Several Complex Variables
MSRI Publications
Volume **37**, 1999

Recent Developments in Seiberg–Witten Theory and Complex Geometry

CHRISTIAN OKONEK AND ANDREI TELEMAN

We dedicate this paper to our wives Christiane and Roxana
for their invaluable help and support during the past two years.

Abstract. In this article, written at the end of 1996, we survey some of the most important results in Seiberg–Witten Theory which are directly related to Algebraic or Kählerian Geometry. We begin with an introduction to abelian Seiberg–Witten Theory, with special emphasis on the generalized Seiberg–Witten invariants, which take also into account 1-homology classes of the base manifold. The more delicate case of manifolds with $b_+ = 1$ is discussed in detail; we present our universal wall-crossing formula which shows that, crossing a wall in the parameter space, produces jumps of the invariants which are of a purely topological nature.

Next we introduce nonabelian Seiberg–Witten equations associated with very general compact Lie groups, and we describe in detail some of the properties of the moduli spaces of PU(2)-monopoles. The latter play an important role in our approach to prove Witten's conjecture. Then we specialize to the case where the base manifold is a Kähler surface, and we present the complex geometric interpretation of the corresponding moduli spaces of monopoles. This interpretation is another instance of a Kobayashi–Hitchin correspondence, which is based on the analysis of various types of vortex equations. Finally we explain our strategy for a proof of Witten's conjecture in an abstract setting, using the algebraic geometric "coupling principle" and "master spaces" to relate the relevant correlation functions.

Contents

The authors were partially supported by AGE (Algebraic Geometry in Europe) contract no. ERBCHRXCT940557 (BBW 93.0187), and by SNF grant no. 21-36111.92.

Introduction

In October 1994, E. Witten revolutionized the theory of 4-manifolds by introducing the now famous Seiberg–Witten invariants [Witten 1994]. These invariants are defined by counting gauge equivalence classes of solutions of the Seiberg–Witten monopole equations, a system of nonlinear PDE's which describe the absolute minima of a Yang–Mills–Higgs type functional with an abelian gauge group.

Within a few weeks after Witten's seminal paper became available, several long-standing conjectures were solved, many new and totally unexpected results were found, and much simpler and more conceptional proofs of already established theorems were given.

Among the most spectacular applications in this early period are the solution of the Thom conjecture [Kronheimer and Mrowka 1994], new results about Einstein metrics and Riemannian metrics of positive scalar curvature [LeBrun 1995a; 1995b], a proof of a $\frac{10}{8}$ bound for intersection forms of Spin manifolds [Furuta 1995], and results about the $\mathcal{C}^\infty$-classification of algebraic surfaces [Okonek and Teleman 1995a; 1995b; 1997; Friedman and Morgan 1997; Brussee 1996]. The latter include Witten's proof of the $\mathcal{C}^\infty$-invariance of the canonical class of a minimal surface of general type with $b_+ \neq 1$ up to sign, and a simple proof of the Van de Ven conjecture by the authors. Moreover, combining results in [LeBrun 1995b; 1995a] with ideas from [Okonek and Teleman 1995b], P. Lupaşcu recently obtained [1997] the optimal characterization of complex surfaces of Kähler type admitting Riemannian metrics of nonnegative scalar curvature.

In two of the earliest papers on the subject, C. Taubes found a deep connection between Seiberg–Witten theory and symplectic geometry in dimension four: He first showed that many aspects of the new theory extend from the case of Kähler surfaces to the more general symplectic case [Taubes 1994], and then he went on to establish a beautiful relation between Seiberg–Witten invariants and Gromov–Witten invariants of symplectic 4-manifolds [Taubes 1995; 1996].

A report on some papers of this first period can be found in [Donaldson 1996].

Since the time this report was written, several new developments have taken place:

The original Seiberg–Witten theory, as introduced in [Witten 1994], has been refined and extended to the case of manifolds with $b_+ = 1$. The structure of the Seiberg–Witten invariants is more complicated in this situation, since the invariants for manifolds with $b_+ = 1$ depend on a chamber structure. The general theory, including the complex-geometric interpretation in the case of Kähler surfaces, is now completely understood [Okonek and Teleman 1996b].

At present, three major directions of research have emerged:

– Seiberg–Witten theory and symplectic geometry
– Nonabelian Seiberg–Witten theory and complex geometry
– Seiberg–Witten–Floer theory and contact structures

In this article, which had its origin in the notes for several lectures which we gave in Berkeley, Bucharest, Paris, Rome and Zürich during the past two years, we concentrate mainly on the second of these directions.

The reader will probably notice that the nonabelian theory is a subject of much higher complexity than the original (abelian) Seiberg–Witten theory; the difference is roughly comparable to the difference between Yang–Mills theory and Hodge theory. This complexity accounts for the length of the article. In rewriting our notes, we have tried to describe the essential constructions as simply as possible but without oversimplifying, and we have made an effort to explain the most important ideas and results carefully in a nontechnical way; for proofs and technical details precise references are given.

We hope that this presentation of the material will motivate the reader, and we believe that our notes can serve as a comprehensive introduction to an interesting new field of research.

We have divided the article in three chapters. In Chapter 1 we give a concise but complete exposition of the basics of abelian Seiberg–Witten theory in its most general form. This includes the definition of refined invariants for manifolds with $b_1 \neq 0$, the construction of invariants for manifolds with $b_+ = 1$, and the universal wall crossing formula in this situation.

Using this formula in connection with vanishing and transversality results, we calculate the Seiberg–Witten invariant for the simplest nontrivial example, the projective plane.

In Chapter 2 we introduce nonabelian Seiberg–Witten theories for rather general structure groups G. After a careful exposition of Spin^G-structures and G-monopoles, and a short description of some important properties of their moduli spaces, we explain one of the main results of the Habiliationsschrift of the second author [Teleman 1996; 1997]: the fundamental Uhlenbeck type compactification of the moduli spaces of PU(2)-monopoles.

Chapter 3 deals with complex-geometric aspects of Seiberg–Witten theory: We show that on Kähler surfaces moduli spaces of G-monopoles, for unitary structure groups G, admit an interpretation as moduli spaces of purely holomorphic objects. This result is a Kobayashi–Hitchin type correspondence whose proof depends on a careful analysis of the relevant vortex equations. In the abelian case it identifies the moduli spaces of twisted Seiberg–Witten monopoles with certain Douady spaces of curves on the surface [Okonek and Teleman 1995a]. In the nonabelian case we obtain an identification between moduli spaces of PU(2)-monopoles and moduli spaces of stable oriented pairs; see [Okonek and Teleman 1996a; Teleman 1997].

The relevant stability concept is new and makes sense on Kähler manifolds of arbitrary dimensions; it is induced by a natural moment map which is closely related to the projective vortex equation. We clarify the connection between this new equation and the parameter dependent vortex equations which had been studied in the literature [Bradlow 1991]. In the final section we construct moduli spaces of stable oriented pairs on projective varieties of any dimension with GIT methods [Okonek et al. 1999]. Our moduli spaces are projective varieties which come with a natural $\mathbb{C}^*$-action, and they play the role of master spaces for stable pairs. We end our article with the description of a very general construction principle which we call "coupling and reduction". This fundamental principle allows to reduce the calculation of correlation functions associated with vector bundles to a computation on the space of reductions, which is essentially a moduli space of lower rank objects.

Applied to suitable master spaces on curves, our principle yields a conceptional new proof of the Verlinde formulas, and very likely also a proof of the Vafa–Intriligator conjecture. The gauge theoretic version of the same principle can be used to prove Witten's conjecture, and more generally, it will probably also lead to formulas expressing the Donaldson invariants of arbitrary 4-manifolds in terms of Seiberg–Witten invariants.

1. Seiberg–Witten Invariants

1.1. The Monopole Equations. Let (X, g) be a closed oriented Riemannian 4-manifold. We denote by Λ^p the bundle of p-forms on X and by $A^p := A^0(\Lambda^p)$ the corresponding space of sections. Recall that the Riemannian metric g defines a Hodge operator $* : \Lambda^p \longrightarrow \Lambda^{4-p}$ with $*^2 = (-1)^p$. Let $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ be the corresponding eigenspace decomposition.

A <u>Spin^c-structure</u> on (X, g) is a triple $\tau = (\Sigma^\pm, \iota, \gamma)$ consisting of a pair of $U(2)$-vector bundles $\Sigma^\pm$, a unitary isomorphism $\iota : \det \Sigma^+ \longrightarrow \det \Sigma^-$ and an orientation-preserving linear isometry $\gamma : \Lambda^1 \longrightarrow \mathbb{R}\,\mathrm{SU}(\Sigma^+, \Sigma^-)$. Here

$$\mathbb{R}\,\mathrm{SU}(\Sigma^+, \Sigma^-) \subset \mathrm{Hom}_{\mathbb{C}}(\Sigma^+, \Sigma^-)$$

is the subbundle of real multiples of (fibrewise) isometries of determinant 1. The <u>spinor bundles</u> $\Sigma^\pm$ of τ are — up to isomorphism — uniquely determined by their first Chern class $c := c_1(\det \Sigma^\pm)$, the Chern class of the $\mathrm{Spin}^c(4)$-structure τ. This class can be any integral lift of the second Stiefel–Whitney class $w_2(X)$ of X, and, given c, we have

$$c_2(\Sigma^\pm) = \frac{1}{4}(c^2 - 3\sigma(X) \mp 2e(X)).$$

Here $\sigma(X)$ and $e(X)$ denote the signature and the Euler characteristic of X.

The map γ is called the <u>Clifford map</u> of the Spin^c-structure τ. We denote by Σ the total spinor bundle $\Sigma := \Sigma^+ \oplus \Sigma^-$, and we use the same symbol γ also for the induced the map $\Lambda^1 \longrightarrow \mathrm{su}(\Sigma)$ given by

$$u \longmapsto \begin{pmatrix} 0 & -\gamma(u)^* \\ \gamma(u) & 0 \end{pmatrix}.$$

Note that the Clifford identity

$$\gamma(u)\gamma(v)+\gamma(v)\gamma(u)=-2g(u,v)$$

holds, and that the formula

$$\Gamma(u\wedge v):=\tfrac{1}{2}[\gamma(u),\gamma(v)]$$

defines an embedding $\Gamma:\Lambda^2\longrightarrow \mathrm{su}(\Sigma)$ which maps $\Lambda^2_\pm$ isometrically onto $su(\Sigma^\pm)\subset \mathrm{su}(\Sigma)$.

The second cohomology group $H^2(X,\mathbb{Z})$ acts on the set of equivalence classes $\mathfrak{c}$ of $\mathrm{Spin}^c(4)$-structures on (X,g) in a natural way: Given a representative $\tau=(\Sigma^\pm,\iota,\gamma)$ of $\mathfrak{c}$ and a Hermitian line bundle M representing a class $m\in H^2(X,\mathbb{Z})$, the tensor product $(\Sigma^\pm\otimes M,\ \iota\otimes\mathrm{id}_{M^{\otimes 2}},\ \gamma\otimes\mathrm{id}_M)$ defines a Spin^c-structure τ_m. Endowed with the $H^2(X,\mathbb{Z})$-action given by $(m,[\tau])\longmapsto[\tau_m]$, the set of equivalence classes of Spin^c-structures on (X,g) becomes a $H^2(X,\mathbb{Z})$-torsor, which is independent of the metric g up to canonical isomorphism [Okonek and Teleman 1996b]. We denote this $H^2(X,\mathbb{Z})$-torsor by $\mathrm{Spin}^c(X)$.

Recall that the choice of a $\mathrm{Spin}^c(4)$-structure $(\Sigma^\pm,\iota,\gamma)$ defines an isomorphism between the affine space $\mathcal{A}(\det\Sigma^+)$ of unitary connections in $\det\Sigma^+$ and the affine space of connections in $\Sigma^\pm$ which lift the Levi-Civita connection in the bundle $\Lambda^2_\pm\simeq\mathrm{su}(\Sigma^\pm)$. We denote by $\hat a\in\mathcal{A}(\Sigma)$ the connection corresponding to $a\in\mathcal{A}(\det\Sigma^+)$.

The <u>Dirac</u> <u>operator</u> associated with the connection $a\in\mathcal{A}(\det\Sigma^+)$ is the composition

$$\not{D}_a : A^0(\Sigma^\pm)\xrightarrow{\nabla_{\hat a}} A^1(\Sigma^\pm)\xrightarrow{\gamma} A^0(\Sigma^\mp)$$

of the covariant derivative $\nabla_{\hat a}$ in the bundles $\Sigma^\pm$ and the Clifford multiplication $\gamma:\Lambda^1\otimes\Sigma^\pm\longrightarrow\Sigma^\mp$.

Note that, in order to define the Dirac operator, one needs a Clifford map, not only a Riemannian metric and a pair of spinor bundles; this will later become important in connection with transversality arguments. The Dirac operator $\not{D}_a : A^0(\Sigma^\pm)\longrightarrow A^0(\Sigma^\mp)$ is an elliptic first order operator with symbol $\gamma:\Lambda^1\longrightarrow\mathbb{R}\,\mathrm{SU}(\Sigma^\pm,\Sigma^\mp)$. The direct sum-operator $\not{D}_a : A^0(\Sigma)\longrightarrow A^0(\Sigma)$ on the total spinor bundle is selfadjoint and its square has the same symbol as the rough Laplacian $\nabla^*_{\hat a}\nabla_{\hat a}$ on $A^0(\Sigma)$.

The corresponding <u>Weitzenböck</u> <u>formula</u> is

$$\not{D}_a^2=\nabla^*_{\hat a}\nabla_{\hat a}+\tfrac{1}{2}\Gamma(F_a)+\tfrac{1}{4}s\,\mathrm{id}_\Sigma,$$

where $F_a\in iA^2$ is the curvature of the connection a, and s denotes the scalar curvature of (X,g) [Lawson and Michelsohn 1989].

To write down the Seiberg–Witten equations, we need the following notations: For a connection $a\in\mathcal{A}(\det\Sigma^+)$ we let $F_a^\pm\in iA^2_\pm$ be the (anti) selfdual components of its curvature. Given a spinor $\Psi\in A^0(\Sigma^+)$, we denote by $(\Psi\bar\Psi)_0\in A^0(\mathrm{End}_0(\Sigma^\pm))$ the trace free part of the Hermitian endomorphism

$\Psi\otimes\bar\Psi$. Now fix a $\mathrm{Spin}^c(4)$-structure $\tau=(\Sigma^\pm,\iota,\gamma)$ for (X,g) and a closed 2-form $\beta\in A^2$. The β-twisted monopole equations for a pair $(a,\Psi)\in\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)$ are

$$\begin{aligned} \not{D}_a\Psi &= 0,\\ \Gamma(F_a^+ + 2\pi i\beta^+) &= (\Psi\bar\Psi)_0. \end{aligned} \tag{SW^τ_β}$$

These β-twisted Seiberg–Witten equations should not be regarded as perturbations of the equations (SW^τ_0) since later the cohomology class of β will be fixed. The twisted equations arise naturally in connection with nonabelian monopoles (see Section 2.2). Using the Weitzenböck formula one easily gets the following fact:

LEMMA 1.1.1. *Let β be a closed 2-form and $(a,\Psi)\in\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)$. Then*

$$\begin{aligned} &\|\not{D}_a\Psi\|^2+\tfrac14\|(F_a^+ + 2\pi i\beta^+)-(\Psi\bar\Psi)_0\|^2\\ &\qquad =\|\nabla_{A_a}\Psi\|^2+\tfrac14\|F_a^+ + 2\pi i\beta^+\|^2+\tfrac18\|\Psi\|^4_{L^4}+\int_X\big((\tfrac14 s\,\mathrm{id}_{\Sigma^+}-\Gamma(\pi i\beta^+))\Psi,\Psi\big). \end{aligned}$$

COROLLARY 1.1.2 [Witten 1994]. *On manifolds (X,g) with nonnegative scalar curvature s the only solutions of (SW^τ_0) are pairs $(a,0)$ with $F_a^+=0$.*

1.2. Seiberg–Witten Invariants for 4-Manifolds with $b_+>1$. Let (X,g) be a closed oriented Riemannian 4-manifold, and let $\mathfrak{c}\in\mathrm{Spin}^c(X)$ be an equivalence class of Spin^c-structures of Chern class c, represented by the triple $\tau=(\Sigma^\pm,\iota,\gamma)$. The configuration space for Seiberg–Witten theory is the product $\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)$ on which the gauge group $\mathcal{G}:=\mathcal{C}^\infty(X,S^1)$ acts by

$$f\cdot(a,\Psi):=(a-2f^{-1}df,\ f\Psi).$$

Let $\mathcal{B}(c):=\big(\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)\big)/\mathcal{G}$ be the orbit space; up to homotopy equivalence, it depends only on the Chern class c. Since the gauge group acts freely in all points (a,Ψ) with $\Psi\neq0$, the open subspace

$$\mathcal{B}(c)^*:=\big(\mathcal{A}(\det\Sigma^+)\times\big(A^0(\Sigma^+)\setminus\{0\}\big)\big)/\mathcal{G}$$

is a classifying space for $\mathcal{G}$. It has the weak homotopy type of a product $K(\mathbb{Z},2)\times K(H^1(X,\mathbb{Z}),1)$ of Eilenberg–Mac Lane spaces and there is a natural isomorphism

$$\nu:\mathbb{Z}[u]\otimes\Lambda^*(H_1(X,\mathbb{Z})/\,\mathrm{Tors})\longrightarrow \mathrm{H}^*(\mathcal{B}(\mathrm{c})^*,\mathbb{Z}),$$

where the generator u is of degree 2. The $\mathcal{G}$-action on $\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)$ leaves the subset $[\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)]^{\mathrm{SW}^\tau_\beta}$ of solutions of (SW^τ_β) invariant; the orbit space

$$\mathcal{W}^\tau_\beta:=[\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)]^{\mathrm{SW}^\tau_\beta}/\mathcal{G}$$

is the moduli space of β-twisted monopoles. It depends, up to canonical isomorphism, only on the metric g, on the closed 2-form β, and on the class $\mathfrak{c}\in\mathrm{Spin}^c(X)$ [Okonek and Teleman 1996b].

Let $\mathcal{W}^{\tau\,*}_\beta \subset \mathcal{W}^\tau_\beta$ be the open subspace of monopoles with nonvanishing spinor-component; it can be described as the zero-locus of a section in a vector-bundle over $\mathcal{B}(c)^*$. The total space of this bundle is

$$\left[\mathcal{A}(\det\Sigma^+) \times (A^0(\Sigma^+)\setminus\{0\})\right] \times_{\mathcal{G}} \left[iA^2_+ \oplus A^0(\Sigma^-)\right],$$

and the section is induced by the $\mathcal{G}$-equivariant map

$$\mathrm{SW}^\tau_\beta : \mathcal{A}(\det\Sigma^+) \times \left(A^0(\Sigma^+)\setminus\{0\}\right) \longrightarrow iA^2_+ \oplus A^0(\Sigma^-)$$

given by the equations (SW^τ_β).

Completing the configuration space and the gauge group with respect to suitable Sobolev norms, we can identify $\mathcal{W}^{\tau\,*}_\beta$ with the zero set of a real analytic Fredholm section in the corresponding Hilbert vector bundle on the Sobolev completion of $\mathcal{B}(c)^*$, hence we can endow this moduli space with the structure of a finite dimensional real analytic space. As in the instanton case, one has a Kuranishi description for local models of the moduli space around a given point $[a,\Psi] \in \mathcal{W}^\tau_\beta$ in terms of the first two cohomology groups of the elliptic complex

$$0 \longrightarrow iA^0 \xrightarrow{D^0_p} iA^1 \oplus A^0(\Sigma^+) \xrightarrow{D^1_p} iA^2_+ \oplus A^0(\Sigma^-) \longrightarrow 0 \qquad (\mathcal{C}_p)$$

obtained by linearizing in $p = (a,\Psi)$ the action of the gauge group and the equivariant map SW^τ_β. The differentials of this complex are

$$\begin{aligned} D^0_p(f) &= (-2df, f\Psi), \\ D^1_p(\alpha,\psi) &= \left(d^+\alpha - \Gamma^{-1}[(\Psi\bar\psi)_0 + (\psi\bar\Psi)_0], \not{D}_a(\psi) + \gamma(\alpha)(\Psi)\right), \end{aligned}$$

and its <u>index</u> w_c depends only on the Chern class c of the Spin^c-structure τ and on the characteristic classes of the base manifold X:

$$w_c = \tfrac{1}{4}(c^2 - 3\sigma(X) - 2e(X)).$$

The moduli space $\mathcal{W}^\tau_\beta$ is compact. This follows, as in [Kronheimer and Mrowka 1994], from the following consequence of the Weitzenböck formula and the maximum principle.

Proposition 1.2.1 (A priori $\mathcal{C}^0$-bound of the spinor component). *If* (a,Ψ) *is a solution of* (SW^τ_β), *then*

$$\sup|\Psi|^2 \le \max\left(0,\ \sup_X(-s + |4\pi\beta^+|)\right).$$

Moreover, let $\tilde{\mathcal{W}}^\tau$ be the moduli space of triples

$$(a,\Psi,\beta) \in \mathcal{A}(\det\Sigma^+) \times A^0(\Sigma^+) \times Z^2_{\mathrm{DR}}(X)$$

solving the Seiberg–Witten equations above now regarded as equations for the triple (a,Ψ,β). Two such triples define the same point in $\tilde{\mathcal{W}}^\tau$ if they are congruent modulo the gauge group $\mathcal{G}$ acting trivially on the third component. Using the proposition above and arguments of [Kronheimer and Mrowka 1994], one can easily see that the natural projection $\tilde{\mathcal{W}}^\tau \xrightarrow{p} Z^2_{\mathrm{DR}}(X)$ is proper. Moreover, one has the following transversality results:

LEMMA 1.2.2. *After suitable Sobolev completions the following results hold*:

1 [Kronheimer and Mrowka 1994]. *The open subspace* $[\tilde{\mathcal{W}}^\tau]^* \subset \tilde{\mathcal{W}}^\tau$ *of points with nonvanishing spinor component is smooth.*
2 [Okonek and Teleman 1996b]. *For any de Rham cohomology class* $b \in H^2_{\mathrm{DR}}(X)$ *the moduli space* $[\tilde{\mathcal{W}}^\tau_b]^* := [\tilde{\mathcal{W}}^\tau]^* \cap p^{-1}(b)$ *is also smooth.*

Now let $c \in H^2(X,\mathbb{Z})$ be a <u>characteristic</u> <u>element</u>, that is, an integral lift of $w_2(X)$. A pair $(g,b) \in \mathcal{M}\mathrm{et}(X) \times H^2_{\mathrm{DR}}(X)$ consisting of a Riemannian metric g on X and a de Rham cohomology class b is called <u>c-good</u> when the g-harmonic representant of $c-b$ is not g-antiselfdual. This condition guarantees that $\mathcal{W}^\tau_\beta = \mathcal{W}^{\tau\,*}_\beta$ for every Spin^c-structure τ of Chern class c and every 2-form β in b. Indeed, if $(a,0)$ would solve (SW^τ_β), then the g-antiselfdual 2-form $\frac{i}{2\pi}F_a - \beta$ would be the g-harmonic representant of $c-b$.

In particular, using the transversality results above, one gets:

THEOREM 1.2.3 [Okonek and Teleman 1996b]. *Let* $c \in H^2(X,\mathbb{Z})$ *be a characteristic element and suppose* $(g,b) \in \mathcal{M}\mathrm{et}(X) \times H^2_{\mathrm{DR}}(X)$ *is c-good. Let* τ *be a* Spin^c*-structure of Chern class* c *on* (X,g), *and* $\beta \in b$ *a general representant of the cohomology class* b. *Then the moduli space* $\mathcal{W}^\tau_\beta = \mathcal{W}^{\tau\,*}_\beta$ *is a closed manifold of dimension* $w_c = \frac{1}{4}(c^2 - 3\sigma(X) - 2e(X))$.

Fix a maximal subspace $H^2_+(X,\mathbb{R})$ of $H^2(X,\mathbb{R})$ on which the intersection form is positive definite. The dimension $b_+(X)$ of such a subspace is the number of positive eigenvalues of the intersection form. The moduli space $\mathcal{W}^\tau_\beta$ can be oriented by the choice of an orientation of the line $\det H^1(X,\mathbb{R}) \otimes \det H^2_+(X,\mathbb{R})^\vee$.

Let $[\mathcal{W}^\tau_\beta]_{\mathcal{O}} \in H_{w_c}(\mathcal{B}(c)^*,\mathbb{Z})$ be the <u>fundamental</u> <u>class</u> associated with the choice of an orientation $\mathcal{O}$ of the line $\det H^1(X,\mathbb{R}) \otimes \det H^2_+(X)^\vee$.

The <u>Seiberg</u>-<u>Witten</u> <u>form</u> associated with the data $(g,b,\mathfrak{c},\mathcal{O})$ is the element $\mathrm{SW}^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c}) \in \Lambda^* H^1(X,\mathbb{Z})$ defined by

$$\mathrm{SW}^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c})(l_1 \wedge \cdots \wedge l_r) := \left\langle \nu(l_1) \cup \cdots \cup \nu(l_r) \cup u^{(w_c - r)/2}, [\mathcal{W}^\tau_\beta]_{\mathcal{O}} \right\rangle$$

for decomposable elements $l_1 \wedge \cdots \wedge l_r$ with $r \equiv w_c \pmod 2$. Here τ is a Spin^c-structure on (X,g) representing the class $\mathfrak{c} \in \mathrm{Spin}^c(X)$, and β is a general form in the class b.

One shows, using again transversality arguments, that the Seiberg–Witten form $\mathrm{SW}^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c})$ is well defined, independent of the choices of τ and β. Moreover, if any two c-good pairs (g_0,b_0), (g_1,b_1) can be joined by a smooth path of c-good pairs, then $\mathrm{SW}^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c})$ is also independent of (g,b) [Okonek and Teleman 1996b].

Note that the condition "(g,b) is not c-good" is of codimension $b_+(X)$ for a fixed class c. This means that for manifolds with $b_+(X) > 1$ we have a well defined map

$$\mathrm{SW}_{X,\mathcal{O}} : \mathrm{Spin}^c(X) \longrightarrow \Lambda^* H^1(X,\mathbb{Z})$$

which associates to a class of Spin^c-structures $\mathfrak{c}$ the form $\mathrm{SW}^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c})$ for any $b \in H^2_{\mathrm{DR}}(X)$ such that (g,b) is c-good. This map, which is functorial with respect to orientation preserving diffeomorphisms, is the Seiberg–Witten invariant.

Using the identity in Lemma 1.1.1, one can easily prove the next result:

REMARK 1.2.4 [Witten 1994]. *Let X be an oriented closed 4-manifold with $b_+(X) > 1$. Then the set of classes $\mathfrak{c} \in \mathrm{Spin}^c(X)$ with nontrivial Seiberg–Witten invariant is finite.*

In the special case $b_+(X) > 1$, $b_1(X) = 0$, $\mathrm{SW}_{X,\mathcal{O}}$ is simply a function

$$\mathrm{SW}_{X,\mathcal{O}} : \mathrm{Spin}^c(X) \longrightarrow \mathbb{Z}\,.$$

The values $\mathrm{SW}_{X,\mathcal{O}}(\mathfrak{c}) \in \mathbb{Z}$ are refinements of the numbers $n^{\mathcal{O}}_c$ defined by Witten [1994]. More precisely:

$$n^{\mathcal{O}}_c = \sum_{\mathfrak{c}} \mathrm{SW}_{X,\mathcal{O}}(\mathfrak{c}),$$

the summation being over all classes of Spin^c-structures $\mathfrak{c}$ of Chern class c. It is easy to see that the indexing set is a torsor for the subgroup $\mathrm{Tors}_2\, H^2(X,\mathbb{Z})$ of 2-torsion classes in $H^2(X,\mathbb{Z})$.

The structure of the Seiberg–Witten invariants for manifolds with $b_+(X) = 1$ is more complicated and will be described in the next section.

1.3. The Case $b_+ = 1$ and the Wall Crossing Formula. Let X be a closed oriented differentiable 4-manifold with $b_+(X) = 1$. In this situation the Seiberg–Witten forms depend on a chamber structure: Recall first that in the case $b_+(X) = 1$ there is a natural map $\mathcal{M}\mathrm{et}(X) \longrightarrow \mathbb{P}(H^2_{\mathrm{DR}}(X))$ which sends a metric g to the line $\mathbb{R}[\omega_+] \subset H^2_{\mathrm{DR}}(X)$, where ω_+ is any nontrivial g-selfdual harmonic form. Let

$$\mathbf{H} := \{h \in H^2_{\mathrm{DR}}(X) : h^2 = 1\}$$

be the hyperbolic space. This space has two connected components, and the choice of one of them orients the lines $\mathbb{H}^2_{+,g}(X)$ of selfdual g-harmonic forms, for all metrics g. Furthermore, once we fix a component $\mathbf{H}_0$ of $\mathbf{H}$, every metric defines a unique g-selfdual form ω_g of length 1 with $[\omega_g] \in \mathbf{H}_0$; see Figure 1.

Let $c \in H^2(X,\mathbb{Z})$ be characteristic. The wall associated with c is the hypersurface

$$c^{\perp} := \{(h,b) \in \mathbf{H} \times H^2_{\mathrm{DR}}(X) : (c-b)\cdot h = 0\}\,,$$

and the connected components of $[\mathbf{H} \times H^2_{\mathrm{DR}}(X)] \setminus c^{\perp}$ are called chambers of type c.

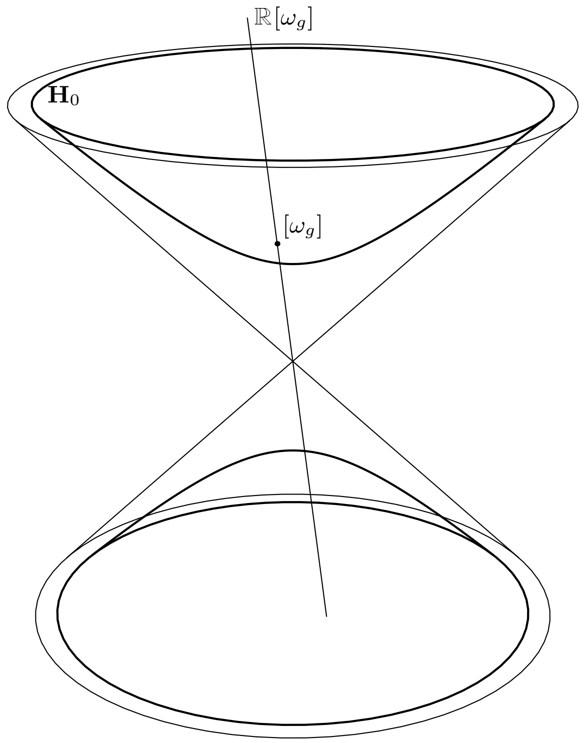

Figure 1.

Notice that the walls are nonlinear. Each characteristic element c defines precisely four chambers of type c, namely

$$C_{\mathbf{H}_0,\pm} := \{(h,b) \in \mathbf{H} \times H^2_{\mathrm{DR}}(X) : \pm(c-b)\cdot h < 0\},\ \mathbf{H}_0 \in \pi_0(\mathbf{H}),$$

and each of these four chambers contains elements of the form $([\omega_g], b)$ with $g \in \mathcal{M}\mathrm{et}(X)$.

Let $\mathcal{O}_1$ be an orientation of $H^1(X,\mathbb{R})$. The choice of $\mathcal{O}_1$ together with the choice of a component $\mathbf{H}_0 \in \pi_0(\mathbf{H})$ defines an orientation $\mathcal{O} = (\mathcal{O}_1, \mathbf{H}_0)$ of $\det(H^1(X,\mathbb{R})) \otimes \det(H^2_+(X,\mathbb{R})^\vee)$. Set

$$\mathrm{SW}^{\pm}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) := \mathrm{SW}^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c}),$$

where (g,b) is a pair such that $([\omega_g], b)$ belongs to the chamber $C_{\mathbf{H}_0,\pm}$. The map

$$\mathrm{SW}_{X,(\mathcal{O}_1,\mathbf{H}_0)} : \mathrm{Spin}^c(X) \longrightarrow \Lambda^* H^1(X,\mathbb{Z}) \times \Lambda^* H^1(X,\mathbb{Z})$$

which associates to a class $\mathfrak{c}$ of Spin^c-structures on the oriented manifold X the pair of forms $(\mathrm{SW}^{+}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}), \mathrm{SW}^{-}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}))$ is the <u>Seiberg</u>-<u>Witten</u> <u>invariant</u> of X with respect to the orientation data $(\mathcal{O}_1, \mathbf{H}_0)$. This invariant is functorial with respect to orientation-preserving diffeomorphisms and behaves as follows with respect to changes of the orientation data:

$$\mathrm{SW}_{X,(-\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) = -\,\mathrm{SW}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}), \quad \mathrm{SW}^{\pm}_{X,(\mathcal{O}_1,-\mathbf{H}_0)}(\mathfrak{c}) = -\,\mathrm{SW}^{\mp}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}).$$

More important, however, is the fact that the difference

$$\mathrm{SW}^{+}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) - \mathrm{SW}^{-}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})$$

is a topological invariant of the pair (X, c). To be precise, consider the element $u_c \in \Lambda^2\bigl(H_1(X,\mathbb{Z})/\operatorname{Tors}\bigr)$ defined by

$$u_c(a\wedge b) := \tfrac{1}{2}\bigl\langle a\cup b\cup c,\,[X]\bigr\rangle$$

for elements $a, b \in H^1(X,\mathbb{Z})$. The following universal wall-crossing formula generalizes results of [Witten 1994; Kronheimer and Mrowka 1994; Li and Liu 1995].

THEOREM 1.3.1 (WALL-CROSSING FORMULA [Okonek and Teleman 1996b]). *Let $l_{\mathcal{O}_1} \in \Lambda^{b_1}H^1(X,\mathbb{Z})$ be the generator defined by the orientation $\mathcal{O}_1$, and let $r \geq 0$ with $r \equiv w_c \pmod 2$. For every $\lambda \in \Lambda^r\bigl(H_1(X,\mathbb{Z})/\operatorname{Tors}\bigr)$ we have*

$$\bigl[\mathrm{SW}^{+}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) - \mathrm{SW}^{-}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})\bigr](\lambda) = \frac{(-1)^{(b_1-r)/2}}{\bigl(\frac{1}{2}(b_1-r)/2\bigr)!}\langle \lambda\wedge u_c^{(b_1-r)/2}, l_{\mathcal{O}_1}\rangle$$

when $r \leq \min(b_1, w_c)$, and the difference vanishes otherwise.

We illustrate these results with the simplest possible example, the projective plane.

EXAMPLE. Let $\mathbb{P}^2$ be the complex projective plane, oriented as a complex manifold, and denote by h the first Chern class of $\mathcal{O}_{\mathbb{P}^2}(1)$. Since $h^2 = 1$, the hyperbolic space $\mathbf{H}$ consists of two points $\mathbf{H} = \{\pm h\}$. We choose the component $\mathbf{H}_0 := \{h\}$ to define orientations.

An element $c \in H^2(\mathbb{P}^2,\mathbb{Z})$ is characteristic if and only if $c \equiv h \pmod 2$. In Figure 2 we have drawn (as vertical intervals) the two chambers

$$C_{\mathbf{H}_0,\pm} = \{(h,b)\in \mathbf{H}_0\times H^2_{\mathrm{DR}}(\mathbb{P}^2) : \pm(c-b)\cdot h < 0\}$$

of type c, for every $c \equiv h \pmod 2$.

The set $\mathrm{Spin}^c(\mathbb{P}^2)$ can be identified with the set $(2\mathbb{Z}+1)h$ of characteristic elements under the map which sends a Spin^c-structure $\mathfrak{c}$ to its Chern class c. The corresponding virtual dimension is $w_c = \frac{1}{4}(c^2-9)$. Note that, for any metric g, the pair $(g, 0)$ is c-good for all characteristic elements c. Also recall that the Fubini–Study metric g is a metric of positive scalar curvature which can be normalized such that $[\omega_g] = h$. We can now completely determine the Seiberg–Witten invariant $\mathrm{SW}_{\mathbb{P}^2,\mathbf{H}_0}$ using three simple arguments:

(i) For $c = \pm h$ we have $w_c < 0$, hence $\mathrm{SW}^{\pm}_{\mathbb{P}^2,\mathbf{H}_0}(c) = 0$, by the transversality results of Section 1.2.

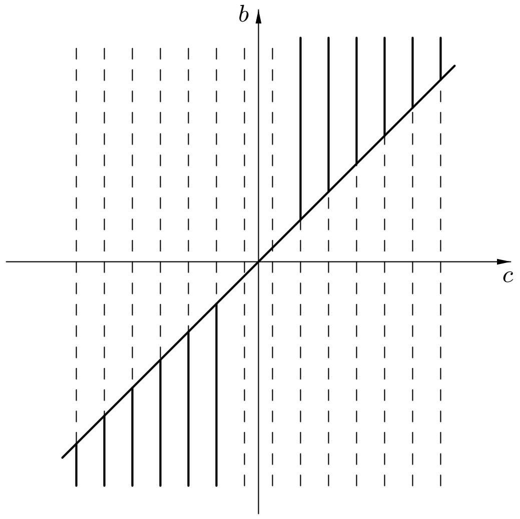

Figure 2.

(ii) Let c be a characteristic element with $w_c \geq 0$. Since the Fubini–Study metric g has positive scalar curvature and $(g, 0)$ is c-good, we have $\mathcal{W}_0^\tau = \mathcal{W}_0^{\tau *} = \varnothing$ by Corollary 1.1.2. But this moduli space can be used to compute $\mathrm{SW}^{\pm}_{\mathbb{P}^2,\mathbf{H}_0}(c)$ for characteristic elements c with $\pm c{\cdot}h < 0$. Thus we find $\mathrm{SW}^{\pm}_{\mathbb{P}^2,\mathbf{H}_0}(c) = 0$ when $w_c \geq 0$ and $\pm c{\cdot}h < 0$.

(iii) The remaining values, $\mathrm{SW}^{\mp}_{\mathbb{P}^2,\mathbf{H}_0}(c) = 0$ for classes satisfying $w_c \geq 0$ and $\pm c{\cdot}h < 0$, are determined by the wall-crossing formula. Altogether we get

$$\mathrm{SW}^{+}_{\mathbb{P}^2,\mathbf{H}_0}(c) = \begin{cases} 1 & \text{if } c{\cdot}h \geq 3, \\ 0 & \text{if } c{\cdot}h < 3, \end{cases} \qquad \mathrm{SW}^{-}_{\mathbb{P}^2,\mathbf{H}_0}(c) = \begin{cases} -1 & \text{if } c{\cdot}h \leq -3, \\ 0 & \text{if } c{\cdot}h > -3. \end{cases}$$

2. Nonabelian Seiberg–Witten Theory

2.1. G-Monopoles. Let V be a Hermitian vector space, and let $U(V)$ be its group of unitary automorphisms. For any closed subgroup $G \subset U(V)$ which contains the central involution $-\mathrm{id}_V$, we define a new Lie group by

$$\mathrm{Spin}^G(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}_2} G.$$

By construction one has the exact sequences

$$1 \longrightarrow \mathrm{Spin} \longrightarrow \mathrm{Spin}^G \xrightarrow{\delta} G/\mathbb{Z}_2 \longrightarrow 1,$$
$$1 \longrightarrow G \longrightarrow \mathrm{Spin}^G \xrightarrow{\pi} \mathrm{SO} \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^G \xrightarrow{(\pi,\delta)} \mathrm{SO} \times G/\mathbb{Z}_2 \longrightarrow 1,$$

where Spin, Spin^G, SO denote one of the groups $\mathrm{Spin}(n)$, $\mathrm{Spin}^G(n)$, $\mathrm{SO}(n)$, respectively.

Given a Spin^G-principal bundle P^G over a topological space, we form the following associated bundles:

$$\delta(P^G) := P^G \times_\delta (G/\mathbb{Z}_2), \quad \mathbb{G}(P^G) := P^G \times_{\mathrm{Ad}} G, \quad \mathfrak{g}(P^G) := P^G \times_{\mathrm{ad}} \mathfrak{g},$$

where $\mathfrak{g}$ stands for the Lie algebra of G. The group $\mathcal{G}$ of sections of the bundle $\mathbb{G}(P^G)$ can be identified with the group of automorphism of P^G over the associated SO-bundle $P^G \times_\pi \mathrm{SO}$.

Consider now an oriented manifold (X, g), and let P_g be the SO-bundle of oriented g-orthonormal coframes. A <u>Spin^G-structure</u> in P_g is a principal bundle morphism $\sigma : P^G \longrightarrow P_g$ of type π [Kobayashi and Nomizu 1963]. An isomorphism of Spin^G-structures σ, σ' in P_g is a bundle isomorphism $f : P^G \longrightarrow P'^G$ with $\sigma' \circ f = \sigma$. One shows that the data of a Spin^G-structure in (X, g) is equivalent to the data of a linear, orientation-preserving isometry $\gamma : \Lambda^1 \longrightarrow P^G \times_\pi \mathbb{R}^n$, which we call the <u>Clifford map</u> of the Spin^G-structure [Teleman 1997].

In dimension 4, the spinor group Spin(4) splits as

$$\mathrm{Spin}(4) = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- = \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- .$$

Using the projections

$$p_\pm : \mathrm{Spin}(4) \longrightarrow \mathrm{SU}(2)_\pm$$

one defines the adjoint bundles

$$\mathrm{ad}_\pm(P^G) := P^G \times_{\mathrm{ad}_\pm} \mathrm{su}(2).$$

Coupling $p_\pm$ with the natural representation of G in V, we obtain representations $\lambda_\pm : \mathrm{Spin}^G(4) \longrightarrow U(\mathbb{H}_\pm \otimes_\mathbb{C} V)$ and associated <u>spinor bundles</u>

$$\Sigma^\pm(P^G) := P^G \times_{\lambda_\pm} (\mathbb{H}_\pm \otimes_\mathbb{C} V).$$

The Clifford map $\gamma : \Lambda^1 \longrightarrow P^G \times_\pi \mathbb{R}^4$ of the Spin^G-structure yields identifications

$$\Gamma : \Lambda^2_\pm \longrightarrow \mathrm{ad}_\pm(P^G).$$

An interesting special case occurs when V is a Hermitian vector space over the quaternions and G is a subgroup of $\mathrm{Sp}(V) \subset U(V)$. Then one can define <u>real spinor bundles</u>

$$\Sigma^\pm_\mathbb{R}(P^G) := P^G \times_{\rho_\pm} (\mathbb{H}_\pm \otimes_\mathbb{H} V),$$

associated with the representations

$$\rho_\pm : \mathrm{Spin}^G(4) \longrightarrow \mathrm{SO}(\mathbb{H}_\pm \otimes_\mathbb{H} V).$$

Examples. Let (X, g) be a closed oriented Riemannian 4-manifold with coframe bundle P_g.

$G = S^1$: A Spin^{S^1}-structure is just a Spin^c-structure as described in Chapter 1 [Teleman 1997].

$G = \mathrm{Sp}(1)$: $\mathrm{Spin}^{\mathrm{Sp}(1)}$-structures have been introduced in [Okonek and Teleman 1996a], where they were called Spin^h-structures. The map which associates to a $\mathrm{Spin}^{\mathrm{Sp}(1)}$-structure $\sigma : P^h \longrightarrow P_g$ the first Pontrjagin class $p_1(\delta(P^h))$ of the associated SO(3)-bundle $\delta(P^h)$, induces a bijection between the set of isomorphism classes of $\mathrm{Spin}^{\mathrm{Sp}(1)}$-structures in (X, g) and the set

$$\{p \in H^4(X, \mathbb{Z}) : p \equiv w_2(X)^2 \pmod 4\}.$$

There is a 1-to-1 correspondence between isomorphism classes of $\mathrm{Spin}^{\mathrm{Sp}(1)}$-structures in (X, g) and equivalence classes of triples $(\tau : P^{S^1} \longrightarrow P_g, E, \iota)$ consisting of a Spin^{S^1}-structure τ, a unitary vector bundle E of rank 2, and an unitary isomorphism $\iota : \det \Sigma^+_\tau \longrightarrow \det E$. The equivalence relation is generated by tensorizing with Hermitian line bundles [Okonek and Teleman 1996a; Teleman 1997]. The associated bundles are — in terms of these data — given by

$$\delta(P^h) = P_E/S^1, \quad \mathbb{G}(P^h) = \mathrm{SU}(E), \quad \mathfrak{g}(P^h) = su(E),$$
$$\Sigma^{\pm}(P^h) = (\Sigma^{\pm}_\tau)^\vee \otimes E, \quad \Sigma^{\pm}_{\mathbb{R}}(P^h) = \mathbb{R}\,\mathrm{SU}(\Sigma^{\pm}_\tau, E),$$

where P_E denotes the principal $U(2)$-frame bundle of E.

$G = U(2)$: In this case $G/\mathbb{Z}_2$ splits as $G/\mathbb{Z}_2 = \mathrm{PU}(2) \times S^1$, and we write δ in the form $(\bar\delta, \det)$. The map which associates to a $\mathrm{Spin}^{U(2)}$-structure $\sigma : P^u \longrightarrow P_g$ the characteristic classes $p_1(\bar\delta(P^u))$, $c_1(\det P^u)$ identifies the set of isomorphism classes of $\mathrm{Spin}^{U(2)}$-structures in (X, g) with the set

$$\{(p, c) \in H^4(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) : p \equiv (w_2(X) + \bar c)^2 \pmod 4\}.$$

There is a one-to-one correspondence between isomorphism classes of $\mathrm{Spin}^{U(2)}$-structures in (X, g) and equivalence classes of pairs $(\tau : P^{S^1} \longrightarrow P_g, E)$ consisting of a Spin^{S^1}-structure τ and a unitary vector bundle of rank 2. Again the equivalence relation is given by tensorizing with Hermitian line bundles [Teleman 1997]. If $\sigma : P^u \longrightarrow P_g$ corresponds to the pair $(\tau : P^{S^1} \longrightarrow P_g, E)$, the associated bundles are now

$$\bar\delta(P^u) = P_E/S^1, \quad \det P^u = \det[\Sigma^+_\tau]^\vee \otimes \det E,$$
$$\mathbb{G}(P^u) = U(E), \quad \mathfrak{g}(P^u) = u(E), \quad \Sigma^{\pm}(P^u) = (\Sigma^{\pm}_\tau)^\vee \otimes E.$$

We will later also need the subbundles $\mathbb{G}_0(P^u) := P^u \times_{\mathrm{Ad}} \mathrm{SU}(2) \simeq \mathrm{SU}(E)$ and $\mathfrak{g}_0 := P^u \times_{\mathrm{ad}} \mathrm{su}(2) \simeq \mathrm{su}(E)$. The group of sections $\Gamma(X, \mathbb{G}_0)$ can be identified with the group of automorphisms of P^u over $P_g \times_X \det(P^u)$.

Now consider again a general Spin^G-structure $\sigma : P^G \longrightarrow P_g$ in the 4-manifold (X, g). The spinor bundle $\Sigma^{\pm}(P^G)$ has $\mathbb{H}_{\pm} \otimes_{\mathbb{C}} V$ as standard fiber, so that the standard fiber $su(2)_{\pm} \otimes \mathfrak{g}$ of the bundle $\mathrm{ad}_{\pm}(P^G) \otimes \mathfrak{g}(P^G)$ can be viewed as real subspace of $\mathrm{End}(\mathbb{H}_{\pm} \otimes_{\mathbb{C}} V)$. We define a quadratic map

$$\mu_{0G} : \mathbb{H}_{\pm} \otimes_{\mathbb{C}} V \longrightarrow \mathrm{su}(2)_{\pm} \otimes \mathfrak{g}$$

by sending $\psi \in \mathbb{H}_\pm \otimes_{\mathbb{C}} V$ to the orthogonal projection $pr_{su(2)_\pm \otimes \mathfrak{g}}(\psi \otimes \bar{\psi})$ of the Hermitian endomorphism $(\psi \otimes \bar{\psi}) \in \mathrm{End}(\mathbb{H}_\pm \otimes_{\mathbb{C}} V)$. One can show that $-\mu_{0G}$ is the total (hyperkähler) moment map for the G-action on the space $\mathbb{H}_\pm \otimes_{\mathbb{C}} V$ endowed with the natural hyperkähler structure given by left multiplication with quaternionic units [Teleman 1997].

These maps give rise to quadratic bundle maps

$$\mu_{0G} : \Sigma^\pm(P^G) \longrightarrow \mathrm{ad}_\pm(P^G) \otimes \mathfrak{g}(P^G).$$

In the case $G = U(2)$ one can project $\mu_{0U(2)}$ on $\mathrm{ad}_\pm(P^G) \otimes \mathfrak{g}_0(P^G)$ and gets a map

$$\mu_{00} : \Sigma^\pm(P^u) \longrightarrow \mathrm{ad}_\pm(P^u) \otimes \mathfrak{g}_0(P^u).$$

Note that a fixed Spin^G-structure $\sigma : P^G \longrightarrow P_g$ defines a bijection between connections $A \in \mathcal{A}(\delta(P^G))$ in $\delta(P^G)$ and connections $\hat{A} \in \mathcal{A}(P^G)$ in the Spin^G-bundle P^G which lift the Levi-Civita connection in P_g via σ. This follows immediately from the third exact sequence above. Let

$$\not{D}_A : A^0(\Sigma^\pm(P^G)) \longrightarrow A^0(\Sigma^\mp(P^G))$$

be the associated <u>Dirac</u> <u>operator</u>, defined by

$$\not{D}_A : A^0(\Sigma^\pm(P^G)) \xrightarrow{\nabla_{\hat{A}}} A^1(\Sigma^\pm(P^G)) \xrightarrow{\gamma} A^0(\Sigma^\mp(P^G)).$$

Here $\gamma : \Lambda^1 \otimes \Sigma^\pm(P^G) \longrightarrow \Sigma^\mp(P^G)$ is the Clifford multiplication corresponding to the embeddings $\gamma : \Lambda^1 \longrightarrow P^G \times_\pi \mathbb{R}^4 \subset \mathrm{Hom}_{\mathbb{C}}(\Sigma^\pm(P^G), \Sigma^\mp(P^G))$.

Definition 2.1.1. Let $\sigma : P^G \longrightarrow P_g$ be a Spin^G-structure in the Riemannian manifold (X, g). The <u>G-monopole</u> <u>equations</u> for a pair (A, Ψ), with $A \in \mathcal{A}(\delta(P^G))$ and $\Psi \in A^0(\Sigma^+(P^G))$, are

$$\begin{aligned} \not{D}_A \Psi &= 0, \\ \Gamma(F_A^+) &= \mu_{0G}(\Psi). \end{aligned} \tag{SW$^\sigma$}$$

The solutions of these equations will be called <u>G-monopoles</u>. The symmetry group of the G-monopole equations is the gauge group $\mathcal{G} := \Gamma(X, \mathbb{G}(P^G))$. If the Lie algebra of G has a nontrivial center $z(\mathfrak{g})$, then one has a family of $\mathcal{G}$-equivariant "twisted" G-monopole equations $(\mathrm{SW}^\sigma_\beta)$ parameterized by $iz(\mathfrak{g})$-valued 2-forms $\beta \in A^2(iz(\mathfrak{g}))$:

$$\begin{aligned} \not{D}_A \Psi &= 0, \\ \Gamma((F_A + 2\pi i \beta)^+) &= \mu_{0G}(\Psi). \end{aligned} \tag{SW$^\sigma_\beta$}$$

We denote by $\mathcal{M}^\sigma$ and $\mathcal{M}^\sigma_\beta$, respectively, the corresponding moduli spaces of solutions modulo the gauge group $\mathcal{G}$.

Since in the case $G = U(2)$ there exists the splitting

$$U(2)/\mathbb{Z}_2 = \mathrm{PU}(2) \times S^1,$$

the data of a connection in $\delta(P^u) = \bar{\delta}(P^u) \times_X \det P^u$ is equivalent to the data of a pair of connections $(A, a) \in \mathcal{A}(\bar{\delta}(P^u)) \times \mathcal{A}(\det P^u)$. This can be used to introduce new important equations, obtained by fixing the abelian connection $a \in \mathcal{A}(\det P^u)$ in the $U(2)$-monopole equations, and regarding it as a parameter. One gets in this way the equations

$$\begin{aligned} \not{D}_{A,a}\Psi &= 0, \\ \Gamma(F_A^+) &= \mu_{00}(\Psi) \end{aligned} \tag{SW_a^σ}$$

for a pair

$$(A, \Psi) \in \mathcal{A}(\bar{\delta}(P^u)) \times A^0(\Sigma^+(P^u)),$$

which will be called the <u>PU(2)-monopole equations</u>. These equations should be regarded as a twisted version of the quaternionic monopole equations introduced in [Okonek and Teleman 1996a], which coincide in our present framework with the SU(2)-monopole equations. Indeed, a $\mathrm{Spin}^{U(2)}$-structure $\sigma : P^u \longrightarrow P_g$ with trivialized determinant line bundle can be regarded as $\mathrm{Spin}^{\mathrm{SU}(2)}$-structure, and the corresponding quaternionic monopole equations are ($\mathrm{SW}_\theta^\sigma$), where θ is the trivial connection in $\det P^u$.

The PU(2)-monopole equations are only invariant under the group $\mathcal{G}_0 := \Gamma(X, \mathbb{G}_0)$ of automorphisms of P^u over $P_g \times_X \det P^u$. We denote by $\mathcal{M}_a^\sigma$ the <u>moduli space</u> of PU(2)-monopoles modulo this gauge group. Note that $\mathcal{M}_a^\sigma$ comes with a natural <u>S^1-action</u> given by the formula $\zeta \cdot [A, \Psi] := [A, \zeta^{\frac{1}{2}}\Psi]$.

<u>Comparing with other formalisms:</u>

1. For $G = S^1$, $V = \mathbb{C}$ one recovers the original abelian Seiberg–Witten equations and the twisted abelian Seiberg–Witten equations of [LeBrun 1995b; Brussee 1996; Okonek and Teleman 1996b].
2. For $G = S^1$, $V = \mathbb{C}^{\oplus k}$ one gets the so called “multimonopole equations” studied by J. Bryan and R. Wentworth [1996].
3. In the case $G = U(2)$, $V = \mathbb{C}^2$ one obtains the $U(2)$-monopole equations which were studied in [Okonek and Teleman 1995a] (see also Chapter 3).
4. In the case of a Spin-manifold X and $G = \mathrm{SU}(2)$ the corresponding monopole equations were introduced in [Okonek and Teleman 1995c]; they have been studied from a physical point of view in [Labastida and Mariño 1995].
5. If X is simply connected, the S^1-quotient $\mathcal{M}_a^\sigma / S^1$ of a moduli space of PU(2)-monopoles can be identified with a moduli space of “nonabelian monopoles” as defined in [Pidstrigach and Tyurin 1995]. Note that in the general non-simply connected case, one has to use our formalism.

Remark 2.1.2. Let $G = \mathrm{Sp}(n) \cdot S^1 \subset U(\mathbb{C}^{2n})$ be the Lie group of transformations of $\mathbb{H}^{\oplus n}$ generated by left multiplication with quaternionic matrices in $\mathrm{Sp}(n)$ and by right multiplication with complex numbers of modulus 1. Then $G/\mathbb{Z}_2$ splits as $\mathrm{PSp}(n) \times S^1$. In the same way as in the PU(2)-case one defines the

$\mathrm{PSp}(n)$-monopole equations (SW^σ_a) associated with a $\mathrm{Spin}^{\mathrm{Sp}(n)\cdot S^1}(4)$-structure $\sigma : P^G \longrightarrow P_g$ in (X, g) and an abelian connection a in the associated S^1-bundle.

The solutions of the (twisted) G- and $\mathrm{PU}(2)$-monopole equations are the absolute minima of certain gauge invariant functionals on the corresponding configuration spaces $\mathcal{A}(\delta(P^G)) \times A^0(\Sigma^+(P^G))$ and $\mathcal{A}(\bar\delta(P^u)) \times A^0(\Sigma^+(P^G))$.

For simplicity we describe here only the case of nontwisted G-monopoles. The Seiberg-Witten functional $\mathrm{SW}^\sigma : \mathcal{A}(\delta(P^G)) \times A^0(\Sigma^+(P^G)) \longrightarrow \mathbb{R}$ associated to a Spin^G-structure is defined by

$$\mathrm{SW}^\sigma(A, \Psi) := \|\nabla_{\hat A}\Psi\|^2 + \tfrac14\|F_A\|^2 + \tfrac12\|\mu_{0G}(\Psi)\|^2 + \tfrac14\int_X s|\Psi|^2.$$

The Euler-Lagrange equations describing general critical points are

$$\begin{aligned} d_A^* F_A + J(A, \Psi) &= 0,\\ \Delta_{\hat A}\Psi + \mu_{0G}(\Psi)(\Psi) + \tfrac14 s\Psi &= 0, \end{aligned}$$

where the current $J(A, \Psi) \in A^1(\mathfrak{g}(P^G))$ is given by $\sqrt{32}$ times the orthogonal projection of the $\mathrm{End}(\Sigma^+(P^G))$-valued 1-form $\nabla_{\hat A}\Psi \otimes \bar\Psi \in A^1(\mathrm{End}(\Sigma^+(P^G)))$ onto $A^1(\mathfrak{g}(P^G))$.

In the abelian case $G = S^1$, $V = \mathbb{C}$, a closely related functional and the corresponding Euler–Lagrange equations have been investigated in [Jost et al. 1996].

2.2. Moduli Spaces of $\mathrm{PU}(2)$-Monopoles. We retain the notations of the previous section. Let $\sigma : P^u \longrightarrow P_g$ be a $\mathrm{Spin}^{U(2)}$-structure in a closed oriented Riemannian 4-manifold (X, g), and let $a \in \mathcal{A}(\det P^u)$ be a fixed connection. The $\mathrm{PU}(2)$-monopole equations

$$\begin{aligned} \not{D}_{A,a}\Psi &= 0,\\ \Gamma(F_A^+) &= \mu_{00}(\Psi) \end{aligned} \qquad (\mathrm{SW}^\sigma_a)$$

associated with these data are invariant under the action of the gauge group $\mathcal{G}_0$, and hence give rise to a closed subspace $\mathcal{M}^\sigma_a \subset \mathcal{B}(P^u)$ of the orbit space $\mathcal{B}(P^u) := \big(\mathcal{A}(\bar\delta(P^u)) \times A^0(\Sigma^+(P^u))\big)/\mathcal{G}_0$.

The moduli space $\mathcal{M}^\sigma_a$ can be endowed with the structure of a ringed space with local models constructed by the well-known Kuranishi method [Okonek and Teleman 1995a; 1996a; Donaldson and Kronheimer 1990; Lübke and Teleman 1995]. More precisely: The linearization of the $\mathrm{PU}(2)$-monopole equations in a solution $p = (A, \Psi)$ defines an elliptic deformation complex

$$0 \to A^0(\mathfrak{g}_0(P^u)) \xrightarrow{D_p^0} A^1(\mathfrak{g}_0(P^u)) \oplus A^0(\Sigma^+(P^u)) \xrightarrow{D_p^1} A^2_+(\mathfrak{g}_0(P^u)) \oplus A^0(\Sigma^-(P^u)) \to 0$$

whose differentials are given by $D_p^0(f) = (-d_A f, f\Psi)$ and

$$D_p^1(\alpha, \psi) = (d_A^+\alpha - \Gamma^{-1}[m(\psi, \Psi) + m(\Psi, \psi)], \not{D}_{A,a}\psi + \gamma(\alpha)\Psi).$$

Here m denotes the sesquilinear map associated with the quadratic map μ_{00}. Let $\mathbb{H}^i_p$, for $i=0,1,2$, denote the harmonic spaces of the elliptic complex above. The stabilizer $\mathcal{G}_{0p}$ of the point $p\in\mathcal{A}(\bar\delta(P^u))\times A^0(\Sigma^+(P^u))$ is a finite dimensional Lie group, isomorphic to a closed subgroup of SU(2), which acts in a natural way on the spaces $\mathbb{H}^i_p$.

PROPOSITION 2.2.1 [Okonek and Teleman 1996a; Teleman 1997]. *For every point $p\in\mathcal{M}^\sigma_a$ there exists a neighborhood $V_p\subset\mathcal{M}^\sigma_a$, a $\mathcal{G}_{0p}$-invariant neighborhood U_p of $0\in\mathbb{H}^1_p$, an $\mathcal{G}_{0p}$-equivariant map $K_p:U_p\longrightarrow\mathbb{H}^2_p$ with $K_p(0)=0$ and $dK_p(0)=0$, and an isomorphism of ringed spaces*

$$V_p\simeq Z(K_p)/\mathcal{G}_{0p}$$

sending p to $[0]$.

The local isomorphisms $V_p\simeq Z(K_p)/\mathcal{G}_{0p}$ define the structure of a smooth manifold on the open subset

$$\mathcal{M}^\sigma_{a,\mathrm{reg}}:=\{[A,\Psi]\in\mathcal{M}^\sigma_a:\mathcal{G}_{0p}=\{1\},\ \mathbb{H}^2_p=\{0\}\},$$

and a real analytic orbifold structure in the open set of points $p\in\mathcal{M}^\sigma_a$ with $\mathcal{G}_{0p}$ finite. The dimension of $\mathcal{M}^\sigma_{a,\mathrm{reg}}$ coincides with the expected dimension of the PU(2)-monopole moduli space, which is given by the index $\chi(\mathrm{SW}^\sigma_a)$ of the elliptic deformation complex:

$$\chi(\mathrm{SW}^\sigma_a)=\tfrac12(-3p_1(\bar\delta(P^u))+c_1(\det P^u)^2)-\tfrac12(3e(X)+4\sigma(X)).$$

Our next goal is to describe the fixed point set of the S^1-action on $\mathcal{M}^\sigma_a$ introduced above.

First consider the closed subspace $\mathcal{D}(\bar\delta(P^u))\subset\mathcal{M}^\sigma_a$ of points of the form $[A,0]$. It can be identified with the Donaldson moduli space of anti-selfdual connections in the PU(2)-bundle $\bar\delta(P^u)$ modulo the gauge group $\mathcal{G}_0$. Note however, that if $H^1(X,\mathbb{Z}_2)\neq\{0\}$, $\mathcal{D}(\bar\delta(P^u))$ does not coincide with the usual moduli space of PU(2)-instantons in $\bar\delta(P^u)$ but is a finite cover of it.

The stabilizer $\mathcal{G}_{0p}$ of a Donaldson point $(A,0)$ contains always $\{\pm\mathrm{id}\}$, hence $\mathcal{M}^\sigma_a$ has at least $\mathbb{Z}_2$-orbifold singularities in the points of $\mathcal{D}(\bar\delta(P^u))$.

Secondly, consider S^1 as a subgroup of PU(2) via the standard embedding $S^1\ni\zeta\longmapsto\left[\left(\begin{smallmatrix}\zeta&0\\0&1\end{smallmatrix}\right)\right]\in\mathrm{PU}(2)$. Note that any S^1-reduction $\rho:P\longrightarrow\bar\delta(P^u)$ of $\bar\delta(P^u)$ defines a reduction $\tau_\rho:P^\rho:=P^u\times_{\bar\delta(P^u)}P\longrightarrow P^u\xrightarrow{\sigma}P_g$ of the $\mathrm{Spin}^{U(2)}$-structure σ to a $\mathrm{Spin}^{S^1\times S^1}$-structure, hence a pair of Spin^c-structures $\tau^i_\rho:P^{\rho_i}\longrightarrow P_g$. One has natural isomorphisms

$$\det P^{\rho_1}\otimes\det P^{\rho_2}=(\det P^u)^{\otimes 2},\quad \det P^{\rho_1}\otimes(\det P^{\rho_2})^{-1}=P^{\otimes 2},$$

and natural embeddings $\Sigma^\pm(P^{\rho_i})\longrightarrow\Sigma^\pm(P^u)$ induced by the bundle morphism $P^{\rho_i}\longrightarrow P^u$. A pair (A,Ψ) will be called abelian if it lies in the image of $\mathcal{A}(P)\times A^0(\Sigma^+(P^{\rho_1}))$ for a suitable S^1-reduction ρ of $\bar\delta(P^u)$.

PROPOSITION 2.2.2. *The fixed point set of the S^1-action on $\mathcal{M}^\sigma_a$ is the union of the Donaldson locus $\mathcal{D}(\bar\delta(P^u))$ and the locus of abelian solutions. The latter can be identified with the disjoint union*

$$\coprod_\rho \mathcal{W}^{\tau^1_\rho}_{\frac{i}{2\pi}F_a},$$

where the union is over all S^1-reductions of the $\mathrm{PU}(2)$*-bundle* $\bar\delta(P^u)$.

This result suggests to use the S^1-quotient of $\mathcal{M}^\sigma_a \setminus (\mathcal{M}^\sigma_a)^{S^1}$ for the comparison of Donaldson invariants and (twisted) Seiberg–Witten invariants, as explained in [Okonek and Teleman 1996a].

Note that only using moduli spaces $\mathcal{M}^\sigma_\theta$ of quaternionic monopoles one gets, by the proposition above, moduli spaces of <u>non-twisted</u> abelian monopoles in the fixed point locus of the S^1-action. This was one of the motivations for studying the quaternionic monopole equations in [Okonek and Teleman 1996a]. There it has been shown that one can use the moduli spaces of quaternionic monopoles to relate certain Spin^c-polynomials to the original nontwisted Seiberg–Witten invariants.

The remainder of this section is devoted to the description of the <u>Uhlenbeck compactification</u> of the moduli spaces of PU(2)-monopoles [Teleman 1996].

First of all, the Weitzenböck formula and the maximum principle yield a bound on the spinor component, as in the abelian case. More precisely, one has the a priori estimate

$$\sup_X |\Psi|^2 \le C_{g,a} := \max\bigl(0,\ C\sup(-\frac{1}{2}s + |F_a^+|)\bigr)$$

on the space of solutions of (SW^σ_a), where C is a universal positive constant.

The construction of the Uhlenbeck compactification of $\mathcal{M}^\sigma_a$ is based, as in the instanton case, on the following three essential results.

1. A <u>compactness</u> theorem for the subspace of solutions with suitable bounds on the curvature of the connection component.
2. A <u>removable singularities</u> theorem.
3. Controlling <u>bubbling</u> phenomena for an arbitrary sequence of points in the moduli space $\mathcal{M}^\sigma_a$.

1. A compactness result.

THEOREM 2.2.3. *There exists a positive number $\delta > 0$ such that for every oriented Riemannian manifold (Ω, g) endowed with a $\mathrm{Spin}^{U(2)}(4)$-structure $\sigma : P^u \longrightarrow P_g$ and a fixed connection $a \in \mathcal{A}(\det P^u)$, the following holds*:

If (A_n, Ψ_n) is a sequence of solutions of (SW^σ_a), *such that any point $x \in \Omega$ has a geodesic ball neighborhood D_x with*

$$\int_{D_x} |F_{A_n}|^2 < \delta^2$$

for all large enough n, then there is a subsequence $(n_m) \subset \mathbb{N}$ and gauge transformations $f_m \in \mathcal{G}_0$ such that $f_m^*(A_{n_m}, \Psi_{n_m})$ converges in the $\mathcal{C}^\infty$-topology on Ω.

2. Removable singularities. Let g be a metric on the 4-ball B, and let

$$\sigma : P^u = B \times \mathrm{Spin}^{U(2)}(4) \longrightarrow P_g \simeq B \times \mathrm{SO}(4)$$

be a $\mathrm{Spin}^{U(2)}$-structure in (B, g). Fix $a \in iA^1_B$ and put $B^\bullet := B \setminus \{0\}$, $\sigma^\bullet := \sigma|_{B^\bullet}$.

THEOREM 2.2.4. *Let (A_0, Ψ_0) be a solution of the equations $(\mathrm{SW}^{\sigma^\bullet}_a)$ on the punctured ball such that*

$$\|F_{A_0}\|^2_{L^2} < \infty .$$

Then there exists a solution (A, Ψ) of (SW^σ_a) on B and a gauge transformation $f \in \mathcal{C}^\infty(B^\bullet, \mathrm{SU}(2))$ such that $f^*(A|_{B^\bullet}, \Psi|_{B^\bullet}) = (A_0, \Psi_0)$.

3. Controlling bubbling phenomena. The main point is that the selfdual components $F^+_{A_n}$ of the curvatures of a sequence of solutions $([A_n, \Psi_n])_{n\in\mathbb{N}}$ in $\mathcal{M}^\sigma_a$ cannot bubble.

DEFINITION 2.2.5. Let $\sigma : P^u \longrightarrow P_g$ be a $\mathrm{Spin}^{U(2)}$-structure in (X, g) and fix $a \in \mathcal{A}(\det P^u)$. An <u>ideal</u> <u>monopole</u> of type (σ, a) is a pair $([A', \Psi'], \{x_1, \dots, x_l\})$ consisting of a point $[A', \Psi'] \in \mathcal{M}^{\sigma'_l}_a$, where $\sigma'_l : P'^u \longrightarrow P_g$ is a $\mathrm{Spin}^{U(2)}$-structure satisfying

$$\det P'^u = \det P^u , \quad p_1(\bar\delta(P'^u)) = p_1(\bar\delta(P^u)) + 4l ,$$

and $\{x_1, \dots, x_l\} \in S^l X$. The set of ideal monopoles of type (σ, a) is

$$I\mathcal{M}^\sigma_a := \coprod_{l\geq 0} \mathcal{M}^{\sigma'_l}_a \times S^l X .$$

THEOREM 2.2.6. *There exists a metric topology on $I\mathcal{M}^\sigma_a$ such that the moduli space $\mathcal{M}^\sigma_a$ becomes an open subspace with compact closure $\overline{\mathcal{M}^\sigma_a}$.*

SKETCH OF PROOF. Given a sequence $([A_n, \Psi_n])_{n\in\mathbb{N}}$ of points in $\mathcal{M}^\sigma_a$, one finds a subsequence $([A_{n_m}, \Psi_{n_m}])_{m\in\mathbb{N}}$, a finite set of points $S \subset X$, and gauge transformations f_m such that $(B_m, \Phi_m) := f_m^*(A_{n_m}, \Psi_{n_m})$ converges on $X \setminus S$ in the $\mathcal{C}^\infty$-topology to a solution (A_0, Ψ_0). This follows from the compactness theorem above, using the fact that the total volume of the sequence of measures $|F_{A_n}|^2$ is bounded. The set S consists of points in which the measure $|F_{A_{n_m}}|^2$ becomes concentrated as m tends to infinity.

By the Removable Singularities theorem, the solution (A_0, Ψ_0) extends after gauge transformation to a solution (A, Ψ) of $(\mathrm{SW}^{\sigma'}_a)$ on X, for a possibly different $\mathrm{Spin}^{U(2)}$-structure σ' with the same determinant line bundle. The curvature of A satisfies

$$|F_A|^2 = \lim_{m\to\infty} |F_{A_{n_m}}|^2 - 8\pi^2 \sum_{x\in S} \lambda_x \delta_x ,$$

where δ_x is the Dirac measure of the point x. Now it remains to show that the λ_x's are natural numbers and

$$\sum_{x\in S}\lambda_x = \tfrac{1}{4}(p_1(\bar\delta(P'^u)) - p_1(\bar\delta(P^u))).$$

This follows as in the instanton case, if one uses the fact that the measures $|F^+_{A_{n_m}}|^2$ cannot bubble in the points $x \in S$ as $m \to \infty$ and that the integral of $|F^-_{A_n}|^2 - |F^+_{A_n}|^2$ is a topological invariant of $\bar\delta(P^u)$. In this way one gets an ideal monopole $\mathfrak{m} := ([A,\Psi],\{\lambda_1 x_1,\dots,\lambda_k x_k\})$ of type (σ,a). With respect to a suitable topology on the space of ideal monopoles, one has $\lim\limits_{m\to\infty}[A_{n_m},\Psi_{n_m}] = \mathfrak{m}$. $\square$

3. Seiberg–Witten Theory and Kähler Geometry

3.1. Monopoles on Kähler Surfaces. Let (X,J,g) be an almost Hermitian surface with associated Kähler form ω_g. We denote by Λ^{pq} the bundle of (p,q)-forms on X and by A^{pq} its space of sections. The Hermitian structure defines an orthogonal decomposition

$$\Lambda^2_+\otimes\mathbb{C} = \Lambda^{20}\oplus\Lambda^{02}\oplus\Lambda^{00}\omega_g$$

and a canonical Spin^c-structure τ. The spinor bundles of τ are

$$\Sigma^+ = \Lambda^{00}\oplus\Lambda^{02},\quad \Sigma^- = \Lambda^{01},$$

and the Chern class of τ is the first Chern class $c_1(T^{10}_J) = c_1(K_X^\vee)$ of the complex tangent bundle. The complexification of the canonical Clifford map γ is the standard isomorphism

$$\gamma:\Lambda^1\otimes\mathbb{C}\longrightarrow \mathrm{Hom}(\Lambda^{00}\oplus\Lambda^{02},\Lambda^{01}),\quad \gamma(u)(\varphi+\alpha) = \sqrt{2}(\varphi u^{01} - i\Lambda_g u^{10}\wedge\alpha),$$

and the induced isomorphism $\Gamma:\Lambda^{20}\oplus\Lambda^{02}\oplus\Lambda^{00}\omega_g\longrightarrow \mathrm{End}_0(\Lambda^{00}\oplus\Lambda^{02})$ acts by

$$(\lambda^{20},\lambda^{02},f\omega_g)\overset{\Gamma}{\longmapsto} 2\begin{bmatrix} -if & -*(\lambda^{20}\wedge\cdot)\\ \lambda^{02}\wedge\cdot & if\end{bmatrix}\in \mathrm{End}_0(\Lambda^{00}\oplus\Lambda^{02}).$$

Recall from Section 1.1 that the set Spin$^c(X)$ of equivalence classes of Spin^c-structures in (X,g) is a $H^2(X,\mathbb{Z})$-torsor. Using the class of the canonical Spin^c-structure $\mathfrak{c} := [\tau]$ as base point, Spin$^c(X)$ can be identified with the set of isomorphism classes of S^1-bundles: When M is an S^1-bundle with $c_1(M) = m$, the Spin^c-structure τ_m has spinor bundles $\Sigma^\pm\otimes M$ and Chern class $2c_1(M) - c_1(K_X)$. Let $\mathfrak{c}_m$ be the class of τ_m.

Suppose now that (X,J,g) is Kähler, and let $k\in\mathcal{A}(K_X)$ be the Chern connection in the canonical line bundle. In order to write the (abelian) Seiberg–Witten equations associated with the Spin^c-structure τ_m in a convenient form, we make the variable substitution $a = k\otimes e^{\otimes 2}$ for a connection $e\in\mathcal{A}(M)$ in the S^1-bundle M, and we write the spinor Ψ as a sum $\Psi = \varphi+\alpha\in A^0(M)\oplus A^{02}(M)$.

LEMMA 3.1.1 [Witten 1994; Okonek and Teleman 1996b]. *Let* (X, g) *be a Kähler surface,* $\beta \in A^{11}_{\mathbb{R}}$ *a closed real* (1,1)*-form in the de Rham cohomology class* b, *and let* M *be a* S^1*-bundle with* $(2c_1(M) - c_1(K_X) - b)\cdot[\omega_g] < 0$. *The pair* $(k \otimes e^{\otimes 2}, \varphi + \alpha) \in \mathcal{A}(\det(\Sigma^+ \otimes M)) \times A^0(\Sigma^+ \otimes M)$ *solves the equations* $(\mathrm{SW}^{\tau_m}_{\beta})$ *if and only if* $\alpha = 0$, $F^{20}_e = F^{02}_e = 0$, $\bar\partial_e \varphi = 0$, *and*

$$i\Lambda_g F_e + \tfrac{1}{4}\varphi\bar\varphi + (\tfrac{1}{2}s - \pi\Lambda_g\beta) = 0. \qquad (*)$$

Note that the conditions $F^{20}_e = F^{02}_e = 0$, $\bar\partial_e\varphi = 0$ mean that e is the Chern connection of a holomorphic structure in the Hermitian line bundle M and that φ is a holomorphic section with respect to this holomorphic structure. Integrating the relation $(*)$ and using the inequality in the hypothesis, one sees that φ cannot vanish identically.

To interpret the condition $(*)$ consider an arbitrary real valued function function $t : X \longrightarrow \mathbb{R}$, and let

$$m_t : \mathcal{A}(M) \times A^0(M) \longrightarrow iA^0$$

be the map defined by

$$m_t(e, \varphi) := \Lambda_g F_e - \tfrac{1}{4} i\varphi\bar\varphi + it.$$

It is easy to see that (after suitable Sobolev completions) $\mathcal{A}(M) \times A^0(M)$ has a natural symplectic structure, and that m_t is a moment map for the action of the gauge group $\mathcal{G} = \mathcal{C}^\infty(X, S^1)$. Let $\mathcal{G}^{\mathbb{C}} = \mathcal{C}^\infty(X, \mathbb{C}^*)$ be the complexification of $\mathcal{G}$, and let $\mathcal{H} \subset \mathcal{A}(M) \times A^0(M)$ be the closed set

$$\mathcal{H} := \{(e, \varphi) \in \mathcal{A}(M) \times A^0(M) : F^{02}_e = 0,\ \bar\partial_e\varphi = 0\}$$

of integrable pairs. For any function t put

$$\mathcal{H}_t := \{(e, \varphi) \in \mathcal{H} : \mathcal{G}^{\mathbb{C}}(e, \varphi) \cap m_t^{-1}(0) \neq \varnothing\}.$$

Using a general principle in the theory of symplectic quotients, which also holds in our infinite dimensional framework, one can prove that the $\mathcal{G}^{\mathbb{C}}$-orbit of a point $(e, \varphi) \in \mathcal{H}_t$ intersects the zero set $m_t^{-1}(0)$ of the moment map m_t precisely along a $\mathcal{G}$-orbit (see Figure 3). In other words, there is a natural bijection of quotients

$$[m_t^{-1}(0) \cap \mathcal{H}]/\mathcal{G} \simeq \mathcal{H}_t/\mathcal{G}^{\mathbb{C}}. \qquad (1)$$

Now take $t := -(\frac{1}{2}s - \pi\Lambda_g\beta)$ and suppose again that the assumptions in Lemma 3.1.1 hold. We have seen that $m_t^{-1}(0) \cap \mathcal{H}$ cannot contain pairs of the form $(e, 0)$, hence $\mathcal{G}$ ($\mathcal{G}^{\mathbb{C}}$) acts freely on $m_t^{-1}(0) \cap \mathcal{H}$ ($\mathcal{H}_t$). Using this fact one can show that $\mathcal{H}_t$ is open in the space $\mathcal{H}$ of integrable pairs, and endowing the two quotients in (1) with the natural real analytic structures, one proves that (1) is a real analytic isomorphism. By the lemma, the first quotient is precisely the moduli space $\mathcal{W}^{\tau_m}_{\beta}$. The second quotient is a complex-geometric object, namely an open subspace in the moduli space of simple holomorphic pairs $\mathcal{H} \cap \{\varphi \neq 0\}/\mathcal{G}^{\mathbb{C}}$. A point in this moduli space can be regarded as an

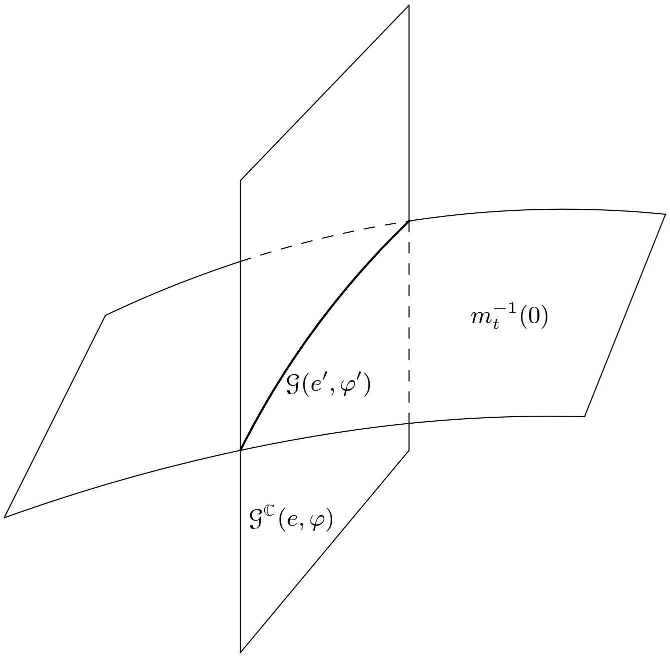

Figure 3.

isomorphism class of pairs $(\mathcal{M}, \varphi)$ consisting of a holomorphic line bundle $\mathcal{M}$ of topological type M, and a holomorphic section in $\mathcal{M}$. Such a pair defines a point in $\mathcal{H}_t/\mathcal{G}^{\mathbb{C}}$ if and only if $\mathcal{M}$ admits a Hermitian metric h satisfying the equation

$$i\Lambda F_h + \tfrac{1}{4}\varphi\bar{\varphi}^h = t.$$

This equation for the unknown metric h is the <u>vortex</u> <u>equation</u> associated with the function t: it is solvable if and only if the <u>stability</u> <u>condition</u>

$$(2c_1(M) - c_1(K_X) - b)\cdot[\omega_g] < 0$$

is fulfilled. Let $\mathcal{D}\mathrm{ou}(m)$ be the Douady space of effective divisors $D \subset X$ with $c_1(\mathcal{O}_X(D)) = m$. The map $Z : \mathcal{H}_t/\mathcal{G}^{\mathbb{C}} \longrightarrow \mathcal{D}\mathrm{ou}(m)$ which associates to an orbit $[e, \varphi]$ the zero-locus $Z(\varphi) \subset X$ of the holomorphic section φ is an isomorphism of complex spaces.

Putting everything together, we have the following interpretation for the monopole moduli spaces $\mathcal{W}_\beta^{\tau_m}$ on Kähler surfaces.

THEOREM 3.1.2 [Okonek and Teleman 1995a; 1996b]. *Let (X, g) be a compact Kähler surface, and let τ_m be the* Spin^c*-structure defined by the S^1-bundle M. Let $\beta \in A^{11}_{\mathbb{R}}$ be a closed 2-form representing the de Rham cohomology class b such that*

$$(2c_1(M) - c_1(K_X) - b)\cdot[\omega_g] < 0 \quad (\textit{alternatively},\ > 0).$$

If $c_1(M) \notin \mathrm{NS}(X)$, *were* $\mathrm{NS}(X)$ *is the Néron–Severi group of* X, *then* $\mathcal{W}_\beta^{\tau_m} = \varnothing$. *When* $c_1(M) \in \mathrm{NS}(X)$, *there is a natural real analytic isomorphism*

$$\mathcal{W}_\beta^{\tau_m} \simeq \mathcal{D}\mathrm{ou}(m) \quad (\textit{respectively}, \ \mathcal{D}\mathrm{ou}(c_1(K_X) - m)).$$

A moduli space $\mathcal{W}_\beta^{\tau_m} \neq \varnothing$ is smooth at the point corresponding to $D \in \mathcal{D}\mathrm{ou}(m)$ if and only if $h^0(\mathcal{O}_D(D)) = \dim_D \mathcal{D}\mathrm{ou}(m)$. This condition is always satisfied when $b_1(X) = 0$. If $\mathcal{W}_\beta^{\tau_m}$ is smooth at a point corresponding to $D \in \mathcal{D}\mathrm{ou}(m)$, then it has the expected dimension in this point if and only if $h^1(\mathcal{O}_D(D)) = 0$.

The natural isomorphisms $\mathcal{W}_\beta^{\tau_m} \simeq \mathcal{D}\mathrm{ou}(m)$ respects the orientations induced by the complex structure of X when $(2c_1(M) - c_1(K_X) - b)\cdot[\omega_g] < 0$. If $(2c_1(M) - c_1(K_X) - b)\cdot[\omega_g] > 0$, then the isomorphism $\mathcal{W}_\beta^{\tau_m} \simeq \mathcal{D}\mathrm{ou}(c_1(K_X) - m)$ multiplies the complex orientations by $(-1)^{\chi(M)}$ [Okonek and Teleman 1996b].

Example. Consider again the complex projective plane $\mathbb{P}^2$, polarized by

$$h = c_1(\mathcal{O}_{\mathbb{P}^2}(1)).$$

The expected dimension of $\mathcal{W}_\beta^{\tau_m}$ is $m(m + 3h)$. The theorem above yields the following explicit description of the corresponding moduli spaces:

$$\mathcal{W}_\beta^{\tau_m} \simeq \begin{cases} |\mathcal{O}_{\mathbb{P}^2}(m)| & \text{if } (2m + 3h - [\beta])\cdot h < 0, \\ |\mathcal{O}_{\mathbb{P}^2}(-(m+3))| & \text{if } (2m + 3h - [\beta])\cdot h > 0. \end{cases}$$

E. Witten [1994] has shown that on Kählerian surfaces X with geometric genus $p_g > 0$ all nontrivial Seiberg–Witten invariants $\mathrm{SW}_{X,\mathcal{O}}(\mathfrak{c})$ satisfy $w_c = 0$.

In the case of Kählerian surfaces with $p_g = 0$ one has a different situation. Suppose for instance that $b_1(X) = 0$. Choose the standard orientation $\mathcal{O}_1$ of $H^1(X, \mathbb{R}) = 0$ and the component $\mathbf{H}_0$ containing Kähler classes to orient the moduli spaces of monopoles. Then, using the previous theorem and the wall-crossing formula, we get:

Proposition 3.1.3. *Let* X *be a Kähler surface with* $p_g = 0$ *and* $b_1 = 0$. *If* $m \in H^2(X, \mathbb{Z})$ *satisfies* $m(m - c_1(K_X)) \geq 0$, *that is*, *the expected dimension* $w_{2m - c_1(K_X)}$ *is nonnegative*, *then*

$$\mathrm{SW}^+_{X,\mathbf{H}_0}(\mathfrak{c}_m) = \begin{cases} 1 & \textit{if } \mathcal{D}\mathrm{ou}(m) \neq \varnothing, \\ 0 & \textit{if } \mathcal{D}\mathrm{ou}(m) = \varnothing, \end{cases}$$

$$\mathrm{SW}^-_{X,\mathbf{H}_0}(\mathfrak{c}_m) = \begin{cases} 0 & \textit{if } \mathcal{D}\mathrm{ou}(m) \neq \varnothing, \\ -1 & \textit{if } \mathcal{D}\mathrm{ou}(m) = \varnothing. \end{cases}$$

Our next goal is to show that the PU(2)-monopole equations on a Kähler surface can be analyzed in a similar way. This analysis yields a complex geometric description of the moduli spaces whose S^1-quotients give formulas relating the Donaldson invariants to the Seiberg–Witten invariants. If the base is projective, one also has an algebro-geometric interpretation [Okonek et al. 1999], which leads to explicitly computable examples of moduli spaces of PU(2)-monopoles

[Teleman 1996]. Such examples are important, because they illustrate the general mechanism for proving the relation between the two theories, and help to understand the geometry of the ends of the moduli spaces in the more difficult $\mathcal{C}^\infty$-category.

Recall that, since (X,g) comes with a canonical Spin^c-structure τ, the data of of a $\mathrm{Spin}^{U(2)}$-structure in (X,g) is equivalent to the data of a Hermitian bundle E of rank 2. The bundles of the corresponding $\mathrm{Spin}^{U(2)}$-structure $\sigma : P^u \longrightarrow P_g$ are given by $\bar\delta(P^u) = P_E \big/ S^1$, $\det P^u = \det E \otimes K_X$, and $\Sigma^\pm(P^u) = \Sigma^\pm \otimes E \otimes K_X$.

Suppose that $\det P^u$ admits an <u>integrable</u> connection $a \in \mathcal{A}(\det P^u)$. Let $k \in \mathcal{A}(K_X)$ be the Chern connection of the canonical bundle, and let $\lambda := a \otimes k^\vee$ be the induced connection in $L := \det E$. We denote by $\mathcal{L} := (L, \bar\partial_\lambda)$ the holomorphic structure defined by λ. Now identify the affine space $\mathcal{A}(\bar\delta(P^u))$ with the space $\mathcal{A}_{\lambda\otimes k^{\otimes 2}}(E \otimes K_X)$ of connections in $E \otimes K_X$ which induce $\lambda \otimes k^{\otimes 2} = a \otimes k$ in $\det(E \otimes K_X)$, and identify $A^0(\Sigma^+(P^u))$ with

$$A^0(E \otimes K_X) \oplus A^0(E) = A^0(E \otimes K_X) \oplus A^{02}(E \otimes K_X).$$

Proposition 3.1.4. *Fix an integrable connection* $a \in \mathcal{A}(\det E \otimes K_X)$. *A pair* $(A, \varphi + \alpha) \in \mathcal{A}_{\lambda\otimes k^{\otimes 2}}(E \otimes K_X) \times [A^0(E \otimes K_X) \oplus A^{02}(E \otimes K_X)]$ *solves the* PU(2)-*monopole equations* (SW^σ_a) *if and only if* A *is integrable and one of the following conditions is satisfied*:

(I) $\alpha = 0$, $\bar\partial_A \varphi = 0$, *and* $i\Lambda_g F_A^0 + \frac{1}{2}(\varphi\bar\varphi)_0 = 0$.
(II) $\varphi = 0$, $\partial_A \alpha = 0$, *and* $i\Lambda_g F_A^0 - \frac{1}{2} * (\alpha \wedge \bar\alpha)_0 = 0$.

Note that solutions (A, φ) of type I give rise to holomorphic pairs $(\mathcal{F}_A, \varphi)$, consisting of a holomorphic structure in $F := E \otimes K$ and a holomorphic section φ in $\mathcal{F}_A$. The remaining equation $i\Lambda_g F_A^0 + \frac{1}{2}(\varphi\bar\varphi)_0 = 0$ can again be interpreted as the vanishing condition for a moment map for the $\mathcal{G}_0$-action in the space of pairs $(A, \varphi) \in \mathcal{A}_{\lambda\otimes k^{\otimes 2}}(F) \times A^0(F)$. We shall study the corresponding stability condition in the next section.

The analysis of the solutions of type II can be reduced to the investigation of the type I solutions: Indeed, if $\varphi = 0$ and $\alpha \in A^{02}(E \otimes K_X)$ satisfies $\partial_A \alpha = 0$, we see that the section $\psi := \bar\alpha \in A^0(\bar E)$ must be holomorphic, that is, it satisfies $\bar\partial_{A\otimes[a^\vee]}\psi = 0$. On the other hand one has $- * (\alpha \wedge \bar\alpha)_0 = *(\bar\alpha \wedge \bar{\bar\alpha})_0 = (\psi\bar\psi)_0$.

3.2. Vortex Equations and Stable Oriented Pairs. Let (X,g) be a compact Kähler manifold of arbitrary dimension n, and let E be a differentiable vector bundle of rank r, endowed with a fixed holomorphic structure $\mathcal{L} := (L, \bar\partial_{\mathcal{L}})$ in $L := \det E$.

An <u>oriented</u> <u>pair</u> of type $(E, \mathcal{L})$ is a pair $(\mathcal{E}, \varphi)$, consisting of a holomorphic structure $\mathcal{E} = (E, \bar\partial_{\mathcal{E}})$ in E with $\bar\partial_{\det \mathcal{E}} = \bar\partial_{\mathcal{L}}$, and a holomorphic section $\varphi \in H^0(\mathcal{E})$. Two oriented pairs are isomorphic if they are equivalent under the natural action of the group $\mathrm{SL}(E)$ of differentiable automorphisms of E with determinant 1.

An oriented pair $(\mathcal{E},\varphi)$ is <u>simple</u> if its stabilizer in $\mathrm{SL}(E)$ is contained in the center $\mathbb{Z}_r\cdot \mathrm{id}_E$ of $\mathrm{SL}(E)$; it is <u>strongly</u> <u>simple</u> if this stabilizer is trivial.

PROPOSITION 3.2.1 [Okonek and Teleman 1996a]. *There exists a (possibly non-Hausdorff) complex analytic orbifold $\mathcal{M}^{\mathrm{si}}(E,\mathcal{L})$ parameterizing isomorphism classes of simple oriented pairs of type $(E,\mathcal{L})$. The open subset $\mathcal{M}^{\mathrm{ssi}}(E,\mathcal{L})\subset \mathcal{M}^{\mathrm{si}}(E,\mathcal{L})$ of classes of strongly simple pairs is a complex analytic space, and the points $\mathcal{M}^{\mathrm{si}}(E,\mathcal{L})\setminus \mathcal{M}^{\mathrm{ssi}}(E,\mathcal{L})$ have neighborhoods modeled on $\mathbb{Z}_r$-quotients.*

Now fix a Hermitian background metric H in E. In this section we use the symbol $(\mathrm{SU}(E))$ $U(E)$ for the groups of (special) unitary automorphisms of (E,H), and not for the bundles of (special) unitary automorphisms.

Let λ be the Chern connection associated with the Hermitian holomorphic bundle $(\mathcal{L},\det H)$. We denote by $\bar{\mathcal{A}}_{\bar\partial_\lambda}(E)$ the affine space of semiconnections in E which induce the semiconnection $\bar\partial_\lambda=\bar\partial_{\mathcal{L}}$ in $L=\det E$, and we write $\mathcal{A}_\lambda(E)$ for the space of unitary connections in (E,H) which induce λ in L. The map $A\longmapsto\bar\partial_A$ yields an identification $\mathcal{A}_\lambda(E)\longrightarrow\bar{\mathcal{A}}_{\bar\partial_\lambda}(E)$, which endows the affine space $\mathcal{A}_\lambda(E)$ with a complex structure. Using this identification and the Hermitian metric H, the product $\mathcal{A}_\lambda(E)\times A^0(E)$ becomes — after suitable Sobolev completions — an infinite dimensional Kähler manifold. The map

$$m:\mathcal{A}_\lambda(E)\times A^0(E)\longrightarrow A^0(su(E))$$

defined by $m(A,\varphi):=\Lambda_g F_A^0-\frac{i}{2}(\varphi\bar\varphi)_0$ is a <u>moment</u> <u>map</u> for the $\mathrm{SU}(E)$-action on the Kähler manifold $\mathcal{A}_\lambda(E)\times A^0(E)$.

We denote by $\mathcal{H}_\lambda(E):=\{(A,\varphi)\in\mathcal{A}_\lambda(E)\times A^0(E)|\ F_A^{02}=0,\ \bar\partial_A\varphi=0\}$ the space of integrable pairs, and by $\mathcal{H}_\lambda(E)^{\mathrm{si}}$ the open subspace of pairs

$$(A,\varphi)\in\mathcal{H}_\lambda(E)$$

with $(\bar\partial_A,\varphi)$ simple. The quotient

$$\mathcal{V}_\lambda(E):=\big(\mathcal{H}_\lambda(E)\cap m^{-1}(0)\big)/\,\mathrm{SU}(E)$$

is called the moduli space of <u>projective</u> <u>vortices</u>, and

$$\mathcal{V}^*_\lambda(E):=\big(\mathcal{H}^{\mathrm{si}}_\lambda(E)\cap m^{-1}(0)\big)/\,\mathrm{SU}(E)$$

is called the moduli space of <u>irreducible</u> <u>projective</u> <u>vortices</u>. Note that a vortex (A,φ) is irreducible if and only if $\mathrm{SL}(E)_{(A,\varphi)}\subset\mathbb{Z}_r\,\mathrm{id}_E$. Using again an infinite dimensional version of the theory of symplectic quotients (as in the abelian case), one gets a homeomorphism

$$j:\mathcal{V}_\lambda(E)\xrightarrow{\simeq}\mathcal{H}^{\mathrm{ps}}_\lambda(E)/\,\mathrm{SL}(E)$$

where $\mathcal{H}^{\mathrm{ps}}_\lambda(E)$ is the subspace of $\mathcal{H}_\lambda(E)$ consisting of pairs whose $\mathrm{SL}(E)$-orbit meets the vanishing locus of the moment map. $\mathcal{H}^{\mathrm{ps}}_\lambda(E)$ is in general not open, but $\mathcal{H}^s_\lambda(E) := \mathcal{H}^{\mathrm{ps}}_\lambda(E) \cap \mathcal{H}^{\mathrm{si}}_\lambda(E)$ is open, and restricting j to $\mathcal{V}^*_\lambda(E)$ yields an isomorphism of real analytic orbifolds

$$\mathcal{V}^*_\lambda(E) \xrightarrow{\simeq} \mathcal{H}^s_\lambda(E)\big/\operatorname{SL}(E) \subset \mathcal{M}^{\mathrm{si}}(E,\mathcal{L}).$$

The image

$$\mathcal{M}^s(E,\mathcal{L}) := \mathcal{H}^s_\lambda(E)\big/\operatorname{SL}(E)$$

of this isomorphism can be identified with the set of isomorphism classes of simple oriented holomorphic pairs $(\mathcal{E},\varphi)$ of type $(E,\mathcal{L})$, with the property that $\mathcal{E}$ admits a Hermitian metric with $\det h = \det H$ which solves the <u>projective vortex equation</u>

$$i\Lambda_g F^0_h + \tfrac{1}{2}(\varphi\bar{\varphi}^h)_0 = 0.$$

Here F_h is the curvature of the Chern connection of $(\mathcal{E},h)$.

The set $\mathcal{M}^s(E,\mathcal{L})$ has a purely holomorphic description as the subspace of elements $[\mathcal{E},\varphi] \in \mathcal{M}^{\mathrm{si}}(E,\mathcal{L})$ which satisfy a suitable <u>stability</u> condition.

This condition is rather complicated for bundles E of rank $r > 2$, but it becomes very simple when $r = 2$.

Recall that, for any torsion free coherent sheaf $\mathcal{F} \neq 0$ over a n-dimensional Kähler manifold (X,g), one defines the g-<u>slope</u> of $\mathcal{F}$ by

$$\mu_g(\mathcal{F}) := \frac{c_1(\det\mathcal{F}) \cup [\omega_g]^{n-1}}{\operatorname{rk}(\mathcal{F})}.$$

A holomorphic bundle $\mathcal{E}$ over (X,g) is called <u>slope-stable</u> if $\mu_g(\mathcal{F}) < \mu_g(\mathcal{E})$ for all proper coherent subsheaves $\mathcal{F} \subset \mathcal{E}$. The bundle $\mathcal{E}$ is <u>slope-polystable</u> if it decomposes as a direct sum $\mathcal{E} = \oplus\mathcal{E}_i$ of slope-stable bundles with $\mu_g(\mathcal{E}_i) = \mu_g(\mathcal{E})$.

Definition 3.2.2. Let $(\mathcal{E},\varphi)$ be an oriented pair of type $(E,\mathcal{L})$ with $\operatorname{rk} E = 2$ over a Kähler manifold (X,g). The pair $(\mathcal{E},\varphi)$ is <u>stable</u> if $\varphi = 0$ and $\mathcal{E}$ is slope-stable, or $\varphi \neq 0$ and the divisorial component D_φ of the zero-locus $Z(\varphi) \subset X$ satisfies $\mu_g(\mathcal{O}_X(D_\varphi)) < \mu_g(E)$. The pair $(\mathcal{E},\varphi)$ is <u>polystable</u> if it is stable or $\varphi = 0$ and $\mathcal{E}$ is slope-polystable.

Example. Let $D \subset X$ be an effective divisor defined by a section

$$\varphi \in H^0(\mathcal{O}_X(D)) \setminus \{0\},$$

and put $\mathcal{E} := \mathcal{O}_X(D) \oplus [\mathcal{L} \otimes \mathcal{O}_X(-D)]$. The pair $(\mathcal{E},\varphi)$ is stable if and only if $\mu_g(\mathcal{O}_X(2D)) < \mu_g(\mathcal{L})$.

The following result gives a metric characterization of polystable oriented pairs.

Theorem 3.2.3 [Okonek and Teleman 1996a]. *Let E be a differentiable vector bundle of rank 2 over (X,g) endowed with a Hermitian holomorphic structure $(\mathcal{L},l)$ in $\det E$. An oriented pair of type $(E,\mathcal{L})$ is polystable if and only if $\mathcal{E}$ admits a Hermitian metric h with $\det h = l$ which solves the projective vortex equation*

$$i\Lambda_g F^0_h + \tfrac{1}{2}(\varphi\bar{\varphi}^h)_0 = 0.$$

If $(\mathcal{E},\varphi)$ is stable, then the metric h is unique.

This result identifies the subspace $\mathcal{M}^s(E,\mathcal{L}) \subset \mathcal{M}^{\mathrm{si}}(E,\mathcal{L})$ as the subspace of isomorphism classes of <u>stable</u> <u>oriented</u> <u>pairs</u>.

Theorem 3.2.3 can be used to show that the moduli spaces $\mathcal{M}^\sigma_a$ of PU(2)-monopoles on a Kähler surface have a natural complex geometric description when the connection a is integrable. Recall from Section 3.1 that in this case $\mathcal{M}^\sigma_a$ decomposes as the union of two Zariski-closed subspaces

$$\mathcal{M}^\sigma_a = (\mathcal{M}^\sigma_a)_I \cup (\mathcal{M}^\sigma_a)_{II}$$

according to the two conditions I, II in Proposition 3.1.4. By this proposition, both terms of this union can be identified with moduli spaces of projective vortices. Using again the symbol * to denote subsets of points with central stabilizers, one gets the following Kobayashi–Hitchin type description of $(\mathcal{M}^\sigma_a)^*$ in terms of stable oriented pairs.

Theorem 3.2.4 [Okonek and Teleman 1996a; 1997]. *If $a \in \mathcal{A}(\det P^u)$ is integrable, the moduli space $\mathcal{M}^\sigma_a$ decomposes as a union $\mathcal{M}^\sigma_a = (\mathcal{M}^\sigma_a)_I \cup (\mathcal{M}^\sigma_a)_{II}$ of two Zariski closed subspaces isomorphic with moduli spaces of projective vortices, which intersect along the Donaldson moduli space $\mathcal{D}(\bar\delta(P^u))$. There are natural real analytic isomorphisms*

$$(\mathcal{M}^\sigma_a)^*_I \xrightarrow{\simeq} \mathcal{M}^s(E\otimes K_X, \mathcal{L}\otimes\mathcal{K}_X^{\otimes 2}), \quad (\mathcal{M}^\sigma_a)^*_{II} \xrightarrow{\simeq} \mathcal{M}^s(E^\vee, \mathcal{L}^\vee),$$

where $\mathcal{L}$ denotes the holomorphic structure in $\det E = \det P^u \otimes K_X^\vee$ defined by $\bar\partial_a$ and the canonical holomorphic structure in K_X.

Example (R. Plantiko). On $\mathbb{P}^2$, endowed with the standard Fubini–Study metric g, we consider the $\mathrm{Spin}^{U(2)}(4)$-structure $\sigma : P^u \longrightarrow P_g$ defined by the standard $\mathrm{Spin}^c(4)$-structure $\tau : P^c \longrightarrow P_g$ and the $U(2)$-bundle E with $c_1(E) = 7$, $c_2(E) = 13$, and we fix an integrable connection $a \in \mathcal{A}(\det P^u)$. This $\mathrm{Spin}^{U(2)}(4)$-structure is characterized by $c_1(\det(P^u)) = 4$, $p_1(\bar\delta(P^u)) = -3$, and the bundle $F := E \otimes K_{\mathbb{P}^2}$ has Chern classes $c_1(F) = 1$, $c_2(F) = 1$. It is easy to see that every stable oriented pair $(\mathcal{F},\varphi)$ of type $(F, \mathcal{O}_{\mathbb{P}^2}(1))$ with $\varphi \neq 0$ fits into an exact sequence of the form

$$0 \longrightarrow \mathcal{O} \xrightarrow{\varphi} \mathcal{F} \longrightarrow J_{Z(\varphi)} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0,$$

where $\mathcal{F} = \mathcal{T}_{\mathbb{P}^2}(-1)$ and the zero locus $Z(\varphi)$ of φ consists of a simple point $z_\varphi \in \mathbb{P}^2$. Two such pairs $(\mathcal{F},\varphi)$, $(\mathcal{F},\varphi')$ define the same point in the moduli space $\mathcal{M}^s(F, \mathcal{O}_{\mathbb{P}^2}(1))$ if and only if $\varphi' = \pm\varphi$. The resulting identification

$$\mathcal{M}^s(F, \mathcal{O}_{\mathbb{P}^2}(1)) = H^0(\mathcal{T}_{\mathbb{P}^2}(-1))\big/\{\pm\,\mathrm{id}\}$$

is a complex analytic isomorphism.

Since every polystable pair of type $(F, \mathcal{O}_{\mathbb{P}^2}(1))$ is actually stable, and since there are no polystable oriented pairs of type $(E^\vee, \mathcal{O}_{\mathbb{P}^2}(-7))$, Theorem 3.2.4 yields a real analytic isomorphism

$$\mathcal{M}_a^\sigma = H^0(\mathcal{T}_{\mathbb{P}^2}(-1))\big/\{\pm \mathrm{id}\},$$

where the origin corresponds to the unique stable oriented pair of the form $(\mathcal{T}_{\mathbb{P}^2}(-1), 0)$. The quotient $H^0(\mathcal{T}_{\mathbb{P}^2}(-1))\big/\{\pm \mathrm{id}\}$ has a natural algebraic compactification $\mathcal{C}$, given by the cone over the image of $\mathbb{P}(H^0(\mathcal{T}_{\mathbb{P}^2}(-1)))$ under the Veronese map to $\mathbb{P}(S^2 H^0(\mathcal{T}_{\mathbb{P}^2}(-1)))$. This compactification coincides with the Uhlenbeck compactification $\overline{\mathcal{M}_a^\sigma}$ (see Section 2.2 and [Teleman 1996]). More precisely, let $\sigma' : P'^u \longrightarrow P_g$ be the $\mathrm{Spin}^{U(2)}(4)$-structure with $\det P'^u = \det P^u$ and $p_1(P'^u) = 1$. This structure is associated with τ and the $U(2)$-bundle E' with Chern classes $c_1(E') = 7$, $c_2(E') = 12$. The moduli space $\mathcal{M}_a^{\sigma'}$ consists of one (abelian) point, the class of the abelian solution corresponding to the stable oriented pair $(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1), \mathrm{id}_{\mathcal{O}_{\mathbb{P}^2}})$ of type $(E' \otimes K_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))$. $\mathcal{M}_a^{\sigma'}$ can be identified with the moduli space

$$\mathcal{W}^\tau_{\frac{i}{2\pi}F_a}$$

of $[\frac{i}{2\pi}F_a]$-twisted abelian Seiberg–Witten monopoles. Under the identification $\mathcal{C} = \overline{\mathcal{M}_a^\sigma}$, the vertex of the cone corresponds to the unique Donaldson point which is given by the stable oriented pair $(\mathcal{T}_{\mathbb{P}^2}(-1), 0)$. The base of the cone corresponds to the space $\mathcal{M}_a^{\sigma'} \times \mathbb{P}^2$ of ideal monopoles concentrated in one point.

We close this section by explaining the stability concept which describes the subset $\mathcal{M}_X^{\mathrm{s}}(E, \mathcal{L}) \subset \mathcal{M}_X^{\mathrm{si}}(E, \mathcal{L})$ in the general case $r \geq 2$. This stability concept does not depend on the choice of parameter and the corresponding moduli spaces can be interpreted as "master spaces" for holomorphic pairs (see next section); in the projective framework they admit Gieseker type compactifications [Okonek et al. 1999].

We shall find this stability concept by relating the $\mathrm{SU}(E)$-moment map

$$m : \mathcal{A}_\lambda(E) \times A^0(E) \longrightarrow A^0(su(E))$$

to the universal family of $U(E)$-moment maps $m_t : \mathcal{A}(E) \times A^0(E) \longrightarrow A^0(u(E))$ defined by

$$m_t(A, \varphi) := \Lambda_g F_A - \tfrac{1}{2} i (\varphi\bar\varphi) + \tfrac{1}{2} i t\, \mathrm{id}_E,$$

where $t \in A^0$ is an arbitrary real valued function. Given t, we consider the system of equations

$$\begin{aligned} F_A^{02} &= 0, \\ \bar\partial_A \varphi &= 0, \\ i\Lambda_g F_A + \tfrac{1}{2}(\varphi\bar\varphi) &= \tfrac{1}{2} t\, \mathrm{id}_E \end{aligned} \qquad (V_t)$$

for pairs $(\mathcal{A},\varphi)\in A(E)\times A^0(E)$. Put

$$\rho_t := \frac{1}{4\pi n}\int_X t\omega_g^n .$$

To explain our first result, we have to recall some classical stability concepts for holomorphic pairs.

For any holomorphic bundle $\mathcal{E}$ over (X,g) denote by $\mathcal{S}(\mathcal{E})$ the set of reflexive subsheaves $\mathcal{F}\subset\mathcal{E}$ with $0<\mathrm{rk}(\mathcal{F})<\mathrm{rk}(\mathcal{E})$, and for a fixed section $\varphi\in H^0(\mathcal{E})$ put

$$\mathcal{S}_\varphi(\mathcal{E}) := \{\mathcal{F}\in\mathcal{S}(\mathcal{E}) : \varphi\in H^0(\mathcal{F})\}.$$

Define real numbers $\underline{m}_g(\mathcal{E})$ and $\overline{m}_g(\mathcal{E},\varphi)$ by

$$\underline{m}_g(\mathcal{E}) := \max\bigl(\mu_g(\mathcal{E}), \sup_{\mathcal{F}'\in\mathcal{S}(\mathcal{E})}\mu_g(\mathcal{F}')\bigr),\ \overline{m}_g(\mathcal{E},\varphi) := \inf_{\mathcal{F}\in\mathcal{S}_\varphi(\mathcal{E})}\mu_g(\mathcal{E}/\mathcal{F}).$$

A bundle $\mathcal{E}$ is φ-<u>stable</u> in the sense of S. Bradlow when $\underline{m}_g(\mathcal{E})<\overline{m}_g(\mathcal{E},\varphi)$. Let $\rho\in\mathbb{R}$ be any real parameter. A holomorphic pair $(\mathcal{E},\varphi)$ is called ρ-<u>stable</u> if ρ satisfies the inequality

$$\underline{m}_g(\mathcal{E})<\rho<\overline{m}_g(\mathcal{E},\varphi).$$

The pair $(\mathcal{E},\varphi)$ is ρ-<u>polystable</u> if it is ρ-stable or $\mathcal{E}$-splits holomorphically as $\mathcal{E}=\mathcal{E}'\oplus\mathcal{E}''$ such that $\varphi\in H^0(\mathcal{E}')$, $(\mathcal{E}',\varphi)$ is ρ-stable and $\mathcal{E}''$ is a slope-polystable vector bundle with $\mu_g(\mathcal{E}'')=\rho$ [Bradlow 1991]. Let $\mathrm{GL}(E)$ be the group of bundle automorphisms of E. With these definitions one proves [Okonek and Teleman 1995a] (see [Bradlow 1991] for the case of a constant function t):

Proposition 3.2.5. *The complex orbit* $\mathrm{GL}(E)\cdot(A,\varphi)$ *of an integrable pair* $(A,\varphi)\in\mathcal{A}(E)\times A^0(E)$ *contains a solution of* (V_t) *if and only if the pair* $(\mathcal{E}_A,\varphi)$ *is* ρ_t*-polystable.*

Now fix again a Hermitian metric H in E and an integrable connection λ in the Hermitian line bundle $L:=(\det E,\det H)$. Consider the system of equations

$$\begin{aligned} F_A^{02}&=0,\\ \bar\partial_A\varphi&=0,\\ i\Lambda_g F_A^0+\tfrac12(\varphi\bar\varphi)_0&=0 \end{aligned}\qquad (V^0)$$

for pairs $(A,\varphi)\in\mathcal{A}_\lambda(E)\times A^0(E)$. Then one can prove

Proposition 3.2.6. *Let* $(A,\varphi)\in\mathcal{A}_\lambda(E)\times A^0(E)$ *be an integrable pair. The following assertions are equivalent*:

(i) *The complex orbit* $\mathrm{SL}(E)\cdot(A,\varphi)$ *contains a solution of* (V^0).
(ii) *There exists a function* $t\in A^0$ *such that the* $\mathrm{GL}(E)$*-orbit* $\mathrm{GL}(E)\cdot(A,\varphi)$ *contains a solution of* (V_t).
(iii) *There exists a real number* ρ *such that the pair* $(\mathcal{E}_A,\varphi)$ *is* ρ*-polystable.*

COROLLARY 3.2.7. *The open subspace $\mathcal{M}^s(E,\mathcal{L}) \subset \mathcal{M}^{\rm si}(E,\mathcal{L})$ is the set of isomorphism classes of simple oriented pairs which are ρ-polystable for some $\rho \in \mathbb{R}$.*

REMARK 3.2.8. There exist stable oriented pairs $(\mathcal{E},\varphi)$ whose stabilizer with respect to the $\mathrm{GL}(E)$-action is of positive dimension. Such pairs cannot be ρ-stable for any $\rho \in \mathbb{R}$.

Note that the moduli spaces $\mathcal{M}^s(E,\mathcal{L})$ have a natural $\mathbb{C}^*$-action defined by $z\cdot[\mathcal{E},\varphi] := [\mathcal{E}, z^{1/r}\varphi]$. This action is well defined since r-th roots of unity are contained in the complex gauge group $\mathrm{SL}(E)$.

There exists an equivalent definition for stability of oriented pairs, which does not use the parameter dependent stability concepts of [Bradlow 1991]. The fact that it is expressible in terms of ρ-stability is related to the fact that the moduli spaces $\mathcal{M}^s(E,\mathcal{L})$ are master spaces for moduli spaces of ρ-stable pairs.

3.3. Master Spaces and the Coupling Principle. Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth complex projective variety with hyperplane bundle $\mathcal{O}_X(1)$. All degrees and Hilbert polynomials of coherent sheaves will be computed corresponding to these data.

We fix a torsion-free sheaf $\mathcal{E}_0$ and a holomorphic line bundle $\mathcal{L}_0$ over X, and we choose a Hilbert polynomial P_0. By $P_{\mathcal{F}}$ we denote the Hilbert polynomial of a coherent sheaf $\mathcal{F}$. Recall that any nontrivial torsion free coherent sheaf $\mathcal{F}$ admits a unique subsheaf $\mathcal{F}_{\max}$ for which $P_{\mathcal{F}'}/\mathrm{rk}\,\mathcal{F}'$ is maximal and whose rank is maximal among all subsheaves $\mathcal{F}'$ with $P_{\mathcal{F}'}/\mathrm{rk}\,\mathcal{F}'$ maximal.

An $\mathcal{L}_0$-oriented pair of type $(P_0,\mathcal{E}_0)$ is a triple $(\mathcal{E},\varepsilon,\varphi)$ consisting of a torsion free coherent sheaf $\mathcal{E}$ with determinant isomorphic to $\mathcal{L}_0$ and Hilbert polynomial $P_{\mathcal{E}} = P_0$, a homomorphism $\varepsilon : \det\mathcal{E} \longrightarrow \mathcal{L}_0$, and a morphism $\varphi : \mathcal{E} \longrightarrow \mathcal{E}_0$. The homomorphisms ε and φ will be called the orientation and the framing of the oriented pair. There is an obvious equivalence relation for such pairs. When $\ker\varphi \neq 0$, we set

$$\delta_{\mathcal{E},\varphi} := P_{\mathcal{E}} - \frac{\mathrm{rk}\,\mathcal{E}}{\mathrm{rk}\big[\ker(\varphi)_{\max}\big]} P_{\ker(\varphi)_{\max}}.$$

An oriented pair $(\mathcal{E},\varepsilon,\varphi)$ is semistable if either

1. φ is injective, or
2. ε is an isomorphism, $\ker\varphi \neq 0$, $\delta_{\mathcal{E},\varphi} \geq 0$, and for all nontrivial subsheaves $\mathcal{F} \subset \mathcal{E}$ the inequality
$$\frac{P_{\mathcal{F}}}{\mathrm{rk}\,\mathcal{F}} - \frac{\delta_{\mathcal{E},\varphi}}{\mathrm{rk}\,\mathcal{F}} \leq \frac{P_{\mathcal{E}}}{\mathrm{rk}\,\mathcal{E}} - \frac{\delta_{\mathcal{E},\varphi}}{\mathrm{rk}\,\mathcal{E}}.$$
holds.

The corresponding stability concept is slightly more complicated; see [Okonek et al. 1999]. Note that the (semi)stability definition above does not depend on a parameter. It is, however, possible to express (semi)stability in terms of the parameter dependent Gieseker-type stability concepts of [Huybrechts and Lehn 1995]. For example, $(\mathcal{E},\varepsilon,\varphi)$ is semistable if and only if φ is injective, or $\mathcal{E}$ is Gieseker semistable, or there exists a rational polynomial δ of degree smaller than $\dim X$ with positive leading coefficient, such that $(\mathcal{E},\varphi)$ is δ-semistable in the sense of [Huybrechts and Lehn 1995].

For all stability concepts introduced so far there exist analogous notions of slope-(semi)stability. In the special case when the reference sheaf $\mathcal{E}_0$ is the trivial sheaf $\mathcal{O}_X$, slope stability is the algebro-geometric analog of the stability concept associated with the projective vortex equation.

THEOREM 3.3.1 [Okonek et al. 1999]. *There exists a <u>projective</u> scheme*

$$\mathcal{M}^{\mathrm{ss}}(P_0,\mathcal{E}_0,\mathcal{L}_0)$$

whose closed points correspond to gr*-equivalence classes of Gieseker semistable* $\mathcal{L}_0$*-oriented pairs of type* $(P_0,\mathcal{E}_0)$. *This scheme contains an open subscheme* $\mathcal{M}^s(P_0,\mathcal{E}_0,\mathcal{L}_0)$ *which is a coarse moduli space for stable* $\mathcal{L}_0$*-oriented pairs.*

It is also possible to construct moduli spaces for stable oriented pairs where the orienting line bundle is allowed to vary [Okonek et al. 1999]. This generalization is important in connection with Gromov–Witten invariants for Grassmannians [Bertram et al. 1996].

Note that $\mathcal{M}^{\mathrm{ss}}(P_0,\mathcal{E}_0,\mathcal{L}_0)$ possesses a natural $\mathbb{C}^*$-action, given by

$$z\cdot[\mathcal{E},\varepsilon,\varphi] := [\mathcal{E},\varepsilon,z\varphi],$$

whose fixed point set can be explicitly described. The fixed point locus

$$[\mathcal{M}^{\mathrm{ss}}(P_0,\mathcal{E}_0,\mathcal{L}_0)]^{\mathbb{C}^*}$$

contains two distinguished subspaces, $\mathcal{M}_0$ defined by the equation $\varphi = 0$, and $\mathcal{M}_\infty$ defined by $\varepsilon = 0$. $\mathcal{M}_0$ can be identified with the Gieseker scheme $\mathcal{M}^{\mathrm{ss}}(P,\mathcal{L}_0)$ of equivalence classes of semistable $\mathcal{L}_0$-oriented torsion free coherent sheaves with Hilbert polynomial P_0. The subspace $\mathcal{M}_\infty$ is the Grothendieck Quot-scheme $\mathrm{Quot}^{\mathcal{E}_0,\mathcal{L}_0}_{P_{\mathcal{E}_0}-P_0}$ of quotients of $\mathcal{E}_0$ with fixed determinant isomorphic with $(\det\mathcal{E}_0)\otimes\mathcal{L}_0^\vee$ and Hilbert polynomial $P_{\mathcal{E}_0}-P_0$.

In the terminology of [Białynicki-Birula and Sommese 1983], $\mathcal{M}_0$ is the source $\mathcal{M}_{\mathrm{source}}$ of the $\mathbb{C}^*$-space $\mathcal{M}^{\mathrm{ss}}(P_0,\mathcal{E}_0,\mathcal{L}_0)$, and $\mathcal{M}_\infty$ is its sink when nonempty.

The remaining subspace of the fixed point locus

$$\mathcal{M}_R := [\mathcal{M}^{\mathrm{ss}}(P_0,\mathcal{E}_0,\mathcal{L}_0)]^{\mathbb{C}^*}\setminus[\mathcal{M}_0\cup\mathcal{M}_\infty],$$

the so-called space of <u>reductions</u>, consists of objects which are of the same type but essentially of lower rank.

Note that the Quot scheme $\mathcal{M}_\infty$ is empty if $\mathrm{rk}(\mathcal{E}_0)$ is smaller than the rank r of the sheaves $\mathcal{E}$ under consideration, in which case the sink of the moduli space is a closed subset of the space of reductions.

Recall from [Białynicki-Birula and Sommese 1983] that the closure of a general $\mathbb{C}^*$-orbit connects a point in $\mathcal{M}_{\text{source}}$ with a point in $\mathcal{M}_{\text{sink}}$, whereas closures of special orbits connect points of other parts of the fixed point set.

The flow generated by the $\mathbb{C}^*$-action can therefore be used to relate data associated with $\mathcal{M}_0$ to data associated with $\mathcal{M}_\infty$ and $\mathcal{M}_R$.

The technique of computing data on $\mathcal{M}_0$ in terms of $\mathcal{M}_R$ and $\mathcal{M}_\infty$ is a very general principle which we call coupling and reduction. This principle has already been described in a gauge theoretic framework in Section 2.2 for relating monopoles and instantons. However, the essential ideas may probably be best understood in an abstract Geometric Invariant Theory setting, where one has a very simple and clear picture.

Let G be a complex reductive group, and consider a linear representation $\rho_A : G \longrightarrow \mathrm{GL}(A)$ in a finite dimensional vector space A. The induced action $\bar{\rho}_A : G \longrightarrow \mathrm{Aut}(\mathbb{P}(A))$ comes with a natural linearization in $\mathcal{O}_{\mathbb{P}(A)}(1)$, hence we have a stability concept, and thus we can form the GIT quotient

$$\mathcal{M}_0 := \mathbb{P}(A)^{\text{ss}}/\!/G.$$

Suppose we want to compute "correlation functions"

$$\Phi_I := \langle \mu_I, [\mathcal{M}_0] \rangle;$$

that is, we want to evaluate suitable products of canonically defined cohomology classes μ_i on the fundamental class $[\mathcal{M}_0]$ of $\mathcal{M}_0$. Usually the μ_i's are slant products of characteristic classes of a "universal bundle" $\mathfrak{E}_0$ on $\mathcal{M}_0 \times X$ with homology classes of X. Here X is a compact manifold, and $\mathfrak{E}_0$ comes from a tautological bundle $\tilde{\mathfrak{E}}_0$ on $A \times X$ by applying Kempf's Descend Lemma.

The main idea is now to couple the original problem with a simpler one, and to use the $\mathbb{C}^*$-action which occurs naturally in the resulting GIT quotients to express the original correlation functions in terms of simpler data. More precisely, consider another representation $\rho_B : G \longrightarrow \mathrm{GL}(B)$ with GIT quotient $\mathcal{M}_\infty := \mathbb{P}(B)^{\text{ss}}/\!/G$. The direct sum $\rho := \rho_A \oplus \rho_B$ defines a naturally linearized G-action on the projective space $\mathbb{P}(A \oplus B)$. We call the corresponding quotient

$$\mathcal{M} := \mathbb{P}(A \oplus B)^{\text{ss}}/\!/G$$

the master space associated with the coupling of ρ_A to ρ_B.

The space $\mathcal{M}$ comes with a natural $\mathbb{C}^*$-action, given by

$$z \cdot [a, b] := [a, z \cdot b],$$

and the union $\mathcal{M}_0 \cup \mathcal{M}_\infty$ is a closed subspace of the fixed point locus $\mathcal{M}^{\mathbb{C}^*}$.

Now make the simplifying assumptions that $\mathcal{M}$ is smooth and connected, the $\mathbb{C}^*$-action is free outside $\mathcal{M}^{\mathbb{C}^*}$, and suppose that the cohomology classes μ_i extend to $\mathcal{M}$. This condition is always satisfied if the μ_i's were obtained by the procedure described above, and if Kempf's lemma applies to the pull-back bundle $p_A^*(\mathfrak{E}_0)$ and provides a bundle on $\mathcal{M} \times X$ extending $\mathfrak{E}_0$.

Under these assumptions, the complement

$$\mathcal{M}_R := \mathcal{M}^{\mathbb{C}^*} \setminus (\mathcal{M}_0 \cup \mathcal{M}_\infty)$$

is a closed submanifold of $\mathcal{M}$, disjoint from $\mathcal{M}_0$, and $\mathcal{M}_\infty$. We call $\mathcal{M}_R$ the manifold of reductions of the master space. Now remove a sufficiently small S^1-invariant tubular neighborhood U of $\mathcal{M}^{\mathbb{C}^*} \subset \mathcal{M}$, and consider the S^1-quotient $W := [\mathcal{M} \setminus U]/S^1$. This is a compact manifold whose boundary is the union of the projectivized normal bundles $\mathbb{P}(N_{\mathcal{M}_0})$ and $\mathbb{P}(N_{\mathcal{M}_\infty})$, and a differentiable projective fiber space P_R over $\mathcal{M}_R$. Note that in general P_R has no natural holomorphic structure. Let n_0, n_∞ be the complex dimensions of the fibers of $\mathbb{P}(N_{\mathcal{M}_0})$, $\mathbb{P}(N_{\mathcal{M}_\infty})$, and let $u \in H^2(W, \mathbb{Z})$ be the first Chern class of the S^1-bundle dual to $\mathcal{M} \setminus U \longrightarrow W$. Let μ_I be a class as above. Then, taking into account orientations, we compute:

$$\begin{aligned}\Phi_I := \langle \mu_I,\, [\mathcal{M}_0] \rangle &= \langle \mu_I \cup u^{n_0},\, [\mathbb{P}(N_{\mathcal{M}_0})] \rangle \\ &= \langle \mu_I \cup u^{n_0},\, [\mathbb{P}(N_{\mathcal{M}_\infty})] \rangle - \langle \mu_I \cup u^{n_0},\, [P_R] \rangle .\end{aligned}$$

In this way the coupling principle reduces the calculation of the original correlation functions on $\mathcal{M}_0$ to computations on $\mathcal{M}_\infty$ and on the manifold of reductions $\mathcal{M}_R$. A particular important case occurs when the GIT problem given by ρ_B is trivial, that is, when $\mathbb{P}(B)^{\mathrm{ss}} = \varnothing$. Under these circumstances the functions Φ_I are completely determined by data associated with the manifold of reductions $\mathcal{M}_R$.

Of course, in realistic situations, our simplifying assumptions are seldom satisfied, so that one has to modify the basic idea in a suitable way.

One of the realistic situations which we have in mind is the coupling of coherent sheaves with morphisms into a fixed reference sheaf $\mathcal{E}_0$. In this case, the original problem is the classification of stable torsion-free sheaves, and the corresponding Gieseker scheme $\mathcal{M}^{\mathrm{ss}}(P_0, \mathcal{L}_0)$ of $\mathcal{L}_0$-oriented semistable sheaves of Hilbert polynomial P_0 plays the role of the quotient $\mathcal{M}_0$. The corresponding master spaces are the moduli spaces $\mathcal{M}^{\mathrm{ss}}(P_0, \mathcal{E}_0, \mathcal{L}_0)$ of semistable $\mathcal{L}_0$-oriented pairs of type $(P_0, \mathcal{E}_0)$.

Coupling with $\mathcal{E}_0$-valued homomorphisms $\varphi : \mathcal{E} \longrightarrow \mathcal{E}_0$ leads to two essentially different situations, depending on the rank r of the sheaves $\mathcal{E}$ under consideration:

1. When $\mathrm{rk}(\mathcal{E}_0) < r$, the framings $\varphi : \mathcal{E} \longrightarrow \mathcal{E}_0$ can never be injective, i.e. there are no semistable homomorphisms. This case correspond to the GIT situation $\mathcal{M}_\infty = \varnothing$.

2. As soon as $\mathrm{rk}(\mathcal{E}_0) \geq r$, the framings φ can become injective, and the Grothendieck schemes $\mathrm{Quot}^{\mathcal{E}_0, \mathcal{L}_0}_{P_{\mathcal{E}_0} - P_0}$ appear in the master space $\mathcal{M}^{\mathrm{ss}}(P_0, \mathcal{E}_0, \mathcal{L}_0)$. These Quot schemes are the analoga of the quotients $\mathcal{M}_\infty$ in the GIT situation.

In both cases the spaces of reductions are moduli spaces of objects which are of the same type but essentially of lower rank.

Everything can be made very explicit when the base manifold is a curve X with a trivial reference sheaf $\mathcal{E}_0 = \mathcal{O}_X^{\oplus k}$. In the case $k < r$, the master spaces relate correlation functions of moduli spaces of semistable bundles with fixed determinant to data associated with reductions. When $r = 2$, $k = 1$, the manifold of reductions are symmetric powers of the base curve, and the coupling principle can be used to prove the Verlinde formula, or to compute the volume and the characteristic numbers (in the smooth case) of the moduli spaces of semistable bundles.

The general case $k \geq r$ leads to a method for the computation of Gromov-Witten invariants for Grassmannians. These invariants can be regarded as correlation functions of suitable Quot schemes [Bertram et al. 1996], and the coupling principle relates them to data associated with reductions and moduli spaces of semistable bundles. In this case one needs a master space $\mathcal{M}^{\rm ss}(P_0, \mathcal{E}_0, \mathcal{L})$ associated with a Poincaré line bundle $\mathcal{L}$ on $\mathrm{Pic}(X) \times X$ which set theoretically is the union over $\mathcal{L}_0 \in \mathrm{Pic}(X)$ of the master spaces $\mathcal{M}^{\rm ss}(P_0, \mathcal{E}_0, \mathcal{L}_0)$ [Okonek et al. 1999]. One could try to prove the Vafa–Intriligator formula along these lines.

Note that the use of master spaces allows us to avoid the sometimes messy investigation of chains of flips, which occur whenever one considers the family of all possible $\mathbb{C}^*$-quotients of the master space [Thaddeus 1994; Bradlow et al. 1996].

The coupling principle has been applied in two further situations.

Using the coupling of vector bundles with twisted endomorphisms, A. Schmitt has recently constructed projective moduli spaces of Hitchin pairs [Schmitt 1998]. In the case of curves and twisting with the canonical bundle, his master spaces are natural compactifications of the moduli spaces introduced in [Hitchin 1987].

Last but not least, the coupling principle can also be used in certain gauge theoretic situations:

The coupling of instantons on 4-manifolds with Dirac-harmonic spinors has been described in detail in Chapter 2. In this case the instanton moduli spaces are the original moduli spaces $\mathcal{M}_0$, the Donaldson polynomials are the original correlation functions to compute, and the moduli spaces of PU(2)-monopoles are master spaces for the coupling with spinors. One is again in the special situation where $\mathcal{M}_\infty = \varnothing$, and the manifold of reductions is a union of moduli spaces of twisted abelian monopoles. In order to compute the contributions of the abelian moduli spaces to the correlation functions, one has to give explicit descriptions of the master space in an S^1-invariant neighborhood of the abelian locus.

Finally consider again the Lie group $G = \mathrm{Sp}(n)\cdot S^1$ and the $\mathrm{PSp}(n)$-monopole equations (SW^σ_a) for a $\mathrm{Spin}^{\mathrm{Sp}(n)\cdot S^1}(4)$-structure $\sigma : P^G \longrightarrow P_g$ in (X, g) and an abelian connection a in the associated S^1-bundle (see Remark 2.1.2). Regarding the compactification of the moduli space $\mathcal{M}^\sigma_a$ as master space associated with the coupling of $\mathrm{PSp}(n)$-instantons to harmonic spinors, one should get a relation between Donaldson $\mathrm{PSp}(n)$-theory and Seiberg–Witten type theories.

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Christian Okonek
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
CH-8057 Zürich
Switzerland
okonek@math.unizh.ch

Andrei Teleman
Facultatea de Matematică
Universitatea den Bucureşti
Strada Academici 14
70109 Bucarest
Rumania
teleman@math.unizh.ch