reductive actions:

Identity Principle. *Let G be a reductive group, K a maximal compact subgroup and $G \times X \to X$ a holomorphic action. It follows that $\mathcal{O}(X)^G = \mathcal{O}(X)^K$.*

Proof. For $f \in \mathcal{O}(X)^K$ regard $B_x(g) := f(g(x))$ as a holomorphic K-invariant function on G. Since the submanifold K is totally real in G with $\dim_{\mathbb{R}} K = \dim_{\mathbb{C}} G$ and K has non-empty intersection with every component of G, it follows that $B_x(g) \equiv B_x(e)$, i.e., $f(g(x)) = f(x)$ for all $x \in X$. □

The *averaging process* is a useful way of constructing invariant holomorphic functions. For this, let G, K and $G \times X \to X$ be as above. For dk an invariant probability measure on K, define $A : \mathcal{O}(X) \to \mathcal{O}(X)$ by $f \mapsto \int_K k^*(f)dk$. It follows from the identity principle that $A : \mathcal{O}(X) \to \mathcal{O}(X)^G$ is a projection.

Extension Principle. *Let X be Stein and Y be a closed G-invariant complex subspace. Then the restriction map $r : \mathcal{O}(X)^G \to \mathcal{O}(Y)^G$ is surjective.*

PROOF. Given $f \in \mathcal{O}(Y)$ define $\tilde{f} := A(F)$, where $F \in \mathcal{O}(X)$ is an arbitrary holomorphic extension of f. It follows that $r(\tilde{f}) = f$. $\square$

THEOREM 2.2.9 (MATSUSHIMA–ONITSHICK). *A complex homogeneous $X = G/H$ of a reductive group G is Stein if and only if H is reductive.*

PROOF. If X is Stein, then by Proposition 2.2.8 it follows that H is an algebraic subgroup of G and in particular has only finitely many components. An application of Corollary 2.2.4 for the case $D = X$ shows that H is reductive.

Conversely, suppose that $H = L^{\mathbb{C}}$ is reductive. For a discrete sequence of points $\{x_n\}$ in X, let $Y := \dot{\bigcup} Y_n$ be the preimage $\pi^{-1}\{x_n\}$ in G via the canonical quotient $\pi : G \to G/H$. Define $\tilde{f} \in \mathcal{O}(Y)^H$ by $f|Y_n \equiv n$. Regard $\tilde{f} \in \mathcal{O}(X)$ and observe that $\tilde{f}(x_n) = n$. This proves both the holomorphic convexity of X and the fact that $\mathcal{O}(X)$ separates points. Thus X is Stein. $\square$

c. Exhaustions associated to Ad**-invariant inner products.** Let K be a connected compact Lie group. If it is semi-simple, then the Killing form $b : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$, $(\xi, \eta) \mapsto \mathrm{Tr}(\mathrm{ad}(\xi) \cdot \mathrm{ad}(\eta))$, is an $\mathrm{Ad}(K)$-invariant, negative-definite inner product; see [Helgason 1978]. Of course, for an arbitrary compact group, the degeneracy of b is exactly the center $\mathfrak{z}$.

Recall that $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_{\mathrm{ss}}$, where $\mathfrak{k}_{\mathrm{ss}}$ is the Lie algebra of a maximal (compact) semi-simple subgroup K_{ss}. Furthermore, if Z is the connected component of the center $Z(K)$ at the identity, then $K = Z \cdot K_{\mathrm{ss}}$ and $Z \cap K_{\mathrm{ss}}$ is finite.

Given a positive definite Ad-invariant bilinear form on k_{ss}, any extension to $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}_{\mathrm{ss}}$ is Ad-invariant. Thus there exist $\mathrm{Ad}(K)$-invariant inner products $B : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$. We now show how to associate a $(K \times K)$-invariant strictly plurisubharmonic exhaustion $\rho : G \to \mathbb{R}$ to such an inner product.

The natural map $(K \times K) \times \mathfrak{k} \to G$, $(k_1, k_2, \xi) \mapsto k_1 \exp(i\xi) k_2^{-1}$, factors through the quotient $(K \times K) \times_K \mathfrak{k}$ by the free diagonal K-action $k(k_1, k_2, \xi) := (k_1 k^{-1}, k_2 k^{-1}, \mathrm{Ad}(k)(\xi))$. In fact, using $KP = G$, a direct computation shows that in this way G is $(K \times K)$-equivariantly identified with the total space of the $(K \times K)$-vector bundle $(K \times K) \times_K \mathfrak{k} \to (K \times K)/K \cong K$, where the base is diffeomorphic to the group K equipped with the standard $(K \times K)$-action, $k \mapsto k_1 k k_2^{-1}$.

Now the representation of the isotropy $K \hookrightarrow K \times K$, $k \mapsto (k, k^{-1})$, on the fiber $\mathfrak{k}$ over the identity e is simply the adjoint representation. Since the definition of $\mathrm{Ad} : K \to \mathrm{GL}(\mathfrak{k})$, $\mathrm{Ad}(k) = \mathrm{int}_*(k)$, is given by the natural induced action on the tangent space $T_e K \cong \mathfrak{k}$, it follows that $(K \times K) \times_K \mathfrak{k} \to (K \times K)/K$ is just the tangent bundle TK.

Summarizing, we have the following result.

PROPOSITION 2.2.10. *Via the KP-decomposition the reductive group G is $(K \times K)$-equivariantly identifiable with the tangent bundle TK.*

If $B : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ is $\mathrm{Ad}(K)$-invariant, then it defines a $(K \times K)$-invariant metric on $TK = (K \times K) \times_K \mathfrak{k}$; on the neutral fiber this is given by $(\xi, \eta) \mapsto B(\xi, \eta)$

and it is extended to the full space by the $(K \times K)$-action. Let $\rho : TK \to \mathbb{R}$, $\xi \mapsto B(\xi, \xi)$, denote the associated norm function.

The main goal of this section is to prove the following observation.

PROPOSITION 2.2.11. *Under the canonical identification $TK \cong G$, the function $\rho : G \to \mathbb{R}$ is a $(K \times K)$-invariant, strictly plurisubharmonic exhaustion.*

In fact we prove a slightly more general statement which is a special case of Loeb's variation on a theme of Lassalle. For this recall the KAK-decomposition and the fact that a $(K \times K)$-invariant function on G is uniquely determined by a Weyl-group invariant function on $\mathfrak{a} = \mathrm{Lie}\, A$.

PROPOSITION 2.2.12. *A $(K \times K)$-invariant function $\rho : G \to \mathbb{R}$ determined by a W-invariant strictly convex function $\rho_{\mathfrak{a}} : \mathfrak{a} \to \mathbb{R}$ is strictly plurisubharmonic.*

The proof follows immediately from an elementary lemma in the 3-dimensional case. This requires some notational preparation.

Let S be a 3-dimensional complex semi-simple Lie group, K_S a maximal compact subgroup and let $L := K_S \times K_S$ act on S by left- and right multiplication. With one exception, an orbit $\Sigma = Ls$ is a strictly pseudoconvex hypersurface in S. The exception Σ_0 is totally real with $\dim_{\mathbb{R}} \Sigma_0 = \dim_{\mathbb{C}} S$.

Let T be a maximal torus in K. If we regard it in L via the diagonal embedding, we write T_Δ. The connected component Y of the 1-dimensional complex submanifold $\mathrm{Fix}(T_\Delta)$ which contains the identity $e \in S$, is the subgroup $T^{\mathbb{C}} < S$.

Let $\rho : S \to \mathbb{R}$ be a smooth, L-invariant function such that $\rho|Y$ is a strictly plurisubharmonic exhaustion. Since ρ is in particular invariant by left multiplication by elements of T, it follows from the strict convexity of the pull-back $t \mapsto \rho(\exp(it\xi)y)$, $\langle \xi \rangle = \mathfrak{t}$, that $\rho|Y$ has an absolute minimum along exactly one orbit $Y_0 = Ty_0$ and otherwise $d\rho|Y \neq 0$.

Suppose $\Sigma_0 = Ly_0$ is a hypersurface. If $y_0 \in Y_0$, then, moving along the curve $y_t := \rho(\exp(it\xi))y_0$, the Levi form of Σ_t changes signs at Σ_0. This is contrary to the strong pseudoconvexity of Σ_t for all t. Thus $\{d\rho = 0\} = \Sigma_0$ is the totally real L-orbit.

LEMMA 2.2.13. *The function $\rho : S \to \mathbb{R}$ is strictly plurisubharmonic exhaustion.*

PROOF. Let $y \in Y$ and suppose $\Sigma = Ly$ is a hypersurface. Then $d\rho \neq 0$ in some neighborhood of Σ. Since $\rho(\exp(i\xi t)y)$ is increasing, ρ defines Σ as a strictly pseudoconvex hypersurface. Thus $\omega = dd^c\rho$ is positive-definite on the complex tangent space V of Σ at y.

Of course V is T_Δ-invariant and thus its complement $V^{\perp\alpha}$ in T_yS is likewise T_Δ-invariant. But the only possibility for this is the tangent space T_yY, where α is known to be positive-definite.

If $y = y_0 \in Y_0$, then $\Sigma = \Sigma_0$; in particular, L_{y_0} is 3-dimensional and acts irreducibly on $T_{y_0}G$. Since ω is non-degenerate on $T_{y_0}Y$, it follows that it is non-degenerate on the full space $T_{y_0}G$. Since mixed signature is an open condition, it follows from the preceding discussion along generic L-orbits that ω is positive-definite.

The exhaustion property follows from the fact that from the $(K \times K)$-invariance and the fact that $\rho|Y$ is an exhaustion. □

Proof of Proposition 2.2.12. We begin by showing that $\omega = dd^c\rho$ is non-degenerate. It is enough to do so at points $t_0 \in T^{\mathbb{C}}$, the maximal torus of G associated to the Lie algebra $\mathfrak{t}^{\mathbb{C}} = \langle \mathfrak{a} \rangle_{\mathbb{C}}$.

Since ρ is strictly plurisubharmonic on $T^{\mathbb{C}}$, it follows that $E := (T_{t_0}G)^{\perp\omega}$ is a direct sum $l_*(t_0)\big(\sum_{\alpha \in I} \mathfrak{g}_\alpha\big)$ of certain root spaces when regarded as an $\mathrm{Int}(T)$-module.

Arguing by contradiction, suppose that $E \neq \{0\}$ and let $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ be a standard 3-dimensional subalgebra associated to any one of the $\mathfrak{g}_\alpha$'s. Let S be the associated (algebraic) subgroup of G.

Let $\rho : S \to \mathbb{R}$ denote the restriction of the given function to the S-orbit St_0. By assumption $\omega = dd^c\rho$ is degenerate along $l_*(t_0)(\mathfrak{g}_\alpha)$ which is tangent to this S-orbit. But this is contrary to Lemma 2.2.13, i.e., ω is indeed non-degenerate.

Since ω is non-degenerate, it suffices to prove the positive definiteness at the identity $e \in G$. For this observe that the decomposition $T_eG = \mathfrak{t}^{\mathbb{C}} \oplus \sum_\alpha \mathfrak{g}_\alpha$ is α-orthogonal. By assumption $\omega > 0$ on $\mathfrak{t}^{\mathbb{C}}$ and by Lemma 2.2.13 we have the same conclusion on each root space $\mathfrak{g}_\alpha$. Thus ω is positive-definite. □

2.3. Moment Maps at the Group Level. In the previous sections we observed that an $\mathrm{Ad}(K)$-invariant bilinear form B leads in a canonical way to a $(K \times K)$-invariant strictly plurisubharmonic exhaustion $\rho : G \to \mathbb{R}^{\geq 0}$ of the reductive group G. Here we give a precise description of the moment map $\mu : X \to (\mathrm{Lie}\, K)^*$ of the group manifold $X := G$ equipped with the Kählerian structure $\omega = dd^c\rho$ which is defined by right- multiplication by elements of K.

a. Generalities on moment maps. For the moment let (M, ω) be an arbitrary connected symplectic manifold equipped with a Hamiltonian action of a connected Lie group K, i.e., there exists an equivariant moment map $\mu : M \to (\mathrm{Lie}\, K)^* = \mathfrak{k}^*$.

The basic formula. For $\xi \in \mathfrak{k}$ let $\xi_M \in \mathrm{Vect}_\omega(M)$ be the associated vector field and μ_ξ the associated momentum function. Since $d\mu_\xi(x) = \omega(x)(\xi_M(x), \cdot)$, it follows that the differential μ_* can be calculated by

$$\mu_*(x)(v_x)(\xi) = \omega(x)(\xi_M(x), v_x),$$

where $v_x \in T_xM$ and $\mu_*(x)(v_x) \in T_{\mu(x)}\mathfrak{k}^* \cong \mathfrak{k}^*$.

In other words, this basic formula shows how $\mu_*(x)(v_x)$ acts as a functional on $\mathfrak{k}$.

There are several direct consequences:

(i): $\mathrm{Ker}(\mu_*(x)) = (T_xKx)^{\perp\omega}$.

(ii): If $\dim_{\mathbb{R}} Kx =: k$ is constant, then $\mathrm{Rank}(\mu) = k$ is likewise constant. In particular, if the action $K \times M \to M$ is locally free, then μ is an open immersion.

(iii): $\mathrm{Im}(\mu_*(x)) = \mathfrak{k}^0_{\mu(x)}$, i.e., the annihilator of the algebra of the isotropy group $K_{\mu(x)}$.

The moment maps under consideration in this section are defined by a strictly plurisubharmonic potential function ρ of a Kähler-form α, i.e., $\alpha = dd^c\rho$ and $\mu_\xi := -\frac{1}{2}d^c\rho(\xi_M) = -\frac{1}{2}J\xi_M(\rho)$. The choice of the coefficient $-\frac{1}{2}$ only puts us in tune with classical mechanics. Moment maps of this type are denoted by $\mu : X \to \mathfrak{k}^*$. We leave it as an exercise to check that such moment maps are equivariant.

b. The moment map associated to an Ad**-invariant bilinear form.** Let G be a connected reductive group, K a maximal compact subgroup and $B : \mathfrak{k}\times\mathfrak{k} \to \mathbb{R}$ an $\mathrm{Ad}(K)$-invariant symmetric, positive-definite bilinear form. Using the canonical identification of G and the tangent bundle $TK = (K \times K) \times_K \mathfrak{k}$, the norm-function $\xi \mapsto B(\xi, \xi)$ defines a $(K \times K)$-invariant function $\rho : G = KP \to \mathbb{R}^{\geq 0}$ by $\rho(k \exp(i\xi)) := B(\xi, \xi)$.

It follows from Proposition 2.2.12 that ρ is a strictly plurisubharmonic exhaustion of the group manifold $X = G$ with minimum set $\{\rho = 0\} = K$.

Our goal here is to compute the moment map μ^ρ associated to the action of K which is defined by right multiplication.

Proposition 2.3.1. *Let K act on the group manifold by right multiplication and ρ be the strictly plurisubharmonic function associated to an* $\mathrm{Ad}(K)$*-invariant, symmetric positive- definite bilinear form B. Then*

$$\mu^\rho_\xi(\exp(i\eta)) = B(\xi, \eta).$$

Proof.

$$\begin{aligned}\mu^\rho_\xi(\exp(i\eta)) &= -\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\rho(\exp(i\eta)\exp(-i\xi t)) = -\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\rho(\exp(i(\eta - \xi t))) \\ &= -\frac{1}{2}\frac{d}{dt}\Big|_{t=0}B(\eta - \xi t, \eta - \xi t) = B(\xi, \eta).\end{aligned}$$

For the second equality we use the Campbell–Hausdorff formula:

$$\exp(i\eta)\exp(-i\xi t) = g(t)\exp(i(\eta - \xi t))),$$

where, due to K-invariance,

$$\rho(g(t)\exp(-i(\eta - \xi t))) = \rho(\tilde{g}(t)\exp(i(\eta - \xi t)))$$

for

$$\tilde{g}(t) = \exp(-[\xi, \eta]t)g(t) = \exp(O(t^2))\exp(O(t^2)).$$

The result, i.e., the second equation, follows from the fact that the curves $\gamma_{\mathfrak{p}}(t) := \exp(i(\eta - \xi t))$ and $\gamma(t) := \tilde{g}(t)\gamma_{\mathfrak{p}}(t)$ are tangent at $\exp(i\eta)$. $\square$

2.4. Symplectic Reduction and Consequences for Stein Homogeneous Spaces

Generalities on reductions. We consider the following situation: X is Kählerian with respect to $\omega = dd^c\rho$, where ρ is a strictly plurisubharmonic potential function which is invariant by the action of a connected compact group of holomorphic automorphisms, and $\mu^\xi : X \to \mathfrak{k}^*$ is the associated moment map.

Note that if $L < K$ is a subgroup, connected or not, we have the L-equivariant moment map

$$\mu_L^\rho : X \to \mathfrak{l}^*$$

defined by the inclusion $\mathfrak{l} \hookrightarrow \mathfrak{k}$. Unless it might lead to confusion we drop the dependency on ρ and L in the notation and simply write $\mu : X \to \mathfrak{l}^*$.

As is the case for the group manifold $X = G$, we assume in addition that the K-action extends to a holomorphic action $G \times X \to X$ of its complexification $G = K^{\mathbb{C}}$. Assume that $X_0 := \mu^{-1}\{0\} \neq \varnothing$ and define the associated set of semi-stable points by

$$X(\mu) := \{x \in X : \overline{Gx} \cap X_0 \neq \varnothing\}.$$

In the setting of analytic Hilbert quotients (Section 4), the condition $\mu = \mu^\rho$ for some Kählerian potential can only be locally achieved on a covering of the Kempf–Ness set X_0. However, for all practical purposes this is adequate.

The goal is to show that $X(\mu)$ is an open subset of X (in many situations it is in fact Zariski open), and that an equivalence relation is defined on $X(\mu)$ by $x \sim y$ if and only if $\overline{Gx} \cap \overline{Gy} \neq \varnothing$. We refer to the resulting quotient $X(\mu) \to X(\mu)/\!/G := X(\mu)/\!\sim$ as the analytic Hilbert quotient, because its structure sheaf as a reduced complex space is constructed locally on G-invariant Stein neighborhoods of X_0 by invariant theoretic means.

While the complex analytic structure of $X(\mu)/\!/G$ is described in terms of holomorphic invariant theory, the Kählerian structure arises via the symplectic reduction $X_0 \to X_0/K$. Here we are dealing with simple orbit space quotient by the compact group K, but of course the singularities of X_0 present difficulties. One of the main points is to show that the embedding $X_0 \hookrightarrow X(\mu)$ induces a homeomorphism $X_0/K \cong X(\mu)/\!/G$.

In the case where $\omega = dd^c\rho$ for a K-invariant strictly plurisubharmonic function $\rho : X \to \mathbb{R}$, one is handed a quotient structure on a silver platter: Define ρ_{red} by pushing down the K-invariant restriction $\rho|X_0$.

It can be shown that ρ_{red} is a continuous strictly plurisubharmonic function which is smooth on a natural stratification of $X(\mu)/\!/G$ [Heinzner et al. 1994]. The induced singular form $\omega_{\mathrm{red}} = dd^c\rho_{\mathrm{red}}$ is a Kählerian version of the singular symplectic reduced structure [Sjamaar and Lerman 1991].

As we indicated above, these matters are discussed in substantial detail in Section 4. In the present section we consider the case of the right K-action on the group manifold $X = G$, where ρ is a $(K\times K)$-invariant, strictly plurisubharmonic exhaustion associated to an $\mathrm{Ad}(K)$-invariant form. For brevity we refer to this as the group manifold setting.

LEMMA 2.4.1. *In the group manifold setting let $L < K$ be a compact subgroup and $\mu : X \to \mathfrak{l}^*$ the associated moment map. Then*:

(i) *The moment map μ is an open immersion of rank equal to $l := \dim_{\mathbb{R}} L$.*
(ii) *The Kempf–Ness set X_0 is a smooth, generic Cauchy–Riemann submanifold of X.*
(iii) *For $x_0 \in X_0$ there exists the canonical splitting $T_{x_0}X_0 = T_{x_0}(Lx_0) \oplus T^{\mathbb{C}}_{x_0}X_0$, where $T^{\mathbb{C}}_{x_0}X_0 = T_{x_0}X_0 \cap J(T_{x_0}X_0)$.*
(iv) *The orbits of L in the Kempf–Ness set are isotropic and the complex tangent space $T^{\mathbb{C}}_{x_0}X_0$ is the Riemannian orthogonal complement $(T_{x_0}(Lx_0))^{\perp_g}$.*

PROOF. The point (i) has been discussed previously as an immediate consequence of the basic formula. It is therefore clear that X_0 is a smooth submanifold.

Since L-orbits in X_0 are mapped by μ to $0 \in \mathfrak{l}$, it follows that they are isotropic and are therefore totally real. The splitting in (iii) and the genericity statement in (ii) therefore follow by a dimension count.

The Riemannian complement to $T_{x_0}(Lx_0)$ is the Hermitian complement to $T_{x_0}(L^{\mathbb{C}}x_0)$ and is therefore a complex subspace. Again using the basic formula, one observes that this is also in $\mathrm{Ker}(\mu_*(x_0))$. Consequently, $T_{x_0}(Lx_0)^{\perp g}$ is a complex subspace of $T^{\mathbb{C}}_{x_0}X_0$. Equality follows from a dimension count. □

Let $H := L^{\mathbb{C}}$ be the smallest complex Lie subgroup of G which contains L; in fact, H is an affine algebraic subgroup. We regard H as acting by right multiplication, $h(x) = xh^{-1}$, i.e., the natural extension of the L-action.

LEMMA 2.4.2. *For an arbitrary point $x \in X$ it follows that the H-orbit Hx intersects X_0 in exactly one L-orbit*: *$C_x := Hx \cap X_0 \neq \varnothing$ and for $x_0 \in C_x$ it follows that $C_x = Lx_0$. Furthermore, $C_x = \{p \in Hx : d(\rho|Hx)(p) = 0\} = \{p \in Hx : \rho(p) = \min(\rho|Hx)\}$.*

PROOF. It follows from the invariance of ρ that $\xi_M(\rho) \equiv 0$ and, by the definition of μ^ρ, $J\xi_M(x_0)(\rho) = 0$ for all $\xi \in \mathfrak{l}$.

Thus $x_0 \in C_x$ if and only if $d(\rho|Hx)(x_0) = 0$. From the exhaustion theorem at the group level, Proposition 2.2.12, it follows that C_x is exactly the set where $\rho|Hx$ takes on its minimum. (In this case we know already that ρ is an exhaustion. However, if we did not — and this is a key point for general potential functions — it would nevertheless follow at this point.) Furthermore, the exhaustion theorem also guarantees us that this minimum set consists of exactly one L-orbit. □

Let $\pi : G \to G/H =: X/\!/L$ be the natural projection. It follows from the preceding lemma that $\pi|X_0$ induces a bijective continuous map $i_x : X_0/L \to X/\!/L$.

LEMMA 2.4.3. *The mapping $\pi \big| X_0$ is proper, i.e., $i_x : X_0/L \to X//L$ is a homeomorphism.*

PROOF. Since $\rho : G \to \mathbb{R}^{\geq 0}$ is a proper exhaustion, this is clear: Suppose $\{x_n\} \subset X_0$ is a divergent sequence. It follows that $\{\rho(x_n)\}$ is divergent. If $y_n := \pi(x_n) \to y_0 \in X//L$, let $X_0 \in C_x$ be a critical point in $\pi^{-1}(y_0) = Hx$. For $n \gg 0$ it follows that $\rho(C_{x_n}) \leq \rho(x_0) + \varepsilon$ contrary to $\{\rho(x_n)\}$ being divergent. $\square$

Exhaustions and associated Kählerian structures on reductions. In the setting above the map $\pi|X_0 : X_0 \to X_0/L = X(\mu)//H = X_{\mathrm{red}}$ induces a surjective algebra morphism

$$\pi_* : \mathcal{E}(X)^L \to \mathcal{E}(X_{\mathrm{red}}), \quad f \mapsto f_{\mathrm{red}} := f \big| X_0.$$

The L-invariant function $f|X_0$ is interpreted as a smooth function on the base X_{red}.

For the analogous statement for differential forms the following result is of use.

LEMMA 2.4.4. *Let $\pi : M \to N$ be a surjective immersion of smooth manifolds with connected fibers. Then*

$$\pi^*(\mathcal{E}^k(N)) = \{\eta \in \mathcal{E}^k(M) : i_V\eta = 0 \textit{ and } \mathcal{L}_V\eta = 0 \textit{ for all vertical } V \in \mathrm{Vect}(M)\}.$$

PROOF. It is only a matter of pushing down forms which satisfy $i_V\eta = 0$ and $\mathcal{L}_V\eta = 0$ for all vertical fields V. For $q \in N$ and $p \in M$ with $\pi(p) = q$, define $\pi_*(\eta)(q)(v_1, \dots, v_n) := \eta(p)(\tilde{v}_1, \dots, \tilde{v}_k)$. The first condition shows that this is a well-defined independent of the choice of $\tilde{v}_j \in T_pM$ with $\pi_*(\tilde{v}_j) = v_j$, for $j = 1, \dots, k$. Since the π-fibers are connected and any two points p_1, p_2 can be connected by a curve which is piecewise the integral curve of a vertical field, the second condition guarantees that the definition does not depend on the choice of p with $\pi(p) = q$.

Finally, the smoothness of $\pi_*(\eta)$ is proved by identifying it with $\sigma^*(\eta)$, where σ is a local section. $\square$

If in the context above a Lie group L acts smoothly and transitively on the π-fibers, then the second condition can be replaced by invariance.

COROLLARY 2.4.5. *Let $L \times M \to M$ be a smooth Lie group action, $\pi : M \to N$ a surjective immersion with not necessarily connected fibers which are L-orbits. Then*

$$\pi^*(\mathcal{E}^k(N)) = \{\eta \in \mathcal{E}^k(M)^L : i_V\eta = 0 \textit{ for all vertical fields } V\}.$$

PROOF. The independence of definition of $\pi_*(\eta)(q)$ on the choice of $p \in \pi^{-1}\{q\}$ follows from the L-invariance. $\square$

Now we return to our concrete context: $X = G$ is the group manifold of the complex reductive group $G, \rho : X \to \mathbb{R}$ is a strictly plurisubharmonic function which is invariant by the L-action defined by right multiplication and $X_{\mathrm{red}} = X_0/L = X(\mu)//H = G/H$, where $\mu = \mu^\rho$.

Proposition 2.4.6. *The function $\pi_*(\rho) = \rho_{\mathrm{red}} : G/H \to \mathbb{R}$ is strictly plurisubharmonic and $\omega_{\mathrm{red}} := dd^c\rho_{\mathrm{red}}$ is the reduced symplectic structure of Marsden–Weinstein.*

Proof. Let $i : X_0 \hookrightarrow X$ be the canonical injection. Define $\tilde{\omega} = i^*(\omega)$, where $\omega = dd^c\rho$ is the associated Kählerian structure on X. Since the L-orbits in X_0 are isotropic and $\tilde{\omega}$ is L-invariant, it follows that $\tilde{\omega} = \pi^*(\omega_{\mathrm{red}})$, where ω_{red} is a smooth 2-form on the base.

Note that the distribution of complex tangent spaces $T^{\mathbb{C}}_{x_0}X_0$ of the Cauchy–Riemann manifold X_0 serves as an invariant connection for $\pi : X_0 \to X_0/L = X_{\mathrm{red}}$. Since $\omega = dd^c\rho$ is positive definite on these horizontal complex vector spaces, it follows that ω is non-degenerate. In fact, except that they do not have a canonical choice for the connection, this is exactly the reduction of Marsden and Weinstein [1974]. To complete the proof we show that $\omega_{\mathrm{red}} = dd^c\rho_{\mathrm{red}}$.

For this, for $y \in X_{\mathrm{red}}$ and $x_0 \in X_0$ with $\pi(x_0) = y$, let $\sigma : \Delta \to G$ be a local holomorphic section of $G = X(\mu) \to X(\mu)//H = X_0/L$ defined near y with $\sigma(y) = x_0$ and with $\sigma_* : T_y\Delta \xrightarrow{\sim} T^{\mathbb{C}}_{x_0}X_0$.

For $v \in T_y\Delta$ compute

$$\begin{aligned} d^c\rho_{\mathrm{red}}(v) &= J_{\mathrm{red}}v(\rho_{\mathrm{red}}) = d\rho_{\mathrm{red}}(J_{\mathrm{red}}v) = d\rho(\sigma_*(J_{\mathrm{red}}v)) \\ &= d\rho(J\sigma_*(v)) = d^c\rho(\sigma_*(v)) = \pi_*(d^c\rho(v)). \end{aligned}$$

Thus

$$dd^c\rho_{\mathrm{red}} = d\pi_*(d^c\rho) = \pi_*(dd^c\rho) = \pi_*(\omega) = \omega_{\mathrm{red}}. \qquad \square$$

If ρ is an exhaustion of X, as is the case for those functions which are associated to $\mathrm{Ad}(K)$-invariant bilinear forms, then ρ_{red} is likewise an exhaustion. Note furthermore that, since the action of K by right multiplication commutes with the L-action, it follows that X_0 is K-invariant, $\pi : X_0 \to X_0/L = G/H$ is K-equivariant and ρ_{red} is K-invariant.

Corollary 2.4.7. *To every $\mathrm{Ad}(K)$-invariant, positive-definite, symmetric bilinear form $B : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ is canonically associated to a K-invariant, strictly plurisubharmonic exhaustion $\rho_{\mathrm{red}} : G/H \to \mathbb{R}^{\geq 0}$.*

c. The Mostow fibration. An explicit computation of $X_0 := \mu^{-1}\{0\}$ yields the description of the G-homogeneous space G/H as a K-vector bundle over the orbit $Kx_0 = K/L$ of the neutral point in X.

Lemma 2.4.8. *Let $\mu = \mu^\rho : G \to \mathfrak{l}^*$ be the moment map which is defined by the strictly plurisubharmonic exhaustion ρ associated to an $\mathrm{Ad}(K)$-invariant bilinear form $B : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$. Let $\mathfrak{m} := \mathfrak{l}^{\perp B}$. Then $X_0 = K\exp(i\mathfrak{m})$.*

Proof. It is enough to compute $\mu^{-1}\{0\} \cap P$. But, by Proposition 2.3.1, $\mu_\xi(\exp(i\eta)) = B(\xi, \eta)$. Thus

$$\mu^{-1}\{0\} \cap P = \{\exp(i\eta) : B(\xi, \eta) = 0 \quad \text{for all } \xi \in \mathfrak{l}\} = \exp(i\mathfrak{m}). \qquad \square$$

Of course the decomposition $X_0 = K \exp(i\mathfrak{m})$ is lined up with the KP-decomposition and the action of L by right multiplication in the identification $X_0 = K \times \mathfrak{m}$ is given by $l(k, \xi) = (kl^{-1}, \mathrm{Ad}(l)(\xi))$. Thus we have described a realization of G/H as a K-vector bundle.

Theorem 2.4.9 (Mostow Fibration). *Let G be a connected complex reductive Lie group, K a maximal compact subgroup, L a closed subgroup of K and $H = L^{\mathbb{C}}$. Let $B : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ be an* $\mathrm{Ad}(K)$*-invariant, positive-definite, symmetric bilinear form and $\mathfrak{m} := \mathfrak{l}^{\perp B}$, then, via the mapping $K \times \mathfrak{m} \to X_0 \hookrightarrow G$, $(k, \xi) \mapsto k \exp(i\xi)$, Kählerian reduction of G realizes the Stein homogeneous G/H as the K-vector bundle $K \times_L \mathfrak{m} \to K/L$ over the K-orbit $Kx_0 = K/L$ of the neutral point.*

The argument for the exhaustion theorem at the level of groups can now be carried out for Stein homogeneous spaces G/H by replacing $\mathfrak{p}$ by the Mostow fiber $\mathfrak{m}$; see [Azad and Loeb 1993].

Proposition 2.4.10 (Exhaustion Theorem). *Let G be a (not necessarily connected) reductive complex Lie group, K a maximal compact subgroup, H a reductive subgroup of G and $X = G/H$ be associated Stein homogeneous space. A K-invariant, strictly plurisubharmonic function $\rho : X \to \mathbb{R}$ is a proper exhaustion $\rho : X \to [m, \infty)$, $m := \min\{x \in X : \rho(x)\}$, if and only if $\{d\rho = 0\} \neq \varnothing$. In this case $\{d\rho = 0\} = Kx_0$, where $\rho(x_0) = m$. Furthermore, $G_{x_0} = (K_{x_0})^{\mathbb{C}}$.*

Proof. It is enough to prove this for the case where G is connected. Furthermore, if $\{d\rho = 0\} = \varnothing$, then ρ is clearly not a proper exhaustion. Thus we may let $x_0 \in \{d\rho = 0\}$ and recall that near x_0 the function ρ takes on its absolute minimum $m = \rho(x_0)$ exactly on the orbit Kx_0 (see Proposition 2.2.4). Without loss of generality we may assume that $H = G_{x_0}$. Let $L_1 = K_{x_0}$ and note that $H_1 := L_1^{\mathbb{C}}$ is of finite index in H.

Let $\rho_1 : X_1 := G/H_1 \to \mathbb{R}$ be the induced function on the finite covering space X_1 of X and $x_1 \in X_1$ be a neutral point over x_0. Since $L_1^{\mathbb{C}} = H_1$, we may apply the Mostow-fibration to X_1 with base point x_1. In complete analogy to the case of groups, since $\tilde{\rho}_1 : \mathfrak{m} \to \mathbb{R}$ is strictly convex along the lines through $0 \in \mathfrak{m}$ and has a local minimum at 0, it follows that $\tilde{\rho}_1$ is a proper exhaustion with absolute minimum at its only critical point $0 \in \mathfrak{m}$. Consequently, the same can be said of ρ_1: It is a proper exhaustion with $\{d\rho_1 = 0\} = Kx_1$ the set where it takes on its minimum.

Since ρ_1 is the lift of $\rho : X \to \mathbb{R}$ via $\pi : X_1 \to X$ and its critical set is the lift of the critical set $\{d\rho = 0\}$, it follows that

$$\pi^{-1}(Kx_0) = Kx_1.$$

In other words, the entire fiber $\pi^{-1}(x_0)$ is contained in Kx_1. Consequently Kx_0 is a minimal K-orbit in X and $L_1^{\mathbb{C}} = H$. □

3. Local Models and Exhaustions by Kählerian Potentials

Here we carry out the basic preparatory work for the construction of analytic Hilbert quotients. The goal is to prove the existence of an étale Stein local model where the potential function of the given Kählerian structure is a proper exhaustion along the fibers of the invariant theoretic quotient. This is a group action version with parameters of the exhaustion theorem, Proposition 2.2.12, for Stein homogeneous spaces.

We begin with a discussion of actions on Stein spaces, where, for holomorphic actions of reductive groups, due to the density of the G-finite functions, the situation is quite close to that of algebraic invariant theory.

3.1. Actions on Stein Spaces

Local algebraicity. If X is a complex space equipped with a holomorphic action $G \times X \to X$ of a complex Lie group, then an orbit Gx is said to be Zariski open in its closure $\overline{Gx} = W$ whenever W is a closed complex subspace and the difference $W \setminus Gx$ is the (locally finite) union of nowhere dense analytic subsets. For G connected this is equivalent to G having an open orbit in W.

Proposition 3.1.1. *If X is Stein, G is reductive and $G \times X \to X$ is holomorphic, then every G-orbit in X is Zariski open in its closure.*

Proof. Since G has only finitely many components, it is enough to prove this in the case where it is connected. Let x be given, and, just as in the proof of Proposition 2.2.8, construct a holomorphic equivariant map $F : X \to V^*$ to a representation space which is biholomorphic in a neighborhood of x. Of course F may not be injective on Gx, but it is finite-to-one.

Let h be a G-finite, holomorphic function which separates some fiber of $F\big|Gx$, $V_1 = \langle g(f)\rangle_{g\in G}$ and $F_1 : X \to V_1^*$ the associated map. It follows that $F \oplus F_1$ is injective on Gx and locally biholomorphic at each $z \in Gx$. By changing notation we may assume that F already had this property.

Since G is reductive, its representation on V^* is algebraic and therefore $G(F(x))$ is Zariski open in its closure Z. Let W be the irreducible of $F^{-1}(Z)$ which contains x. By construction it follows that Gx is open in W. □

b. Invariant theoretic quotients. If G is reductive, X is affine and $G \times X \to X$ is an algebraic action, then the ring $\mathcal{O}_{\text{alg}}(X)^G$ of invariant regular functions is finitely generated; see [Kraft 1984], for example. Thus $X/\!/G := \operatorname{Spec}(\mathcal{O}_{\text{alg}}(X)^G)$ is affine and there is a canonical surjective, invariant, regular morphism $\pi : X \to X/\!/G$. We view this as a complex analytic quotient.

Since the G-representation on $\mathcal{O}_{\text{alg}}(X)$ is locally finite, it follows that X can be algebraically and equivariantly realized as a G-invariant subvariety of a representation space V. Now, by restriction, the ring of polynomials $\mathbb{C}[V]$ is dense in the ring of holomorphic functions $\mathcal{O}(X)$. Thus, by averaging, $\mathcal{O}_{\mathrm{alg}}(X)^G$ is dense in $\mathcal{O}(X)^G$. Consequently, the Hilbert quotient $\pi : X \to X/\!/G$ is also defined (at least at the set-theoretic level) by the equivalence relation $x \sim y$ if and only if $f(x) = f(y)$ for all invariant holomorphic function $f \in \mathcal{O}(X)^G$ and the points of $X/\!/G$ can be regarded as maximal ideals $\mathfrak{m} < \mathcal{O}(X)^G$.

The above equivalence relation is defined for any action of a group of holomorphic transformations on a complex space. If X is Stein, G is reductive and $G \times X \to X$ is holomorphic, then, if there is no confusion, we also denote this by $\pi : X \to X/\!/G$. In fact, even in the non-reduced case, equipped with the direct image sheaf $U \mapsto \mathcal{O}_X(\pi^{-1}(U))^G$, $X/\!/G$ is a Stein space [Snow 1982; Heinzner 1988; 1989; Hausen and Heinzner 1999]. In the reduced case this is the analytic Hilbert quotient associated to the moment map μ^ρ of any K-invariant strictly plurisubharmonic exhaustion $\rho : X \to \mathbb{R}$ (see Section 4).

Of course at this point $X/\!/G$ is only a Hausdorff topological space. The complex structure will be constructed in the sequel. In situations where the complex structure is known to exist with the preceding properties, we refer to $X \to X/\!/G$ as the holomorphic invariant theoretic quotient.

Since at this point the quotient $X/\!/G$ only carries the structure of a topological space, we temporarily refer to $\pi : X \to X/\!/G =: Q$ as the formal invariant theory quotient. The π-fibers are of course closed analytic subspaces of X.

Proposition 3.1.2. *Let G be reductive, X Stein, $G \times X \to X$ a holomorphic action and $\pi : X \to Q$ the formal invariant theoretic quotient. Then every π-fiber contains a unique closed G-orbit.*

Proof. Let $Z = \pi^{-1}(\pi\{x\})$ be a π-fiber. It follows from Proposition 3.1.1 that orbits of minimal dimension are closed. Thus Z contains at least one closed orbit $Y = Gy$.

If $\tilde{Y} = G\tilde{x}$ is an additional closed orbit in X, then, by using the extension principle (see 2.2), one can construct $f \in \mathcal{O}(X)^G$ with $f|Y \equiv 0$ and $f|\tilde{Y} \equiv 1$. In particular $\tilde{Y} \cap Z = \varnothing$, i.e., Z contains exactly one closed orbit. □

From the point of view of simply constructing a Hausdorff quotient, it is natural to attempt to define an equivalence relation by $x \sim y$ if and only if $\overline{G}x \cap \overline{G}y \neq \varnothing$. This is in fact the equivalence relation of the analytic Hilbert quotient. The invariant theoretic quotient of Stein spaces is also of this type.

Corollary 3.1.3. *Let G be reductive, X Stein and $G \times X \to X$ be a holomorphic G-action. Then $f(x) = f(y)$ for all $f \in \mathcal{O}(X)^G$ if and only if $\overline{G}x \cap \overline{G}y \neq \varnothing$.*

Proof. In every equivalence class Z there is exactly one closed orbit Y, i.e., $x \sim y$ if and only if $\overline{G}x \cap \overline{G}y$ contains such a Y. □

Remark. In fact the π-fibers have canonical affine algebraic structure with G acting algebraically; see 3.3.7.

Even if it is connected and smooth, a Stein space X equipped with a holomorphic action of a reductive group G may not be holomorphically, equivariantly embeddable in a G-representation space; see [Heinzner 1988]. Locally, however, there always exist G-equivariant, closed holomorphic embeddings (3.3.14). As a result, the following is of particular use.

THEOREM 3.1.4. *If X is a G-invariant closed complex subspace of a G-representation space V with Hilbert quotient $\pi : V \to V/\!/G$, then the restriction $\pi|X$ has closed complex analytic image $\pi(X) =: X/\!/G$ and $\pi : X \to X/\!/G$ is the holomorphic invariant theoretic quotient.*

PROOF. Let $S := \{v \in V : \overline{Gv} \cap X \neq \varnothing\}$. It follows that S is a closed complex subspace defined by the ideal $I(X)^G := \{f \in \mathcal{O}(V)^G : f|X = 0\}$ and the image $\pi(X)$ is the zero set of that ideal regarded as a space of functions on $X/\!/G$. In particular, $\pi(X)$ is a closed complex subspace of $V/\!/G$. Using coherence arguments [Heinzner 1991], one shows that $\mathcal{O}(X)^G = \mathcal{O}(Z)^G/I(X)^G$ and thus the image is equipped with the right set of functions. □

c. The Kempf–Ness set. Let X be a complex space, G a reductive group $K < G$ a maximal compact subgroup and $\rho : X \to \mathbb{R}$ a K-invariant, smooth strictly plurisubharmonic function. An equivariant moment map $\mu^\rho : X \to \mathfrak{k}$ is defined by $\mu^\rho_\xi = -J\xi_X(\rho)$, where, for $\xi \in \mathfrak{k}$, ξ_X denotes the associated vector field on X. We refer to $X_0 := \mu^{-1}\{0\}$ as the Kempf–Ness set associated to ρ and the action.

The next result is essential.

PROPOSITION 3.1.5. *Let X be Stein and $\rho : X \to \mathbb{R}$ a K-invariant, strictly plurisubharmonic function. For $x_0 \in X_0$ it follows that Gx_0 is closed and $Gx_0 \cap X_0 = Kx_0$. If ρ is an exhaustion, then every closed orbit intersects X_0 and the inclusion $X_0 \hookrightarrow X$ induces a bijective continuous map $X_0/K \to Q$ to the invariant theoretic quotient.*

PROOF. If $x_0 \in X_0$, then x_0 is a critical point of $\rho\big|Gx_0$. Consequently $\rho\big|Gx_0$ is an exhaustion of Gx_0 with its only critical points being along the K-orbit Kx_0 (Proposition 2.4.10). Thus Gx_0 is closed and $Gx_0 \cap X_0 = Kx_0$.

If ρ is an exhaustion, then $\rho\big|Gx$ clearly has critical points along any closed orbit Gx. □

It is of basic importance to show, e.g., in the Stein case for an exhaustion ρ, that the induced map $X_0/K \to Q$ is in fact a homeomorphism. A key notion for the discussion of such questions is that of orbit convexity.

DEFINITION. *Let G be a reductive group acting on a set X and K be a maximal compact subgroup. A K-invariant subset Y in X is said to be orbit convex if for every $y \in Y$ and $\xi \in \mathfrak{k}$ with $\exp(i\xi)y \in Y$ it follows that $\exp(i\xi t)y \in Y$ for $t \in [0,1]$.*

We now will prove a useful technical result on the existence of orbit convex neighborhood bases at points of the Kempf–Ness set. For later applications we carry this out in a more general setting than that of the formal invariant theoretic quotient of Stein spaces.

Let X be a complex space equipped with a holomorphic action of a reductive group G. A surjective, continuous, G-invariant map $\pi : X \to Q$ to a locally compact Hausdorff topological space is called a Hausdorff quotient if it defines the equivalence relation $x \sim y$ if and only if $\overline{G}x \cap \overline{G}y \neq \varnothing$.

In the Kählerian setting (see Section 4) we deal with strictly plurisubharmonic potentials $\rho : X \to \mathbb{R}$ which are exhaustions along π-fibers. The appropriate properness is defined in terms of the join: ρ is said to be a relative exhaustion if its restriction to every fiber is bounded from below and $\pi \times \rho$ is proper.

If $\pi : X \to Q$ is a Hausdorff quotient and $\rho : X \to \mathbb{R}$ is a K-invariant, strictly plurisubharmonic function whose restriction to each fiber is bounded from below and an exhaustion, then, as was shown above, $\pi\big|X_0$ is surjective and induces a continuous bijective map $X_0/K \to Q$. If ρ is a relative exhaustion, then it is in fact a homeomorphism. For this we prove the result mentioned above on the existence of orbit convex neighborhoods.

Let $q \in Q$ and $x_q \in X_0$ with $\pi(x_q) = q$. Define $r_q := \rho(x_q)$. For $r \in \mathbb{R}$ let $D_\rho(r) = \{x \in X : \rho(x) < r\}$.

Proposition 3.1.6. *The set $D_\rho(r)$ is orbit convex. If the restriction of ρ to every π-fiber is bounded from below and an exhaustion, then $GD_\rho(r)$ is π-saturated. If ρ is a relative exhaustion, then sets of the form $\pi^{-1}(V) \cap \{x : r_q - \varepsilon < \rho(x) < r_q + \varepsilon\}$, where $V = V(q)$ is an open neighborhood of q and $\varepsilon > 0$, form a neighborhood basis of $\pi^{-1}\{q\} \cap X_0 = Kx_q$.*

Proof. If $x \in D_\rho(r)$, $\xi \in \mathfrak{k}$, then we consider the $\mathbb{R}$-invariant plurisubharmonic function $\rho_\xi(z) := \rho(\exp(\xi z)x)$. It follows that $\rho_\xi\big|i\mathbb{R}$ is convex and thus, if $\rho_\xi(i) < r$, then $\rho_\xi(it) < r$ for $t \in [0, 1]$.

To prove that $GD_\rho(r)$ is π-saturated, observe that if a π-fiber Z has non-empty intersection with $D_\rho(r)$, then since $\rho\big|Z$ is a proper exhaustion, it achieves its minimum in $D_\rho(r)$, i.e., $X_0 \cap Z \subset D_\rho(r)$ and in particular the unique closed orbit $Y \subset Z$ satisfies $Y \cap D_\rho(r) \neq \varnothing$. Since every G-orbit in Z has Y in its closure, it follows that

$$G(Z \cap D_\rho(r)) = Z.$$

Now suppose that ρ is a relative exhaustion and let U be an open neighborhood of Kx_q in X. For convenience, replace Q by the compact closure of an open neighborhood of q; in particular we may assume that ρ is a relative exhaustion and therefore $(\pi \times \rho)^{-1}(q, r_q) = Kx_q$.

Thus there exists open neighborhoods $V = V(q) \subset Q$ (open also in the original Hausdorff quotient Q) and a number $\varepsilon > 0$ so that

$$\pi^{-1}(V) \cap \{x \in X : r_q - \varepsilon < \rho(x) < r_q + \varepsilon\} \subset U.$$

If $V \subset Q \setminus \overline{\pi(D_\rho(r_q - \frac{\varepsilon}{2}))}$, then in fact

$$\pi^{-1}(V) \cap D_\rho(r_q + \varepsilon) = \pi^{-1}(V) \cap \rho^{-1}((r_q - \varepsilon, r_q + \varepsilon)),$$

which would complete the proof.

We prove the existence of such a V by contradiction. Thus, suppose that there exists a sequence $\{y_n\} \subset X$ such that $q = \lim \pi(y_n)$ and $\rho(y_n) < r_q - \frac{\varepsilon}{2}$. By the properness of $\pi \times \rho$ we may assume that $y_n \to y_q \in \pi^{-1}(q)$. But

$$r_q - \frac{\varepsilon}{2} \geq \rho(y_q) \geq \rho(x_q) = r_q$$

which is a contradiction. □

Corollary 3.1.7. *If $\pi : X \to Q$ is a Hausdorff quotient, $\rho : X \to \mathbb{R}$ is a K-invariant strictly plurisubharmonic function and ρ is a relative exhaustion, then the induced map $X_0/K \to Q$ is a homeomorphism.*

Remark. The main goal of this chapter is to show that, in a certain local setting related to the analytic Hilbert quotient, ρ is a relative exhaustion. In this case ρ is a certain Kählerian potential. The local model for these considerations is the étale Stein covering which is constructed in section 3.2, and the main result of section 2.3 is the relative exhaustion property for ρ.

3.2. Étale Stein Coverings. If $\mu : X \to \mathfrak{k}$ is a moment map of a Kählerian G-space with respect to a K-invariant Kähler form ω, then there exists a covering $\mathcal{U} = \{U_\alpha\}$ of X_0 by G-invariant neighborhoods so that $\omega|U_\alpha$ has a strictly plurisubharmonic K-invariant, potential function ρ_α with $\mu = \mu^{\rho_\alpha}$ (see Section 4) Thus, for $x_0 \in X_0 \cap U_\alpha$, Gx_0 is a closed affine homogeneous space, and in particular $G_{x_0} = K^{\mathbb{C}}_{x_0}$ is reductive.

The goal of the present section is, in the context of actions of reductive group actions, i.e., independent of a Kählerian setting, to construct equivariant étale Stein neighborhoods of points with reductive isotropy.

a. Local product structure. Throughout this paragraph X is a complex space, G is a complex Lie group acting holomorphically on X and L is a compact subgroup. The analysis takes place at a point $x_0 \in X$, where $G_{x_0} =: H = L^{\mathbb{C}}$

It follows that the natural representation of H on the Zariski tangent space $T_{x_0}X$ is holomorphic and there exists an L-invariant neighborhood $U = U(x_0)$ which can be biholomorphically, equivariantly identified with a closed analytic subset A of an L-invariant ball $B \subset T_{x_0}X$ [Kaup 1967]. Let $i : U \to A$ be this identification; of course $i(x_0) = 0$.

Proposition 3.2.1 (Local Product decomposition). *Let*

$$T_{x_0}X = T_{x_0}Gx_0 \oplus V$$

be an H-invariant decomposition, $S_{\text{loc}} := i^{-1}(A \cap V)$ and N an $\text{Int}(L)$*-invariant local submanifold at $e \in G$ so that $T_eG = T_eN \oplus T_eH$. Then, after shrinking* *all sets appropriately, it follows that* $N \times S_{\text{loc}} \to X$, $(g,s) \mapsto g(s)$, *is an* L-*equivariant biholomorphic map onto an open neighborhood of* x_0 *in* X.

PROOF. Let $A_V := A \cap V$. After shrinking N and A to sets N_1 and A_1, we have the local holomorphic action $N_1 \times A_1 \to A$. Replacing N and A by N_1 and A_1 respectively, we have the induced holomorphic map $\varphi : N \times A_V \to U$. Since the local N-orbit of $0 \in T_{x_0}X$ is transversal to V, by shrinking even further we may assume that φ is biholomorphic onto its image.

Now the desired result is local, and N is connected. Thus we may argue one component at a time to show that locally $\operatorname{Im}(\varphi) = A$.

So assume A is irreducible and let A_V^0 be a component of A_V at 0. Since A_V^0 is a component of an analytic set in A which is defined by the linear functions which define V in $T_{x_0}X$, it follows that

$$\dim A \le \dim A_V^0 + \operatorname{codim} V = \dim \varphi(N \times A_V^0).$$

Since $\operatorname{Im}\varphi \subset A$, it follows that $\operatorname{Im}\varphi$ is a neighborhood of $0 \in A$. □

b. Local complexification. In the previous section we constructed an L-invariant local slice S_{loc} transversal to an orbit Gx_0 where $G_{x_0} = L^{\mathbb{C}} = H$. Of course S_{loc} is in general not H-invariant and $H \cdot S_{\text{loc}} \subset X$ is possibly quite wild. Thus, in order to globalize the H-action on S_{loc}, we regard S_{loc} as an L-invariant subvariety of a ball in T_{x_0} where the H is much easier to control. As usual, the controlling device is a strictly plurisubharmonic function.

We formulate these general results on complexification in the original notation, i.e., for a reductive group G and a fixed maximal compact subgroup K. These are very special cases of the results in [Heinzner 1991; Heinzner and Iannuzzi 1997].

The basic question here is, given a real form K of a complex Lie group G and an action of K as a group of holomorphic transformations on a complex space X, does there exist a complex space $X^{\mathbb{C}}$, a holomorphic G-action $G \times X^{\mathbb{C}} \to X^{\mathbb{C}}$ and an open, K-equivariant embedding $i : X \hookrightarrow X^{\mathbb{C}}$. Optimally, this should have the obvious universality property: Holomorphic K-equivariant maps $X \to Y$ to holomorphic G-spaces should factor through $i : X \hookrightarrow X^{\mathbb{C}}$. If this universality property is fulfilled, then $X^{\mathbb{C}}$ is referred to as a G-complexification of X. If X is Stein [Heinzner 1991] or holomorphically convex [Heinzner and Iannuzzi 1997], such a complexification indeed exists.

In this paragraph we develop enough of this theory to construct $S := S_{\text{loc}}^{\mathbb{C}}$.

A convenient notion for these considerations is that of *orbit connectedness*: Let $G \times X \to X$ be a holomorphic action of a reductive group and K a maximal compact subgroup. A set K-invariant subset $U \subset X$ is said to be orbit connected if for every $x \in U$ and $B_x : G \to X$, $g \mapsto g(x)$, the preimage $B_x^{-1}(U)$ is K-connected, i.e., $B_x^{-1}(U)/K$ is connected.

LEMMA 3.2.2. *If $G \times X \to X$ is a holomorphic action of the reductive group G and U is a K-invariant, orbit connected, open subset, then GU is a G-complexification.*

PROOF. Let $\varphi : U \to Y$ be a K-equivariant holomorphic map to a complex space Y equipped with a holomorphic G-action. We must show that it extends to a G-equivariant map $\varphi^{\mathbb{C}} : GU \to Y$. Since the map $G \times U \to Y$, $(g, u) \mapsto g\varphi(u)$, is holomorphic, it is only a question if it factors through $G \times U \to GU$.

For $x \in U$ and $U_x := B_x^{-1}(U)$ it follows from the orbit connectedness and the identity principle that $\psi_1 : G \to Y$, $g \mapsto g(\varphi(x))$, gives a well-defined holomorphic extension of $\psi_2 : U_x \to Y$, $g \mapsto \varphi(g(x))$. So for $y = g_1(x_1) = g_2(x_2)$, with $x_1, x_2 \in U$, it follows that $g_2^{-1}g_1 \in U_{x_1}$ and therefore

$$g_1(\varphi(x_1)) = (g_2(g_2^{-1}g_1))(\varphi(x_1)) = g_2(\varphi(g_2^{-1}g_1(x_1))) = g_2(\varphi(x_2)). \qquad \square$$

If U is orbit convex (a fortiori orbit connected), then the preceding result holds for K-invariant analytic subsets.

LEMMA 3.2.3. *If U is orbit convex and $A \subset U$ is a K-invariant analytic subset, then GA is an analytic subset of $U^{\mathbb{C}} = GU$ and is a G-complexification $A^{\mathbb{C}}$.*

PROOF. The orbit convexity of U is clearly inherited by A. Thus it is enough to show that GA is an analytic subset of GU. In that case, since A is orbit connected in GA, the result follows from the previous lemma.

Note that $GA \subset \bigcup_{g \in G} g(U)$. Thus, to prove that GA is an analytic subset of GU, it is enough to show that $(GA) \cap g(U) = g(A)$ or equivalently that $(GA) \cap U = A$. However, this is just the orbit connectedness of A. $\square$

We now come to the result which will allow us to complexify an appropriately chosen local slice S_{loc}.

PROPOSITION 3.2.4. *Let $G \times X \to X$ a holomorphic action of a reductive group G on a Stein complex space which has a holomorphic Hausdorff quotient $\pi : X \to Q$, i.e., Q is itself a complex space and π is holomorphic. Assume in addition that $\rho : X \to \mathbb{R}$ is a K-invariant strictly plurisubharmonic relative exhaustion function. If x_q is a point in the Kempf–Ness set X_0 and A is a K-invariant analytic subset of some open subset of X_0 with $x_q \in A$, then there is a basis of K-invariant Stein neighborhoods U of Kx_q such that $G(A \cap U)$ is a Stein G-complexification of A.*

PROOF. Let V run through a Stein neighborhood basis of $q = \pi(x_q)$. In the notation of Proposition 3.1.6 let $U = \pi^{-1}(V) \cap \rho^{-1}((r_q - \varepsilon, r_q + \varepsilon))$ be orbit convex with GU being π-saturated. We may choose V sufficiently small so that the π-saturation GU is just $\pi^{-1}(V)$.

If $\{x_n\}$ is a divergent sequence in $\pi^{-1}(V)$ which is not divergent in X, then $\{\pi(x_n)\}$ is divergent in V. In the former case there exists $f \in \mathcal{O}(X)$ with

$\lim|f(x_n)| = \infty$ and in the latter $f \in \mathcal{O}(\pi^{-1}(V))^G$ with the same property. Thus $\pi^{-1}(V) = GU$ is Stein.

Now, by the previous lemma the orbit convexity of U implies that $G(U \cap A)$ is a complex subspace of GU which is a G-complexification of $U \cap A$. Since $GU = \pi^{-1}(V)$ is Stein, it is likewise Stein. □

Our main application is the complexification of local slice of Proposition 3.2.1.

PROPOSITION 3.2.5. *Let $G \times X \to X$ be the holomorphic action of a complex Lie group, $x_0 \in X$ and assume that $G_{x_0} = L^{\mathbb{C}} =: H$ is a reductive group. Then there exists a Stein local slice S_{loc} with a Stein H-complexification S.*

PROOF. In fact S_{loc} is constructed as a K-invariant analytic subset of a ball B about $0 \in S_{\mathrm{loc}}$ in an H- representation space $V \hookrightarrow T_{x_0}X$. The algebraic Hilbert quotient $\pi : V \to V//H = Q$ satisfies the conditions of the previous proposition. □

c. The local model as an étale Stein covering. The étale Stein model is constructed as a quotient $G \times_H S$ of the Stein space $G \times S$ by the diagonal H-action defined by

$$h(g, s) := (gh^{-1}, h(s)).$$

Since this action is free and proper, the following is of use.

LEMMA 3.2.6. *Let $G \times X \to X$ be a free, proper holomorphic action of a complex Lie group G on a complex space X. Then, equipped with the quotient topology, the orbit space X/G has a unique structure of a complex space so that $\pi : X \to X/G$ is holomorphic.*

PROOF. Let $x \in X$ and $Y = Gx$. Since the action is holomorphic and proper, Y is a closed complex submanifold of X. Let S_{loc} be the local slice of Proposition 3.2.1. (In this case the isotropy is trivial.)

The map $\alpha : G \times S_{\mathrm{loc}} \to X$ is biholomorphic in some neighborhood $N \times S_{\mathrm{loc}}$. Since it is equivariant, α is therefore everywhere locally biholomorphic with open G-invariant image $U \subset X$.

If $\alpha(g_1, s_1) = \alpha(g_2, s_2)$ with $g := g_1^{-1}g_2$, then $g(s_2) = s_1$. If $g \neq e$, then, since $\alpha|N \times S_{\mathrm{loc}}$ is biholomorphic, it follows that $g \in G \setminus N$. Now, if we could construct such pairs of points for S_{loc} arbitrarily small, then we would have a sequence $\{s_n\} \subset S_{\mathrm{loc}}$ with $s_n \to x_0$ and $g_n \in G\setminus N$ with $g_n(s_n) \to x_0$ as well. By the properness of the action this would imply that, after going to a subsequence, $g_n \to g \in G \setminus N$ with $g(x_0) = x_0$. This is contrary to G acting freely.

Thus, $\alpha : G \times S_{\mathrm{loc}} \to U$ is biholomorphic and S_{loc} can be identified with a neighborhood of Gx_0 in the orbit space X/G. Since $\pi : X \to X/G$ is required to be holomorphic, this realization of S_{loc} in X/G must be a holomorphic chart.

Finally, given two local sections S_α and S_β over the same chart in X/G, the change of coordinates $\varphi_{\alpha\beta} : S_\alpha \to S_\beta$ is given by a G-valued holomorphic map $S_\alpha \to G$. □

REMARK. Of course this is just another description of a G-principal bundle. There would be little difference in the discussion if the action were only required to be locally free: the local models in that case are the finite group quotients S_{loc}/Γ, where $\Gamma = G_{x_0}$ is the possibly non-trivial isotropy group.

COROLLARY 3.2.7. *Let $G \times X \to X$ be a free, proper, holomorphic action of a reductive group on a Stein complex space X. Then X/G is Stein and $\pi : X \to X/G$ is an invariant theoretic quotient.*

PROOF. The proof that X/G is Stein goes exactly as the proof that G/H is Stein in the Theorem of Matsushima–Onitshick on reductive pairs (see 2.2).

Universality means that every invariant holomorphic map $F : X \to Y$ factors through $\pi : X \to X/G$. By localizing to coordinate neighborhoods in Y this is reduced to the same question for invariant functions and this follows immediately from the fact that $\pi^* : \mathcal{O}(X/G) \to \mathcal{O}(X)^G$ is an isomorphism. □

As an application, let G be a complex Lie group, H a closed subgroup and suppose that $H \times S \to S$ is a holomorphic action on a complex space S. Let $G \times_H S$ denote the quotient by the free, proper H-action. Leaning on the language of representation theory, we might refer to the holomorphic fiber bundle $G \times_H S \to G/H$ with fiber S, base G/H and structure group H as a sort of geometric induction of going from an H-action on S to a G-action on $G \times_H S$.

We now turn to the description of the étale Stein local model. In the following S_{loc} refers to the local slice of Proposition 3.2.1.

PROPOSITION 3.2.8. *Let $G \times X \to X$ be a holomorphic action of a Stein complex Lie group on a complex space. For $x_0 \in X$ suppose that $G_{x_0} = L^{\mathbb{C}} = H$ is a reductive group. Let S be a Stein H-complexification of the local slice S_{loc} and let $i : S \to X$ be the H-equivariant holomorphic mapping guaranteed by the universality property. Then the canonical holomorphic map $\tilde{\alpha} : G \times S \to X$, $(g, s) \mapsto g(i(s))$ factors through the Stein space $G \times_H S$ and the induced map $\alpha : G \times_H S \to X$ is everywhere locally biholomorphic.*

REMARKS. (1) We refer to $\alpha : G \times_H S \to X$ as an étale Stein model of X at x_0. Although locally biholomorphic, the map α could be very complicated.
(2) The assumption that G is Stein is only needed to insure that $G \times_H S$ is Stein.
(3) Whether or not a complex Lie group is Stein is well understood. For example, if G is connected, then there is a uniquely defined closed, connected, central subgroup M with $\mathcal{O}(M) \cong \mathbb{C}$ so that G/M is Stein.

PROOF OF THE PROPOSITION. Since

$$\tilde{\alpha}(gh^{-1}, i(h(s))) = (gh^{-1})(h(i(s))) = g(i(s)) = \tilde{\alpha}(g, s),$$

it is clear that $\tilde{\alpha}$ is H-invariant and therefore factors through the Stein quotient $G \times_H S$.

Now, when restricted to $N \times S_{\rm loc} \hookrightarrow G \times_H S$, this map is in fact biholomorphic. By its G-equivariance and the fact that $G(N \times S_{\rm loc}) = G \times_H S$, it follows that α is everywhere locally biholomorphic. □

3.3. The Relative Exhaustion Property. The main goal of this section is to prove that if $\pi : X \to Q$ is a Hausdorff quotient for the action of a reductive group G with maximal compact subgroup K and $\rho : X \to \mathbb{R}$ is a K-invariant, strictly plurisubharmonic function which, when restricted to a fiber $\pi^{-1}\{q_0\}$ has a local minimum at x_0, then, after replacing Q by an appropriately chosen neighborhood of q_0 and X by its saturation, it follows that ρ is a relative exhaustion. This then shows that, after localizing, the restriction $\pi|X_0$ is proper. In other words, $\pi|X_0$ is an open mapping. In Section 4 this will be applied to a potential of a Kähler form lifted to the étale Stein local model.

a. The Hilbert Lemma for algebraic actions. We must to analyze the fibers of a Hausdorff quotient using a strictly plurisubharmonic function to control the action. In the end (item c on page 342) we will show that such a fiber possesses a natural structure of an affine algebraic variety where the reductive group at hand is acting algebraically. We begin by analyzing the algebraic case.

Let X be an affine algebraic variety equipped with an algebraic action $G \times X \to X$ of a reductive group. Assume that $\mathcal{O}(X)^G \cong \mathbb{C}$ and let $Y \hookrightarrow X$ be the unique closed G-orbit. It follows that $Y \subset \overline{Gx}$ for every $x \in X$. The Hilbert Lemma provides an organized way for finding limit points in Y.

For this let T be a fixed maximal torus in a fixed maximal compact subgroup K of G. In this context a 1-parameter subgroup $\lambda \in \Lambda(T)$ is an algebraic morphism[1] $\lambda : \mathbb{C}^* \to G$ with $\lambda(S^1) \subset T$, i.e., after lifting to the Lie algebra level, $z \mapsto \exp(\xi z)$ for $\xi \in \mathfrak{t}$ with integral periods with respect to $\exp : \mathfrak{t} \to T$.

Hilbert Lemma. *Let X be affine, $G \times X \to X$ an algebraic action of a reductive group with $\mathcal{O}(X)^G \cong \mathbb{C}$ and K a maximal compact subgroup with a fixed maximal torus T. Let Y be the unique closed G-orbit in X. Then there exist finitely many 1-parameter subgroups $\lambda_1, \dots, \lambda_m \in \Lambda(T)$ so that, for any $x \in X$ there exists $\lambda \in \{\lambda_1, \dots, \lambda_m\}$ and $k \in K$ such that $\lim_{t \to 0} \lambda(t)(k(x)) = y \in Y$.*

The proof requires a bit of preparation. First, notice that $\Lambda(T)$ is countable and, for any given $\lambda \in \Lambda(T)$, the saturation

$$\mathcal{S}_\lambda(Y) := \{x : \overline{\lambda(\mathbb{C}^*)}(x) \cap Y \neq \varnothing\}$$

is the preimage $\pi^{-1}(\pi(Y))$ defined via the algebraic Hilbert quotient $X \to X/\!/\mathbb{C}^*$, where $\mathbb{C}^*$ acts via the morphism $\lambda : \mathbb{C}^* \to G$. In particular, $\mathcal{S}_\lambda(Y)$ is a closed, algebraic subvariety.

[1]Holomorphic morphisms with values in an affine algebraic group are automatically algebraic.

It is convenient to formulate the conclusion of the Hilbert Lemma as follows:

$$X = K \bigcup_{j=1}^{m} \mathcal{S}_{\lambda_j}(Y).$$

The essential step in the proof of the Hilbert Lemma is a reduction to closures of $T^{\mathbb{C}}$-orbits.

Proposition 3.3.1. *Under the assumptions of the Hilbert Lemma it follows that*

$$\overline{T^{\mathbb{C}} Kx} \cap Y \neq \varnothing.$$

Proof. Suppose to the contrary that $\overline{T^{\mathbb{C}} k(x)} \cap Y = \varnothing$ for all $k \in K$. Therefore, for every $k \in K$, there exists a $T^{\mathbb{C}}$-invariant regular function $f_k \in \mathcal{O}(X)^{T^{\mathbb{C}}}$ such that $f_k | \overline{T^{\mathbb{C}} k(x)} \equiv 1$ and $f_k | Y \equiv 0$. Consequently there are finitely many such functions $f_{k_1}, \ldots, f_{k_m}$ such that

$$f := \sum_{j=1}^{m} |f_{k_j}|^2$$

satisfies $f \geq m > 0$ on Kx and $f|Y \equiv 0$. Thus $f \geq m$ on $\overline{T^{\mathbb{C}} Kx}$ as well.

Hence it follows that $\overline{T^{\mathbb{C}} Kx} \cap Y = \varnothing$ which implies that $K.\overline{T^{\mathbb{C}}.Kx} \cap Y = \varnothing$ as well. But $K.\overline{T^{\mathbb{C}}.Kx} = \overline{Gx}$ and every G-orbit in X has Y in its closure. $\square$

Remark. For this and related information on algebraic actions, see [Kraft 1984]. The proof above is due to Richardson.

The next step is to go from closures of $T^{\mathbb{C}}$-orbits to closures of orbits in $\Lambda(T)$. We only state this result. The proof (see [Kraft 1984]) amounts to proving it for toral groups of matrices with $Y = \{0\}$.

Proposition 3.3.2.

$$\mathcal{S}_{T^{\mathbb{C}}}(Y) = \bigcup_{\lambda \in \Lambda(T)} \mathcal{S}_{\lambda}(Y).$$

Proof of the Hilbert Lemma. By Proposition 3.3.1 and Lemma 3.3.2 it follows that

$$X = K \mathcal{S}_{T^{\mathbb{C}}}(Y) = K \bigcup_{\lambda \in \Lambda(T)} \mathcal{S}_{\lambda}(Y).$$

From the countability of $\Lambda(T)$ and the fact that $\mathcal{S}_{\lambda}(Y)$ is a closed analytic subset, it follows that there exist $\lambda_j \in \Lambda(T)$, $j = 1, \ldots, m$, so that

$$\bigcup_{\lambda \in \Lambda(T)} \mathcal{S}_{\lambda}(Y) = \bigcup_{j=1}^{m} \mathcal{S}_{\lambda_j}(Y).$$

$\square$

b. The exhaustion property in the case $\mathcal{O}(X)^G \cong \mathbb{C}$. Throughout this paragraph $G \times X \to X$ denotes the holomorphic action of a reductive group on a Stein space. It will always be assumed that $\mathcal{O}(X)^G \cong \mathbb{C}$ and therefore that X possesses a unique closed G-orbit Y.

Definition. Under the preceding assumptions, a G-space X is said to have the exhaustion property if, for every maximal compact subgroup K in G, every K-invariant, strictly plurisubharmonic function $\rho : X \to \mathbb{R}$ which possesses a local minimum $x_0 \in X$ is a proper exhaustion.

Remark. It follows that if X has the exhaustion property, then any K-invariant, strictly plurisubharmonic function $\rho : X \to \mathbb{R}$ with $\rho(x_0) = m_0$ as local minimum is a proper exhaustion $\rho : X \to [m_0, \infty)$ and the differential $d(\rho|Gx)$ vanishes only on Gx_0 and there it vanishes exactly along Kx_0. In the language of momentum geometry, $X_0 = Kx_0$ is the Kempf–Ness set.

The main goal of this section is to prove the following result:

Proposition 3.3.3. *A holomorphic action $G \times X \to X$ of a reductive group on a Stein space with $\mathcal{O}(X)^G \cong \mathbb{C}$ has the exhaustion property.*

For this it is important to understand the connection to the Hilbert Lemma in the analytic setting.

Proposition 3.3.4. *A Stein space X equipped with the action $G \times X \to X$ of a reductive group with $\mathcal{O}(X)^G \cong \mathbb{C}$ has the exhaustion property if and only if the Hilbert Lemma is valid.*

Remarks. (1) The validity of the Hilbert Lemma in this context has the obvious meaning: For K and T fixed there exist finitely many $\lambda_j \in \Lambda(T)$, $j = 1, \dots, m$, so that

$$X = K \bigcup_{j=1}^{n} \mathcal{S}_{\lambda_j}(Y).$$

(2) In [Heinzner and Huckleberry 1996] we showed that, if X and the action are algebraic, then X has the exhaustion property; in fact, only the Hilbert Lemma was used.

Suppose the Hilbert Lemma is valid and let $\{x_n\}$ be a divergent sequence. Since the Hilbert Lemma only requires finitely many 1-parameter subgroups, we may assume without loss of generality that there exists $\lambda \in \Lambda(T)$ such that $\lim_{t\to 0} \lambda(t)(x_n) \in Y$ for all n.

Of course we wish to control the region where these limit points land.

Monotonicity Lemma. *Let $\mathbb{C}^* \times X \to X$ be a holomorphic $\mathbb{C}^*$-action, and let $x \in X$ with $\mathbb{C}^* x$ Zariski open in its closure Z. If $K = S^1 < \mathbb{C}^*$ is the maximal compact subgroup, $\rho : X \to \mathbb{R}$ is a K-invariant, plurisubharmonic function and $z_0 := \lim_{t\to 0} \mathbb{C}^* x$, then $\rho(z_0) \le \rho(x)$.*

Proof. The normalization of Z yields a $\mathbb{C}^*$-equivariant holomorphic map $\varphi : \mathbb{C} \to Z$ from the standard $\mathbb{C}^*$-representation with $\varphi(0) = z_0$. The function $\varphi^*(\rho)$ is S^1-invariant and plurisubharmonic. Thus the result follows from the mean-value theorem. $\square$

Now we return to the setting Proposition 3.3.4. From the Monotonicity Lemma it follows that if $\{\rho(x_n)\}$ is bounded, then so is the collection of limit points $\{y_n\}$, $y_n := \lim_{t\to 0} \lambda(t)(x_n)$. It should not be forgotten that these are $\mathbb{C}^*$-fixed points.

Consider for a moment a linear $\mathbb{C}^*$-action on a vector space V. Let W be the set of $v \in V$ so that $\lim_{t\to 0} t(v) = w_0$ exists and let $W_0 \subset W$ be the fixed points.

By using the idea of approximation by G-finite functions, given an action $G \times X \to X$ of a reductive group on a Stein space and a relatively compact domain $D \subset X$, there is a G-equivariant holomorphic map $F : X \to V$ with $F|D$ biholomorphic onto its image (see Section 2.2b).

For our purposes choose D to contain the closure of the set of fixed points $\{y_n\}$.

Let $\mathbb{C}^*$ act on V by transporting the λ-induced action by the equivariant map F and $\Sigma(r)$ be an S^1-invariant normal sphere bundle of radius $r > 0$ in W around the fixed point set W_0, i.e., $\Sigma(r)$ is the boundary of a tubular neighborhood of W_0 in W.

Choose $r > 0$ sufficiently small so that every $\mathbb{C}^*$-orbit which has a point in the closure of the set $F(\{y_n\})$ in its closure has non-empty intersection with $\Sigma(r) \cap D$. In fact we may choose a compact set $C \subset \Sigma(r) \cap D$ with this property and regard it in X via the mapping F; in particular, there exists $t_n \in \mathbb{C}^*$ such that $\lambda(t_n)(x_n) = c_n \in C$ for all n.

Proof of Proposition 3.3.4. First, suppose that the Hilbert Lemma is valid and set things up as above. Consider the plurisubharmonic functions $\rho_n : \mathbb{C}^* \to \mathbb{R}$, $t \mapsto \rho(\lambda(t)c_n)$. These functions are S^1-invariant and yield strictly convex functions $\tilde{\rho}_n : \mathbb{R} \to \mathbb{R}$, $s \mapsto \rho(\lambda(e^s)c_n)$, with $\tilde{\rho}_n'(0) =: m_n > 0$.

By the construction of C, i.e., its compactness in $D \setminus Fix(\mathbb{C}^*)$, it follows that there exists $m > 0$ with $m_n \geq m$. Consequently, $\rho_n(t_n) \geq M + mt_n$, where $M := \min_C \rho$, and therefore, contrary to assumption, $\rho(x_n) = \rho_n(t_n) \to \infty$.

Conversely, suppose that X has the exhaustion property. The usual techniques with G-finite functions yield the existence equivariant holomorphic map $F : X \to V$ to a representation space which is locally biholomorphic along the closed open Y and such that $F|Y$ is injective (see Section 2.2b).

By averaging we may assume that V is a unitary representation for the compact group K. Let η be an invariant norm function and $\rho := \eta \circ F$. Since every G-orbit has Y in its closure and F is locally biholomorphic along Y, it follows that ρ is strictly plurisubharmonic. By explicit construction, it is possible to insure that the restriction $\rho|Y$ has critical points along a minimal K-orbit, e.g., Kx_0, where x_0 is a base point of the Mostow fibration.

Now X has the exhaustion property and therefore $\rho : X \to \mathbb{R}$ is a proper exhaustion. Using this, the Hilbert Lemma on the algebraic closure $\overline{F(X)}$ can be transported back to X.

Note that $F^{-1}(F(Y)) = Y$, because, if not, there would be an additional K-orbit of minima, i.e., another closed orbit. Thus if, $\lim_{t\to 0} \lambda(t)F(x) = z \in F(Y)$, we must only show that $\{\lambda(t)(x)\}$ is not divergent. However, if it were divergent, then, by the exhaustion property, $\lim_{t\to 0} \rho(\lambda(t)(x)) = \infty$. But $\rho = \eta \circ F$ and consequently this is not the case. □

In this proof we made use of a holomorphic G-equivariant map $F : X \to Z$ to an affine G-space with the property that $F|Y$ is a closed embedding and F is locally biholomorphic along Y. In that situation, and under the assumption that $\mathcal{O}(X)^G \cong \mathbb{C}$, so that in particular $F^{-1}(F(Y)) = Y$, we say that X has the *embedding property* if and only if every such map is in fact an embedding $F : X \hookrightarrow Z$ onto a closed subvariety.

THEOREM 3.3.5. *Let $G \times X \to X$ be a holomorphic action of a reductive group on a Stein complex space with $\mathcal{O}(X)^G \cong \mathbb{C}$. Then the following are equivalent*:

(i) *X has the embedding property.*
(ii) *The Hilbert Lemma is valid.*
(iii) *X has the exhaustion property.*

PROOF. In the previous proposition we proved the equivalence *(ii)*$\Leftrightarrow$*(iii)*. Since there is always an equivariant holomorphic map $F : X \to V$ to a representation space with the desired conditions along Y, if X has the embedding property, then F is an embedding onto an algebraic subvariety of V and the Hilbert Lemma is obviously valid.

Conversely, suppose that $F : X \to Z$ satisfies the conditions along Y and (ii) and (iii) are fulfilled. Now Z can be equivariantly embedded in a representation space. Hence, by pulling back a K-invariant norm from that space and the further pulling this back to X, we obtain a K-invariant, strictly plurisubharmonic function $\rho = \eta \circ F$ which attains its minimum at some point $x_0 \in Y$.

Consequently, $\rho : X \to \mathbb{R}$ is a proper exhaustion and it follows that the image $F(X)$ is closed.

It therefore remains to prove the injectivity of F. For this, suppose that $F(x_1) = F(x_2) = z \in Z$ and choose $\lambda \in \Lambda(T)$ so that $\lim_{t\to 0} \lambda(t)(z) =: z_0 \in F(Y)$. From the exhaustion property, applied to the restriction $\rho|\lambda(t)(x_j)$, it follows that $\lim_{t\to 0} \lambda(t)(x_j) =: y_j \in Y$, $j = 1, 2$.

But $F(y_1) = F(y_2)$. Thus $y_1 = y_2 =: y$. Since F is biholomorphic near y, it follows that, for t sufficiently small, $\lambda(t)(x_1) = \lambda(t)(x_2)$. Hence, this holds for all $t \in \mathbb{C}^*$ and as a result $x_1 = x_2$. □

In fact, as we shall now show, the Hilbert Lemma is valid.

THEOREM 3.3.6. *Let $G \times X \to X$ be a holomorphic action of a reductive group on a Stein complex space with $\mathcal{O}(X)^G \cong \mathbb{C}$. Then the Hilbert Lemma is valid.*

Proof. (By induction on $\dim X$) The case of $\dim X = 0$ is clear. Thus we assume that the result is valid for all complex spaces with dimension at most $n-1$.

Let X be given with $\dim X = n$. We must only show that given $x \in X$ there exists $\lambda \in \Lambda(T)$ so that $\lim_{t\to 0} \lambda(t)(x) = y \in Y$, where Y is the unique closed G-orbit.

Since G-orbits are Zariski open in their closures and $Y \subset \overline{Gx}$, it is enough to consider the case where $X = \overline{Gx}$, i.e., if Gx were not locally open, then the desired result would follow from the induction assumption.

Furthermore, by the induction assumption it follows that the Hilbert Lemma is valid on $E := \overline{Gx} \setminus Gx$. Now let $F : X \to V$ be a holomorphic equivariant map to a representation space which is every locally biholomorphic and embeds Y. By adding a map F_1 generated by G-finite functions which separate the finite fibers of the map $F|Gx$, we may assume that $F|Gx$ is injective.

Now let $\lambda \in \Lambda(T)$ be such that $\lim_{t\to 0} \lambda(t)(F(x)) =: z_0 = F(y_0)$ for $y_0 \in Y$. Since F is biholomorphic near y_0, there is a unique local complex curve C_{y_0} through y_0 with $F|C_{y_0}$ biholomorphic onto a piece of the closure of $F(\mathbb{C}^*)(x)$ through z_0. Thus by equivariance and identity principle, it follows that there exists $x_1 \in \mathbb{C}^* C_{y_0}$ such that $F(x_1) = F(x)$. By the injectivity of F, it follows that $x_1 = x$ and therefore $\lim_{t\to 0} \lambda(t)(x) = y_0 \in Y$ as desired. □

It of course follows that all three properties of Theorem 3.3.5 are fulfilled. As a consequence we have the following basic result of Snow [1982].

Corollary 3.3.7. *Let $G \times X \to X$ be a holomorphic action of a reductive group on a Stein complex space with $\mathcal{O}(X)^G \cong \mathbb{C}$. Then X possesses the structure of an affine algebraic variety with algebraic G-action.*

Proof. It follows from the preceding discussion that X can be holomorphically, equivariantly embedded as a closed complex analytic subvariety of a G-representation space V. Let Z be the Zariski closure of X in such an embedding

For essentially the same reason as that for the algebraicity of a holomorphic G-representation, since $\mathcal{O}(Z)^G \cong \mathbb{C}$, it follows that the G-finite holomorphic functions on Z are algebraic. Since the G-finite functions are dense in $I(X) := \{f \in \mathcal{O}(X) : f|X = 0\}$, it follows that X is defined as the zero set of algebraic functions. □

The following result takes its name from its usefulness in applications to the study of fibers of analytic Hilbert quotients.

Corollary 3.3.8 (Fiber exhaustion). *Let $G \times X \to X$ be a holomorphic action of a reductive group on a Stein space with $\mathcal{O}(X)^G \cong \mathbb{C}$. If K is a maximal compact subgroup of G and $\rho : X \to \mathbb{R}$ is a K-invariant, strictly subharmonic function which has a local minimum value $\rho(x_0) = m_0$, then $\rho : X \to [m_0, \infty)$ is a proper exhaustion with $\{x : \rho(x) = m_0\} = Kx_0$. Furthermore, the differential*

$d(\rho|Gx)$ of the restriction to an orbit vanishes only on Gx_0 and there only along Kx_0.

c. The relative exhaustion property of ρ. The exhaustion property for Stein G-spaces with $\mathcal{O}(X)^G \cong \mathbb{C}$ should be regarded as a properness statement for $\pi \times \rho$, where $\pi : X \to (*)$ is a map to a point. Here we extend this to a local theorem for invariant maps with values in a complex space.

Let $\pi : X \to Q$ be a Hausdorff quotient and $\rho_0 : X \to \mathbb{R}$ a K-invariant, strictly plurisubharmonic (background) function which is a relative exhaustion. In this situation we call $\pi : X \to Q$ a *gauged* quotient.

In applications we are interested in the behavior of a K-invariant, strictly plurisubharmonic function ρ which is known to be an exhaustion of the neutral fiber $F_{q_0} := \pi^{-1}(q_0)$.

THEOREM 3.3.9 (RELATIVE EXHAUSTION PROPERTY). *Let $G \times X \to X$ be a holomorphic action of a reductive action with a gauged Hausdorff quotient $\pi : X \to Q$. Let $\rho : X \to \mathbb{R}$ be a K-invariant, strictly plurisubharmonic function such that $\rho|F_{q_0}$ has a local minimum for a π-fiber F_{q_0}. Then, after shrinking Q to an appropriate neighborhood of q_0, it follows that ρ is a relative exhaustion.*

After the preparation in the previous sections the proof is almost immediate. We begin by making several reductions.

First, note that by replacing ρ_0 by $\rho_0 + c$, where $c > 0$ is an appropriate constant, we may assume that $\rho_0 > 0$.

Secondly, if e^ρ is a relative exhaustion, then so is ρ. Of course the function e^ρ also satisfies the assumptions of the theorem. Thus we may assume in addition that $\rho : X \to \mathbb{R}^{\geq 0}$.

In the sequel $\chi : \mathbb{R} \to \mathbb{R}^{\geq 0}$ denotes a smooth function with $\operatorname{supp}(\chi) = [M, \infty)$ and which is strictly convex on (M, ∞). If D is a relatively compact domain in X and $M = \max_D \rho_0$, then the function $\rho + \chi \circ \rho_0$ is still a gauge for $\pi : X \to Q$.

LEMMA 3.3.10. *After replacing ρ_0 by a function of the type $\rho + \chi \circ \rho_0$, it may be assumed that the Kempf–Ness sets agree in a neighborhood of x_0. In particular, after replacing Q by an appropriate neighborhood of $q_0 \in Q$, it follows that $\pi|X_0(\rho)$ is proper and induces a homeomorphism $X_0/K \cong Q$.*

PROOF. Choose D to be a relatively compact neighborhood of Kx_0. The result then follows from Corollary 3.1.7. □

This result already shows that the restriction of ρ to the π-fibers near F_{q_0} is bounded from below and an exhaustion. A similar argument yields the properness. For later applications we state the relevant technical results.

LEMMA 3.3.11. *If $G \times X \to X$ is an action of a reductive group on a complex space with Hausdorff quotient $\pi : X \to Q$ and $\rho : X \to \mathbb{R}$ is a K-invariant strictly plurisubharmonic function which is a relative exhaustion, then for every*

$R > 0$ and open connected set $V \subset Q$ the set $T_R := \pi^{-1}(V) \cap \{x : \rho(x) < R\}$ is K-connected.

Proof. Let X_0 be the Kempf–Ness set over V. Since X_0/K is homeomorphic to V, we must only prove that a point in $x \in T_R$ can be connected to X_0. It is therefore sufficient to discuss the case where Q is just a point.

Let Y be the closed G-orbit in X and note that, by the Hilbert Lemma, for a maximal torus T in K there exits a 1-parameter group $\lambda \in \Lambda_T$ so that $\lim_{t \to 0} \lambda(t)(x) = y \in Y$. By the Monotonicity Lemma $\rho(\lambda(t)(x)) < \rho(x) < R$ for $|t| < 1$. In particular we can connect x to $T_R \cap X$ by a curve in T_R.

Now $\rho|(T_R \cap Y)$ has a minimum in each of its components, i.e., X_0 has non-empty intersections with every such component, and therefore x can be connected to X_0 by a curve in T_R. $\square$

Given a radius $R > 0$ we now define a comparison domain D to be used in proving Theorem 3.3.9. As usual let ρ_0 denote the background function for which it is known that $\rho_0 \times \pi$ is proper.

To clarify the notation, let q_0 be the neutral point in Q and, for $q \in Q$, let $F_q := \pi^{-1}(q)$. Let R_1 be the maximum of ρ_0 on the set $\{x \in F_{q_0} : \rho(x) \leq 2R\}$. Let $D := \pi^{-1}(V) \cap \rho_0^{-1}[0, R_1)$ for V a connected neighborhood of q_0 which we will now choose.

Note that $\rho|\partial(D \cap F_{q_0}) \geq 2R$. Thus this condition essential holds for q sufficiency close to q_0: For some small $\varepsilon > 0$ we choose V so that $\rho|\partial(D \cap F_q) \geq 2R - \varepsilon$ for all $q \in V$. For W relatively compact in V it follows that $\pi^{-1}(W) \cap \rho^{-1}[0, R) \cap D = T_R \cap D$ is relatively compact in D.

If D is constructed in this way, in particular containing $T_R \cap D$ as a relatively compact subset, we refer to it as being adapted to ρ and R.

Lemma 3.3.12. *If D is adapted to ρ and R, then $T_R \subset D$.*

Proof. If not, then for some $q \in W$ the set $\{x \in F_q : \rho(x) < R\}$ would be disconnected, contrary to the previous lemma. $\square$

Theorem 3.3.9 is now a consequence of the following properness result.

Lemma 3.3.13. *If for every $q \in Q$ the restriction $\rho|Gx$ of ρ to some orbit in F_q has a critical point, then ρ is a relative exhaustion.*

Proof. The existence of a divergent sequence $x_n \in X$ so that $\pi(x_n) \to q_0$ with $\rho(x_n) < R$ would be in violation to the previous lemma. $\square$

Using the relative exhaustion property it is possible to prove the local embedding theorem for Stein spaces.

Theorem 3.3.14. *Let $G \times X \to X$ be a holomorphic action of a reductive group on a Stein space. Then every point $x \in X$ possesses a G-invariant Stein neighborhood U which is saturated with respect to the formal invariant theoretic quotient and which can be holomorphically and G-equivariantly embedded as a closed complex subspace of a G-representation space.*

Proof. Let $\pi : X \to Q$ be the formal invariant theory quotient and $F := \pi^{-1}(\pi(x))$ its fiber through x. By Corollary 3.3.7 there exits a equivariant, holomorphic embedding $\varphi : F \to V$ onto to a closed (algebraic) subvariety of a G-representation space.

Extend φ to a holomorphic map $\varphi : X \to V$ by the same name which is locally biholomorphic along F. Taking the average

$$A(\varphi)(x) := \int_{k \in K} k\varphi(k^{-1}(x))\, dk,$$

we obtain a G-equivariant extension with the same properties. Let φ denote this extension.

Now let η be a K-invariant norm function on V and $\rho := \eta \circ \varphi$. It follows from Theorem 3.3.9 that, after shrinking Q to a perhaps smaller neighborhood W of $\pi(x)$, the function ρ is a relative exhaustion.

Furthermore, by adding additional G-finite functions and further shrinking if necessary, we may assume that φ is biholomorphic on the (compact) Kempf–Ness set X_0. Since it is therefore a diffeomorphism on the K-orbits $Kx_0 = K/L$ in X_0, and since $H = L^{\mathbb{C}}$, it is injective on the closed G-orbits $Y = Gx_0$ for $x_0 \in X_0$.

Consequently, the restriction $\varphi|Y$ to every closed G-orbit is a closed embedding (Theorem 3.3.5).

Thus, since we may assume that φ is biholomorphic in a neighborhood of the Kempf–Ness set X_0, we have organized a situation where φ is an injective immersion whose restriction to every π-fiber is a closed embedding. Furthermore, since the Kempf–Ness set X_0 is embedded by φ, if W is a sufficiently small neighborhood of $\varphi(F)$ which is $\pi : V \to V/\!/G$ saturated and $U := \pi^{-1}(W)$, then $\varphi : U \to W$ is a closed embedding.

Now we may choose W is the preimage of a Stein open set W_0 in the Hilbert quotient $V/\!/G$ which we embed as a closed complex subspace of the trivial representation V_0 by a holomorphic map F_0. Letting $\varphi_0 := F_0 \circ \pi \circ \varphi$, it follows that $\varphi \times \varphi_0 : U \to V \times V_0$ is a closed holomorphic embedding. □

Corollary 3.3.15. *If $G \times X \to X$ is a holomorphic action of a reductive group on a Stein space, then every $x \in X$ possesses a G-invariant, neighborhood U which is saturated with respect to the formal invariant theoretic quotient $\pi : X \to Q$ and which possesses a Stein holomorphic invariant theoretic quotient $U \to U/\!/G$.*

Proof. This is now an immediate consequence of 3.1.4. □

4. Kähler spaces

4.1. The Slice Theorem. Let X be a holomorphic $G = K^{\mathbb{C}}$ space and assume that K acts on X in a Hamiltonian fashion. The corresponding equivariant moment map $\mu : X \to \mathfrak{k}^*$ defines the set $X(\mu)$ of semistable points. We show in this section that at every point $x_0 \in \mu^{-1}(0)$ there is a slice S, i.e., a local Stein G_{x_0}-stable subvariety of X through x_0 such that the natural map $G \times_{G_{x_0}} S \to X$ is biholomorphic onto its open image.

We already know that the isotropy group $G_{x_0} = (K_{z_0})^{\mathbb{C}}$ is reductive at every point $x_0 \in \mu^{-1}(0)$ and therefore there is a Slice $Z := G \times_{G_{x_0}} S$ at x_0 up to local biholomorphy. After pulling back the Kähler form ω on X to Z we obtain a Kähler form $\tilde{\omega}$ on Z with a moment map $\tilde{\mu}$ which is given by pulling back μ to Z. Let $z_0 = [e, 0]$ be the point in Z which corresponds to x_0. Since S can be chosen such that Gz_0 is a strong deformation retract of Z and Kz_0 is a strong deformation retract of Gz_0, it follows that the cohomology class of $\tilde{\omega}$ is determined by the pull back of $\tilde{\omega}$ to the orbit Kz_0. The moment map condition $d\tilde{\mu}_\xi = \imath_{\xi_X}\tilde{\omega}$ implies that $\tilde{\omega}$ is exact. Moreover, since Z can be chosen to be a Stein space, it follows that $\tilde{\omega} = 2i\partial\bar{\partial}\tilde{\rho}$ for some smooth function $\tilde{\rho} : Z \to \mathbb{R}$. Further, after averaging $\tilde{\rho}$ over the compact group K we may assume $\tilde{\rho}$ to be K-invariant. In this setting,there is the moment map $\mu^{\tilde{\rho}}$ which differs only by a K-invariant constant $a \in \mathfrak{k}^*$ from the original moment map $\tilde{\mu}$. Of course, if the group K is semisimple, then a is zero and we have $\tilde{\mu} = \mu^{\tilde{\rho}}$ (see [Heinzner et al. 1994] for more details). In general, after adding to $\tilde{\rho}$ a pluriharmonic K-invariant function h that we can always arrange that $\tilde{\mu} = \mu^{\tilde{\rho}}$; see [Heinzner et al. 1994]. Now by the previous results in the Stein case we are in the following setup.

- There is a locally biholomorphic G-equivariant map $\phi : Z \to X$ such that $\phi(z_0) = x_0$ and whose restriction to Gz_0 is biholomorphic onto its image Gx_0.
- $\tilde{\mu} = \mu \circ \phi = \mu^{\tilde{\rho}}$ for some K-invariant strictly plurisubharmonic positive function $\tilde{\rho} : Z \to \mathbb{R}$.
- The analytic Hilbert quotient $\pi_Z : Z \to Z/\!/G$ exists and $\pi \times \tilde{\rho} : Z \to Z/\!/G \times \mathbb{R}$ is proper.

In order to show that ϕ is biholomorphic we introduce the following terminology.

Let A be a K-invariant subset in X. Consider for any $a \in A$ and $\xi \in \mathfrak{k}$ the set $I(a, \xi) = \{t \in \mathbb{R} : \exp(it\xi)a \in A\}$. We call A *μ-adapted* if for the closure of any bounded connected component of $I(a, \xi)$, say $[t_-, t_+]$, we have $\mu_\xi(\exp it_-\xi a) < 0$ and $\mu_\xi(\exp it_+\xi a) > 0$. If the closure of the connected component is $[t_-, \infty)$, we require just $\mu_\xi(\exp it_-\xi a) < 0$ and if it is $(-\infty, t_+]$, then we require $\mu_\xi(\exp it_+\xi a) > 0$.

Note that, since $t \to \mu_\xi(\exp it\xi a)$ is increasing, for a μ-adapted subset A of X the set $\{t \in \mathbb{R} : \exp it\xi a \in A\}$ is connected for every $a \in A$. This proves the following

LEMMA 4.1.1. *A μ-adapted subset A of X is orbit convex in X.* □

The next Lemma is the crucial step in the construction of a slice at x_0.

LEMMA 4.1.2. *After replacing Z with a G-stable open neighborhood of z_0 the map ϕ is biholomorphic onto its open image.*

PROOF. Since ϕ is locally biholomorphic and injective on Kz_0 there is an K-stable neighborhood of $\tilde{U}$ of z_0 which is mapped by ϕ biholomorphically onto its image U. If we can show that U can be chosen to be orbit convex, then the map $(\phi|\tilde{U})^{-1} : U \to Z$ extends to GU and gives an inverse to $\phi|G\tilde{U}$. Thus it is sufficient to show that Kx_0 has arbitrary small K-stable μ-adapted open neighborhoods.

Now we may choose $\tilde{U}$ such that $\tilde{\omega} = 2i\partial\bar{\partial}\tilde{\rho}$ and $\tilde{\mu} = \mu^{\tilde{\rho}}$ on $G\tilde{U}$. In particular we have $\omega = 2i\partial\bar{\partial}\rho$ and $\mu = \mu^{\rho}$ on U where $\rho \circ \phi = \tilde{\rho}|\tilde{U}$. Moreover, since $\pi \times \tilde{\rho}$ is a relative exhaustion, we may choose an open neighborhood $Q \subset Z//G$ and a $r \in \mathbb{R}$ so that $\bar{V} := \pi_Z^{-1}(Q) \cap \bar{D}_{\tilde{\rho}}(r)$ is contained in $\tilde{U}$. Here $\bar{D}_{\tilde{\rho}}(r) := \{z \in Z : \tilde{\rho}(z) \leq r\}$. Let $V := D_{\tilde{\rho}}(r) := \{z \in Z : \tilde{\rho}(z) < r\}$. It is sufficient to show this:

CLAIM. *$\phi(V)$ is μ-adapted.*

This is a consequence of the definition of the moment map, as follows:

Consider the smallest $t_+ > 0$ such that $\exp it\xi a \in \phi(V)$ for $t \in [0, t_+)$. If $t_+ \neq +\infty$, then by construction of V we have $\rho(\exp it_+\xi a) = r$. Since

$$\mu_\xi(\exp it_+\xi a) = d\rho(J\xi_X(\exp it\xi a)),$$

this implies that $\mu_\xi(\exp it_+\xi a) > 0$. The function $t \to \mu_\xi(\exp it\xi a)$ is strictly increasing. Thus it follows by using again the equality just displayed that $\{t \in \mathbb{R} : t \geq 0 \text{ and } \exp it\xi a \in \phi(V)\}$ is connected. By a similar argument applied to the smallest t_- such that for all negative t bigger then t_- the curve $\exp it\xi a$ is contained in $\phi(V)$ it follows that $\{t \in \mathbb{R} : \exp it\xi a \in \phi(V)\}$ is connected. □

COROLLARY 4.1.3. *Every point $x_0 \in X_0 = \mu^{-1}(0)$ has a neighborhood basis of open μ-adapted neighborhoods.* □

Now let V be a K-invariant open neighborhood of z_0 such that $\phi_V := \phi|V$ is biholomorphic and maps V biholomorphically onto an μ-adapted open subset $U := \phi(V)$. It follows that the inverse ϕ_V^{-1} extends to a G-equivariant holomorphic map $\psi : U^c \to Z$ where $U^c := GU$. Of course ψ is the inverse of $\phi|GV$. This shows that x_0 has an open G-stable Stein neighborhood U^c which is biholomorphically isomorphic to $G \times_{G_{x_0}} S$, where S is a slice at x_0. Hence we have the following

THEOREM 4.1.4. *At every point $x_0 \in X$ such that $\mu(x_0)$ is a K-fixed point there exists a slice.*

PROOF. This follows from the preceding discussion, since by adding a constant to μ if necessary, we may assume that $\mu(x) = 0$. □

4.2. The Quotient Theorem. Given a moment map $\mu : X \to \mathfrak{k}^*$, then, under the assumption that $G = K^{\mathbb{C}}$ acts holomorphically on X, there is an associated set $X(\mu)$ of semistable points. The goal here is to show that the analytic Hilbert quotient $X(\mu)$ exists. We first show that the relation $x \sim y$ if and only if the intersection of the closures in $X(\mu)$ of the corresponding G orbits is non trivial is in fact an equivalence relation.

Lemma 4.2.1. *Let $V_1, V_2 \subset X$ be μ-adapted subsets of X. Then*

$$G(V_1 \cap V_2) = GV_1 \cap GV_2.$$

Proof. We have to show that $GV_1 \cap GV_2 \subset G(V_1 \cap V_2)$. Thus let $z = g_1 v_1 = g_2 v_2$ with $g_j \in G$ and $v_j \in V_j$ be given. There exist $k \in K$ and $\xi \in \mathfrak{k}$ so that $g_2^{-1} g_1 = k \exp i\xi$. Consider the path $\alpha\colon [0,1] \to X$, $t \to \exp it\xi v_1$. It is sufficient to show that $\alpha(t_0) \in V_1 \cap V_2$ for some $t_0 \in [0,1]$. Since this is obvious if $v_1 \in V_2$ or $v_2 \in V_1$, we may assume that there exists t_1 and t_2 in $[0,1]$ where α leaves V_1 and enters V_2. But μ_ξ is increasing on α. Thus, from $\mu_\xi(\alpha(t_1)) > 0$ and $\mu_\xi(\alpha(t_2)) < 0$ we conclude that $t_1 > t_2$, i.e., $\alpha(t) \in V_1 \cap V_2$ for $t \in [t_2, t_1]$. □

Since every point $x_0 \in X_0 = \mu^{-1}(0)$ has a μ-adapted neighborhood and the union of μ adapted sets remains μ-adapted, the lemma implies the following

Corollary 4.2.2. *If $\overline{Gx} \cap X_0 \neq \varnothing$, then $\overline{Gx} \cap X_0 = Kx_0$ for some $x_0 \in X_0$.* □

Note that $x \sim y$ if and only if $\overline{Gx} \cap \overline{Gy} = Ky_0$ where one has to take the closure in $X(\mu)$.

Corollary 4.2.3. *The Hausdorff quotient $X(\mu)/{\sim}$ exists.* □

Let $X(\mu)/\!/G$ be topologically defined as $X/{\sim}$ and let $\pi : X(\mu) \to X(\mu)/\!/G$ be the quotient map.

Theorem 4.2.4. *The quotient $X(\mu)/\!/G$ is an analytic Hilbert quotient.*

Proof. We already know that $X(\mu)/\!/G$ is well defined as a topological space. In order to endow $X(\mu)/\!/G$ with a complex structure we use the slice theorem.

If we fix a point $x_0 \in X_0$, then, by the slice theorem, we find a K-invariant μ-convex Stein neighborhood U of x_0 such that $U^c = GU$ is a complexification of the K-action on U. Moreover, by construction of U we have $(\mu|U)^{-1}(0) = \mu^{-1}(0) \cap U^c$. In particular, U^c is contained in $X(\mu)$ and is π-saturated, i.e., $\pi^{-1}(\pi(U^c)) = U^c$. By Corollary 3.3.15 the analytic Hilbert quotient $U^c/\!/G$ exists and, since U^c is saturated, is naturally identified with an open subset of $X(\mu)/\!/G$. □

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