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Varieties of Minimal Rational Tangents on Uniruled Projective Manifolds

JUN-MUK HWANG AND NGAIMING MOK

ABSTRACT. On a polarized uniruled projective manifold we pick an irreducible component $\mathcal{K}$ of the Chow space whose generic members are free rational curves of minimal degree. The normalized Chow space of minimal rational curves marked at a generic point is nonsingular, and its strict transform under the tangent map gives a variety of minimal rational tangents, or VMRT. In this survey we present a systematic study of VMRT by means of techniques from differential geometry (distributions, G-structures), projective geometry (the Gauss map, tangency theorems), the deformation theory of (rational) curves, and complex analysis (Hartogs phenomenon, analytic continuation). We give applications to a variety of problems on uniruled projective manifolds, especially on irreducible Hermitian symmetric manifolds S of the compact type and more generally on rational homogeneous manifolds G/P of Picard number 1, including the deformation rigidity of S and the same for homogeneous contact manifolds of Picard number 1, the characterization of S of rank at least 2 among projective uniruled manifolds in terms of G-structures, solution of Lazarsfeld's Problem for finite holomorphic maps from G/P of Picard number 1 onto projective manifolds, local rigidity of finite holomorphic maps from a fixed projective manifold onto G/P of Picard number 1 other than $\mathbb{P}^n$, and a proof of the stability of tangent bundles of certain Fano manifolds.

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Rational curves play a crucial role in the study of Fano manifolds. By Mori's theory, Fano manifolds are uniruled. We consider more generally uniruled projective manifolds. Fixing an ample line bundle and considering only components of the Chow space whose generic members are free rational curves, we introduce the notion of minimal rational curves by minimizing the degree of a generic member. The normalized Chow space of minimal rational curves marked at a generic point is nonsingular, and its strict transform under the tangent map gives the variety of minimal rational tangents. In [Hwang and Mok 1997; 1998a; 1998b; 1999; Hwang 1997; 1998] we have put forth the idea of recapturing complex-analytic properties of Fano manifolds from varieties of minimal rational tangents and holomorphic distributions spanned at generic points by them. In this survey we present a systematic treatment of the fundamental notions, and examine a number of applications primarily in the context of rational homogeneous manifolds of Picard number 1.

The scope of problems we consider covers deformation rigidity, algebro-geometric characterizations (of Grassmannians, etc.), stability of the tangent bundle, and holomorphic mappings. We also give complex-analytic and geometric proofs of results from the theory of geometric structures of Tanaka, as given by Ochiai [1970] as well as Tanaka and Yamaguchi [Yamaguchi 1993], which we needed for various problems, making our presentation essentially self-contained. As to the techniques we employ, an important role is played by holomorphic distributions and the Frobenius condition. Distributions spanned by minimal rational tangents are first of all studied using the deformation theory of rational curves. Then, projective geometry enters the picture in various ways, in the problem of integrability of such distributions, in vanishing theorems related to flatness of G-structures and in the study of stability of tangent bundles. Complex-analytic techniques enter, in the form of analytic continuation and Hartogs extension, in conjunction with the use of the Gauss map, when we study varieties of minimal rational tangents $\mathcal{C}_x$ as x varies. Further study of deformations of curves, in the context of finite holomorphic maps to Fano manifolds, leads to the notion of varieties of distinguished tangents. For the study of rational homogeneous manifolds, we will need basics for graded Lie algebras associated to simple Lie groups, a summary of which will be given.

For the general theory, varieties of minimal rational tangents are first studied as projective subvarieties. The first motivation for studying their projective-geometric properties stemmed from [Hwang and Mok 1998b], where we proved the rigidity of irreducible Hermitian symmetric manifolds of the compact type S under Kähler deformation. There we reduced the problem to the study of the distribution W spanned by varieties of minimal rational tangents $\mathcal{C}_x$ at the central fiber X, and derived a sufficient condition for the integrability in terms of the projective geometry of $\mathcal{C}_x$, namely, W is integrable whenever the variety $\mathcal{T}_x$ of lines tangent to $\mathcal{C}_x$ is linearly nondegenerate in $\mathbb{P}(\bigwedge^2 W_x)$ for a generic point x.

In Section 1 we study $\mathcal{C}_x$ as projective subvarieties, and give first applications of such results and their methods of proof. We prove the algebro-geometric characterization of irreducible Hermitian symmetric manifolds of the compact type and of rank at least 2 as the only uniruled projective manifolds admitting G-structures for some reductive G. After identifying varieties of highest weight tangents $\mathcal{W}_x$ with varieties $\mathcal{C}_x$ of minimal rational tangents $\mathcal{C}_x$, the proof is obtained by vanishing theorems for the obstruction to flatness of G-structures, which reduce to projective-geometric properties of $\mathcal{C}_x$. We further discuss the question of stability of the tangent bundle of Fano manifolds by applying variants of Zak's theorem on tangencies on $\mathcal{C}_x$. In Section 2 we return to deformation rigidity. As we may restrict to the case where S is of rank at least 2, the problem reduces to recovering an S-structure at the central fiber X. The latter is possible, whenever $\mathcal{C}_x$ is linearly nondegenerate at a generic point of X. Otherwise we have a proper distribution $W \subsetneqq T(X)$ spanned at generic points by $\mathcal{C}_x$. We prove the nonexistence of W by studying its integrability in terms of $\mathcal{C}_x$, as mentioned. We give further a generalization [Hwang 1997] of deformation rigidity to the case of homogeneous contact manifolds, where there is the new element of deformations of contact distributions.

In Section 3 we study varieties of minimal rational tangents $\mathcal{C}_x$ as the base points vary, by considering the tautological 1-dimensional multi-foliation $\mathcal{F}$ defined at generic points of $\mathcal{C}$ by the tautological lifting of minimal rational curves. Assuming that the Gauss map on $\mathcal{C}_x$ to be generically finite for generic x, we prove the univalence of the multi-foliations, resulting in the birationality of the tangent map. This uniqueness result implies that a local biholomorphism f preserving the varieties $\mathcal{C}_x$ must also be $\mathcal{F}$-preserving. The latter constitutes the first step towards a complex-analytic and geometric proof of Ochiai's characterization of S (as above) in terms of flat S-structures, which says that f is the restriction of a biholomorphic automorphism F of S. To prove Ochiai's result, we introduce the method of analytic continuation of $\mathcal{F}$-preserving meromorphic maps along minimal rational curves, and exploit the rational connectedness of S.

In Section 4 we move to rational homogeneous manifolds S of Picard number 1. For the nonsymmetric case a new element arises, namely, there exist nontrivial homogeneous holomorphic distributions. In analogy to Ochiai's result we have the results of Tanaka and Yamaguchi in terms of varieties of highest weight tangents $\mathcal{W}_x$. In the nonsymmetric and noncontact case their results go further, stating that a local biholomorphism f must extend to a biholomorphic automorphism, provided that f preserves the minimal homogeneous distribution D. We give a proof of the result of Tanaka and Yamaguchi, by showing that a D-preserving local biholomorphism already preserves $\mathcal{W}$ and by resorting to methods of Section 3.

In the last two sections we have primarily the study of finite holomorphic maps onto Fano manifolds in mind. In Section 5 we introduce the notion of varieties of distinguished tangents. They generalize varieties of minimal rational tangents, and are of particular relevance in the context of finite holomorphic maps into Fano manifolds, since preimages of varieties of minimal rational tangents give varieties of distinguished tangents. We give an application for rational homogeneous target manifolds S of Picard number 1 distinct from the projective space, proving that any finite holomorphic map into S is locally rigid. In Section 6 we consider the case where the domain manifold is S, and prove that any surjective holomorphic map of S onto a projective manifold X distinct from the projective space is necessarily a biholomorphism, resolving Lazarsfeld's problem.

While in the applications we concentrate on rational homogeneous manifolds, the general theory has been developed to be applicable in much wider contexts. Such applications, especially to the case of Fano complete intersections, will constitute one further step towards developing a theory of "variable geometric structures" modeled on varieties of minimal rational tangents.

1. Minimal Rational Curves, Varieties of Minimal Rational Tangents and Associated Distributions

1.1. For the study of Fano manifolds and more generally uniruled manifolds a basic tool is the deformation theory of rational curves. We will only sketch the basic ideas and refer the reader to [Kollár 1996] for a systematic and rigorous treatment of the general theory. Let X be a projective manifold. By a parametrized rational curve we mean a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$. The image of f is called a rational curve. Given a holomorphic family $f_t : \mathbb{P}^1 \to X$ of rational curves, parametrized by $t \in \triangle := \{t \in \mathbb{C} : |t| < 1\}$, the derivative $\frac{d}{dt}\big|_0 f_t$ defines a holomorphic section of $f_0^*T(X)$. However, given a member f_0 of the space $\mathrm{Hol}(\mathbb{P}^1, X)$ of parametrized rational curves in X, and $\sigma \in \Gamma(\mathbb{P}^1, f_0^*T(X))$, it is not always possible to fit f_0 into a holomorphic family of $f_t \in \mathrm{Hol}(\mathbb{P}^1, X)$, such that $\frac{d}{dt}\big|_0 f_t = \sigma$. Setting in power series $f_t = f + \sigma t + g_2t^2 + \cdots$ locally, the obstruction of lifting to higher coefficients lies in $H^1(\mathbb{P}^1, f_0^*T(X))$. In case the latter vanishes, $\mathrm{Hol}(\mathbb{P}^1, X)$ is smooth in a neighborhood U of $[f_0]$, and the tangent space at $[f] \in U$ can be identified with $\Gamma(\mathbb{P}^1, f^*T(X))$.

By the Grothendieck splitting theorem any holomorphic vector bundle on $\mathbb{P}^1$ splits into a direct sum $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$. When $f^*T(X)$ is semipositive, that is, $a_i \geq 0$, then $f^*T(X)$ is spanned by global sections, $H^1(\mathbb{P}^1, f^*T(X)) = 0$, and deformations of f sweeps out some open neighborhood of $C = f(\mathbb{P}^1)$. We call f a free rational curve. A projective manifold is said to be uniruled if it possesses a free rational curve.

Each irreducible component of $\mathrm{Hol}(\mathbb{P}^1, X)$ can be endowed the structure of a quasi-projective variety. It covers some Zariski-open subset of X if and only if some member is a free rational curve. Consequently, there is an at most countable union Z of proper subvarieties of X such that any rational curve passing through $x \notin Z$ is necessarily free. A point x lying outside Z is called a very general point.

Fix an ample line bundle L on X and consider all irreducible components $\mathcal{H}$ of $\mathrm{Hol}(\mathbb{P}^1, X)$ whose generic member is a free rational curve. As degrees of members of a fixed $\mathcal{H}$ with respect to L are the same we may speak of the degree of the component $\mathcal{H}$. A member of a component $\mathcal{H}$ of minimal degree will be called a minimal rational curve. By Mori's break-up trick [1979] a generic member of $\mathcal{H}$ is an immersed rational curve $f : \mathbb{P}^1 \to X$ such that $f^*T(X) \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ (compare [Mok 1988; Hwang and Mok 1998b]); otherwise one can obtain an algebraic one-parameter family of curves in $\mathcal{H}$ fixing a pair of very general points, which must break up in the limit, contradicting minimality. A minimal rational curve with the splitting type as described is called a standard minimal rational curve.

We fix an irreducible component $\mathcal{H}$ of minimal rational curves. At a generic point $x \in X$ all such curves passing through x are free. Consider the subvariety $\mathcal{H}_x \subset \mathcal{H}$ of all $[f] \in \mathcal{H}$ such that $f(o) = x$, where $o \in \mathbb{P}^1$ is a base point. The isotropy subgroup of $\mathbb{P}^1$ at $f(o) \in x$ o acts on $\mathcal{H}_x$, making its normalization into a principal bundle over a nonsingular quasi-projective variety $\mathcal{M}_x$. We called $\mathcal{M}_x$ the normalized Chow space of minimal rational curves marked at x. By minimality $\mathcal{M}_x$ must be compact, that is, a projective manifold which may have several connected components. By Mori's break-up trick a generic member $[f]$ of $\mathcal{H}_x$ is unramified at o. We have therefore a rational map $\Phi_x : \mathcal{M}_x \dashrightarrow \mathbb{P}T_x(X)$ defined by

$$\Phi_x\big([f(\mathbb{P}^1)]\big) = \big[df(T_o(\mathbb{P}^1))\big]$$

at generic points of $\mathcal{M}_x$. We call Φ_x the tangent map at x.

Fix a base point $x \in X$ and consider now the space $\mathrm{Hol}((\mathbb{P}^1, o); (X, x))$ consisting of all parametrized rational curves f sending o to x. For a holomorphic family f_t, $t \in \triangle$, of such curves

$$\frac{d}{dt}\bigg|_0 f_t \in \Gamma(\mathbb{P}^1, f_0^*T(X) \otimes \mathcal{O}(-1)),$$

where $\mathcal{O}(-1)$ corresponds to the maximal ideal sheaf $\mathfrak{m}_o$ of o on $\mathbb{P}^1$. Given σ in the latter space of sections, the obstruction to extending f_o to a holomorphic family f_t, $f_t(o) = x$, lies in $H^1(\mathbb{P}^1, f_o^*T(X) \otimes \mathcal{O}(-1))$, which vanishes whenever f_o is a free rational curve, since $H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0$ whenever $a \geq -1$. In this case $\mathrm{Hol}((\mathbb{P}^1, o), (X, x))$ is smooth in a neighborhood U of $[f_o]$, and the tangent space at $[f] \in U$ can be identified with $\Gamma(\mathbb{P}^1, f^*T(X) \otimes \mathcal{O}(-1))$.

Since $[f] \in \mathcal{H}_x$ is standard for a generic $[f]$, Φ_x is generically finite. Let $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ be the closure of the image of the tangent map. We call $\mathcal{C}_x$ the variety of minimal rational tangents at x. It may have several components.

For C smooth we can identify $T_{[C]}(\mathcal{M}_x)$ with $\Gamma(C, N_{C|X} \otimes \mathfrak{m}_x)$ for the normal bundle $N_{C|X}$ of C in X and for $\mathfrak{m}_x$ denoting the maximal ideal sheaf of x on C. For C standard and smooth, let $T_x(C) = \mathbb{C}\alpha \subset T_x(X)$. From the description of $T_{[C]}(\mathcal{M}_x)$ we see that the tangent space $T_{[\alpha]}(\mathcal{C}_x) = P_\alpha/\mathbb{C}\alpha$, where P_α is the positive part $(\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)_x$ of a Grothendieck decomposition of $T(X)$ over

C. With obvious modifications the preceding discussion applies to all standard minimal rational curves, which are necessarily immersed.

1.2. From now on we assume that for our choice of $\mathcal{H}$ and for a generic point $x \in X$, $\mathcal{C}_x$ is irreducible. For our study of uniruled projective manifolds via the varieties of minimal rational tangents, an important element is the distribution W spanned at generic points by the homogenization $\tilde{\mathcal{C}}_x \subset T_x(X)$ of $\mathcal{C}_x$. For the problem of deformation rigidity, a key question is the question of integrability of such distributions. By Frobenius, W is integrable if the Frobenius form

$$[\ ,\ ] : \bigwedge^2 W \to T(X)/W$$

vanishes, where $\varphi_x(u, v) = [\tilde{u}, \tilde{v}] \mod W_x$ for local holomorphic sections $\tilde{u}$, $\tilde{v}$ such that $\tilde{u}(x) = u$, $\tilde{v}(x) = v$.

A line tangent to $\mathcal{C}_x$ at a generic point $[\alpha] \in \mathcal{C}_x$ defines a point in $\mathbb{P} \bigwedge^2 W_x$. The closure of such points will be denoted by $\mathcal{T}_x$ and will be called the variety of tangent lines. The linear span E_x of the homogenization $\tilde{\mathcal{T}}_x \subset \bigwedge^2 W_x$ is then given by $E_x = \mathrm{Span}\{\alpha \wedge \xi : [\alpha] \in \mathcal{C}_x \text{ smooth point}, \xi \in P_\alpha\}$. One of the main results of [Hwang and Mok 1998b] is this:

PROPOSITION 1.2.1. *The distribution W is integrable if $\mathcal{T}_x$ is linearly nondegenerate in $\mathbb{P} \bigwedge^2 W_x$ for a generic point $x \in X$, that is, if $E_x = \bigwedge^2 W_x$ for x generic.*

PROOF. From the nondegeneracy condition, it suffices to check the vanishing of $[\alpha, \xi]$ for $\alpha \wedge \xi \in \bigwedge^2 W_x$, where $[\alpha] \in \mathcal{C}_x$ is a generic point and $\xi \mod \mathbb{C}\alpha$ is tangent to $\mathcal{C}_x$ at $[\alpha]$. By Frobenius' condition, $[\alpha, \xi] = 0$ if we can find a local surface in X passing through x tangent to the distribution W such that the tangent space of the surface at x is generated by α and ξ. By the definition of $\mathcal{C}_x$ and the description of its tangent spaces, we can find a standard minimal rational curve C which is tangent to α at x such that ξ lies in $(\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)_x = P_\alpha$ in the splitting of $T(X)$ over C. Then, we can choose a point $y \neq x$ on C and deform C with y fixed so that the derivative of this deformation is parallel to ξ at x. The locus Σ of this deformation will give an integral surface Σ of W at x so that α and ξ generate $T_x(\Sigma)$. It follows that we can find W-valued vector fields $\tilde{\alpha}, \tilde{\xi}$ in a neighborhood of x which are tangent to $\Sigma, \tilde{\alpha}(x) = \alpha$, $\tilde{\xi}(x) = \xi$. This implies the desired vanishing $[\alpha, \xi] = 0$ at x. □

If W is integrable, it defines a foliation on X outside a proper subvariety $\mathrm{Sing}(W) \subset X$ of codimension at least 2. Any minimal rational curve which is not contained in $\mathrm{Sing}(W)$ is contained in a leaf of W. Pick a generic point x, then the leaf of W containing x can be compactified to a subvariety of X in the following way. Consider the subvariety covered by all minimal rational curves through x. Enlarge this subvariety by adjoining all minimal rational curves through generic points on it. Repeat this adjoining process. This process must stop after a finite number of steps and the resulting enlarged subvariety gives the compactification of the leaf through x [Hwang and Mok 1998b, Proposition 11]. Using this fact, we have the following topological obstruction to the integrability of W.

PROPOSITION 1.2.2. *Let X be a uniruled projective manifold such that $b_2(X)=1$. Suppose a choice of $\mathcal{H}$ can be made so that the generic variety of minimal rational tangents $\mathcal{C}_x$ is linearly degenerate. Then, the distribution W spanned at generic point by $\tilde{\mathcal{C}}_x$ cannot be integrable.*

PROOF. Assuming integrability, compactified leaves of W define a rational fibration of X over a projective variety X' of smaller dimension. The exceptional locus of this fibration is contained in $\mathrm{Sing}(W)$ and of codimension at least 2. But a generic minimal rational curve is disjoint from $\mathrm{Sing}(W)$ [Hwang and Mok 1998b, Proposition 12]. and is contained in a leaf of the fibration. Taking a very ample divisor on X' and pulling it back to X, we find a hypersurface in X disjoint from a generic minimal rational curve. This is a contradiction, since any effective divisor on X is ample as X is of Picard number 1. $\square$

1.3. In this section we consider meromorphic distributions W spanned by varieties of minimal rational tangents, as in Section 1.2, and give sufficient conditions for the integrability of W, in terms of properties of the generic variety of minimal rational tangents $\mathcal{C}_x$.

PROPOSITION 1.3.1. *Suppose the generic variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}W \subset \mathbb{P}T_x$ is irreducible and the second fundamental form $\sigma_{[\alpha]} : T_{[\alpha]}(\mathcal{C}_x) \times T_{[\alpha]}(\mathcal{C}_x) \longrightarrow N_{\mathcal{C}_x|\mathbb{P}W_x,[\alpha]}$ in the sense of projective geometry is surjective at a generic smooth point $[\alpha]$ of $\mathcal{C}_x$. Then, W is integrable.*

PROOF. Let $\alpha \in \tilde{\mathcal{C}}_x$ be a generic point and let $\{\alpha(t) : t \in \mathbb{C}, |t| < 1\}$ be a local holomorphic curve. We write

$$\alpha(t) = \alpha + t\xi + t^2\zeta + O(t^3).$$

Denote by σ_α the second fundamental form $\sigma_\alpha : T_\alpha(\tilde{\mathcal{C}}_x) \times T_\alpha(\tilde{\mathcal{C}}_x) \longrightarrow N_{\tilde{\mathcal{C}}_x|W_x,\alpha}$ in the sense of Euclidean geometry. Then $\sigma_{[\alpha]}$ is surjective if and only if σ_α is surjective. We have $\xi \in P_\alpha$ and $\sigma_\alpha(\xi,\xi) = \zeta \mod P_\alpha$. From now on we will fix a choice of Euclidean metric on W_x and identify the normal space $N_{\tilde{\mathcal{C}}_x|W_x,\alpha}$ with the orthogonal complement $P_\alpha^\perp$ of P_α in W_x. With this convention we may now choose the expansion for $\alpha(t)$ such that $\zeta \in P_\alpha^\perp$, so that $\sigma_\alpha(\xi,\xi) = \zeta$. We fix α and write σ for σ_α. Now

$$\alpha(t)\wedge\alpha'(t) = \big(\alpha + t\xi + t^2\zeta + O(t^3)\big)\wedge(\xi+2t\zeta+O(t^2)) = \alpha\wedge\xi+2t\alpha\wedge\zeta+O(t^2).$$

It follows that $\alpha\wedge\zeta = \alpha\wedge\sigma(\xi,\xi)$ lies on $\mathrm{Span}\{\beta\wedge P_\beta : \beta \in \tilde{\mathcal{C}}_x\} = E_x \subset \bigwedge^2 W_x$. Since σ is symmetric, by polarization we have $\alpha\wedge\sigma(\xi,\eta) \in E$ for any $\xi,\eta \in P_\alpha$. The hypothesis of Proposition 1.3.1 then implies that $\alpha \wedge W_x \in E$ for any $\alpha \in \tilde{\mathcal{C}}_x$. Varying α we conclude that $E_x = \bigwedge^2 W_x$. Since x is a generic point, W is integrable, by Proposition 1.2.1. $\square$

PROPOSITION 1.3.2. *Suppose the generic cone $\mathcal{C}_x \subset \mathbb{P}W_x \subset \mathbb{P}T_x$ is irreducible and smooth and* $\dim(\mathcal{C}_x) > \frac{1}{2}\operatorname{rank}(W) - 1$. *Then W is integrable.*

For the proof of Proposition 1.3.2, we will need Zak's theorem on tangencies in Projective Geometry, for the case of projective submanifolds.

THEOREM 1.3.3 (SPECIAL CASE OF ZAK'S THEOREM ON TANGENCIES [ZAK 1993]). *Let $Z \subset \mathbb{P}^N$ be a k-dimensional complex submanifold and $\mathbb{P}E \subset \mathbb{P}^N$ be a $p-$dimensional projective subspace, $p \geq k$. Then, the set of points on Z at which $\mathbb{P}E$ is tangent to Z is at most of complex dimension $p - k$.*

PROOF OF PROPOSITION 1.3.2. In the proof of Proposition 1.3.1 we use the expansion of a 1-parameter family of minimal rational tangents $\alpha(t)$. In the notations there consider now a 2-dimensional local complex submanifold of minimal rational tangent vectors $\{\alpha(t,s) : t,s \in \mathbb{C}, |t|,|s| < 1\}$. Write $r^2 = |t|^2 + |s|^2$. We have

$$\alpha(t,s) = \alpha + t\xi + s\eta + t^2\sigma(\xi,\xi) + 2ts\sigma(\xi,\eta) + s^2\sigma(\eta,\eta) + O(r^3).$$

Taking partial derivatives we have

$$\begin{aligned}\alpha(t,s) \wedge \partial_t\alpha(t,s) &= \big(\alpha + t\xi + s\eta + O(r^2)\big) \wedge \big(\xi + 2t\sigma(\xi,\xi) + 2s\sigma(\xi,\eta) + O(r^2)\big)\\ &= \alpha \wedge \xi + s\left(\eta \wedge \xi + 2\alpha \wedge \sigma(\xi,\eta)\right) + 2t\big(\alpha \wedge \sigma(\xi,\xi)\big) + O(r^2)\end{aligned}$$

and

$$\partial_s\big(\alpha(t,s) \wedge \partial_t\alpha(t,s)\big)(o) = \eta \wedge \xi + 2\alpha \wedge \sigma(\xi,\eta) \in E_x.$$

From the proof of Proposition 1.3.1 we know that $\alpha \wedge \sigma(\xi,\eta) \in E_x$, from which we conclude that $\xi \wedge \eta \in E_x$, that is, $\bigwedge^2 P_\alpha \subset E_x$ for each $\alpha \in \tilde{\mathcal{C}}_x$. Suppose now $E_x \subsetneqq \bigwedge^2 W_x$. Then, there exists $\mu \in \bigwedge^2 W_x^*$ such that $\mu(e) = 0$ for any $e \in E_x$. It follows that for each α, $P_\alpha \subset W_x$ is an isotropic subspace with respect to μ.

If $\dim(\mathcal{C}_x) > \frac{1}{2}\operatorname{rank}(W) - 1$, then $\dim(\tilde{\mathcal{C}}_x) > \frac{1}{2}\operatorname{rank}(W)$, and any such μ must be degenerate. We claim that this leads to a contradiction. Let Q be the kernel of μ, so that $\mu(\lambda,\xi) = 0$ for any $\lambda \in Q, \xi \in W_x$. Let $\pi : \mathbb{P}W \dashrightarrow \mathbb{P}(W/Q)$ be the linear projection. For x generic $\mathcal{C}_x$ is linearly nondegenerate in W and $\pi\big|_{\mathcal{C}_x}$ is well-defined as a rational map. Consider a point $[\alpha] \in \mathcal{C}_x, \alpha \notin Q$, where $\pi\big|_{\mathcal{C}_x}$ is of maximal rank and $A = \pi([\alpha])$ is a smooth point of the strict transform $\pi(\mathcal{C}_x)$. Write $T_o \subset \mathbb{P}(W/Q)$ for the projective tangent subspace at A, $T_o = \mathbb{P}S_o$. Let $S \subset W_x$ be the linear subspace such that $S \supset Q$ and $S/Q = S_o$. Then, T is tangent to $\mathcal{C}_x$ along the fiber F of $\pi^{-1}(A)$. μ induces a (nondegenerate) symplectic form $\bar{\mu}$ on W_x/Q, with respect to which S_o is isotropic. Hence

$$\dim S_o \leq \tfrac{1}{2}\dim(W_x/Q),$$

so that

$$\dim F = \dim \mathcal{C}_x - \dim \mathbb{P}S_o > \tfrac{1}{2}\dim W_x - \tfrac{1}{2}\dim(W_x/Q) = \tfrac{1}{2}\dim Q.$$

On the other hand, for $T = \mathbb{P}S$, we have

$$\dim T = \dim T_o + \dim Q = (\dim \mathcal{C}_x - \dim F) + \dim Q.$$

Since the projective space T is tangent to $\mathcal{C}_x$ along F, by Zak's Theorem on Tangencies we have

$$\dim F \leq \dim T - \dim \mathcal{C}_x = \dim Q - \dim F,$$

that is, $\dim F \leq \frac{1}{2} \dim Q$, a contradiction. □

1.4. To illustrate the results of Section 1.3, we look at homogeneous contact manifolds of Picard number 1. Recall that a contact structure on a complex manifold X of odd dimension $n = 2m + 1$, is a holomorphic subbundle $D \subset T(X)$ of rank $2m$ such that the Frobenius bracket tensor $\omega : \bigwedge^2 D \to L := T(X)/D$ defines a symplectic form on D_x for each x. A homogeneous contact manifold is a rational homogeneous manifold with a contact structure. According to Boothby's classification [1961], any homogeneous contact manifold is associated to a complex simple Lie algebra in the following way. For a simple Lie algebra $\mathfrak{g}$, the highest weight orbit in $\mathfrak{g}$ under the adjoint representation has a symplectic structure induced by the Lie bracket of $\mathfrak{g}$, so-called Kostant–Kirillov symplectic structure. This induces a contact structure on the projectivization $X \subset \mathbb{P}\mathfrak{g}$ of the highest weight orbit, making it into a homogeneous contact manifold. When $\mathfrak{g}$ is of type A, X is the projectivized cotangent bundle of a projective space and has Picard number 2. When $\mathfrak{g}$ is of type C, X is an odd dimensional projective space regarded as a homogeneous space of the symplectic group. These two cases are not interesting in our study.

We look at a homogeneous contact manifold X associated to an orthogonal or an exceptional simple Lie algebra. In this case, X has Picard number 1 and the line bundle $L = T(X)/D$ is an ample generator of $\mathrm{Pic}(X)$. In fact, L is the $\mathcal{O}(1)$-bundle of the embedding $X \subset \mathbb{P}\mathfrak{g}$. There are lines of $\mathbb{P}\mathfrak{g}$ lying on X and they are minimal rational curves on X. When we represent X as G/P, where G is the adjoint group of $\mathfrak{g}$ and P is the isotropy subgroup at one point $x \in X$, the set of all lines on X is homogeneous under G and the set of all lines through x is homogeneous under P. Since a minimal rational curve is actually a line under the embedding $X \subset \mathbb{P}g$, we see that $\mathcal{C}_x$ is smooth and homogeneous under the isotropy group P.

Let $\theta : T(X) \to L = T(X)/D$ be the quotient map and $\omega : \bigwedge^2 D \to L$ be the Frobenius bracket. Then $\theta \wedge \omega^m, n = 2m+1$, defines a nowhere vanishing section of $K_X \otimes L^{m+1}$. This shows that for a line $C \subset X$, $T(X)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{m-1} \oplus \mathcal{O}^{m+1}$. The symplectic form θ on D induces an isomorphism $D \cong D^* \otimes L$, which gives $D|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{m-1} \oplus \mathcal{O}^{m-1} \oplus \mathcal{O}(-1)$ using $T(X)/D|_C = L|_C = \mathcal{O}(1)$. This shows that the $\mathcal{O}(2)$-component of $T(X)|_C$ is contained in D. Thus $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is linearly degenerate and is contained in $\mathbb{P}D_x$. In fact, the isotropy representation of P on $T_x(X)$ is irreducible on D_x and $\mathcal{C}_x$ must be the projectivization of the orbit of a highest weight vector because $\mathcal{C}_x$ is compact and homogeneous under P.

Since X has Picard number 1 and $\mathcal{C}_x$ is degenerate $W_x = D_x$, we have $E_x \neq \bigwedge^2 D_x$ from Section 1.2. The dimension of $\mathcal{C}_x$ is $m-1$, which is smaller than half the rank of D, as expected from Section 1.3. In fact, here we have the symplectic form ω and we can see that $\tilde{\mathcal{C}}_x$ is Lagrangian with respect to ω, as follows. Given two tangent vectors u, v to $\tilde{\mathcal{C}}_x$ at a point on $T(C)$ for a line C through x, we can extend them to sections $\tilde{u}, \tilde{v}$ of $(\mathcal{O}(2) \oplus [\mathcal{O}(1)]^{m-1})$-part of $T(X)|_C$ vanishing at some points of C by Section 1.1. Then $\tilde{u}, \tilde{v}$ are sections of $D|_C$ and $\omega(\tilde{u} \wedge \tilde{v})$ is a section of L having two zeros. From $L|_C = \mathcal{O}(1)$, we see $\omega(\tilde{u} \wedge \tilde{v}) = 0$. This explains $E_x \neq \bigwedge^2 D_x$, because $E_x \subset \mathrm{Ker}(\omega) \subset \bigwedge^2 D_x$.

1.5. The splitting type of the tangent bundle restricted to a minimal rational curve can be used to get information about principal bundles associated to the tangent bundle. For this purpose, we need the full statement of Grothendieck's splitting theorem [1957]. Let $\mathcal{O}(1)^*$ be the principal $\mathbb{C}^*$-bundle on $\mathbb{P}^1$, which is just the complement of the zero section of $\mathcal{O}(1)$. Given a connected reductive complex Lie group G, choose a maximal algebraic torus $H \subset G$.

THEOREM 1.5.1 [Grothendieck 1957]. *Let $\mathcal{P}$ be a principal G-bundle on $\mathbb{P}^1$. Then there exists an algebraic one-parameter subgroup $\rho : \mathbb{C}^* \to H$ such that $\mathcal{P}$ is equivalent to the G-bundle associated to $\mathcal{O}(1)^*$ via the action ρ. Furthermore, let $\mathcal{V}$ be a vector bundle associated to $\mathcal{G}$ via a representation $\mu : G \to \mathrm{GL}(V)$ on a finite dimensional vector space V. Then $\mathcal{V}$ splits as the direct sum of line bundles $\mathcal{O}(\langle \mu_i, \rho \rangle)$, where $\mu_i : H \to \mathbb{C}^*$ are the weights of μ and $\langle \mu_i, \rho \rangle$ denotes the integral exponent of the homomorphism $\mu_i \circ \rho : \mathbb{C}^* \to \mathbb{C}^*$.*

This theorem can be used in the following situation. We are given a principal G-bundle $\mathcal{P}$ on a uniruled manifold X and an associated vector bundle $\mathcal{V}$ via a representation $\mu : G \to \mathrm{GL}(V)$ on a vector space V. Usually the vector bundle $\mathcal{V}$ is related to the tangent bundle $T(X)$ so that from the splitting $T(X)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ on a standard minimal rational curve C, we have information about the splitting type of $\mathcal{V}|_C$. This information helps us understand $\mathcal{P}$ and μ by Theorem 1.5.1.

Consider the case when $\mathcal{V}$ is the tangent bundle $T(X)$ itself. The natural $\mathrm{GL}(V)$-principal bundle associated to $T(X)$ is the frame bundle $\mathcal{F}$. Here V is an n-dimensional complex vector space. Theorem 1.5.1 applied to $\mathcal{F}$ does not say much. A more interesting case is when there is a reduction of the structure group of the frame bundle, namely when there exists a subgroup $G \subset \mathrm{GL}(V)$ and a G-subbundle $\mathcal{G}$ of $\mathcal{F}$. In this case, we say that X has a G-structure. By Theorem 1.5.1, the splitting type of $T(X)$ on a minimal rational curve gives a nontrivial restriction on the possibility of $G \subset \mathrm{GL}(V)$. One particularly simple case is when G is a connected reductive proper subgroup and the representation $\mu : G \subset \mathrm{GL}(V)$ is irreducible. In this case, we can get a complete classification of uniruled manifolds admitting a G-structure in the following manner [Hwang and Mok 1997].

The key point here is the coincidence of two subvarieties in $\mathbb{P}T_x(X)$ for a generic $x \in X$, which are *a priori* of different nature. On the one hand we have the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$. On the other hand, the G-structure defines the variety of highest weight tangents $\mathcal{W}_x \subset \mathbb{P}T_x$. Indeed, since G is reductive and μ is irreducible, there is a unique highest weight among μ_i with multiplicity one and the orbit of the highest vector in $\mathbb{P}V$ defines a subvariety $\mathcal{W}_x \subset \mathbb{P}T_x(X)$. These two subvarieties $\mathcal{C}_x$ and $\mathcal{W}_x$ coincide. The proof is as follows.

By Theorem 1.5.1, the existence of a unique highest weight vector and the existence of a unique line subbundle of highest degree $\mathcal{O}(2)$ in the splitting of $\mathcal{V} = T(X)$ on a minimal rational curve C imply that the tangent direction of the curve C belongs to the orbit of a highest weight vector. Thus, $\mathcal{C}_x \subset \mathcal{W}_x$. When G is a proper subgroup of $\mathrm{GL}(V)$, we can easily show that $\mathcal{W}_x$ is a proper nondegenerate subvariety of $\mathbb{P}T_x(X)$. To prove that $\mathcal{C}_x = \mathcal{W}_x$, we need to show that $q \leq \operatorname{codim}(\mathcal{W}_x \subset \mathbb{P}T_x(X))$ where $T(X)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. For this, we look at the splitting type of $\operatorname{End}(T(X))$ over a minimal rational curve C:

$$\operatorname{End}(T(X))|_C = [\mathcal{O}(2)]^q \oplus [\mathcal{O}(1)]^{p(q+1)} \oplus \mathcal{O}^{p^2+q^2+1} \oplus [\mathcal{O}(-1)]^{p(q+1)} \oplus [\mathcal{O}(-2)]^q.$$

From the reductivity of G, we have a direct sum decomposition of the Lie algebra $\mathfrak{g}l(V) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$ with respect to the trace form. This induces a decomposition $\operatorname{End}(T(X)) = U \oplus U^{\perp}$. At a generic point $x \in C$, the $\mathcal{O}(2)_x$-factor in $\operatorname{End}(T(X))_x$ corresponds to endomorphisms with image in $\mathbb{C}\alpha = T_x(C)$. Suppose the bundle $U|_C$ corresponding to $\mathfrak{g}$ contains an $\mathcal{O}(2)$-factor. Then for any $\gamma \in \mathcal{W}_x$, the tangent space $\mathfrak{g}\gamma$ to $\mathcal{W}_x$ contains the point $\alpha \in \mathcal{W}_x$. This is a contradiction to the nondegeneracy of $\mathcal{W}_x$. It follows that all $\mathcal{O}(2)$-factors are in $U^{\perp}$. From the orthogonality of $\mathfrak{g}$ and $\mathfrak{g}^{\perp}$, endomorphisms in $\mathfrak{g}^{\perp}$ which have images in $\mathbb{C}\alpha$ must annihilate $\mathfrak{g}\alpha$, the tangent space to $\mathcal{W}_x$ at α. This means that $[\mathcal{O}(2)]^q_x$ annihilates the tangent space to $\mathcal{W}_x$ at α, implying $q \leq \operatorname{codim}(\mathcal{W}_x \subset \mathbb{P}T_x(X))$.

Now we identify $\mathcal{C}_x = \mathcal{W}_x$ for a generic $x \in X$. $\rho : \mathbb{C}^* \to G$ in Theorem 1.5.1 tells us that for each $\alpha \in \mathcal{C}_x$, there exists a $\mathbb{C}^*$-action on $T_x(X)$ under which $T_x(X)$ decomposes as $\mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$ where $t \in \mathbb{C}^*$ acts as t^2 on $\mathbb{C}\alpha$, as t on $\mathcal{H}_\alpha$, and as 1 on $\mathcal{N}_\alpha$. By rescaling, we get a $\mathbb{C}^*$-action on $\mathcal{V}_x$ preserving $\mathcal{C}_x$ which fixes $\mathbb{C}\alpha$, acts as t on $\mathcal{H}_\alpha$, and as t^2 on $\mathcal{N}_\alpha$. Moreover, $\mathbb{C}\alpha \oplus \mathcal{H}_\alpha$ corresponds to the tangent space of $\mathcal{C}_x$ at α. This fact has an interesting implication on $\mathcal{T}_x$, that the linear span E_x of $\mathcal{T}_x$ contains $\alpha \wedge \mathcal{H}_\alpha$ and $\alpha \wedge \mathcal{N}_\alpha$ for all $\alpha \in \mathcal{C}_x$. In fact, Choose a generic point $\alpha + \xi + \zeta$ on $\mathcal{C}_x$. The orbit of the $\mathbb{C}^*$-action is $\alpha + t\xi + t^2\zeta$. At $t = t_0$, we further consider the curve $\alpha + e^s t_0 \xi + e^{2s} t_0^2 \zeta$. Taking derivative with respect to s, we get the tangent vector $t_0\xi + 2t_0^2\zeta$ to $\mathcal{C}_x$ at the point $\alpha + t_0\xi + t_0^2\zeta$. The corresponding element of $\mathcal{T}_x$ is

$$(\alpha + t_0\xi + t_0^2\zeta) \wedge (t_0\xi + 2t_0^2\zeta) = t_0\alpha \wedge \xi + 2t_0^2\alpha \wedge \zeta + t_0^3\xi \wedge \zeta.$$

It follows that the linear span of $\mathcal{T}_x$ contains vectors of the form $\alpha \wedge \xi, \alpha \wedge \zeta$. As ξ takes values in the tangent space of $\mathcal{C}_x$ at α, ζ takes independent values in $\mathcal{N}_\alpha$. Thus $\mathcal{T}_x$ contains $\alpha \wedge \mathcal{H}_\alpha$ and $\alpha \wedge \mathcal{N}_\alpha$. This implies that $E_x = \bigwedge^2 T_x(X)$.

Moreover, putting $t = -1$ in the $\mathbb{C}^*$-action on $\mathcal{C}_x$ considered above, we see that $\mathcal{C}_x$ is a Hermitian symmetric space of rank at least 2. It is not hard to see from this that on a uniruled manifold, if an irreducible reductive G-structure with $G \neq \mathrm{GL}(n, \mathbb{C})$ is given, then $G = K^{\mathbb{C}}$ where K is the isotropy subgroup of the isometry group of an irreducible Hermitian symmetric space S of the compact type of rank at least 2. Such a G-structure will be called an S-structure.

A G-structure $\mathcal{G}$ is flat if there exists a local coordinate system whose coordinate frames belong to $\mathcal{G}$ regarded as a subbundle of the frame bundle. The following result of Ochiai will be proved in Section 3.2 below.

Proposition 1.5.2 [Ochiai 1970]. *Let S be an irreducible Hermitian symmetric space of the compact type and of rank at least* 2. *Let M be a compact simply-connected complex manifold with a flat S-structure. Then*, *M is biholomorphic to S.*

The flatness of an S-structure is equivalent to the vanishing of certain holomorphic tensors, just as the flatness of a Riemannian metric is equivalent to the vanishing of the Riemannian curvature tensor in Riemannian geometry. By restricting these tensors to minimal rational curves and considering the splitting type, it is easy to show the vanishing of these tensors from $E_x = \bigwedge^2 T_x(X)$; see [Hwang and Mok 1997] for details. As a result:

Theorem 1.5.3 [Hwang and Mok 1997]. *A uniruled projective manifold with an irreducible reductive G-structure*, $G \neq \mathrm{GL}(V)$, *is an irreducible Hermitian symmetric space of the compact type.*

This theorem gives an algebro-geometric characterization of irreducible Hermitian symmetric spaces of the compact type without the assumption of homogeneity. For example, Grassmannians of rank at least 2 can be characterized as the only uniruled manifolds whose tangent bundle can be written as the tensor product of two vector bundles of rank at least 2.

1.6. To construct a reasonable moduli space of vector bundles on projective manifolds, we have to restrict ourselves to a special class of vector bundles, called semistable bundles. On a Fano manifold X of Picard number 1, they can be defined as follows. Fix a component $\mathcal{K}$ of the Chow space of rational curves on X, so that a generic member is a free rational curve. Given a torsion-free sheaf $\mathcal{F}$ on X choose a generic member C of $\mathcal{K}$ so that $\mathcal{F}|_C$ is locally free. We can always make such a choice because the singular loci of a torsion-free sheaf has codimension at least 2 (see [Hwang and Mok 1998b, Proposition 12], for example). Let $\mathcal{F}|_C = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_k), a_1 \geq \cdots \geq a_k$, be the splitting. We define the slope of $\mathcal{F}$ as the rational number $\mu(\mathcal{F}) := \sum a_i/k$. A vector bundle

V of rank r on X is if stable $\mu(\mathcal{F}) < \mu(V)$ for every subsheaf $\mathcal{F}$ of rank k, with $0 < k < r$. It is stable if $\mu(\mathcal{F}) \leq \mu(V)$ for every such $\mathcal{F}$.

A well-known problem in Kähler geometry of Fano manifolds is the Calabi problem on the existence of Kähler–Einstein metrics. The semistability of the tangent bundle is a necessary condition for the existence of Kähler–Einstein metric. As a result, people have been interested in the stability or the semistability of $T(X)$ for a Fano manifold X with Picard number 1. By taking wedge product, the instability of $T(X)$, or equivalently the instability of $T^*(X)$, implies the nonvanishing of $H^0(X, \Omega^r(k))$ for certain r, k. Thus one sufficient condition for the stability of $T(X)$ is the vanishing of these cohomology groups. Peternell and Wiśniewski [1995] proved the stability of $T(X)$ for many examples of X, including all X of dimension at most 4 by using this idea. It looks hard to generalize this method to higher dimensions.

This problem can be studied by relating it to the geometry of $\mathcal{C}_x$ for a generic $x \in X$ [Hwang 1998]. Choose $\mathcal{H}$ as in Section 1.1 and the corresponding Chow space $\mathcal{K}$. Suppose $T(X)$ is not stable. Choose a subsheaf $F \subset T(X)$ with maximal value of $\mu(F) \geq \mu(T(X)) = \frac{p+2}{n}$. The maximality of $\mu(F)$ implies the minimality of $\mu(T(X)/F)$, and the vanishing of the Frobenius bracket tensor $\bigwedge^2 F \to T(X)/F$. Thus, F defines a meromorphic foliation on X. If $\mathcal{C}_x$ is contained in F_x for a generic $x \in X$, we can get a contradiction to the Picard number of X as in Proposition 1.2.2. Thus $\mathbb{P}F_x \subset \mathbb{P}T_x(X)$ is a linear subspace which does not contain $\mathcal{C}_x$. On the other hand, the condition that $\mu(F) \geq \frac{p+2}{n}$ reads $\sum a_i \geq \frac{r(p+2)}{n}$, where $F|_C = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ for a generic minimal rational curve C. This can be rephrased as "the intersection of $F|_C$ with the positive part of $T(X)|_C$ has larger dimension than the one expected from the rank of F". Since the positive part of $T(X)|_C$ corresponds to the tangent space to $\mathcal{C}_x$, the assumption that $T(X)$ is not stable implies that the tangent spaces of $\mathcal{C}_x$ have an excessive intersection with a linear subspace $\mathbb{P}F_x$. More precisely:

Proposition 1.6.1. *Suppose that $T(X)$ is not stable. Then, there exists an integrable meromorphic distribution $F \subset T(X)$ of rank r so that for a generic $x \in X$, the intersection of the projective tangent space at a generic point of $\mathcal{C}_x$ with $\mathbb{P}F_x$ has dimension greater than $\frac{r}{n}(p+2) - 2$.*

Thus by studying the projective geometry of $\mathcal{C}_x$, we can prove the stability of $T(X)$. Although very little is known about the geometry of $\mathcal{C}_x$ in general, there are many cases where Proposition 1.6.1 can be used to show the stability of $T(X)$. For example, in low dimension, the excessive intersection property gives a heavy restriction on $\mathcal{C}_x$ which gives a contradiction easily. The main results of [Hwang 1998] that Fano 5-folds with Picard number 1 have stable tangent bundles and Fano 6-folds with Picard number 1 have semistable tangent bundles were obtained this way.

Another interesting case is when we know that $\mathcal{C}_x$ is smooth and of small codimension in $\mathbb{P}T_x(X)$.

Theorem 1.6.2. *Assume that $\mathcal{C}_x$ is an irreducible submanifold of $\mathbb{P}T_x(X)$ for a generic $x \in X$. If $2p \geq n-2$, then $T(X)$ is stable.*

From the discussion in Section 1.3, our assumption shows that $\mathcal{C}_x$ is nondegenerate in $\mathbb{P}T_x(X)$. Theorem 1.6.2 follows directly from Proposition 1.6.1 using the following lemma, which is a variation of Zak's theorem on tangencies just as in the proof of Proposition 1.3.2.

Lemma 1.6.3. *Let $Y \subset \mathbb{P}^{n-1}$ be a nondegenerate irreducible subvariety of dimension $p \geq \frac{n-2}{2}$. If there exists a linear subspace $E \subset \mathbb{P}^{n-1}$ of dimension $r-1$ such that for a generic point $y \in Y$, the projective tangent space to Y at y intersects E on a subspace of dimension $q-1$ for $q \geq \frac{r}{n}(p+2)$, then Y is not smooth.*

As a corollary of Theorem 1.6.2, we get the stability of $T(X)$ in the following cases, most of which have not been proved previously:

- smooth linear sections of codimension 1 or 2 of Grassmannians of rank 2;
- smooth hyperplane sections of Grassmannian of rank 3 of dimension 9;
- smooth linear sections of dimension at least 10 of the 16-dimensional E_6 symmetric space;
- smooth linear sections of dimension at least 20 of the 27-dimensional E_7 symmetric space.

2. Deformation Rigidity of Irreducible Hermitian Symmetric Spaces and Homogeneous Contact Manifolds

2.1. We propose to study deformation of certain Fano manifolds of Picard number 1 by considering deformations of their bundles of varieties of minimal rational tangents and distributions spanned by them. As a first step we deal with irreducible Hermitian symmetric spaces of the compact type, and proved in [Hwang and Mok 1998b] their rigidity under Kähler deformation, as follows.

Theorem 2.1.1. *Let S be an irreducible Hermitian symmetric space of the compact type. Let $\pi : \mathcal{X} \to \triangle$ be a regular family of compact complex manifolds over the unit disk $\triangle$. Suppose $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$ and the central fiber X_0 is Kähler. Then, X_0 is also biholomorphic to S.*

The case of $S \cong \mathbb{P}^n$ being a consequence of the classical result of Hirzebruch and Kodaira [1957], we restrict ourselves to S of rank at least 2. (The case of the hyperquadric also follows from [Brieskorn 1964].) S is associated to holomorphic G-structures, as explained in Section 1.5. On S the varieties of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ are highest weight orbits of isotropy representations as discussed in Section 1.5. They turn out to be themselves Hermitian symmetric manifolds of the compact type of rank 1 or 2, and irreducible except in the case of Grassmannians $G(p,q)$ of rank at least 2, where $\mathcal{C}_x$ is isomorphic to $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, embedded in $\mathbb{P}^{pq-1}$ by the Segre embedding.

Under the hypothesis of Theorem 2.1.1 we proceed to prove that the central fiber $X_0 \cong S$. Denote by T the relative tangent bundle of $\pi : \mathcal{X} \to \triangle$. Pick some $t_o \neq 0$ and a minimal rational curve C on X_{t_o}. By considering the deformation of C as a curve in $\mathcal{X}$, we obtain a subvariety $\mathcal{C} \subset \mathbb{P}T$ such that for a point y on X_t; $t \neq 0$; the fiber $\mathcal{C}_y \subset \mathbb{P}T_y$ is isomorphic to the embedded standard variety of minimal rational tangents $\mathcal{C}_o \subset \mathbb{P}T_o(S)$. At a generic point x of the central fiber every minimal rational curve is free and the normalized Chow space $\mathcal{M}_x$ of such curves marked at x is projective and nonsingular. Over such a point $\mathcal{C}_x$ is the same as the variety of minimal rational tangents. Here we make use of the hypothesis that X_0 is Kähler, which implies that the deformations of C situated on X_0 are irreducible and of degree 1 with respect to the positive generator of $\mathrm{Pic}(X_0) \cong \mathbb{Z}$.

Suppose we are able to prove that, for a generic point x on X_0,

(A) $\mathcal{M}_x \cong \mathcal{C}_o$ and
(B) $\mathcal{C}_x \subset \mathbb{P}T_x$ is linearly nondegenerate.

Then, it follows readily that the latter is isomorphic to the model $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as projective submanifolds. From this one can readily recover a holomorphic S-structure on the complement of some subvariety E of X_0. As a subvariety of $\mathcal{X}$, E is of codimension at least 2, and a Hartogs-type extension theorem resulting from [Matsushima and Morimoto 1960] allows us to obtain a holomorphic S-structure on X_0. Since flatness of holomorphic G-structures is a closed condition, the S-structure on X_0 is flat, leading to a biholomorphism $X_0 \cong S$, as a consequence of Ochiai's theorem, Proposition 1.5.2.

(A) can be established by induction except for the case of $G(p,q)$; $p,q > 1$ where $\mathcal{C}_x \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$. In the abstract case it is possible to have nontrivial deformation of $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, as exemplified by the deformation of $\mathbb{P}^1 \times \mathbb{P}^1$ to Hirzebrach surfaces. However, in our situation the individual factors $\mathbb{P}^{p-1}$ and $\mathbb{P}^{q-1}$ correspond to projective spaces $\cong \mathbb{P}^p$ and $\mathbb{P}^q$ (respectively) of degree 1 in S. By cohomological considerations we prove that limits of such projective spaces cannot decompose in the central fiber, thus establishing (A) even in the special case of $S = G(p,q)$, with $p,q > 1$. We refer the reader to [Hwang and Mok 1998b, § 3] for details.

2.2. It remains now to prove (B) that on the central fiber X_0, generic varieties of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x$ are linearly nondegenerate. From (A) we can identify $\mathcal{M}_x$ with $\mathcal{C}_o \subset \mathbb{P}T_o(S)$, and realize the tangent map $\Phi_x : \mathcal{M}_x \to \mathcal{C}_x$ as the restriction of a linear projection $\mathbb{P}T_o(S) \to \mathbb{P}W_x$, where $\mathbb{P}W_x$ is the projective-linear span of $\mathcal{C}_x$. We proceed to prove (B) by contradiction. The assignment of W_x to each generic x on X_0 defines a (meromorphic) distribution on X_0. Since X_0 is of Picard number 1 by Proposition 1.2.2 if $W_x \neq T_x$ at generic points the distribution W is not integrable. On the other hand, by Proposition 1.2.1, W is integrable whenever the variety of tangential lines $\mathcal{T}_x \subset \mathbb{P}\bigwedge^2 W_x$

is linearly nondegenerate at generic points x. To prove (B) by contradiction it remains therefore only to check the condition of linear nondegeneracy on $\mathcal{T}_x \subset \mathbb{P} \bigwedge^2 T_x$. As explained in Section 1.5, $\mathcal{T}_o \subset \mathbb{P} \bigwedge^2 T_o(S)$ is nondegenerate for the model $\mathcal{C}_o$. Since the tangent map $\Phi_x : \mathcal{M}_x \to \mathcal{C}_x$; $\mathcal{M}_x \cong \mathcal{C}_o$; is the restriction of a linear projection $\mathbb{P}T_o(S) \to \mathbb{P}W_x$, it follows that $\mathcal{T}_x \subset \mathbb{P} \bigwedge^2 W_x$ is linearly nondegenerate, as desired.

2.3. Deformation rigidity can be asked for other rational homogeneous spaces, too.

CONJECTURE 2.3.1. *Let S be a rational homogeneous space with Picard number 1. Let $\pi : \mathcal{X} \to \Delta$ be a regular family of compact complex manifolds with $X_t \cong S$ for all $t \neq 0$. If X_0 is Kähler, $X_0 \cong S$.*

We expect that the method of Sections 2.1 and 2.2 can be generalized to a general S, although the details will not be straightforward. When S is a homogeneous contact manifold, this was done in [Hwang 1997]. We will sketch the main ideas here.

Let S be a homogeneous contact manifold of dimension $n = 2m + 1$ associated to an orthogonal or an exceptional simple Lie algebra as in Section 1.4. We consider $\pi : \mathcal{X} \to \Delta$ as in Conjecture 2.3.1. There exists a distribution D of rank $2m$ on $\mathcal{X}$ which may have singularity on X_0 and gives the contact distribution on $X_t, t \neq 0$. Just as the deformation rigidity of Hermitian symmetric spaces was obtained by the recovery of the S-structure at a generic point of X_0, the deformation rigidity of homogeneous contact manifolds can be obtained by showing that D defines a contact structure at a generic point of X_0. Here, in place of Ochiai's result, we can just look at the Kodaira–Spencer class [LeBrun 1988].

By the Lagrangian property of $\mathcal{C}_s$, for $s \in S$, discussed in Section 1.4, $\mathcal{T}_s$ is contained in the kernel of

$$\omega : \textstyle\bigwedge^2 D_s \to L_s.$$

Now the key point of the proof of the deformation rigidity of homogeneous contact manifold is the fact that $\mathcal{T}_s$ is nondegenerate in $\mathbb{P}\,\mathrm{Ker}(\omega) \subset \mathbb{P} \bigwedge^2 D_s$. This can be checked case by case as in [Hwang 1997, Section 2]. In other words, the symplectic structure on D_s is completely determined by $\mathcal{C}_s$.

To prove the deformation rigidity, we argue as in Sections 2.1 and 2.2. Let $x \in X_0$ be a generic point and choose a section $\sigma : \Delta \to \mathcal{X}$ with $\sigma(0) = x$. If the family $\mathcal{C}_{\sigma(t)} \subset \mathbb{P}T_{\sigma(t)}(X_t)$ remains unchanged as projective subvarieties as $t \to 0$, then the linear span of $\mathcal{T}_{\sigma(t)}$ is isomorphic to $\mathrm{Ker}(\omega)$ of S. This implies that D defines a contact structure at x and we are done. As in Sections 2.1 and 2.2, an induction argument reduces the proof of the rigidity of $\mathcal{C}_{\sigma(t)}$ to showing that $\mathcal{C}_x$ is linearly nondegenerate in D_x. But linear degeneracy would imply the integrability of the distribution D using the linear nondegeneracy of $\mathcal{T}_s$ in $\mathrm{Ker}(\omega)$, just as in Section 2.2. Integrability of D gives a contradiction to Section 1.2, thus $\mathcal{C}_{\sigma(t)}$ is rigid.

3. Tautological Foliations on Varieties of Minimal Rational Tangents and the Method of Analytic Continuation

3.1. In this section, we consider the following problem. For two Fano manifolds X_1, X_2 with bundles of varieties of minimal rational tangents $\mathcal{C}_1 \subset \mathbb{P}T(X_1), \mathcal{C}_2 \subset \mathbb{P}T(X_2)$ and a biholomorphism f from an open subset $U_1 \subset X_1$ to an open subset $U_2 \subset X_2$ satisfying $df(\mathcal{C}_1) = \mathcal{C}_2$, when can we say that f sends the intersection of a minimal rational curve with U_1 to the intersection of a minimal rational curve with U_2. In other words, when does $\mathcal{C} \subset \mathbb{P}T(X)$ determine the minimal rational curves locally? One sufficient condition is that $\mathcal{C}_x$ has generically finite Gauss map as a projective subvariety of $\mathbb{P}T_x(X)$ for generic x and U_1, U_2 are sufficiently generic. This is contained in Corollary 3.1.4 below.

We start with some notations. In this section, we will view a distribution D on a manifold as a subsheaf of the tangent sheaf. We will be always looking at generic points of the manifold where all distributions concerned are locally free, and we regard the distribution as a subbundle of the tangent bundle at such points. We will not make notational distinction between sheaves and bundles in this case. Given a distribution D on a manifold, the derived system of D is the distribution $\partial D := D + [D, D]$, and its Cauchy characteristic $\mathrm{Ch}(D)$ is the distribution defined by $\mathrm{Ch}(D)(U) := \{f \in D(U), [f, g] \in D(U), \forall g \in D(U)\}$, for any open subset U of the manifold. $\mathrm{Ch}(D)$ is always integrable.

Let X be any complex manifold and $U \subset X$ be a sufficiently general small open set. Given any subvariety $\mathcal{C} \subset \mathbb{P}T(U)$, we consider two distributions $\mathcal{J}$ and $\mathcal{P}$ on the smooth part of $\mathcal{C}$ defined by

$$\mathcal{J}_\alpha := (d\pi)^{-1}(\mathbb{C}\alpha),$$
$$\mathcal{P}_\alpha := (d\pi)^{-1}(P_\alpha),$$

where $d\pi : T_\alpha(\mathcal{C}) \to T_x(U)$ is the differential of the natural projection $\pi : \mathcal{C} \to U$ at $\alpha \in \mathcal{C}$, $x = \pi(\alpha)$, and $P_\alpha \subset T_x(X)$ is the linear tangent space of $\mathcal{C}_x := \pi^{-1}(x) \subset \mathbb{P}T_x(X)$ at α. Both $\mathcal{J}$ and $\mathcal{P}$ are canonically determined by $\mathcal{C}$. $\mathcal{J}$ has rank $p+1$ and $\mathcal{P}$ has rank $2p+1$, where p is the fiber dimension of $\pi : \mathcal{C} \to U$. Also we have the trivial vertical distribution $\mathcal{V}$ of rank p on $\mathcal{C}$ defining the fibers of π. Clearly, $\mathcal{V} \subset \mathcal{J} \subset \mathcal{P}$.

Now assume that X is a uniruled projective manifold and $\mathcal{C}$ is part of the variety of minimal rational tangents on X. Then we have a meromorphic multi-valued foliation $\mathcal{F}$ on $\mathcal{C}$ defined by lifting minimal rational curves. When we work on a small open set of $\mathcal{C}$, we may assume $\mathcal{F}$ is a foliation by curves on that open set, by choosing a specific branch of the multi-valued foliation. We call $\mathcal{F}$ the tautological foliation. Since $\mathcal{F}$ is defined by lifting curves, $\mathcal{J} = \mathcal{V} + \mathcal{F}$ at generic points of $\mathcal{C}$.

Proposition 3.1.1. $\mathcal{P} = \partial \mathcal{J}$.

Proof. For notational simplicity, we will work over $\Gamma = T(X) \setminus (0\text{-section})$. Let $\gamma : \Gamma \to \mathbb{P}T(X)$ be the natural $\mathbb{C}^*$-bundle. Let $\mathcal{C}' := d\gamma^{-1}(\mathcal{C}), \mathcal{J}' := d\gamma^{-1}\mathcal{J}, \mathcal{P}' := d\gamma^{-1}\mathcal{P}, \mathcal{V}' := d\gamma^{-1}\mathcal{V}$, and $\mathcal{F}' := d\gamma^{-1}\mathcal{F}$. It suffices to check that $\mathcal{P}' = \partial\mathcal{J}'$.

We start with $\partial\mathcal{J}' \subset \mathcal{P}'$. It suffices to show $[\mathcal{V}', \mathcal{F}'] \subset \mathcal{P}'$. Let $x_1, \ldots, x_n$ be a local coordinate system on U. Let $v_1 = dx_1, \ldots, v_n = dx_n$ be linear coordinates in the vertical direction of Γ. Let $v = \sum_i e^i \frac{\partial}{\partial v_i}$ be a local section of $\mathcal{V}'$ and $f = \sum_i f^i \frac{\partial}{\partial v_i} + \zeta \sum_j v_j \frac{\partial}{\partial x_j}$ be a local section of $\mathcal{F}'$ over a small open set in $\mathcal{C}'$. Here e^i, f^i, ζ are suitable local holomorphic functions. Dividing by ζ and looking at generic points outside the zero set of ζ, we may assume that $\zeta \equiv 1$. Then $[v, f] = \sum_i e^i \frac{\partial}{\partial x_i}$ modulo $\mathcal{V}'$. But this is precisely the vector v viewed as the tangent vector to X. Hence $[v, f] \in \mathcal{P}'$.

From the above expression of $[v, f]$ modulo $\mathcal{V}'$, we see that the rank of $\partial\mathcal{J}'$ is higher than the rank of $\mathcal{J}'$ by at least p, which shows $\partial\mathcal{J}' = \mathcal{P}'$. $\square$

Proposition 3.1.2. $\mathcal{F} \subset \mathrm{Ch}(\mathcal{P})$.

Proof. Let $\mathcal{D}$ be the Chow space parametrizing minimal rational curves. Let $U \subset \mathcal{C}$ be a sufficiently generic small open set. Locally, we have a morphism $\rho : U \to \mathcal{D}$ whose fibers are leaves of $\mathcal{F}$. In a neighborhood of a generic point $[C_0] \in \mathcal{D}$ corresponding to a minimal rational curve C_0, we have a distribution $\hat{\mathcal{P}}$ on $\mathcal{D}$ defined as follows. Note that the tangent space to $\mathcal{D}$ at $[C]$ near $[C_0]$ is naturally isomorphic to $H^0(C, N_C)$ where N_C is the normal bundle of C in X. We know that $N_C \cong [\mathcal{O}(1)]^p \oplus [\mathcal{O}]^q$. Let $\hat{\mathcal{P}}_{[C]}$ be the subspace of $H^0(C, N_C)$ consisting of sections of $[\mathcal{O}(1)]^p$-part. This gives a distribution $\hat{\mathcal{P}}$ in a neighborhood of $[C_0]$. From Section 1.1, $\mathcal{P} = d\gamma^{-1}\hat{\mathcal{P}}$. So the result follows from the following easy lemma. $\square$

Lemma 3.1.3. *Let $f : W \to Y$ be a submersion between two complex manifolds and $\mathcal{N}$ be a distribution on Y. Let $\mathcal{K}$ be the distribution defined by the fibers of f. Then $\mathcal{K}$ is contained in the Cauchy characteristic of the distribution $df^{-1}\mathcal{N}$.*

Theorem 3.1.4. *Suppose that a component of $\mathcal{C}_x$ has generically finite Gauss map regarded as a projective subvariety in $\mathbb{P}T_x(X)$. Then $\mathcal{F} = \mathrm{Ch}(\mathcal{P})$.*

Proof. Suppose that for some $v \in \mathcal{V}(U)$, $h \in \mathcal{J}(U)$ and $f \in \mathcal{F}(U)$, we have $[v, f] + h \in \mathrm{Ch}(\mathcal{P})(U)$. Then it is easy to see that $v \in \mathrm{Ch}(\mathcal{P})(U)$. In fact, to show $[v, p] \in \mathcal{P}(U)$ for any $p \in \mathcal{P}(U)$, it suffices to show $[v, [w, f]] \in \mathcal{P}(U)$ for any $w \in \mathcal{V}(U)$, by using Proposition 3.1.1. From $[v, [w, f]] = [w, [v, f]] + [f, [w, v]]$ and $[h, w] \in \mathcal{P}(U)$, we see that $[v, [w, f]] \in \mathcal{P}(U)$.

So it suffices to show that there is no nonzero $v \in \mathcal{V}(U) \cap \mathrm{Ch}(\mathcal{P})(U)$. We will work on Γ as in the proof of Proposition 3.1.1. We need to show that any $v \in \mathcal{V}'(\rho^{-1}(U)) \cap \mathrm{Ch}(\mathcal{P}')(\rho^{-1}(U))$ is tangent to a fiber of γ. Suppose there exists such a $v = \sum_i a_i \frac{\partial}{\partial v_i}$. Then $[v, f] \in \mathrm{Ch}(\mathcal{P}')$ where we used the same letter f to denote the section of $\mathcal{F}'$ lifting a section f of $\mathcal{F}$. For any section k of $\mathcal{V}'$, $k = \sum_i e_i \frac{\partial}{\partial v_i}$, we have $[[v, f], k] \in \mathcal{P}'$. Modulo vertical part, $[[v, f], k] = \sum_i k(a_i) \frac{\partial}{\partial x_i}$.

For this to be a section of $\mathcal{P}'$, $\sum_i k(a_i)\frac{\partial}{\partial v_i}$ must be a section of $\mathcal{V}'$. Since $[v,k]$ is a section of $\mathcal{V}'$, we conclude that $\sum_i v(e_i)\frac{\partial}{\partial v_i}$ is a section of $\mathcal{V}'$. In other words, for any vector field $k = \sum_i e_i \frac{\partial}{\partial v_i}$ tangent to $\mathcal{C}'_x$, $\sum_i v(e_i)\frac{\partial}{\partial v_i}$ remains tangent to $\mathcal{C}'_x$. This implies that v is in the kernel of the differential of the Gauss map of $\mathcal{C}'_x$. By assumption on the Gauss map of $\mathcal{C}_x$, such v must be tangent to the fibers of γ on $\mathcal{C}'$. □

Corollary 3.1.5. *Under the assumption of Theorem* 3.1.4 *on* $\mathcal{C}$, $\mathcal{F}$ *is a single-valued meromorphic foliation on* $\mathcal{C}$ *uniquely determined by* $\mathcal{C}$. *In particular*, *the tangent map* $\Phi_x : \mathcal{M}_x \to \mathcal{C}_x$ *is birational.*

We note that the Gauss map on $\mathcal{C}_x$ is generically finite, whenever the latter is irreducible, nonsingular and distinct from the projective space, by Zak's Theorem on tangencies [Zak 1993]. This is in particular the case at generic points for Fano complete intersections $X \subset \mathbb{P}^n$ of dimension at least 3 provided that $c_1(X) \geq 3$ (X being of Picard number 1); see [Kollár 1996, Section V.4].

3.2. In Section 1.5 we have stated in Proposition 1.5.2 the result of Ochiai's characterizing irreducible Hermitian symmetric spaces S of the compact type and of rank at least 2 in terms of flat S-structures. We will prove the result here, which follows from this proposition:

Proposition 3.2.1. *Let* S *be an irreducible Hermitian symmetric manifold of the compact type and of rank at least* 2. *Denote by* $\mathcal{C} \to S$ *the bundle of varieties of minimal rational tangents. Let* U_1, $U_2 \subset S$ *be two connected open sets and* $f_{12} : U_1 \to U_2$ *be a biholomorphism such that* $(f_{12})_*\mathcal{C}|_{U_1} = \mathcal{C}|_{U_2}$. *Then*, f_{12} *extends to a biholomorphic automorphism of* S.

Proposition 3.2.1 implies Proposition 1.5.2, as follows. Since the S-structure on M is flat, given any $x \in M$ there exists a neighborhood U_x of x and a biholomorphism $f : U_x \to S$ of U_x onto some open subset U of S such that $f_*\mathcal{W}|_{U_x} = \mathcal{C}|_U$, where $\mathcal{W} \to M$ is the bundle of varieties of highest weight tangents defined by the S-structures. Starting with one choice of x and f, Proposition 3.2.1 allows us to continue f holomorphically along any continuous curve, by matching different f_y on U_y on intersecting regions using global automorphisms. This leads to a developing map, which is well-defined on M since M is simply connected. The resulting unramified holomorphic map $F : M \to S$ is necessarily a biholomorphism, since S is simply connected.

Ochiai's original proof of 3.2.1 [1970] used harmonic theory of Lie algebra cohomologies. We will give an alternate proof of Proposition 3.2.1, by making use of analytic continuation. The bundle $\pi : \mathcal{C} \to S$ of varieties of minimal rational tangents is equipped with a one-dimensional foliation $\mathcal{F}$, as in Section 3.1. Recall that for $[\alpha] \in \mathcal{C}_x$, $T_{[\alpha]}\mathcal{C}_x = P_\alpha \mod \mathbb{C}\alpha$, by definition. Since f_{12} preserves $\mathcal{C}$, it also preserves $\mathcal{P}_\alpha = (d\pi)^{-1}(P_\alpha)$. By Theorem 3.1.4, $(f_{12})_*\mathcal{F} = \mathrm{Ch}((f_{12})_*\mathcal{P}) = \mathrm{Ch}(\mathcal{P}) = \mathcal{F}$. In other words, f_{12} preserves the holomorphic foliation $\mathcal{F}$, that is, where defined f_{12} sends open sets on lines to open sets on lines.

To explain our approach note that the problem of analytic continuation of $\mathcal{F}$-preserving germs of holomorphic maps also makes sense for the case of $\mathbb{P}^n$. For $n = 2$ and denoting by $B^2 \subset \mathbb{C}^2 \subset \mathbb{P}^2$ the unit ball, any such map $f : B^2 \to \mathbb{P}^2$ defines a holomorphic mapping $f^\#$ on some open subset $D \subset (\mathbb{P}^2)^*$ of the dual projective space, where D is the open set of all lines having nonempty (and automatically connected) intersection with B^2. In this case D is the complement of a closed Euclidean ball in $(\mathbb{P}^2)^*$, and $f^\#$ extends holomorphically to $(\mathbb{P}^2)^*$ by Hartogs extension, from which we can recover an extension of f from B^2 to $\mathbb{P}^2$, by regarding a point $x \in \mathbb{P}^2$ as the intersection of all lines passing through x. In place of implementing this argument in the case of irreducible Hermitian symmetric spaces, we will adopt here a related but more direct argument, by analytically continuing along lines. This approach was adopted in [Mok and Tsai 1992] in a similar context, but the argument there was incomplete due to the possibility of multivalence of analytically continued functions. We will complete the argument by making use of $\mathbb{C}^*$-actions on S.

We start with a lemma concerning analytic continuation along chains of lines. By a chain of lines K on S we mean the union of a finite number of distinct lines $C_1, \dots, C_m$ such that $C_j \cap C_{j+1}$ is a single point for $1 \le j < m - 1$. We write $K = C_1 + C_2 + \cdots + C_m$. We will say that K is nonoverlapping to mean $C_j \cap C_k = \varnothing$ whenever $|j - k| \ge 2$. We will more generally be dealing with $\mathcal{F}$-preserving meromorphic maps $f : \Omega \dashrightarrow S$ on a domain $\Omega \subset S$. By this we mean that at a generic point, f is a local biholomorphism and $\mathcal{F}$-preserving.

Lemma 3.2.2. *Let $K = C_1 + C_2 + \cdots + C_m$ be a nonoverlapping chain of lines on S, $o \in C_1$, and f be a germ of $\mathcal{F}$-preserving meromorphic map at o. Then, there exists a tubular neighborhood U of K and an $\mathcal{F}$-preserving meromorphic map $\hat{f} : U \to S$ such that $\hat{f}$ extends the germ f.*

The assumption that K is nonoverlapping is not essential. In general, one can replace K by a chain $\tilde{K}$ of $\mathbb{P}^1$ and a holomorphic immersion $\pi_o : \tilde{K} \to S$, $\pi_o(\tilde{K}) = K$. The analogue of Lemma 3.2.2 says that there is a Riemann domain $\pi : U \to S$ including $\pi_o : \tilde{K} \to S$ such that f extends to $\hat{f} : U \to S$.

Proof. Let $\Omega_o \Subset \Omega \subset S$ be open subsets and $f : \Omega \dashrightarrow S$ be an $\mathcal{F}$-preserving meromorphic map. Suppose $C_o \subset S$ is a line such that $C_o \cap \Omega_o$ is nonempty and irreducible. Denote by $F(S)$ the Fano variety of lines on S. Since C_o is reduced and irreducible, for $[C] \in F(S)$ sufficiently close to $[C_o]$, $C \cap \Omega_o$ is nonempty and irreducible. The meromorphic map $f : \Omega \to S$ gives rise to a meromorphic map $f^\# : D \to F(S)$ on some open neighborhood D of $[C]$ in $F(S)$. Denote by $\rho : \mathcal{C} \to F(S)$ the universal family of lines on S. Then, $f^\# \circ \rho$ is defined on $\rho^{-1}(D)$.

Over Ω, f can be recovered from $f^\# \circ \rho$, as follows. For $x \in \Omega$, let σ_1 and σ_2 be two germs of holomorphic sections of $\mathcal{C}$ at x, $\sigma_1(x) \ne \sigma_2(x)$. If f is locally biholomorphic at x, then $f(y) = (f^\# \circ \rho)(\sigma_1(y)) \cap (f^\# \circ \rho)(\sigma_2(y))$ for y sufficiently near x, where a point $[\alpha] \in \mathcal{C}$ is identified as a line $\mathbb{C}\alpha \subset T_x(X)$. In other words,

$f(x)$ is simply the point of intersection of image lines of two distinct lines passing through x. For $x \in \Omega$ in general, let Σ_o be $\mathrm{Graph}(f^\# \circ \rho \circ \sigma_1) \cap \mathrm{Graph}(f^\# \circ \rho \circ \sigma_2)$, $\sigma_1(x) \neq \sigma_2(x)$, and Σ be the unique germ of irreducible component of Σ_o which dominates the germ of Ω at x. Then, Σ is the germ of graph of the meromorphic map f at x. But the same procedure can be used to define a meromorphic map $\hat{f}$ for x lying in a neighborhood U of C, provided that $f^\# \circ \rho$ is defined in a neighborhood of $\sigma_1(x)$ and $\sigma_2(x)$. This observation, together with the following obvious Lemma, implies readily that f admits an extension to a meromorphic map $\hat{f} : U \to S$ on some tubular neighborhood U of C, which is necessarily $\mathcal{F}$-preserving, since it is $\mathcal{F}$-preserving on $\Omega \cap U$.

LEMMA 3.2.3. *Let $V_o \subset V \subset S$ be nonempty connected open subsets of S. Let $g : V_o \to S$ be an $\mathcal{F}$-preserving meromorphic map. Suppose $g^\# \circ \rho$ is defined on the graph of two nonintersecting holomorphic sections $\sigma_1, \sigma_2 : V \to \mathcal{C}$ over V. Define now $\Sigma \subset V \times S$ to be the unique irreducible component of* $\mathrm{Graph}(g^\# \circ \rho \circ \sigma_1) \cap \mathrm{Graph}(g^\# \circ \rho \circ \sigma_2)$ *which projects onto V. Then, Σ is the graph of an $\mathcal{F}$-preserving meromorphic map $\hat{g} : V \to S$ such that $\hat{g}|_{V_o} \equiv g$.*

We continue with the proof of Lemma 3.2.2. In the application of Lemma 3.2.3, the important thing is to have some holomorphic section of $\mathcal{C}$ over V. In the application to prove Proposition 3.2.2, there is no difficulty with finding such local sections on tubular neighborhoods of pieces of rational curves C (taking $m = 1$ and $C_1 = C$) since the lift $\hat{C}$ of C to $\mathcal{C}$ already lies in the domain of definition of $f^\# \circ \rho$. □

We remark that in place of Lemma 3.2.3 one can also take intersections of algebraic families of lines, by first extending the domain of definition of $f^\# \circ \rho$ to $\mathcal{C}|_U$ for some tubular neighborhood U of C, using Oka's Theorem on Hartogs radii (see [Mok and Tsai 1992] and the references there).

By Lemma 3.2.2, any $f \in \Omega$ can be analytically continued along tubular neighborhoods of chains of lines. Since S is rationally connected by (nonoverlapping) chains of lines, f can be extended to any point on S. However, it is not obvious that given $y \in S$, the germ $\hat{f}_y$ of an extension $\hat{f}$ at y obtained along a nonoverlapping chain of lines K, $y \in K$, emanating from $x \in \Omega$ (not necessarily o) will be independent of x and independent of the chain of lines. We will show that this is indeed the case, by making use of $\mathbb{C}^*$-actions on S. This will yield the following result:

LEMMA 3.2.4. *In the notation of Proposition* 3.2.1, *$f_{12} : U_1 \to U_2$ extends to a birational map $F : S \to S$.*

PROOF. Let $y \in S$ and K, with $K' \subset S$, be two (nonoverlapping) chains of lines joining x and x' on Ω to y. We may choose a Harish-Chandra chart $\mathbb{C}^n \subset S$ such that $\Omega \Subset \mathbb{C}^n$, $y \in \mathbb{C}^n$, no irreducible component of K or K' lies on $S - \mathbb{C}^n$ and all points $C_i \cap C_{i+1}$ and $C'_j \cap C'_{j+1}$ lie on $\mathbb{C}^n$. We can now join x to y by a continuous path on $\mathbb{C}^n$ consisting of paths on C_i; similarly x' can be joined to y

by a continuous path consisting of paths on C'_j. Joining x' to x by a continuous path on Ω we obtain a closed continuous path $\gamma(t)$, $\gamma : [0,1] \to \mathbb{C}^n \subset S$. f can then be analytically continued along λ to obtain $\hat{f}$, such that the germ $\hat{f}_x$ (by abuse of notations) at $t = 1$ may *a priori* be distinct from the germ f_x at $t = 0$. We are going to exclude the latter possibility by making use of $\mathbb{C}^*$-actions on S. For $\lambda \in \mathbb{C}^*$, f can be analytically continued along the path γ_λ given by $\gamma_\lambda(t) = \lambda(\gamma(t))$. Denote by $\hat{f}^\lambda$, with $\hat{f}^1 = \hat{f}$, the analytic continuation of f as a meromorphic map on a tubular neighborhood of γ_λ. For λ small enough, γ_λ lies on Ω. As f is defined on Ω for the germs of the extended maps at $t = 1$ we have $\hat{f}^\lambda_x = f_x$ for λ small, hence for $\lambda = 1$, by the identity theorem on holomorphic functions. With this we have proven that f can be analytically continued from Ω to S. Applying this to the $\mathcal{F}$-preserving biholomorphism $f_{12} : U_1 \to U_2$ in Proposition 3.2.1 and to its inverse $f_{21} : U_2 \to U_1$, we conclude that f_{12} can be extended to a birational map $F : S \dashrightarrow S$. The proof of Lemma 3.2.4 is complete. $\square$

For the proof of Proposition 3.2.1 it remains to establish one more fact:

Lemma 3.2.5. *Let S be an irreducible Hermitian symmetric manifold and $F : S \dashrightarrow S$ be a birational self-map. Suppose for a generic line C on S, $F|_C$ maps C onto a line C'. Then, F is a biholomorphism.*

Proof. We denote by $B \subset S$ the subvariety on which F fails to be a local biholomorphism and call B the bad locus of F. Suppose B is of codimension at least 2 (and the same applies to F^{-1}), then F induces a linear isomorphism θ on $\Gamma(S, K_S^{-1})$ by pulling back. Identifying S with its image under the projective embedding by K_S^{-1}, F is nothing other than the restriction of the projectivization

$$[\theta^*] : \mathbb{P}\Gamma(S, K_S^{-1})^* \to \mathbb{P}\Gamma(S, K_S^{-1})^*$$

to S, thus a biholomorphism.

It remains to show that the bad locus B of F is of codimension at least 2. Otherwise let $R \subset B$ be an irreducible component of codimension 1. Choose a connected open subset U on which F is an open embedding. Let $x_o \in U$ and C be a line passing through x_o. C is standard and small deformations of C fill up a tubular neighborhood G of C. Write $Z \subset B$ for the set of indeterminacies of F. Since X is of Picard number 1, C must intersect R. Deforming $x_o \in U$ and hence C slightly without loss of generality we may assume that C intersects R at a point $x_1 \in R - Z$. Since F is holomorphic and ramified at x_1 there exists a nonzero tangent vector $\eta \in T_{x_1}(S)$ such that $dF(\eta) = 0$. For any $x \in C$ we denote by $\alpha(x)$ some nonzero vector tangent to C at x.

Either $\eta \notin P_{\alpha(x_1)}$ or $\eta \in P_{\alpha(x_1)}$. In both cases we are going to obtain a contradiction. Since $T(S)|_C$ is semipositive there exists $s \in \Gamma(C, T(S)|_C)$ such that $s(x_1) = \eta$. Suppose $\eta \notin P_{\alpha(x_1)}$, then $s(x) \notin P_{\alpha(x)}$ for a generic $x \in C$. On the other hand, since $F|_C : C \to S$ is a biholomorphism onto a line

$C' = F(C) \subset S$, F_*s is a well-defined holomorphic section in $\Gamma(C', T(S))$ such that for $y_1 = F(x_1)$, $F_*s(y_1) = dF(s(x_1)) = dF(\eta) = 0$. From $F_*s(y_1) = 0$ it follows that $F_*s(y) \in P_{\beta(y)}$ for any $y \in C', y = F(x), \beta(y)$ being a nonzero vector tangent to C' at y. Choosing x generic on C we get a contradiction from $s(x) \notin P_{\alpha(x)}, F_*s(y) \in P_{\beta(y)}$ and from $dF(P_{\alpha(x)}) = P_{\beta(y)}$, which follows from $dF(\tilde{\mathcal{C}}_x) = \tilde{\mathcal{C}}'_y$.

Suppose now $\eta \in P_{\alpha(x_1)}$. Write $T(S)|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. Let $s \in \Gamma(C, \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)$ be such that $s(x_1) = \eta$. We may choose s so that s vanishes at some point $x_0 \in C$. Then, for the corresponding decomposition $T(S)|_{C'} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$, since F preserves $\mathcal{C}$ along C, we have $F_*s \in \Gamma(C', \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)$ such that F_*s is not tangent to C' and such that $F_*s(y_0) = F_*s(y_1) = 0$, a plain contradiction. Since $\mathrm{Ker}(dF(x_1)) \neq 0$ leads in any event to a contradiction, we have proven that the bad set of F is of codimension at least 2 in S. The proof of Lemma 3.2.5 is complete. □

With Lemma 3.2.5 we have completed the proof of Proposition 3.2.1.

4. Minimal Rational Tangents and Holomorphic Distributions on Rational Homogeneous Manifolds of Picard Number 1

4.1. In the study of Fano manifold of Picard number 1 through their varieties of minimal rational tangents, next to irreducible Hermitian symmetric manifolds we have the rational homogeneous manifolds S of Picard number 1. The nonsymmetric ones are distinguished by the existence of nontrivial homogeneous holomorphic distributions.

In Sections 1.4 and 2.3 we studied the case of homogeneous contact manifolds. As will be seen, the contact case is special among the nonsymmetric ones. By the Tanaka–Yamaguchi theory of differential systems on S, Ochiai-type theorems hold for any $S \neq \mathbb{P}^n$. On nonsymmetric S, there is a natural distribution D^1, whose definition will be recalled shortly. In case S is neither of symmetric nor of contact type, it follows from [Yamaguchi 1993] that any D^1-preserving local holomorphic map extends to an automorphism of S. Yamaguchi's proof uses harmonic theory of Lie algebra cohomologies, just as Ochiai's proof of Proposition 1.5.2. We will give an alternate proof of relevant results from Tanaka–Yamaguchi, by the method of analytic continuation as in Section 3. We start by recalling some basic facts concerning S.

Fixing a base point $o \in S$, we may write $S = G/P$, where G is a connected and simply-connected simple complex Lie group and $P \subset G$ is the maximal parabolic subgroup fixing o. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{p}$ be the parabolic subalgebra corresponding to P. Write $\mathfrak{u} \subset \mathfrak{p}$ for the nilpotent radical and let $\mathfrak{p} = \mathfrak{u} + \mathfrak{l}$ be a choice of Levi decomposition. The center $\mathfrak{z}$ of $\mathfrak{l}$ is one-dimensional. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{l}$, which is also a Cartan subalgebra of $\mathfrak{g}$. We have the root system $\triangle \subset \mathfrak{h}^*$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We can uniquely determine a set $\triangle^+$

of positive roots by requiring that $\mathfrak{u}$ is contained in the span of negative root spaces. (Here our sign convention is opposite to the choice in some references, e.g. [Yamaguchi 1993]. As many geometers do, we prefer this choice for the reason that positive roots correspond to positive line bundles.) Fix a system of simple roots $\Sigma = \{\alpha_1, \ldots, \alpha_r\}$. The maximality of $\mathfrak{p}$ implies that there is a unique simple root α_i satisfying $\alpha_i(\mathfrak{z}) \neq 0$. We say that S is of type $(\mathfrak{g}, \alpha_i)$.

Conversely, given a Cartan subalgebra $\mathfrak{h}$, a simple root system of $(\mathfrak{g}, \mathfrak{h})$ and a distinguished simple root α_i, we can recover $\mathfrak{p} \subset \mathfrak{g}$ and hence $S = G/P$, as follows. For an integer k, $-m \leq k \leq m$, we define $\triangle_k$ to be the set of all roots $\sum_{q=1}^r m_q \alpha_q$ with $m_i = k$. Here m is the largest integer such that $\triangle_m \neq 0$. For $\alpha \in \triangle$ we denote by $\mathfrak{g}_\alpha$ the corresponding root space. Write

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \triangle_0} \mathfrak{g}_\alpha,$$
$$\mathfrak{g}_k = \bigoplus_{\alpha \in \triangle_k} \mathfrak{g}_\alpha, \quad \text{for } k \neq 0$$

for the eigenspace decomposition with respect to $ad(\mathfrak{z})$. More precisely, there exists an element $\theta \in \mathfrak{z}$ such that $[\theta, v] = kv$ for $v \in \mathfrak{g}_k$, so that the eigenspace decomposition $\mathfrak{g} = \bigoplus_{k=-m}^m \mathfrak{g}_k$ endows $\mathfrak{g}$ with the structure of a graded Lie algebra. We denote by $(\mathfrak{g}, \alpha_i)$ the Lie algebra $\mathfrak{g}$ with this graded structure and say that $(\mathfrak{g}, \alpha_i)$ (and S) is of depth m. We have

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-m};$$
$$\mathfrak{l} = \mathfrak{g}_0;$$
$$\mathfrak{u} = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-m}.$$

Identify $T_o(S)$ with $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$. For $1 \leq k \leq m$, the translates of $\mathfrak{g}_1 + \cdots + \mathfrak{g}_k$ under G defines a homogeneous holomorphic distribution D^k on S, so that $D^1 \subsetneqq D^2 \subsetneqq \cdots \subsetneqq D^m = T(S)$ defines a filtration of the holomorphic tangent bundle.

For $x \in S$ we denote by $P_x \subset G$ the maximal parabolic subgroup fixing x (so that $P_o = P$). Denote by $U_x \subset P_x$ the unipotent radical, $L_x = P_x/U_x$, and regard D^1_x as an L_x-representation space. Consider the set of all highest weight vectors ξ of D^1_x as an L_x-representation space and denote by $\mathcal{W}_x \subset \mathbb{P}D^1_x$ the collection of projectivizations $[\xi]$. L_x acts transitively on $\mathcal{W}_x$, so that $\mathcal{W}_x \subset \mathbb{P}D^1_x$ is a rational homogeneous projective submanifold. We call $\mathcal{W}_x$ the variety of highest weight tangents. The collection of $\mathcal{W}_x$ as x ranges over S defines a homogeneous holomorphic fiber bundle $\mathcal{W} \to S$. We denote by $\mathcal{L}^1_x$ the image of L_x in the bundle of automorphisms $\mathcal{GL}(D^1_x)$ and denote by $\mathcal{L}^1 \to S$, $\mathcal{L}^1 \subset \mathcal{GL}(D^1)$, the fiber bundle thus obtained.

We proceed to relate varieties of highest weight tangents $\mathcal{W}_x \subset \mathbb{P}D^1_x$ with minimal rational curves. More generally, we discuss the construction of rational curves associated to roots. For $\rho \in \triangle^+$, let $H_\rho \in \mathfrak{h}_\rho = [\mathfrak{g}_\rho, \mathfrak{g}_{-\rho}]$ be such that

$\rho(H_\rho) = 2$. We call H_ρ the coroot of ρ. Basis vectors $E_\rho \in \mathfrak{g}_\rho$, $E_{-\rho} \in \mathfrak{g}_{-p}$ can be chosen such that $[E_\rho, E_{-\rho}] = H_\rho$, $[H_\rho, E_\rho] = 2E_\rho$, $[H, E_{-\rho}] = -2E_\rho$ so that the triple $(H_\rho, E_\rho, E_{-\rho})$ defines an isomorphism of $\mathfrak{s}_\rho = \mathfrak{h}_\rho \oplus \mathfrak{g}_\rho \oplus \mathfrak{g}_{-\rho}$ with $\mathfrak{sl}_2(\mathbb{C})$; see [Serre 1966, VI, Theorem 2, p. 43 ff.]. Let now $C_\rho \subset X$ be the $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ orbit of $o = eP$ under the Lie group $S_\rho \cong \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$, $S_\rho \subset G$ with Lie algebra $\mathfrak{s}_\rho$. Consider S of type $(\mathfrak{g}, \alpha_i)$. Write H_j for H_{α_j} and let ω_i be the i-th fundamental weight with $\omega_i(H_j) = \delta_{ij}$, and E be the underlying vector space of the representation of G, with lowest weight $-\omega_i$, defining the first canonical embedding $\tau : S \hookrightarrow \mathbb{P}E$. We have $Hv = -\omega_i(H)v$ for any $H \in \mathfrak{h}$ and a lowest weight vector $v \in E$. For the rational curve C_ρ with $\omega_i(H_\rho) = s$ we have $H_\rho v = -sv$. Since H_ρ is a generator of the weight lattice of $\mathfrak{s}_\rho$, the pull-back of $\mathcal{O}(1)$ on $\mathbb{P}E$ to C_ρ, which is the dual of the tautological line bundle, gives a holomorphic line bundle $\cong \mathcal{O}(s)$. In particular, for $\rho = \alpha_i$ we have $\omega_i(H_i) = 1$ so that $\tau(C_{\alpha_i})$ is a line, and $C_{\alpha_i} \subset S$ represents a generator of $H_2(S,\mathbb{Z}) \cong \mathbb{Z}$. We have

$$\tau : H_2(S,\mathbb{Z}) \xrightarrow{\cong} H_2(\mathbb{P}E,\mathbb{Z}) \cong \mathbb{Z}.$$

In general, $C_\rho \subset S$ is a rational curve of degree $s = \omega_i(H_\rho)$.

A minimal rational curve $C \subset S$ is of degree 1, and will also be called a line. C is called a highest weight line if and only if $[T_x(C)] \in \mathcal{W}_x$ at every $x \in C$. Since the lowest weight orbit in $\mathbb{P}\mathfrak{g}_1$ agrees with the highest weight orbit, $C_{\alpha_i} \subset S$ is a highest weight line. We will see that in case all roots of $\mathfrak{g}$ are of equal length, any line is a highest weight line. This is not the case in general.

From now on we will assume $S \neq \mathbb{P}^n$. We have the following result of Tanaka [1979] and Yamaguchi [1993] and its immediate corollary (see [Hwang and Mok 1999]).

PROPOSITION 4.1.1. *Let $U \subset S$ be a connected open set. Then a holomorphic vector field on U can be extended to a global holomorphic vector field on S if it preserves $\mathcal{W}|_U$. Furthermore, if S is neither of symmetric type nor of contact type, then a holomorphic vector field on U can be extended to a global holomorphic vector field on S if it preserves $D^1|_U$.*

COROLLARY 4.1.2. *Let U_1, $U_2 \subset S$ be connected open sets and $f_{12} : U_1 \to U_2$ a biholomorphic map preserving the distribution $D^1 \subset T(S)$. If S is neither of symmetric nor of contact type, then f_{12} can be extended to a biholomorphic automorphism of S. When S is of symmetric type or of contact type, f_{12} can be extended to a biholomorphic automorphism of S, if f_{12} preserves the fiber subbundle $\mathcal{W} \subset \mathbb{P}D^1$.*

The proof of Proposition 4.1.1 as given in [Tanaka 1979; Yamaguchi 1993] requires algebraic machinery that are quite distinct from techniques explained in this survey. For S of symmetric type, this is just Ochiai's theorem which we proved in Section 3.2. The same proof works for S of contact type. In Section 4.2 we will give directly a proof of Corollary 4.1.2 for the case when all roots of $\mathfrak{g}$

are of equal length, by showing that a local D^1-preserving holomorphic map necessarily preserves the bundle $\mathcal{W} \subset \mathbb{P}D^1$ of varieties of highest weight tangents and by applying the method of analytic continuation in Sections 3.1 and 3.2. An adaptation of the argument will apply even in the case with roots of unequal lengths. We will need the following obvious interpretation of the Frobenius form.

LEMMA 4.1.3. *Let S be a rational homogeneous manifold of Picard number 1 and of depth $m \geq 2$. Let k be a positive integer $1 \leq k < m$ and write $F^k : D^k_o \otimes D^k_o \to T_o(S)/D^k_o$ for the Frobenius form for the distribution D^k at $o \in S$. Under an identification of $T_o(S)$ with $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$, we have $F^k(\xi, \zeta) = [\xi, \zeta]$ mod $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$.*

Under the hypothesis of Corollary 4.1.2, it follows readily that $f_{12} : U_1 \to U_2$ preserves the set of $\xi \neq 0$ for which the rank of (F^1_ξ) is minimal. For $[\xi] \in \mathcal{W}_o$ it is easy to see that $\operatorname{rank}(F^1_\xi) \leq \operatorname{rank}(F^1_\eta)$ for any nonzero $\eta \in D^1_o$. In the contact case $\operatorname{rank}(F^1_\xi) = 1$ for any $\xi \neq 0$, and f_{12} does not necessarily preserve $\mathcal{W}$. For the noncontact case it is however not straightforward in the case of exceptional Lie algebras $\mathfrak{g} = E_6, E_7, E_8$ to determine $\operatorname{rank}(F^1_\xi)$.

4.2. We consider in what follows the case of simple Lie algebras $\mathfrak{g}$ for which all roots are of equal length, including $\mathfrak{g} = D_n$ $(n \geq 4)$, E_6, E_7, E_8, (for which there are associated $(\mathfrak{g}, \alpha_i)$ neither of symmetric nor of contact type). We start with a discussion of the root space decomposition for $\mathfrak{g}_1$. Consider a highest weight line C, $o \in C$, $T_x(C) = \mathbb{C}E_\mu$ for a root vector E_μ corresponding to a highest weight $\mu \in \mathfrak{h}^*$ of $\mathfrak{g}_1$. Define

$$\begin{aligned}
\triangle_1'(\mu) &= \{\rho \in \triangle_1 : \mu - \rho \in \triangle\},\\
\triangle_1''(\mu) &= \{\rho \in \triangle_1 : \mu + \rho \in \triangle\},\\
\triangle_1^{\perp}(\mu) &= \{\rho \in \triangle_1 : \mu - \rho, \mu + \rho \notin \triangle\}.
\end{aligned}$$

When all roots of $\mathfrak{g}$ are of equal length, any ρ-chain attached to μ is of length at most 2, so that $\triangle_1 = \{\mu\} \cup \triangle_1'(\mu) \cup \triangle_1^{\perp}(\mu) \cup \triangle_1''(\mu)$ is a disjoint union. We have the following corresponding lemma on the Grothendieck splitting of D^1 over C.

LEMMA 4.2.1. *Let S be a rational homogeneous manifold of the above type, and $C \subset S$ be a rational curve tangent to the distribution D^1. Then, $D^1|_C$ is of the form $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^u \oplus \mathcal{O}^v \oplus [\mathcal{O}(-1)]^r$ for some nonnegative integers u, v and r.*

PROOF. Since H_μ is a generator for the weight lattice of $\mathfrak{s}_\mu = \mathfrak{h}_\mu \oplus \mathfrak{g}_\mu \oplus \mathfrak{g}_{-\mu}$, the root space decomposition of D^1_o gives rise to a Grothendieck splitting of $D^1|_C$, with $\mathfrak{g}_\rho$ corresponding to the direct summand $\mathcal{O}(d_\rho)$, where $[H_\mu, E_\rho] = d_\rho E_\rho$, that is, $d_\rho = \rho(H_\mu)$. For $\mathfrak{g}$ with roots of equal length, $d_\rho = 2, 1, 0, -1$, corresponding to the decomposition $\triangle_1 = \{\mu\} \cup \triangle_1'(\mu) \cup \triangle_1^{\perp}(\mu) \cup \triangle_1''(\mu)$. □

We write P_α for the positive part at o, Z_α for $\mathcal{O}^v_o$ and N_α for $[\mathcal{O}(-1)]^r_o$. The proof of Lemma 4.2.1 also shows that $D^k/D^{k-1}|_C$ can have only summands of degree 1, 0 and -1 for $k > 1$. We know that $T(S)|_C$ must be of the form

$\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. The quotient bundle $T(S)/D^1|_C$ is semipositive. From the knowledge of splitting types of $D^k/D^{k-1}|_C$ and using composition series, we see that $T(S)/D^1|_C$ has at most summands of degree 1 and 0, and that the number of $\mathcal{O}(1)$'s in the Grothendieck splitting is exactly r, the number of roots in $\triangle_1''(\mu)$. This implies $u = p$, namely, that every deformation of a highest weight line is a highest weight line. We may thus take $\mathcal{W} \to S$ to be a bundle of varieties of minimal rational tangents.

The Grothendieck decomposition of $D^1|_C$ as in Lemma 4.2.1 implies that $\mathcal{W}_o$ is the closure of the graph of a vector-valued cubic polynomial in p variables. More precisely, let $\Theta \subset \triangle_0$ be the set of positive roots such that $\mu - \theta \in \triangle_1'$. Then $|\Theta| = p$. Write $\Theta = \{\theta_1, \ldots, \theta_p\}$ and E_{-a} for $E_{-\theta_a}$. In a neighborhood of $[\alpha]$, $\alpha = E_\mu$, we have the cubic expansion of $\mathcal{W}_o$ as the closure of the image of $[\Phi] : \mathbb{C}^p \longrightarrow \mathbb{P}T_o(S)$ for the vector-valued cubic polynomial $\Phi : \mathbb{C}^p \longrightarrow T_o(S)$ defined by

$$\Phi(z) = E_\mu + \sum_a [E_\mu, E_{-a}]z^a + \frac{1}{2!}\sum_{a,b} [[E_\mu, E_{-a}], E_{-b}]z^a z^b + \frac{1}{3!}\sum_{a,b,c} \big[[[E_\mu, E_{-a}], E_{-b}], E_{-c}\big] z^a z^b z^c.$$

We are now ready to state the following result which reduces distribution-preserving local maps to those preserving varieties of highest weight tangents.

Proposition 4.2.2. *Let S be a rational homogeneous manifold of Picard number* 1. *Assume that S is neither of the symmetric nor of the contact type, and that it is of type $(\mathfrak{g}, \alpha_i)$ for some simple Lie algebra $\mathfrak{g}$ for which all roots are of equal length. Denote by $D^1 \subset T(S)$ the homogeneous holomorphic distribution corresponding to $\mathfrak{g}_1$, and by $\mathcal{W} \subset \mathbb{P}D^1$ the homogeneous holomorphic fiber bundle of varieties of highest weight tangents. Then, any D^1-preserving germ of holomorphic maps must preserve $\mathcal{W}$.*

For the proof of Proposition 4.2.2 we will need a number of lemmas.

Lemma 4.2.3. *Let S be a rational homogeneous manifold as in Proposition* 4.2.2, *$o \in S$ be a fixed base point, and $F : D_o^1 \times D_o^1 \to D_o^2$ be the Frobenius form. For $\xi \in D_o^1$ denote by $F_\xi : D_o^1 \to D_o^2$ the linear map defined by $F_\xi(\zeta) = F(\xi, \zeta)$. Let α be a highest weight vector of D_o^1 as an L_o-representation space. Then, there exists some nonzero vector $\eta \in D_o^1$ such that* $\mathrm{rank}(F_\eta) > \mathrm{rank}(F_\alpha)$.

Proof. For $k \geq 1$ let μ_k and $\lambda_k \in \triangle_k$ denote, respectively, the highest and lowest weight of $\mathfrak{g}_k$. For any $(\mathfrak{g}, \alpha_i)$ of the contact type, $\mathfrak{g}_2$ is 1-dimensional and $\lambda_2 = \lambda$ is the only weight in $\triangle_2$, while $\lambda - \rho$ is a positive root for any $\rho \in \triangle_1$. For S as in the Lemma, in particular not of the contact type, by a straightforward checking, $\lambda_2 - \mu_1 \notin \triangle_1$. In fact, λ_2 does not even dominate μ_1. Write $\alpha = E_{\mu_1}$ and $\triangle_1''(\mu_1) = \{\rho(1), \ldots, \rho(r)\}$. Then, $\mathrm{rank}(F_\alpha) = r$. Since $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ we have $\lambda_2 = \varphi_1 + \psi_1$ for some $\varphi_1, \psi_1 \in \triangle_1$. There are two possibilities. Either both $\varphi_1, \psi_1 \in \triangle_1^{\perp}(\mu_1)$, or we may take $\varphi_1 \in \triangle_1''(\mu_1)$, $\psi_1 \in \triangle_1'(\mu_1)$. Writing $\beta = E_{\varphi_1}$ and $\gamma = E_{\psi_1}$ we have in both cases $[\alpha, \gamma] = 0$. For $t \in \mathbb{C}$ consider the vector $\eta_t \in \mathfrak{g}_1$ given by $\eta_t = \alpha + t\beta$. We may choose basis vectors E_ρ of $\mathfrak{g}_\rho$ such that $[\alpha, E_{\rho(i)}] = \pm E_{\mu_1+\rho(i)}$; $[\eta_t, \gamma] = [\alpha, \gamma] + t[\beta, \gamma] = \pm t E_{\lambda_2}$. Since $\lambda_2 - \mu_1 \notin \triangle_1$, E_{λ_2} is not proportional to any $E_{\mu_1+\rho(i)}$ and is linearly independent of $[\alpha, \mathfrak{g}_1] = Im(F_\alpha)$. As $E_{\lambda_2} \in Im(F_{\eta_t})$ for $t \neq 0$ it follows that $[\eta_t, \mathfrak{g}_1]$ contains at least $p+1$ linearly independent elements for $t \neq 0$ sufficiently small, so that $\mathrm{rank}(F_{\eta_t}) > \mathrm{rank}(F_\alpha)$, as desired. □

Lemma 4.2.4. *Let S be a rational homogeneous manifold as in Proposition 4.2.2. Suppose there exists some D^1-preserving germ of holomorphic map which does not preserve $\mathcal{W}$. Then, there exists a holomorphic bundle of connected reductive Lie groups $\mathcal{H} \to S$, $\mathcal{L}^1 \subsetneqq \mathcal{H} \subsetneqq \mathcal{GL}(D^1)$ such that, denoting by $\mathcal{V} \subset \mathbb{P}D^1$ the orbit $\mathcal{H} \cdot \mathcal{W}$ under the natural action of $\mathcal{H}$ on $\mathbb{P}D^1$, the canonical map $\mathcal{V} \to S$ realizes $\mathcal{V}$ as a holomorphic fiber bundle for which the fibers $\mathcal{V}_x \subset \mathbb{P}D^1_x$ are rational homogeneous submanifolds conjugate to each other under projective linear transformations. Moreover, $\mathcal{W} \subsetneqq \mathcal{V} \subsetneqq \mathbb{P}D^1$.*

Proof. Consider the group Q of germs of holomorphic maps $f : (S, o) \to (S, o)$ such that f preserves D^1. We can identify the maximal parabolic P as a subgroup of Q. Let $A \subset \mathrm{GL}(D^1_o)$ be the algebraic subgroup consisting of all $d\varphi(o) \in \mathrm{GL}(D^1_o)$, $\varphi \in Q$. Any A-invariant subvariety contains $\mathcal{W}_o$ and hence $A \cdot \mathcal{W}_o$, the orbit of $\mathcal{W}_o$ under A. $A \cdot \mathcal{W}_o$ is a constructible set and its Zariski closure $\overline{A \cdot \mathcal{W}_o}$ is again A-invariant. The complement B of $A \cdot \mathcal{W}_o$ in $\overline{A \cdot \mathcal{W}_o}$ is A-invariant. B is constructible and its Zariski closure $\overline{B} \subset \overline{A \cdot \mathcal{W}_o}$ is a proper A-invariant subvariety. It follows that B must be empty, otherwise $\overline{B}$ would contain $A \cdot \mathcal{W}_o$, so that $\overline{B} = \overline{A \cdot \mathcal{W}_o}$, a plain contradiction. Thus, $\mathcal{V}_o := A \cdot \mathcal{W}_o$ is a closed subvariety in $\mathbb{P}D^1_o$. By assumption there exists $\nu \in A$ such that $\nu(\mathcal{W}_o) \neq \mathcal{W}_o$, so that $\mathcal{W}_o \subsetneqq \mathcal{V}_o$. Since $\mathcal{V}_o$ is homogeneous under A, it must be smooth. As each component of $\mathcal{V}_o$ is P-invariant and contains $\mathcal{W}_o$, we conclude that $\mathcal{V}_o$ is an irreducible homogeneous submanifold of $\mathbb{P}D^1_o$. Let $H_o \subset \mathrm{GL}(D^1_o)$ be the identity component of A. Then, $\mathcal{V}_o = H_o \cdot \mathcal{V}_o \subset \mathbb{P}D^1_o$ is a rational homogeneous manifold equivariantly embedded in $\mathbb{P}D^1_o$. Passing to projectivizations it follows readily from Borel's fixed point theorem that $H_o \subset \mathrm{GL}(D^1_o)$ is reductive.

To prove Lemma 4.2.4 it remains to show that $\mathcal{V}_o \subsetneqq \mathbb{P}D^1_o$. Denote by $Z_o \subset \mathbb{P}D^1_o$ the subset consisting of all $[\eta]$ such that $\mathrm{rank}(F_\eta) = \mathrm{rank}(F_\alpha)$, with $[\alpha] \in \mathcal{W}_o$. By Lemma 4.2.3, $Z_o \subsetneqq \mathbb{P}D^1_o$. On the other hand, for any D^1-preserving $\varphi \in Q$, $\mathrm{rank}(F_{d\varphi(\eta)}) = \mathrm{rank}(F_\eta)$, so that $\mathcal{V}_o = A \cdot \mathcal{W}_o \subset Z_o \subsetneqq \mathbb{P}D^1_o$, as desired. □

We will prove Proposition 4.2.2 by getting a contradiciton to $\mathcal{W} \subsetneqq \mathcal{V} \subsetneqq \mathbb{P}D^1$. The idea is a variation of the proof of $\mathcal{C}_x = \mathcal{W}_x$ in Section 1.5.

Lemma 4.2.5. *Let $\mathcal{W} \subsetneqq \mathcal{V} \subsetneqq \mathbb{P}D^1$ be as given in Lemma 4.2.4. Then, for each $[\alpha] \in \mathcal{W}_o, T_{[\alpha]}(\mathcal{V}_o) \subset (P_\alpha \oplus Z_\alpha)/\mathbb{C}\alpha$.*

Proof. Let $C \subset S$ be a minimal rational curve passing through o. In what follows we write D for D^1. By Lemma 4.2.1 we have

$$(D^*\otimes D)|_C \cong \big([\mathcal{O}(1)]^r\oplus\mathcal{O}^v\oplus[\mathcal{O}(-1)]^p\oplus\mathcal{O}(-2)]\big)\otimes\big(\mathcal{O}(2)\oplus[\mathcal{O}(1)]^p\oplus\mathcal{O}^v\oplus[\mathcal{O}(-1)]^r\big).$$

Write $T_o(C) = \mathbb{C}\alpha$ and take $\omega^* \in D_o^*$ to be a covector annihilating $P_\alpha \oplus Z_\alpha$. Then, $\omega^* \in D_o^*$ lies in the well-defined direct summand $[\mathcal{O}(1)]^r$ of $D^*|_C$. To prove the Lemma it suffices to prove that $\omega^*(\eta) = 0$ whenever $\eta \mod \mathbb{C}\alpha \in T_{[\alpha]}(\mathcal{V}_o)$. As in Section 1.5 let $U \subset D^* \otimes D$ be the holomorphic subbundle where $U_x \subset D_x^* \otimes D_x = \mathfrak{gl}(D_x)$ is the Lie algebra of H_x for any $x \in S$. Consider the direct sum decomposition $D^* \otimes D = U \oplus U^\perp$. The decomposable tensor $\alpha \otimes \omega^*$ lies in $F := \mathcal{O}(2)\otimes[\mathcal{O}(1)]^r \subset (D^*\otimes D)|_C$. Since $F \cong [\mathcal{O}(3)]^r$ and every direct summand of $(D^* \otimes D)|_C$ is of degree at most 3, we have $F = F' \oplus F''$ with $F' \subset U|_C$ and $F'' \subset U^\perp|_C$ from the uniqueness of Grothendieck decompositions. Since every element of F is of the form $\alpha \otimes \tau^*$ for some $\tau^* \in [\mathcal{O}(1)]_x^r$, we must have correspondingly a decomposition $[\mathcal{O}(1)]_x^r = Q' \oplus Q''$ such that $F' = \mathbb{C}\alpha \otimes Q'$ and $F'' = \mathbb{C}\alpha \otimes Q''$. The arguments of Section 1.5 show that U_x contains no nonzero decomposable tensor element, implying therefore that $Q' = 0$ and hence $F = F'' \subset U^\perp|_C$, which means that $\omega^*(\eta) = 0$ whenever $\eta \mod \mathbb{C}\alpha$ is tangent to $\mathcal{V}_o$ at $[\alpha]$, as desired. □

Lemma 4.2.6. *Let $\zeta \in Z_\alpha$ be such that $[\zeta, P_\alpha] = 0$ for the Lie bracket $[\,\cdot\,,\cdot\,]$: $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2$. Then $\zeta = 0$.*

Proof. Write μ for the highest weight in $\mathfrak{g}_1$ and choose $\alpha = E_\mu$. Recall that Θ is the set of positive roots θ in $\triangle_o$ such that $\mu-\theta = \rho \in \triangle_1'(\mu)$. Then, for any $\theta \in \Theta$ we have $[\zeta, E_{-\theta}] = \pm[\zeta, [\bar{\alpha}, E_\rho]]$. Since $[\bar{\alpha}, \zeta] = 0$ and by hypothesis $[\zeta, P_\alpha] = 0$ we conclude from the Jacobi identity that $[\zeta, E_{-\theta}] = 0$. We proceed to deduce $\zeta = 0$ from $[\zeta, P_\alpha] = 0$ by showing that the latter implies $[\zeta, Z_\alpha] = [\zeta, N_\alpha] = 0$, so that $[\zeta, \mathfrak{g}_1] = 0$, implying $\zeta = 0$. To see this, by the cubic expansion of $\mathcal{W}_o$ and writing $\xi_a = [\alpha, E_{-a}]$, Z_α is spanned by $\zeta_{ab} = \big[[\alpha, E_{-a}], E_{-b}\big] = [\xi_a, E_{-b}]$, $a, b \in \Theta$, so that $[\zeta, \zeta_{ab}] = [[\zeta, \xi_a], E_{-b}] - [[\zeta, E_{-b}], \xi_a] = 0$. Similarly, N_α is spanned by $\omega_{abc} = [[[\alpha, E_{-a}], E_{-b}], E_{-c}] = [\zeta_{ab}, E_{-c}]$ so that $[\zeta, \omega_{abc}] = [[\zeta, \zeta_{ab}], E_{-c}] - [[\zeta, E_{-c}], \zeta_{ab}] = 0$, as desired. □

Proof of Proposition 4.2.2. It suffices now to prove that $\mathcal{V}$ as constructed in Lemma 4.2.4, $\mathcal{W} \subsetneqq \mathcal{V} \subsetneqq \mathbb{P}(D)$ cannot possibly exist. Suppose otherwise. Then for $[\alpha] \in \mathcal{W}_o \subsetneqq \mathcal{V}_o$ we have by Lemma 4.2.5, $T_{[\alpha]}(\mathcal{V}_o) = E \mod \mathbb{C}\alpha$ for some vector subspace E of D_o such that

$$P_\alpha \subsetneqq E \subset P_\alpha \oplus Z_\alpha.$$

By the polarization argument of Section 1.3 E is isotropic with respect to the vector-valued symplectic form $[\,\cdot\,,\cdot\,]$. It follows that there exists some nonzero vector $\zeta \in Z_\alpha$ such that $[\zeta, P_\alpha] = 0$, contradicting Lemma 4.2.6. The proof of Proposition 4.2.2 is complete. □

For S of type $(\mathfrak{g}, \alpha_i)$ as in Proposition 4.2.2, the arguments of analytic continuation of Sections 3.1 and 3.2 apply to show that any D^1-preserving germ of holomorphic map extends to an automorphism of S, as in Corollary 4.1.2. The proof of Proposition 4.2.2 is also valid for $\mathfrak{g} = B_n$ and for (F_4, α_4). The remaining cases of (C_n, α_i), (F_4, α_2), (F_4, α_3), (G_2, α_2) are characterized by the fact that $\mathcal{W} \subsetneqq \mathcal{C}$, that is, by the existence of minimal rational curves other than highest weight lines. We will say for short that $(\mathfrak{g}, \alpha_i)$ is of excessive type. We will exclude (G_2, α_2) from our discussion, since the underlying complex manifold is biholomorphic to the 5-dimensional hyperquadric. For the rest, $\mathcal{C}_s$ is the closure of the isotropy orbit of the highest weight vector in $\mathfrak{g}_2$, from which it can be shown that $\mathcal{W} = \mathcal{C} \cap \mathbb{P}D^1$.

For $(\mathfrak{g}, \alpha_i)$ of excessive type with $\mathfrak{g} = C_n$, F_4; we can still apply the method of analytic continuation to prove Corollary 4.1.2, provided that we prove (1) the analogue of Proposition 4.2.2 and (2) that any $\mathcal{W}$-preserving germ of holomorphic map is $\mathcal{E}$-preserving for the foliation $\mathcal{E}$ on $\mathcal{W}$ defined by highest weight lines, $\mathcal{E} = \mathcal{F}|_{\mathcal{W}}$. (1) can be done by a straightforward verification that highest weight vectors ξ are characterized by the minimality of $\mathrm{rank}(F^1_\xi)$, which we omit. (2) can be done by an adaptation of Section 3.1, as follows. We consider the distribution $\mathcal{R}$ on $\mathcal{W}$ defined by $\mathcal{R} = (d\sigma)^{-1}\mathcal{P}$ for $\sigma : \mathcal{W} \to S$ the restriction of $\pi : \mathcal{C} \to S$ to $\mathcal{W}$. Then, $\mathcal{E} \subset \mathrm{Ch}(\mathcal{R})$. If φ is a germ of $\mathcal{W}$-preserving holomorphic map, then $\varphi^*\mathcal{E}$ is a foliation such that $\varphi^*\mathcal{E} \subset \mathrm{Ch}(\mathcal{R})$. If $\mathcal{E} \neq \varphi^*\mathcal{E}$ then at a generic point $[\alpha]$ of $\mathcal{W}$ we have some vertical vector $\eta \neq 0$ tangent to $\mathcal{W}_x$ at $[\alpha]$, $x = \sigma([\alpha])$, such that $\eta \in \mathrm{Ch}(\mathcal{R})$, by comparing leaves of $\mathcal{E}$ and $\varphi^*\mathcal{E}$ through the same point. Writing $\eta = v \mod \mathbb{C}\alpha$ the arguments of Theorem 3.1.3 then shows that for $k = \sum_i e_i \frac{\partial}{\partial v_i}$ tangent to $\mathcal{W}'_x$, $\sum_i v(e_i)\frac{\partial}{\partial v_i}$ remains tangent to $\mathcal{C}'_x$ (not $\mathcal{W}'_x$). However, since $\mathcal{W}'_x \subset \gamma^{-1}D^1_x$, we conclude that

$$\sum_i v(e_i)\frac{\partial}{\partial v_i} \in \gamma^{-1}D^1_x \cap \mathcal{C}'_x = \mathcal{W}'_x,$$

so that η lies in the kernel of the Gauss map of $\mathcal{W}_x$ in $\mathbb{P}D^1_x$. As $\mathcal{W}_x \subsetneqq \mathbb{P}D^1_x$ is linearly nondegenerate this leads to a contradiction.

5. Varieties of Distinguished Tangents and an Application to Finite Holomorphic Maps

5.1. We hope that readers who have followed this note so far would agree, at least partially, that it is quite rewarding to study $\mathcal{C}_x$. It will be very nice to construct something like $\mathcal{C}_x$ using nonrational curves, because general projective manifolds do not have rational curves at all. We can proceed as follows. For a given projective manifold Y, fix a component $\mathcal{M}$ of the Chow space of curves. Let $\mathcal{M}_y$ be the subscheme corresponding to curves through $y \in Y$. We have the tangent map $\Phi_y : \mathcal{M}_y \to \mathbb{P}T_y(Y)$ defined on those points corresponding to curves smooth at y. Then the closure of the image of Φ_y would play the role of $\mathcal{C}_x$. The problem is that quite often this would give the whole $\mathbb{P}T_y(Y)$ and we cannot get anything interesting out of it. However, by taking a special piece of the image of Φ_y, we get an interesting object, which plays an important role when we study generically finite holomorphic maps to uniruled manifolds. Here we will recall some basic definitions and main results of [Hwang and Mok 1999, Section 1].

Let $g : M \to Z$ be a regular map between two quasi-projective complex algebraic varieties. We can stratify M and Z into finitely many nonsingular quasi-projective subvarieties. On the other hand, given $g : M \to Z$ with both M and Z smooth, we can stratify M into finitely many quasi-projective subvarieties on each of which g has constant rank. Applying these two stratifications repeatedly, we can stratify M naturally into finitely many irreducible quasi-projective nonsingular subvarieties $M = M_1 \cup \cdots \cup M_k$, such that for each i, the reduced image $g(M_i)$ is nonsingular and the holomorphic map $g|_{M_i} : M_i \to g(M_i)$ is of constant rank. It will be called the g-stratification of M. The following two properties of this stratification are immediate:

(i) Any tangent vector to $g(M_i)$ can be realized as the image of the tangent vector to a local holomorphic arc in M_i.
(ii) When a connected Lie group acts on M and Z, and g is equivariant, M_i and $g(M_i)$ are invariant under this group action.

For a given projective manifold Y, fix $\mathcal{M}$ as above and let $\Phi_y : \mathcal{M}_y \to \mathbb{P}T_y(Y)$ be the tangent map, which is well-defined on a subset $\mathcal{M}_y^o \subset \mathcal{M}_y$ corresponding to curves smooth at y. Let $\{M_i\}$ be the Φ_y-stratification of $\mathcal{M}_y^o$. A subvariety of $\mathbb{P}T_y(Y)$ will be called a variety of distinguished tangents in $\mathbb{P}T_y(Y)$, if it is the closure of the image $\Phi_y(M_i)$ for some choice of $\mathcal{M}_y$ and M_i. Note that there exist only countably many subvarieties in $\mathbb{P}T_y(Y)$ which can serve as varieties of distinguished tangents, because the Chow space has only countably many components.

Given an irreducible reduced curve l in Y and a smooth point $y \in l$, consider $\mathcal{M}_y$ which parametrizes deformations of l fixing y. $[l]$ is contained in $\mathcal{M}_y^o$, where the tangent map is well-defined. Let M_1 be the component of the stratification of $\mathcal{M}_y^o$ associated to the tangent map, so that $[l] \in M_1$. The variety of distinguished tangents corresponding to M_1 is called the variety of distinguished tangents associated to l at y and is denoted by $\mathcal{D}_y(l)$. It is an irreducible subvariety and $\mathbb{P}T_y(l)$ is a smooth point on it. $\mathcal{D}_y(l)$ is a generalization of $\mathcal{C}_x$ for a general curve l.

Although we do not have Grothendieck splitting for general l, we can get partial information as follows. In the splitting for a standard minimal rational curve C,

$$T(X)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus [\mathcal{O}]^q,$$

the sum of the $\mathcal{O}(1)$-part and the $\mathcal{O}$-part can be replaced by the normal bundle of the general curve l. The $\mathcal{O}$-part alone can be studied as the part generated by sections of the conormal bundle of l. In general we have to be careful about the singularity of the curve. Let $N_l^* = \mathcal{I}/\mathcal{I}^2$ be the conormal sheaf of l, where $\mathcal{I}$ denotes the ideal sheaf of l. We have a natural map $j : N_l^* \to \Omega(Y)|_l$, where $\Omega(Y) = \mathcal{O}(T^*(Y))$. j is injective if l is an immersed curve. In general, $\mathrm{Ker}(j)$ is a sheaf supported on finitely many points. Let N_l' be the image of j in $\Omega(Y)$. If l is a standard minimal rational curve, sections of N_l' correspond to sections of $\mathcal{O}^q$. So the dimension of $\mathcal{C}_x$ is $n-1-h^0(l, N_l')$. Using the property (i) of the stratification and general deformation theory, we can get a partial result for general l (see [Hwang and Mok 1999] for details):

PROPOSITION 5.1.1. *Let $y \in Y$ be a sufficiently general point and l be a curve smooth at y. Then the tangent space of $\mathcal{D}_y(l)$ at the point $\mathbb{P}T_y(l)$ has dimension at most $n-1-h^0(l, N_l')$, where $n = \dim(Y)$.*

In general, the variety of distinguished tangents for a nonrational curve is not as useful as $\mathcal{C}_x$, because we do not have a good choice of "minimal" $\mathcal{M}$ as in the case of uniruled manifolds. So far, their main interest is in connection with the study of finite morphisms to uniruled manifolds by the following theorem.

THEOREM 5.1.2. *Let $f : Y \to X$ be a generically finite morphism from a projective manifold Y to a uniruled manifold X. Choose $x \in X$ and $y \in f^{-1}(x)$ so that y is sufficiently general and $df : T_y(Y) \to T_x(X)$ is an isomorphism. Then each irreducible component of $df^{-1}(\mathcal{C}_x) \subset \mathbb{P}T_y(Y)$ is a variety of distinguished tangents $\mathcal{D}_y(l)$ for a suitable choice of a curve l through y.*

SKETCH OF PROOF. Choose a generic point $x \in X$ and a component $\mathcal{C}_1$ of $\mathcal{C}_x$. Choose a minimal rational curve C through x so that $\mathbb{P}T_x(C)$ is a generic point of $\mathcal{C}_1$. Let l be an irreducible component of $f^{-1}(C)$ through $y \in f^{-1}(x)$. For simplicity, assume that l is smooth so that $N_l' = N_l^*$. A nonzero section of the conormal bundle of C can be lifted to a nonzero section of the conormal bundle of l. Thus $h^0(l, N_l') \geq h^0(C, N_C^*)$.

Obviously $\mathbb{P}df_y^{-1}(T_x(C)) \in \mathcal{D}_y(l)$. Thus each generic point of $df_y^{-1}(\mathcal{C}_1)$ is contained in some $\mathcal{D}_y(l)$ for a suitable choice of a curve l, depending on C, satisfying $h^0(l, N_l') \geq h^0(C, N_C^*)$. Since there are only countably many subvarieties in $\mathbb{P}T_y(Y)$ which can serve as a variety of distinguished tangents, we can assume that $df_y^{-1}(\mathcal{C}_1) \subset \mathcal{D}_y(l)$, by choosing l generically. We have $\dim(\mathcal{C}_1) = n-1-h^0(C, N_C^*)$. Applying the previous proposition,

$$\begin{aligned} n-1-h^0(C, N_C^*) &= \dim(df_y^{-1}(\mathcal{C}_1)) \leq \dim(\mathcal{D}_y(l)) \\ &\leq n-1-h^0(l, N_l') \leq n-1-h^0(C, N_C^*), \end{aligned}$$

which implies $df_y^{-1}(\mathcal{C}_1) = \mathcal{D}_y(l)$. □

5.2. As an application of the results of Section 5.1, we will prove the following rigidity theorem for generically finite holomorphic maps over rational homogeneous spaces of Picard number 1.

THEOREM 5.2.1. *Let S be a rational homogeneous space of Picard number 1 different from $\mathbb{P}^n$, and Y be any n-dimensional compact complex manifold. Given a family of surjective holomorphic maps $f_t : Y \to S$ parametrized by $\Delta = \{t \in \mathbb{C}, |t| < \varepsilon\}$, we have a holomorphic map $g : \Delta \to \mathrm{Aut}_o(S)$ with $g_0 = id_S$ so that $f_t = g_t \circ f_0$.*

PROOF. Choose a sufficiently small open set $U \subset Y$ so that $f_t|_U$ is biholomorphic for any $t \in \Delta$. Let $\mathcal{C} \subset \mathbb{P}T(S)$ be the variety of minimal rational tangents. By Theorem 5.1.2, $df_t^{-1}(\mathcal{C}_{f(y)})$ is a family of varieties of distinguished tangents for each $y \in U$. From the discreteness of varieties of distinguished tangents, we see that $df_t^{-1}(\mathcal{C}_{f_t(y)}) = df_0^{-1}(\mathcal{C}_{f_0(y)})$ for all $t \in \Delta$. Thus the biholomorphic map $g_t := f_t \circ f_0^{-1}$ from $f_0(U)$ to $f_t(U)$ preserves $\mathcal{C}$. By Corollary 4.1.2, g_t can be extended to an automorphism of S. Since $f_t = g_t \circ f_0$ on U, it must hold on the whole Y. □

6. Lazarsfeld's Problem on Rational Homogeneous Manifolds of Picard Number 1

6.1. When we have a surjective holomorphic map $f : Y \to Z$ between two projective manifolds, it is a general principle of complex geometry that the target Z is more positively curved than the source Y in a suitable sense. Among all projective manifolds, the projective space is most positively curved in the sense that projective manifolds with ample tangent bundles are projective spaces, a result of Mori ([Mori 1979]). Combining these two, one may ask: if a projective manifold Z is the image of a projective space under a holomorphic map, is Z itself a projective space? This was a conjecture of Remmert and Van de Ven [1960], proved by Lazarsfeld [1984]. Not surprisingly, Lazarsfeld used this result of Mori:

THEOREM 6.1.1 [Mori 1979]. *Let X be a Fano manifold and $P \in X$ be a point. If the restrictions of $T(X)$ to all minimal rational curves through P are ample, then X is a projective space.*

The idea of Lazarsfeld's proof is as follows. Given $f : \mathbb{P}^n \to Z$, it is immediate that Z is Fano. Choose a generic $P \in Z$ and consider any minimal rational curve C through P. Then $f^{-1}(C)$ must have ample normal sheaf, because it is a curve in $\mathbb{P}^n$. This forces C to have ample normal sheaf, and Z is a projective space by Theorem 6.1.1.

It is expected that rational homogeneous manifolds of Picard number 1 are the next most positively curved manifolds after projective spaces. So the following question of Lazarsfeld is a natural generalization of Remmert and Van de Ven's conjecture:

CONJECTURE 6.1.2 [Lazarsfeld 1984]. *Let S be a rational homogeneous manifold of Picard number 1. For any surjective holomorphic map $f : S \to X$ to a* *projective manifold X, either X is a projective space, or $X \cong S$ and f is a biholomorphism.*

Applying Theorem 6.1.1, as in the case of $S = \mathbb{P}^n$, we see that the problem is to understand the curves on S on which the restrictions of $T(S)$ are not ample. Of course, minimal rational curves are such examples. But in general, there are a lot of other curves with this property. When S is a hyperquadric, Paranjape and Srinivas [1989] showed that minimal rational curves are the only curves on S with this property, and using this, they settled the conjecture. When S is a Hermitian symmetric space, Tsai [1993] had classified certain classes of curves on S where the restrictions of $T(S)$ are not ample, and settled the conjecture. For this, he needed a very detailed study of the global geometry of curves on S, using fine structure theory of Hermitian symmetric spaces [Wolf 1972]. Generalizing his methods to other S looks hopelessly complicated. To start with, very little is known about global structure of curves on S. Furthermore, even the local picture, say the structure of isotropy representation of the parabolic group, has completely different features from the symmetric case. In [Hwang and Mok 1999], we have settled the conjecture in full generality by a different approach. We will survey this work in this section.

6.2. First, we reduce Conjecture 6.1.2 to the following extension problem of holomorphic maps.

THEOREM 6.2.1. *Let S be a rational homogeneous space of Picard number 1 different from $\mathbb{P}^n$ and $f : S \to X$ be a finite morphism to a projective manifold X different from $\mathbb{P}^n$. Let $s, t \in S$ be an arbitrary pair of distinct points such that $f(s) = f(t)$ and f is unramified at s and t. Write φ for the unique germ of holomorphic map at s, with target space S, such that $\varphi(s) = t$ and $f \circ \varphi = f$. Then φ extends to a biholomorphic automorphism of S.*

In fact, once Theorem 6.2.1 is proved, we can use automorphisms of S arising from various choices of φ to conclude that $f : S \to X$ is a quotient map by a finite group action on S. Then Lazarsfeld's conjecture follows from the following.

PROPOSITION 6.2.2. *Let S be a rational homogeneous space of Picard number 1 of dimension $n \geq 3$, different from $\mathbb{P}^n$. Suppose there exists a nontrivial finite cyclic group $F \subset \mathrm{Aut}(S)$ which fixes a hypersurface $E \subset S$ pointwise. Then S is the hyperquadric, E is equal to an $\mathcal{O}(1)$-hypersurface, and the quotient of S by F, endowed with the standard normal complex structure, is a projective space.*

In principle, Proposition 6.2.2 can be checked case by case. It can be proved also using induction on the dimension by showing that a suitable deformation of E is itself homogeneous and preserved by the F-action.

To prove Theorem 6.2.1, it suffices to show that φ preserves $\mathcal{C} \subset \mathbb{P}T(S)$ for S of symmetric type or contact type, and the distribution D^1 for the other S, by Corollary 4.1.2. For simplicity, we will assume that the Fano manifold X has the property that $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is a proper subvariety, namely $q > 0$. In the case $q = 0$ and X different from $\mathbb{P}^n$, essentially the same argument works when combined with the result in [Mok 1988, 2.4].

6.3. We will consider S of symmetric type or of contact type first. We need to show that φ sends $\mathcal{C}_{s_1}$ to $\mathcal{C}_{t_1}$ for s_1 sufficiently close to s. From Theorem 5.1.2, φ sends a variety of distinguished tangents $\mathcal{D}_{s_1}$ in $\mathbb{P}T_{s_1}(S)$ to a variety of distinguished tangents $\mathcal{D}_{t_1}$ in $\mathbb{P}T_{t_1}(S)$. From the property (ii) of g-stratification mentioned in Section 5.1, $\mathcal{D}_{s_1}$ and $\mathcal{D}_{t_1}$ are invariant under the action of isotropy groups at s_1 and t_1 respectively. Moreover, from the countability of varieties of distinguished tangents, we can assume that $\mathcal{D}_{s_1}$ and $\mathcal{D}_{t_1}$ are conjugate under the G-action. After G-conjugation, φ induces an automorphism of $\mathbb{P}T_{s_1}(S)$ preserving $\mathcal{D}_{s_1}$, and we need to show that it preserves $\mathcal{C}_{s_1}$.

When S is of symmetric type, an automorphism of $\mathbb{P}T_s(S)$ preserving an isotropy-invariant proper subvariety must preserve the highest weight orbit $\mathcal{C}_s$ from the fine structure theory of Hermitian symmetric spaces ([Wolf 1972]). In fact, one can show that the highest weight orbit is a singularity stratum of any other isotropy-invariant subvariety. Thus φ must preserve $\mathcal{C}$, and we are done. Alternatively, the argument in Section 6.4 case (1) gives a different proof without using the fine structure theory.

When S is of contact type, we can show that $\mathcal{D}_{s_1}$ must be equal to $\mathcal{C}_{s_1}$ directly. It is easy to see that any proper isotropy-invariant subvariety of $\mathbb{P}T_s(S)$ is contained in $\mathbb{P}D_s$, the contact hyperplane. From the basic structure theory of isotropy orbits, $\mathcal{C}_s \subset \mathcal{D}_s \subset \mathbb{P}D_s$. The Lagrangian property of $\mathcal{C}_s$ in Section 1.4 can be used to show that the variety of tangential lines to $\mathcal{D}_s$ is nondegenerate in $\mathbb{P}\bigwedge^2 D_s$ unless $\mathcal{D}_s = \mathcal{C}_s$. But if the variety of tangential lines to $\mathcal{D}_s$ is nondegenerate in $\mathbb{P}\bigwedge^2 D_s$, then the distribution D must be integrable by arguing as in Section 1.2, since $\mathcal{D}_s$ is the pull back of the variety of minimal rational tangents in X. This is contradictory to the definition of D. This proves Conjecture 6.1.2 in the contact case.

6.4. For the proof of Theorem 6.2.1, it remains to consider the case of S of depth $m \geq 2$ and of noncontact type. We will also again exclude the unnecessary case of (G_2, α_2). By Corollary 4.1.2 it suffices to show that φ preserves D^1, or equivalently that φ preserves the bundle of varieties of highest weight tangents $\mathcal{W}$, by Proposition 4.2.2. Suppose otherwise. By the argument of Lemma 4.2.4 there exists a holomorphic fiber bundle $\mathcal{V} \to S$, $\mathcal{V} \subset \mathbb{P}T(S)$ preserved by φ, such that the fibers $\mathcal{V}_x \subset \mathbb{P}T_x(S)$ are rational homogeneous submanifolds conjugate to each other under projective transformations. Either

(1) $\mathcal{V}_x \subset \mathbb{P}T_x(S)$ is linearly nondegenerate, or
(2) $\mathcal{V} \subset \mathbb{P}D^k$ for some k for $2 \leq k < m$, and $\mathcal{V} \not\subset \mathbb{P}D^1$.

In the linear nondegenerate case (1), since φ preserves some isotropy-invariant proper subvariety as in Section 6.3, $\mathcal{V}_x \subsetneqq \mathbb{P}T_x(S)$ and $\mathcal{V} \to S$ defines a G-

structure over S, with $\mathrm{G} \subset \mathbb{P}\,\mathrm{GL}(T_o(S))$ being the identity component of the group of projective linear transformations on $\mathbb{P}T_o(S)$ leaving $\mathcal{V}_o$ invariant. As $\mathcal{V}_o$ is linearly nondegenerate in $\mathbb{P}T_o(S)$, G is reductive by Borel's fixed point theorem. It follows from Theorem 1.5.3 that S must be biholomorphic to an irreducible Hermitian symmetric manifold of the compact type and of rank at least 2, contradicting the assumption on S.

We are going to rule out the linearly degenerate case (2), $\mathcal{V} \subset \mathbb{P}D^k$ for some k with $2 \leq k < m$, $\mathcal{V} \not\subset \mathbb{P}D^1$. Only the cases of depth at least 3 matter, with $\mathfrak{g} = F_4, E_6, E_7$ or E_8. By Corollary 4.1.2 it suffices to show that if φ is D^k-preserving for some $k \geq 2$, then it must already be D^1-preserving. Consider the Frobenius form $F^k : D^k_o \times D^k_o \to T_o(S)/D^k_o$. From Lemma 4.1.3, φ must preserve the subvariety of $[\xi] \in \mathbb{P}D^k_o$ for which $\mathrm{rank}(F^k_\xi)$ is minimum. To deduce that φ is necessarily D^1-preserving it remains therefore to establish this result:

Proposition 6.4.1. *Let η_1 be a highest weight vector of $\mathfrak{g}_1 = D^1_o$ as an L_o-representation space and $\xi \in D^k_o - \mathfrak{g}_1$. Then*, $\mathrm{rank}\, F^k_\xi > \mathrm{rank}\, F^k_{\eta_1}$.

The rank of F^k_ξ is constant along the P-orbit of ξ and is lower semicontinuous in $\xi \in D^k_o$. Consider the $\mathbb{C}^*$-action on D^k_o defined by the centre of L_o, given by

$$t \cdot \xi = t\xi_1 + t^2\xi_2 + \cdots + t^k\xi_k$$

according to the decomposition $\xi = \xi_1 + \xi_2 + \cdots + \xi_k$, $\xi_j \in \mathfrak{g}_j$. From lower semicontinuity we conclude that $\mathrm{rank}(F^k_\xi) \geq \mathrm{rank}(F^k_{\xi_i})$ whenever i is the largest index for which $\xi_i \neq 0$. Consider $\mathfrak{g}_j$ as an L_o-representation space. Noting that the highest weight orbit in $\mathbb{P}\mathfrak{g}_j$ lies in the Zariski closure of any orbit in $\mathbb{P}\mathfrak{g}_j$ to prove the Proposition it suffices to show that $\mathrm{rank}(F^k_{\eta_j}) > \mathrm{rank}(F^k_{\eta_1})$ for highest weight vectors η_j of $\mathfrak{g}_j$, $j = 2, \ldots, k$. Furthermore, as η_j lies on the same P-orbit of some $\eta_j + \theta_{j-1}$, $0 \neq \theta_{j-1} \in \mathfrak{g}_{j-1}$, using the $\mathbb{C}^*$-action as described it follows readily that $\mathrm{rank}(F^k_{\eta_k}) \geq \cdots \geq \mathrm{rank}(F^k_{\eta_2})$, thus reducing the proof of Proposition 6.4.1 to the special case of $\xi = \eta_2$. For the case of $\mathfrak{g} = F_4$ a straight-forward checking shows that indeed

$$\mathrm{rank}(F^k_{\eta_2}) > \mathrm{rank}(F^k_{\eta_1})$$

is always valid. For the exceptional cases $\mathfrak{g} = E_6, E_7, E_8$, for which roots are of equal length, in place of tedious checking we have the following statement with a uniform proof.

Proposition 6.4.2. *Let $\mathfrak{g} = E_6, E_7, E_8$ and $(\mathfrak{g}, \alpha_i)$ be of depth at least* 3. *For $k \geq 2$ and for η_j highest weight vectors of $\mathfrak{g}_j$ as an L_o-representation space*, *we have* $\mathrm{rank}(F^k_{\eta_2}) = 2\,\mathrm{rank}(F^k_{\eta_1})$.

Proof. We will interpret $\mathrm{rank}(F^k_{\eta_s})$, for $s = 1, 2$, as Chern numbers. In the notations of Section 4.1, for S of type $(\mathfrak{g}, \alpha_i)$ and for $\rho \in \triangle^+$, the rational curve $C_\rho \subset S$ is of degree $\omega_i(H_\rho)$. When all roots of $\mathfrak{g}$ are of equal length, the root system is self-dual, and for $\rho = \sum_{j=1}^m m_j \alpha_j$, we have $H_\rho = \sum_{j=1}^m m_j H_{\alpha_j}$, so that for $\rho \in \triangle_s$, C_ρ is of degree s. For $s = 1, 2$ write C_s for C_{μ_s}. We have

$$\operatorname{rank}(F^k_{\eta_s}) = \dim\Bigl(\Bigl[\eta_s, \textstyle\bigoplus_{j=1}^k \mathfrak{g}_j\Bigr] \mod \textstyle\bigoplus_{j=1}^k \mathfrak{g}_j\Bigr). \tag{$*$}$$

Denote by $\mathcal{E}^k$ the holomorphic vector bundle D^m/D^k. We claim that

$$\operatorname{rank}(F^k_{\eta_s}) = c_1(\mathcal{E}^k) \cdot C_s,$$

from which $\operatorname{rank}(F^k_{\eta_2}) = 2 \operatorname{rank}(F^k_{\eta_1})$ follows, since C_s is of degree s. $\mathcal{E}^k$ admits a composition series with factors D^l/D^{l-1}, $k < l \le m$. To prove the claim by the argument on splitting types of D^l/D^{l-1} as in Section 4.2, we have

$$c_1(D^l/D^{l-1}) \cdot \mathcal{C}_s = \bigl|\{\rho_l \in \triangle_l : \rho_l - \mu_s \in \triangle_{l-s}\}\bigr| - \bigl|\{\rho_l \in \triangle_l : \rho_l + \mu_s \in \triangle_{l+s}\}\bigr|.$$

Observe that for $l > k$, whenever $\rho_l + \mu_s = \rho_{l+s} \in \triangle_{l+s}$ we also have $\rho_{l+s} - \mu_s = \rho_l \in \triangle_l$. From this and adding up Chern numbers we have

$$c_1(\mathcal{E}^k) \cdot \mathcal{C}_s = \bigl|\{\rho \in \triangle_1 \cup \cdots \cup \triangle_{k-1} : \rho + \mu_s \in \triangle_k \cup \cdots \cup \triangle_m\}\bigr| = \operatorname{rank}(F^k_{\eta_s}),$$

by $(*)$, as claimed. The proof of Proposition 6.4.2 is complete. $\square$

We remark that, using the composition series, splitting types of D^l/D^{l-1} over C_s, and the fact that D^m/D^l is nonnegative, one can easily verify that $\mathcal{E}^k|_{C_s} \cong [\mathcal{O}(1)]^{r_s} \oplus \mathcal{O}^{q_s}$, $r_s = \operatorname{rank}(F^k_{\eta_s})$, and $q_s = \operatorname{rank}(\mathcal{E}^k) - r_s$.

Proposition 6.4.1 follows from Proposition 6.4.2. From this we also rule out alternative (2) (that $\mathcal{V} \subset \mathbb{P}D^k$ for $2 \le k < m$ but $\mathcal{V} \not\subset \mathbb{P}D^1$). We have thus proven by contradiction that in the nonsymmetric and noncontact case, the local biholomorphism φ on S must preserve varieties of highest weight tangents. By Corollary 4.1.2 we conclude that φ extends to a biholomorphic automorphism on S. By Proposition 6.2.2 the finite map $f : S \to X$ must be a biholomorphism unless $X \cong \mathbb{P}^n$. With this we have resolved Conjecture 6.1.2 of Lazarsfeld's.

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Jun-Muk Hwang
Seoul National University
Seoul 151-742
Korea
jmhwang@math.snu.ac.kr

Ngaiming Mok
The University of Hong Kong
Pokfulam Road
Hong Kong
nmok@hkucc.hku.hk