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Recent Techniques in Hyperbolicity Problems

YUM-TONG SIU

Abstract. We explain the motivations and main ideas regarding the new techniques in hyperbolicity problems recently introduced by the author and Sai-Kee Yeung and by Michael McQuillan. Streamlined proofs and alternative approaches are given for previously known results.

We say that a complex manifold is *hyperbolic* if there is no nonconstant holomorphic map from $\mathbb{C}$ to it. This paper discusses the new techniques in hyperbolicity problems introduced in recent years in a series of joint papers which I wrote with Sai-Kee Yeung [Siu and Yeung 1996b; 1996a; 1997] and in a series of papers by Michael McQuillan [McQuillan 1996; 1997]. The goal is to explain the motivations and the main ideas of these techniques. In the process we examine known results using new approaches, providing streamlined proofs for them.

The paper consists of three parts: an Introduction, Chapter 1, and Chapter 2. The Introduction provides the necessary background, states the main problems, and discusses the motivations and the main ideas of the recent new techniques. Chapter 1 presents a proof of the following theorem, using techniques from diophantine approximation.

Theorem 0.0.1. *Let $\hat{m}$ be a positive integer. Let V_λ $(1 \leq \lambda \leq \Lambda)$ be regular complex hypersurfaces in $\mathbb{P}_n$ of degree δ in normal crossing. Let $\varphi : \mathbb{C}^{\hat{m}} \to \mathbb{P}_n$ be a holomorphic map whose image is not contained in any hypersurface of $\mathbb{P}_n$. Then the sum of the defects $\sum_{\lambda=1}^{\Lambda} \operatorname{Defect}(\varphi, V_\lambda)$ is no more than ne for any $\delta \geq 1$ and is no more than $n+1$ for $\delta = 1$.*

Chapter 2 presents a streamlined proof of the following result:

Theorem 0.0.2 [Siu and Yeung 1996a]. *The complement in $\mathbb{P}_2$ of a generic curve of sufficiently high degree is hyperbolic.*

An overview of the proof of these two theorems can be found in Section 0.10 (page 446).

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Introduction

0.1. Statement of Hyperbolicity Problems. Hyperbolicity problems have two aspects, the qualitative aspect and the quantitative aspect. The easier qualitative aspect of the hyperbolicity problems is to prove that certain classes of complex manifolds are hyperbolic in the following sense. A complex manifold is *hyperbolic* if there is no nonconstant holomorphic map from $\mathbb{C}$ to it. There are two classes of manifolds which are usually used to test techniques introduced to prove hyperbolicity. One class is the complement of an ample divisor in an abelian variety, or a submanifold of an abelian variety containing no translates of abelian subvarieties. The second class is the complement of a generic hypersurface of high degree (at least $2n+1$) in the n-dimensional projective space $\mathbb{P}_n$ or a generic hypersurface of high degree (at least $2n-1$ for $n \geq 3$) in $\mathbb{P}_n$. The general conjecture is that any holomorphic map from $\mathbb{C}$ to a compact complex manifold with ample canonical line bundle (or even of general type) must be algebraically degenerate in the sense that its image is contained in a complex hypersurface of the manifold.

The harder quantitative aspect of the hyperbolicity problems is to get a defect relation. The precise definition of defect will be given below. Again there are two situations which are usually used to test new techniques to get defect relations. The first situation is to show that the defect for an ample divisor in an abelian variety is zero. The second situation is to show that for any algebraically nondegenerate holomorphic map from $\mathbb{C}$ to $\mathbb{P}_n$ the sum of the defects for a collection of hypersurfaces of degree δ in normal crossing is no more than $(n+1)/\delta$. The general conjecture is that, for any algebraically nondegenerate holomorphic map from $\mathbb{C}$ to a compact complex manifold M and for a positive line bundle L on M, the sum of the defects for a collection of hypersurfaces in normal crossing is no more than γ if each hypersurface is the divisor of a holomorphic section of L and if $(\gamma+\varepsilon)L + K_M$ is positive for any positive rational number ε. Here K_M means the canonical line bundle of M and the positivity of the $\mathbb{Q}$-bundle $(\gamma+\varepsilon)L + K_M$ means that some high integral multiple of $(\gamma+\varepsilon)L + K_M$ is a positive line bundle.

So far as hyperbolicity problems are concerned, whatever can be done for abelian varieties can also usually be done, with straightforward modifications, for semi-abelian varieties. So we will confine ourselves in this paper only to abelian varieties and not worry about the seemingly more general situation of semi-abelian varieties.

We now state more precisely what has been recently proved and what conjectures remain unsolved. We do not include here a number of very recent results available in preprint form whose proofs are still in the process of being studied and verified.

Since at this point the major difficulties of the hyperbolicity problems already occur in the case of abelian varieties and the complex projective space, we will confine ourselves to abelian varieties and the complex projective space and will not elaborate further on the case of a general compact projective algebraic manifold.

THEOREM 0.1.1 [McQuillan 1996; Siu and Yeung 1996b; 1997]. *The defect of an ample divisor in an abelian variety is zero. In particular, the complement of an ample divisor in an abelian variety is hyperbolic.*

CONJECTURE 0.1.2. *The complement in $\mathbb{P}_n$ of a generic hypersurface of degree at least $2n+1$ is hyperbolic.*

CONJECTURE 0.1.3. *A generic hypersurface of degree at least $2n-1$ in $\mathbb{P}_n$ is hyperbolic for $n \geq 3$.*

For dimensions higher than 1, one known case for Conjecture 0.1.2 is the following.

THEOREM 0.1.4 [Siu and Yeung 1996a]. *The complement in $\mathbb{P}_2$ of a generic curve of sufficiently high degree is hyperbolic.*

There are many partial results in cases when the hypersurface in Conjecture 0.1.2 or Conjecture 0.1.3 is not generic and either has many components or is of a special form such as defined by a polynomial of high degree and few nonzero terms. Since there are already quite a number of survey papers about such partial results for non generic hypersurfaces (for example [Siu 1995]), we will not discuss them here.

In the formal analogy between Nevanlinna theory and diophantine approximation [Vojta 1987], Conjecture 0.1.2 corresponds to the theorem of Roth [Roth 1955; Schmidt 1980] and Conjecture 0.1.3 corresponds to the Mordell Conjecture [Faltings 1983; 1991; Vojta 1992]. For that reason very likely a proof of Conjecture 0.1.3 may require some techniques different from those used in a proof of Conjecture 0.1.2. For example, the analog of Theorem 0.1.4 for the setting of Conjecture 0.1.3 is still open. The most difficult step in the proof of Theorem 0.1.4, which involves the argument of log-pole jet differentials and touching order, uses in an essential way the disjointness of the entire holomorphic curve from the generic curve of sufficiently high degree (see Remarks 0.3.1 and 0.3.2 and also Section 2.8).

For quantitative results involving defects the basic conjecture in the complex projective space is the following.

CONJECTURE 0.1.5. *Let V_λ $(1 \leq \lambda \leq \Lambda)$ be regular complex hypersurfaces in $\mathbb{P}_n$ of degree δ in normal crossing. Let $\varphi : \mathbb{C} \to \mathbb{P}_n$ be a holomorphic map whose image is not contained in any hypersurface of $\mathbb{P}_n$. Then the sum of defects $\sum_{\lambda=1}^{\Lambda} \operatorname{Defect}(\varphi, V_\lambda)$ is no more than $(n+1)/\delta$.*

The main difficulty of the conjecture occurs already for a single hypersurface. If there is a method to handle the case of a single hypersurface for Conjecture

0.1.5, very likely the same method works for the general case of a collection of hypersurfaces in normal crossing. Though the conjecture for a single hypersurface does not imply immediately Conjecture 0.1.2, it is very likely that its proof can be modified to give Conjecture 0.1.2. An example by Biancofiore [1982] shows that the algebraic nondegeneracy condition in Conjecture 0.1.5 cannot be replaced by the weaker condition that the image of φ is not contained in any hypersurface of degree δ.

0.2. Characteristic Functions, Counting Functions, Proximity Functions, and Defects. We now give certain definitions needed for precise discussion. Let M be a compact complex manifold with a positive holomorphic line bundle L whose positive definite curvature form is θ. Let s be a holomorphic section of L over M whose zero-divisor is W. Let $\varphi : \mathbb{C} \to M$ be a holomorphic map. We multiply the metric of L by a sufficiently large positive constant so that the pointwise norm $\|s\|$ of s with respect to the metric of L is less than 1 at every point of M. The characteristic function is defined by

$$T(r, \varphi, \theta) = \int_{\rho=0}^{r} \frac{d\rho}{\rho} \int_{|\zeta|<\rho} \varphi^*\theta$$

which changes by a bounded term as $r \to \infty$ when another positive definite curvature form of L is used. Let $n(\rho, \varphi^*W)$ denote the number of zeroes (with multiplicities) of the divisor φ^*W in $\{|\zeta| < \rho\}$. The counting function is defined as

$$N(r, \varphi, W) = \int_{\rho=0}^{r} n(\rho, \varphi^*W) \frac{d\rho}{\rho}$$

which we also denote by $N(r, \varphi, s)$. When Z is a divisor in $\mathbb{C}$, we also denote by $n(\rho, Z)$ the number of zeroes (with multiplicities) of the divisor Z in $\{|\zeta| < \rho\}$ and define

$$N(r, Z) = \int_{\rho=0}^{r} n(\rho, Z) \frac{d\rho}{\rho}.$$

Let $\oint_{|\zeta|=r}$ denote the average over the circle $\{|\zeta| = r\}$. The proximity function is defined by

$$m(r, \varphi, s) = \oint_{|\zeta|=r} \log \frac{1}{\|\varphi^* s\|}$$

which changes by a bounded term as $r \to \infty$ when another metric of L is used. We will denote $m(r, \varphi, s)$ also by $m(r, \varphi, W)$. The defect is defined as

$$\operatorname{Defect}(\varphi, s) = \liminf_{r\to\infty} \frac{m(r, \varphi, s)}{T(r, \varphi, \theta)}$$

which we also denote by $\operatorname{Defect}(\varphi, W)$. Let σ be a positive number and let $\tilde{\varphi}_\sigma(\zeta) = \varphi(\sigma\zeta)$. Then from the definitions we have

$$T(r,\varphi,\theta)=T\Big(\frac{r}{\sigma},\tilde{\varphi}_\sigma,\theta\Big), \quad N(r,\varphi,s)=N\Big(\frac{r}{\sigma},\tilde{\varphi}_\sigma,s\Big), \quad m(r,\varphi,s)=m\Big(\frac{r}{\sigma},\tilde{\varphi}_\sigma,s\Big).$$

When $M = \mathbb{P}_n$ and L is the hyperplane section line bundle of $\mathbb{P}_n$ and θ is the Fubini–Study form, we simply denote $T(r, \varphi, \theta)$ by $T(r, \varphi)$. In the case a holomorphic map from $\mathbb{C}^{\hat{m}}$ to M, its characteristic function, counting function and proximity function is defined by computing those of the restriction of the map to a complex line in the complex vector space $\mathbb{C}^{\hat{m}}$ and then averaging over all such complex lines. Its defect is defined in the same way from its proximity function and its characteristic function as in the case $\hat{m} = 1$.

There is an alternative description of the characteristic function in the case of the complex projective space and we need this alternative description for the dimension one case later. For a holomorphic map φ from $\mathbb{C}$ to $\mathbb{P}_n$ we can use the homogeneous coordinates of $\mathbb{P}_n$ and represent φ in the form $[\varphi_0, \ldots, \varphi_n]$ by $n+1$ holomorphic functions φ_j $(0 \le j \le n)$ without common zeroes on $\mathbb{C}$. Let θ be the Fubini–Study form on $\mathbb{P}_n$. Then

$$\varphi^*\theta = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\sum_{j=0}^{n}|\varphi_j|^2)$$

and two integrations give

$$T(r, \varphi, \theta) = \oint_{|\zeta|=r} \tfrac{1}{2}\log\bigg(\sum_{j=0}^{n}|\varphi_j|^2\bigg) - \tfrac{1}{2}\log\bigg(\sum_{j=0}^{n}|\varphi_j(0)|^2\bigg).$$

Since

$$\begin{aligned}\max_{0\le j\le n}\log|\varphi_j| \le \tfrac{1}{2}\log\bigg(\sum_{j=0}^{n}|\varphi_j|^2\bigg) &\le \tfrac{1}{2}\log\big((n+1)\max_{0\le j\le n}\log|\varphi_j|^2\big)\\ &\le \max_{0\le j\le n}\log|\varphi_j| + \tfrac{1}{2}\log(n+1),\end{aligned}$$

it follows that up to a bounded term the characteristic function $T(r, \varphi, \theta)$ can be described by $\oint_{|\zeta|=r}\max_{0\le j\le n}\log|\varphi_j|$.

Consider the special case $n = 1$. The characteristic function $T(r, \varphi)$ up to a bounded term is equal to

$$\begin{aligned}\oint_{|\zeta|=r}\max\left(|\varphi_0|, |\varphi_1|\right) &= \oint_{|\zeta|=r}\log|\varphi_0| + \oint_{|\zeta|=r}\max\bigg(1, \log\bigg|\frac{\varphi_1}{\varphi_0}\bigg|\bigg)\\ &= \log|\varphi(0)| + N(r, \varphi_0, 0) + \oint_{|\zeta|=r}\log^+\bigg|\frac{\varphi_1}{\varphi_0}\bigg|.\end{aligned}$$

Here $\log^+$ means the maximum of log and 0. Thus for a single meromorphic function F the characteristic function for the map $\mathbb{C} \to \mathbb{P}_1$ defined by F is equal to

$$\oint_{|\zeta|=r}\log^+|F| + N(r, F, \infty)$$

up to a bounded term.

0.3. The Approach of Jet Differentials. There are two different approaches to proving hyperbolicity. One originated with Bloch [1926], who introduced the use of holomorphic jet differentials vanishing on some ample divisor. Another has its origin from the theory of diophantine approximation. From our present understanding of the so-called Ahlfors–Schwarz lemma for jet differentials, the technique of jet differentials and the technique of diophantine approximation share the same origin of using meromorphic functions of low pole order with high vanishing order, as explained later in this section by means of the logarithmic derivative lemma.

A holomorphic (respectively meromorphic) k-jet differential ω of total weight m on a complex manifold M with local coordinates $z_1, \dots, z_n$ is locally a polynomial, with holomorphic (respectively meromorphic) functions as coefficients, in the variables $d^l z_j$ $(1 \le l \le k,\ 1 \le j \le n)$ and of homogeneous weight m when $d^l z_j$ is given the weight l. A meromorphic k-jet differential M is said to be a log-pole k-jet differential M if it is locally a polynomial, with holomorphic functions as coefficients, in the variables $d^l z_j, d^\nu \log g_\lambda$ $(1 \le l \le k,\ 1 \le j \le n,$ $1 \le \nu \le k,\ 1 \le \lambda \le \Lambda)$, where the g_λ $(1 \le \lambda \le \Lambda)$ are local holomorphic functions whose zero-divisors are contained in a finite number of global nonnegative divisors of M.

The key step in the approach using holomorphic jet differentials is what is usually referred to as the Ahlfors–Schwarz lemma or simply as the Schwarz lemma which says the following. If φ is a holomorphic map from $\mathbb{C}$ to a complex manifold M and if ω is a holomorphic (or log-pole) k-jet differential on M which vanishes on an ample divisor of M (and the image of φ is disjoint from the log-pole of ω), then $\varphi^*\omega$ is identically zero on $\mathbb{C}$.

Remark 0.3.1. In the Schwarz lemma for log-pole jet differentials, the image of the map has to be disjoint from the log-pole of the jet differential. This is one of the main reasons why Conjecture 0.1.3 may require some techniques different from those used in a proof of Conjecture 0.1.2. It is the same reason why the proof of Theorem 0.1.4 cannot be readily modified to yield its analog in the setting of Conjecture 0.1.3.

Remark 0.3.2. Nevanlinna's original theory already makes use of the log-pole differential

$$\left(\prod_{j=1}^{m} 1/(z - a_j)\right) dz$$

on $\mathbb{P}_1$ with affine coordinate z for $m \ge 3$. Note that, in the Schwarz lemma, the vanishing of the pullback of a meromorphic jet differential vanishing on some ample divisor requires the following two key ingredients. The first one is that only log-pole singularities are allowed. Other kinds of pole orders are not allowed. The second one is that the image of the map has to be disjoint from the log-pole. Since the two key ingredients are already essential in the case $M = \mathbb{P}_1$, one cannot weaken the two requirements by simply assuming that the poles of the meromorphic jet differential are in some normal form.

We denote by $J_k(M)$ the bundle of all k-jets of M so that $J_1(M)$ is simply the tangent bundle of M. An element of $J_k(M)$ at a point P of M is defined by a holomorphic map $\gamma : U \to M$ for some open neighborhood U of 0 in $\mathbb{C}$ with $\gamma(0) = P$ and another $\tilde{\gamma}$ defines the same element of $J_k(M)$ if γ and $\tilde{\gamma}$ agree up to order k at 0.

Define the map $d^k\varphi : \mathbb{C} \to J_k(M)$ so that its value at $\zeta \in \mathbb{C}$ is the k-jet at $\varphi(\zeta) \in M$ defined by the curve $\varphi : \mathbb{C} \to M$. The Schwarz lemma means that the image of $\mathbb{C}$ under φ satisfies the differential equation $\omega = 0$. For this, it suffices to have the k-jet differential ω defined as a function on $(d^k\varphi)(\mathbb{C})$ instead of on all of $J_k(M)$. When we have enough independent differential equations of such a kind, we can eliminate the derivatives of φ from the differential equations to get the constancy of the map φ and conclude hyperbolicity. An equivalent way of looking at it is to get hyperbolicity by constructing a holomorphic (or log-pole) k-jet differential on the Zariski closure in $J_k(M)$ of $\varphi(\mathbb{C})$ which vanishes on an ample divisor. It suffices also to construct a collection of local holomorphic (or log-pole) k-jet differential on M vanishing on an ample divisor so that they can be pieced together to give a well defined function on the Zariski closure of $(d^k\varphi)(\mathbb{C})$ in $J_k(M)$. Here the Zariski closure of $(d^k\varphi)(\mathbb{C})$ in $J_k(M)$ means the intersection with $J_k(M)$ of the Zariski closure of $(d^k\varphi)(\mathbb{C})$ in the compactification of $J_k(M)$.

The geometric reason for the Schwarz lemma can be heuristically explained as follows. The existence of a holomorphic section ω of the k-jet bundle $J_k(M)$ which vanishes on an ample divisor D means that $J_k(M)$ carries certain positivity. The pullback $\varphi^*\omega$ is a holomorphic section of $J_k(\mathbb{C})$ and vanishes on the pullback of the zero divisor of ω. On the other hand, since the bundle $J_k(\mathbb{C})$ over $\mathbb{C}$ is globally trivial, there is no positivity of $J_k(\mathbb{C})$ to support the zero divisor of the holomorphic section $\varphi^*\omega$ which contains φ^*D if $\varphi^*\omega$ is not identically zero.

A so-called pointwise version of the Schwarz lemma could be formulated and proved by using arguments involving curvature or some generalized notion of it (see for example [Siu and Yeung 1997]). Such a pointwise version implies the Schwarz lemma just stated. However, the most natural proof of the Schwarz lemma is from the use of the logarithmic derivative lemma in Nevanlinna theory. Let $F(\zeta)$ denote the value of ω at $(d^k\varphi)(\zeta) \in J_k(M)$. Assume that $\varphi^*\omega$ is not identically zero and we will get a contradiction. For some suitable coordinate ζ of $\mathbb{C}$, the holomorphic function $F(\zeta)$ is not identically zero. The characteristic function $T(r, F)$ of F is computed by

$$T(r, F) = \oint_{|\zeta|=r} \log^+ |F(\zeta)|.$$

The key point is that ω is dominated in absolute value by a polynomial with constant coefficients of a finite number of variables of the form $d^l \log g$ with

$1 \leq l \leq k$ for some meromorphic functions g on M. The logarithmic derivative lemma says that

$$\oint_{|\zeta|=r} \log^+ |d^l \log g(\varphi(\zeta))| = O(\log T(r,\varphi))$$

for $l \geq 1$. (Note that later on, when we have inequalities derived from the logarithmic derivative lemma, they will hold only outside a set of finite measure with respect to dr/r. This is not made explicit in the notation, but it should not cause confusion.) Hence $T(r,F) = O(\log T(r,\varphi))$. On the other hand, since ω vanishes on an ample divisor of M, we must have $T(r,F) \geq N(r,F,0) \geq cT(r,\varphi)$ for some positive c, giving $T(r,\varphi) = O(\log T(r,\varphi))$ which contradicts φ being a nonconstant map. This proof works also when ω is a k-jet differential with at most log-pole singularities vanishing on an ample divisor if the image of φ is disjoint from the log-pole. The idea of this proof in the case of an abelian variety was already in [Bloch 1926] and for the case of a general complex manifold was already in [Ru and Wong 1995]. The proof can be interpreted by the pole-order and the vanishing order in the spirit of the method of diophantine approximation as follows. The pullback of the holomorphic 1-jet differential when regarded as a holomorphic function must vanish because the logarithmic derivative lemma takes care of the differentials so that the characteristic function is less than the case of the pole order of any ample divisor but the counting function is like the case of the vanishing order of an ample divisor.

When it comes to the quantitative aspect involving defects, the approach of jet differentials uses jet differentials with low pole-order but high vanishing order along the hypersurfaces whose defects are under consideration. There are two difficulties, the first difficulty is to construct a jet differential with low pole order but high vanishing order along the hypersurfaces. The second difficulty is to make sure that the pullback, to the entire holomorphic curve, of the constructed jet differential is not identically zero.

To handle the first difficulty, when we construct jet differentials we can adjoin many variables of the form $d^l \log g$, with $l \geq 1$ and g holomorphic, to increase the available degrees of freedom to get more vanishing order along the hypersurfaces, without essentially increasing the growth order of the pullback of the constructed jet differential. What makes this possible is the logarithmic derivative lemma. The troublesome point is that we have to make sure that, after adjoining variables of the form $d^l \log g$, the counting function for the pole order is somehow still under control. The situation is much easier in the case of an abelian variety, because we can use the differentials

$$d^l z_j = d^l \exp z_j$$

of coordinates of $\mathbb{C}^n$ as $d^l \log g$ and the nowhere vanishing of the exponential function $\exp z_j$ makes it unnecessary for us to worry about the difficulty of the increased growth of the counting function for the pole order.

When the difficulty of constructing a jet differential with low pole order and high vanishing order along the hypersurfaces and the difficulty of making sure that its pullback to the entire holomorphic curve is not identically zero are both overcome, the above proof of the Schwarz lemma by Nevanlinna theory is easily adapted to give a defect relation.

The second difficulty of making sure the non identical vanishing of the pullback of the jet differential to the entire holomorphic curve corresponds to the step in the proof of Roth's theorem [Roth 1955; Schmidt 1980] of making sure that the constructed polynomial of low degree and high vanishing order has low vanishing order at a point whose components are all equal to the given algebraic number. In the proof of Roth's theorem it was originally done by using Roth's lemma [Roth 1955; Schmidt 1980] and could also be handled by methods introduced later such as the product theorem of Faltings [1991].

For function theory, so far there are two ways of handling the difficulty. One is the use of the translational invariance of the Zariski closure of the differential of a Zariski dense entire curve [Siu and Yeung 1996a; 1997]. Another is the independent slight rescaling of the parameters of the component functions of an entire curve in a product of copies of an abelian variety [McQuillan 1997] which we will discuss more in Section 0.4. Both were introduced to prove Theorem 0.1.1.

Probably the correct way of handling the situation is to use the product theorem of Faltings [1991], but so far there is no way to overcome the following difficulty of adapting Faltings's product theorem to the function theory case. For the application of Faltings's product formula, the ratio of the degrees of the constructed polynomial in consecutive sets of variables has to be greater than some appropriate constant. For diophantine approximation the sequence of approximating rational numbers are chosen to have heights and proximities corresponding to the degrees. An analogous situation for function theory is that, for the component functions of an entire curve in a product of copies of the target manifold, one chooses a rescaling of the parameters to make the characteristic functions and *at the same time* the proximity functions correspond to the degrees of the constructed polynomial in various sets of variables. However, unlike the case of diophantine approximation where a finite sum is used for the corresponding situation, in function theory the proximity function is defined by an integral, which gives rise to a more complicated technical difficulty, so far not overcome.

0.4. The Approach Motivated by Diophantine Approximation. Now we discuss the second approach of using techniques motivated from those of diophantine approximation. The key feature of this second approach is that the k-jet bundle $J_k(M)$ of the target manifold M in the jet differential approach is replaced by a product $M^{\times(k+1)}$ of $k+1$ copies of M. A jet differential in the first approach is replaced by a section of a certain positive line bundle L over

$M^{\times(k+1)}$ in the second approach. For example, in the case where M is an abelian variety A, one can use as L the pullback under

$$A^{\times(k+1)} \to A^{\times(k+1)},$$
$$(x_0, \ldots, x_k) \mapsto (x_0, x_1 - x_0, \ldots, x_k - x_{k-1})$$

of the tensor product of appropriate ample line bundles on the factors of $A^{\times(k+1)}$. For the defect of a hypersurface D in M or the hyperbolicity of $M - D$, this approach involves constructing holomorphic sections s of L over $M^{\times(k+1)}$ so that the sections vanish to high order along $D^{\times(k+1)}$ and yet the characteristic function, with respect to the positive curvature form of L, of the diagonal map $\tilde{\varphi} : \mathbb{C} \to M^{\times(k+1)}$ of the holomorphic map $\varphi : \mathbb{C} \to M$ has slow growth.

For the abelian variety A, the use of $x_j - x_k$ in the approach of diophantine approximation corresponds to the use of dx_j in the approach of jet differentials. It gives us more available degrees of freedom to get more vanishing order, without essentially increasing the growth order of the pullback of the constructed section by the diagonal map, because $x_j - x_k$ vanishes on the diagonal map.

As in the approach of jet differentials, there are in the approach of diophantine approximation the same two major difficulties. The first difficulty is to construct a holomorphic section of a line bundle on the product space with high vanishing order along certain subvarieties so that its pullback to the entire holomorphic curve has low pole order (*i.e.* small characteristic function). The second difficulty is to make sure that the pullback $\tilde{\varphi}^* s$ of the section s to the entire holomorphic curve is not identically zero.

One advantage of the approach of diophantine approximation is that it is easier to use the assumption of algebraic nondegeneracy of the map φ to handle the difficulty of the identical vanishing of $\tilde{\varphi}^* s$. When M is an abelian variety A, for this step McQuillan [1996; 1997] introduced the technique of considering the map $\mathbb{C}^{k+1} \to A^{\times(k+1)}$ induced by φ and rescaling separately the variable of each factor of $\mathbb{C}^{k+1}$. He chose the difference between the rescaling factors and 1 to be of the order of the reciprocal of some high power of the characteristic function at r when integration over the circle $\{|\zeta| = r\}$ is considered.

On the other hand, for the approach of diophantine approximation it can be very hard to construct a holomorphic section of a line bundle on the product space with high vanishing order along certain subvarieties whose pullback to the entire holomorphic curve has low pole order. How hard it is depends on which subvarieties the section is required to vanish along to high order. For example, in the case of the complex projective space it is not possible to require vanishing to high order along the product $D^{\times(k+1)}$ of one single hypersurface D, but it is easy to require vanishing to high order only along the diagonal of $D^{\times(k+1)}$. In order to use rescaling techniques to rule out identical vanishing of the pullback to the entire holomorphic curve, the vanishing along $D^{\times(k+1)}$, instead of merely its diagonal, is needed. That is the reason why for Theorem 0.0.1 only the case of many hypersurfaces gives nontrivial results. For the case of many hypersurfaces $D = \bigcup_\lambda V_\lambda$, the argument goes through also when vanishing to high order along $\bigcup_\lambda V_\lambda^{\times(k+1)}$ is used instead of $D^{\times(k+1)}$.

The abelian case is special in that there is an addition so that for a holomorphic map φ from $\mathbb{C}$ to an abelian variety, the rescaled map $\varphi_\lambda(\zeta) := \varphi(\lambda\zeta)$ gives the following inequality concerning the characteristic function of the difference of two rescaled maps:

$$T(\varphi_\lambda - \varphi_\mu, r) \leq \frac{|\lambda - \mu| r}{(R - |\lambda| r)(R - |\mu| r)} T(\varphi, R) + O(1)$$

when $\max(|\lambda|, |\mu|) r < R$, which enables one to control the characteristic function after separate rescaling. Note that, when one has a holomorphic map $\varphi : \mathbb{C} \to \mathbb{C}^n$, this inequality for the characteristic functions of the difference of two rescaled maps does not hold for the difference operation in $\mathbb{C}^n$. In the case of the abelian variety A we can use the difference operation in A to construct a holomorphic section of a line bundle on $A^{\times(k+1)}$ with high vanishing order along $D^{\times(k+1)}$ whose pullback to the entire holomorphic curve has low pole order. The above inequality makes sure that after the perturbation by rescaling, there is no essential increase in the pole order of the pullback.

One also has to control the effect of the separate rescaling on the counting function which was worked out in [McQuillan 1997]. That particular control works in the case of the projective variety as well as for the abelian variety and it is explained in Section 1.3.

For the first approach of jet differentials, Pit-Mann Wong with his collaborators Min Run and Julie Wang also started introducing the perturbation of $(d^k\varphi)(\mathbb{C})$ to handle the difficulty that $\varphi^*\omega$ is identically zero. The difficulties with such perturbation methods for the approach of jet differentials are the same as those occurring in the approach of diophantine approximation when one requires a constructed section to vanish to high order only along the diagonal of $D^{\times(k+1)}$. So far such difficulties are essential and cannot yet be overcome. We will explain more about them later in Section 0.8.

To see how the techniques mentioned above are applied to hyperbolicity problems and to understand the major obstacles for further progress, we discuss the hyperbolicity problems of the abelian variety which by now have been completely proved and understood. The starting point is the following theorem of Bloch.

Theorem 0.4.1 [Bloch 1926; Green and Griffiths 1980; Ochiai 1977; Wong 1980; Kawamata 1980; Noguchi and Ochiai 1990]. *Let A be an abelian variety and $\varphi : \mathbb{C} \to A$ be a holomorphic map. Let X be the Zariski closure of the image of φ. Then X is the translate of an abelian subvariety of A.*

0.5. Proof of Bloch's Theorem. Denote by $\mathcal{X}$ the Zariski closure of $(d^k\varphi)(\mathbb{C})$ in $J_k(A)$. Here and in the rest of this discussion the Zariski closure in $J_k(A)$

means the intersection with $J_k(A)$ of the Zariski closure of $(d^k\varphi)(\mathbb{C})$ in the compactification $A \times \mathbb{P}_{nk}$ of $J_k(A) = A \times \mathbb{C}^{nk}$. Consider the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\sigma_k} & \mathbb{C}^{nk} \\ \tau\big\downarrow & & \\ A & & \end{array}$$

where σ_k is induced by the natural projection map $J_k(A) = A \times \mathbb{C}^{nk} \to \mathbb{C}^{nk}$ and τ comes from the composite of the map $J_k(X) \to X$ and $X \hookrightarrow A$.

The proof of Bloch's theorem depends on two observations of Bloch.

Observation 0.5.1 (Bloch). *For $k \geq n$ if the map $\sigma_k : \mathfrak{X} \to \mathbb{C}^{nk}$ is not generically finite onto its image, then X is invariant under the translation by some nonzero element of A.*

Proof. Take a point $\zeta_0 \in \mathbb{C}$ so that $\varphi(\zeta_0)$ is a regular point of X. Let N be the complex codimension of X in A. Let $\omega_1, \dots, \omega_N$ be local holomorphic 1-forms on A whose common zero-set is the tangent bundle of X near $\varphi(\zeta_0)$. There is a tangent vector ξ to $J_k(X)$ at the point $(d^k\varphi)(\zeta_0)$ which is mapped to zero by σ_k. The tangent vector ξ is given by a one-parameter local perturbation $\Phi(\zeta, t)$ of the curve φ inside X defined near the point $(\zeta, t) = (\zeta_0, 0)$. The vanishing of $\sigma_k(\xi)$ means that the tangent vector field $\frac{\partial\Phi}{\partial t}(\zeta, 0)$ has zero derivative up to order k along $\varphi(\mathbb{C})$ at $\varphi(\zeta_0)$. Here the differentiation of a tangent vector field of A is with respect to the flat connection for A. Then the fact that $\xi \in J_k(X)$ implies that the value of the derivatives of ω_j up to order k along $\varphi(\mathbb{C})$ vanishes at the tangent vector $\frac{\partial\Phi}{\partial t}(\zeta_0, 0)$. Thus the $((k+1)N) \times n$ matrix formed by the derivatives up to order k, of $\omega_j(\frac{\partial}{\partial z_\nu})$ $(1 \leq \nu \leq n,\ 1 \leq j \leq N)$ along $\varphi(\mathbb{C})$ at ζ_0 has rank less than n. Since this holds when ζ_0 is replaced by an arbitrary ζ near ζ_0, it follows from the standard Wronskian argument that there is a nonzero constant tangent vector η on A such that $\omega_j(\eta)$ is identically zero along $\varphi(\zeta)$ near $\zeta = \zeta_0$. The Zariski density of the image of φ in X implies that X is invariant under the translation in the direction of the tangent vector η. □

Observation 0.5.2 (Bloch). *If $\sigma_k : \mathfrak{X} \to \mathbb{C}^{nk}$ is generically finite onto its image, then for any ample divisor D of A there exists some polynomial of $d^l z_j$ $(1 \leq l \leq k,\ 1 \leq j \leq n)$ with constant coefficients which vanishes on $\tau^{-1}(D)$ but does not vanish identically on $\mathfrak{X}$.*

Proof. The existence of P is verified as follows. For q sufficiently large, there exists a meromorphic function F on A whose divisor is $E - qD$ so that $E \cap X$ and $D \cap X$ do not have any common branch. Since τ is surjective and σ_k is generically finite onto its image, $F \circ \tau$ belongs to a finite extension of the field of all rational functions of $\mathbb{C}^{nk}$. Thus there exist polynomials P_j $(0 \leq j \leq p)$ with constant coefficients in the variables $d^l z_\nu$ $(1 \leq l \leq k, 1 \leq \nu \leq n)$ such that

$$\sum_{j=0}^{p} (\sigma_k^* P_j)(\tau^* F)^j = 0$$

on $\mathcal{X}$ and $\sigma_k^* P_p$ is not identically zero on $\mathcal{X}$. Then P_p must vanish on $\tau^{-1}(D)$ and the holomorphic jet differential P_p on X must vanish on $\tau^{-1}(D)$. We need only set $P = P_p$. □

Bloch's theorem now follows easily from the two observations in the following way. Assume that X is not a translate of an abelian subvariety of A. Let A' be the quotient of A by the subgroup of all elements whose translates leave X invariant. By replacing φ by its composite with the quotient map $A \to A'$, we can assume without loss generality that X is not invariant by the translation of any element of A. From Bloch's first observation σ_k is generically finite onto its image. From Bloch's second observation and the Schwarz's lemma $\varphi^* P$ is identically zero, which contradicts the non identical vanishing of P on $\mathcal{X}$.

In Observation 0.5.1 the significance of the number n in the inequality $k \geq n$ is that there are n coefficients in each $\omega_1, \ldots, \omega_N$, which means that $k \geq$ the dimension of X. The zero-dimensionality of the generic fiber of σ_k corresponds to the following statement used in diophantine approximation [Vojta 1996, Lemma 5.1].

PROPOSITION 0.5.3. *Suppose A is an abelian variety and X is a subvariety of A which is not invariant under the translation of any nonzero element of A. Then for any $m > \dim X$ the map $X^{\times m} \to A^{\times(m(m-1)/2)}$ defined by $(x_j)_{1 \leq j \leq m} \mapsto (x_j - x_k)_{1 \leq j < k \leq m}$ is generically finite onto its image.*

0.6. Proof of Hyperbolicity of Complement of an Ample Divisor in an Abelian Variety. Bloch's argument is modified in [Siu and Yeung 1996a] with the introduction of a log-pole jet differential to give the hyperbolicity of the complement of $A - D$ for any ample divisor D of the abelian variety A. Suppose there is a nonconstant holomorphic map $\varphi : \mathbb{C} \to A - D$ and we will derive a contradiction. By Bloch's theorem we can assume that the image of φ is Zariski dense in A. Let E be the largest subspace of $\mathbb{C}^n$ such that the lifting of φ to $\mathbb{C} \to \mathbb{C}^n$ is contained in a translate of E. A basis of E is given by $\partial/\partial z_{\nu_1}, \ldots, \partial/\partial z_{\nu_q}$. Let $k = q + 1$. Let θ be a theta function defining the ample divisor D. The locally defined k-jet differential

$$\det \begin{pmatrix} d \log \theta & dz_{\nu_1} & dz_{\nu_2} & \cdots & dz_{\nu_q} \\ d^2 \log \theta & d^2 z_{\nu_1} & d^2 z_{\nu_2} & \cdots & d^2 z_{\nu_q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d^{q+1} \log \theta & d^{q+1} z_{\nu_1} & d^{q+1} z_{\nu_2} & \cdots & d^{q+1} z_{\nu_q} \end{pmatrix}$$

gives a well-defined function Θ on the Zariski closure $\mathcal{X}$ of $(d^k\varphi)(\mathbb{C})$ in $J_k(A)$. Now add the function Θ to the nk coordinates of the map $\sigma_k : \mathcal{X} \to \mathbb{C}^{nk}$ to form $\tilde{\sigma}_k : \mathcal{X} \to \mathbb{C}^{nk+1}$. We now use $\tilde{\sigma}_k$ instead of σ_k in Bloch's two observations. Bloch's second observation shows that the map σ_k cannot be generically finite onto its image. Bloch's first observation shows that there exists some nonzero constant direction $\sum_{\alpha=1}^n c_\alpha \frac{\partial}{\partial z_\alpha}$ such that $\varphi^*\left(\sum_{\alpha=1}^n c_\alpha \frac{\partial}{\partial z_\alpha}\right)\Theta$ is identically zero. The standard Wronskian argument then shows that $\varphi^*\left(\sum_{\alpha=1}^n c_\alpha \frac{\partial}{\partial z_\alpha}\right)^2 \log\theta$ is identically zero on $\mathbb{C}$. Because of the Zariski density of $\varphi(\mathbb{C})$ in A, this implies that $\left(\sum_{\alpha=1}^n c_\alpha \frac{\partial}{\partial z_\alpha}\right)^2 \log\theta$ is identically zero on A, which is a contradiction.

0.7. Proof of the Defect Relation for Ample Divisors of Abelian Varieties. The defect relation in Theorem 0.1.1 for an ample divisor D in an abelian variety A was proved in [Siu and Yeung 1997] by using the following generalization of Bloch's theorem. If the image of a holomorphic map $\varphi : \mathbb{C} \to A$ is Zariski dense in an abelian variety A, then the Zariski closure $\overline{(d^k\varphi)(\mathbb{C})}$ of $(d^k\varphi)(\mathbb{C})$ in $J_k(A) = A \times \mathbb{P}_{nk}$ is invariant under the translation by any element of A.

The translational invariance of $\overline{(d^k\varphi)(\mathbb{C})}$ by elements of A means that $\overline{(d^k\varphi)(\mathbb{C})}$ is of the form $A \times W$ for some complex subvariety $W \subset \mathbb{P}_{nk}$ of complex dimension d. When $k \geq n$, since the dimension of $\overline{J_k(D)} \cap (A \times W)$ is at most $(n+d) - (k+1) \leq d-1$ which is less than the complex dimension of W, by the theorem of Riemann–Roch, for any $\varepsilon > 0$ we obtain the following. There exist positive integers p, q with $p/q < \varepsilon$ and there exist pD-valued holomorphic k-jet differentials on A vanishing to order at least q on $J_k(D)$ so that they give a non identically zero well-defined function on $\overline{(d^k\varphi)(\mathbb{C})}$. Then the following standard application of the First Main Theorem technique and the logarithmic derivative lemma yields the upper bound ε for the defect $\mathrm{Defect}(\varphi, D)$ of the map $\varphi : \mathbb{C} \to A$ and the ample divisor D.

Let $\mathcal{A}_r(\cdot)$ denote the operator which averages over the circle in $\mathbb{C}$ of radius r centered at the origin. Let $A = \mathbb{C}^n/\Lambda$ for some lattice Λ and let the divisor D be defined by the theta function θ on $\mathbb{C}^n$ which satisfies the transformation equation

$$\theta(z+u) = \theta(z)\exp\left(\pi H(z,u) + \frac{\pi}{2}H(u,u) + 2\pi\sqrt{-1}\,K(u)\right)$$

for some positive definite Hermitian form $H(z,w)$ and some real-valued function $K(u)$ for $u \in \Lambda$ so that $\exp\left(2\pi\sqrt{-1}K(u)\right)$ is a character on the lattice Λ. Let L_θ be the line bundle on A associated to the divisor D. We choose the global trivialization of the pullback of L_θ to $\mathbb{C}^n$ so that the theta function θ on $\mathbb{C}^n$ corresponds to a holomorphic section of L_θ whose divisor is D. We give L_θ a Hermitian metric so that with respect to our global trivialization of the pullback of L_θ to $\mathbb{C}^n$, it is given by $\exp(-\pi H(z,z))$. The connection from the Hermitian metric is given by $\mathcal{D}g = \partial g - \pi H(dz,z)g$ on $\mathbb{C}^n$. In particular,

$$\mathcal{D}\theta = d\theta + \sum_{\mu,\nu=1}^n h_{\mu,\bar\nu}\bar z_\nu dz_\mu \theta,$$

where $H(z,z) = \sum_{\mu,\nu=1}^{n} h_{\mu,\bar\nu} z_\mu \bar z_\nu$, and

$$\frac{\mathcal{D}\theta}{\theta} = d\log\theta + \sum_{\mu,\nu=1}^{n} h_{\mu,\bar\nu} \bar z_\nu d\log\exp z_\mu.$$

Let

$$\vec\nu = (\nu_{\alpha,\beta})_{1\le\alpha\le k, 1\le\beta\le n}, \qquad \text{weight}(\vec\nu) = \sum_{\substack{1\le\alpha\le k\\ 1\le\beta\le n}} \alpha\nu_{\alpha,\beta},$$

and

$$d^{\vec\nu} z = \prod_{\substack{1\le\alpha\le k\\ 1\le\beta\le n}} (d^\alpha z_\beta)^{\nu_{\alpha,\beta}}.$$

An pD-valued holomorphic k-jet differential on A vanishing to order at least q on $J_k(D)$ means

$$P = \sum_{\text{weight}(\vec\nu)=p} \tau_{\vec\nu}\left(d^{\vec\nu} z\right),$$

where $\tau_{\vec\nu}$ is an entire function on $\mathbb{C}^n$ so that $\frac{\tau_{\vec\nu}}{\theta^p}$ defines a meromorphic function on the abelian variety A. In other words, $\tau_{\vec\nu}$ defines a holomorphic section of $p\,L_\theta$ over A. Moreover, P vanishes to order at least q along

$$\{\theta = d\theta = \cdots = d^k\theta = 0\},$$

which means that we can write

$$P = \sum_{\nu_0+\nu_1+\cdots+\nu_k=q} a_{\nu_0,\nu_1,\dots,\nu_k} \theta^{\nu_0} (\mathcal{D}\theta)^{\nu_1} \cdots \left(\mathcal{D}^k\theta\right)^{\nu_k}$$

with smooth functions $a_{\nu_0,\nu_1,\dots,\nu_k}$ on $\mathbb{C}^n$ so that $\frac{a_{\nu_0,\nu_1,\dots,\nu_k}}{\theta^{p-q}}$ defines a function on A. In other words, $a_{\nu_0,\nu_1,\dots,\nu_k}$ is a smooth section of $(p-q)L_\theta$ over A. Then

$$\frac{P}{\theta^q} = \sum_{\nu_0+\nu_1+\cdots+\nu_k=q} a_{\nu_0,\nu_1,\dots,\nu_k} \left(\frac{\mathcal{D}\theta}{\theta}\right)^{\nu_1} \cdots \left(\frac{\mathcal{D}^k\theta}{\theta}\right)^{\nu_k}$$

Let $\tilde\varphi$ be the lifting of φ to $\mathbb{C}\to\mathbb{C}^n$. Now we compute the characteristic function of $\tilde\varphi^* P$ which is regarded as a meromorphic function on $\mathbb{C}$ (by identifying it with the coefficient of $(d\zeta)^m$ with $\zeta\in\mathbb{C}$. By the logarithmic derivative lemma

$$\mathcal{A}_r\left(\log^+ |\tilde\varphi^*(dz_\nu)|\right) = O\left(\log r + \log T(r,\varphi)\right).$$

Since

$$|\tau_{\vec\nu}|^2 \exp\left(-p\pi H(z,z)\right)$$

is smooth bounded function on $\mathbb{C}^n$, it follows that

$$\mathcal{A}_r\left(\log^+ |\tilde\varphi^*\tau_{\vec\nu}|\right) = \mathcal{A}_r\left(\frac{p\pi}{2} H(z,z)\right) \le pT(r,\varphi)$$

and

$$T(r,\tilde\varphi^* P) = p\,T(r,\varphi) + O\left(\log r + \log T(r,\varphi)\right). \tag{0.7.1}$$

We also need the estimate for $\mathcal{A}_r\left(\log^+\left|\tilde{\varphi}^*\left(\frac{P}{\theta^q}\right)\right|\right)$. From

$$\frac{P}{\theta^q}=\sum_{\nu_0+\nu_1+\cdots+\nu_k=q} a_{\nu_0,\nu_1,\ldots,\nu_k}\left(\frac{\mathcal{D}\theta}{\theta}\right)^{\nu_1}\cdots\left(\frac{\mathcal{D}^k\theta}{\theta}\right)^{\nu_k} \tag{0.7.2}$$

and

$$\frac{\mathcal{D}\theta}{\theta}=d\log\theta+\sum_{\mu,\nu=1}^{n} h_{\mu,\bar{\nu}}\bar{z}_\nu d\log\exp z_\mu \tag{0.7.3}$$

it follows that

$$\mathcal{A}_r\left(\log^+\left|\tilde{\varphi}^*\left(\frac{\mathcal{D}\theta}{\theta}\right)\right|\right)=O\left(\log r+\log T(r,\varphi)\right).$$

Since

$$\left|a_{\nu_0,\nu_1,\ldots,\nu_k}\right|^2\exp\left(-p\pi H(z,z)\right)$$

is smooth bounded function on $\mathbb{C}^n$, it follows that

$$\mathcal{A}_r\left(\log^+\left|\tilde{\varphi}^* a_{\nu_0,\nu_1,\ldots,\nu_k}\right|\right)=\mathcal{A}_r\left(\frac{p\pi}{2}H(z,z)\right)\le pT(r,\varphi).$$

Thus

$$\mathcal{A}_r\left(\log^+\left|\tilde{\varphi}^*\left(\frac{P}{\theta^q}\right)\right|\right)\le pT(r,\varphi)+O\left(\log r+\log T(r,\varphi)\right). \tag{0.7.4}$$

The vanishing of the defect $\mathrm{Defect}(\varphi,D)$ now follows from $\frac{p}{q}<\varepsilon$ and from

$$\begin{aligned} q\,m\left(r,\theta,0\right)=\mathcal{A}_r\left(\log^+\left|\tilde{\varphi}^*\left(\frac{1}{\theta^q}\right)\right|\right)&\le\mathcal{A}_r\left(\log^+\left|\tilde{\varphi}^*\left(\frac{P}{\theta^q}\right)\right|\right)+T\left(r,\tilde{\varphi}^*\left(\frac{1}{P}\right)\right)\\ &\le\mathcal{A}_r\left(\log^+\left|\tilde{\varphi}^*\left(\frac{P}{\theta^q}\right)\right|\right)+T\left(r,\tilde{\varphi}^*P\right)\end{aligned}$$

which by Equations (0.7.1) and (0.7.4) is no more than

$$2pT(r,\varphi)+O\left(\log r+\log T(r,\varphi)\right).$$

S.-K. Yeung observed that the proof in [Siu and Yeung 1997] could be slightly refined as follows to give the following stronger Second Main Theorem for an ample divisor D in an abelian variety A and for any positive number ε.

$$m\left(r,\varphi,D\right)+\left(N\left(r,\varphi,D\right)-N_n\left(r,\varphi,D\right)\right)\le\varepsilon\,T(r,\varphi)+O\left(\log r+\log T(r,\varphi)\right),$$

where $N_n\left(r,\varphi,D\right)$ is defined in the same as the counting function $N\left(r,\varphi,D\right)$ except that the counting is truncated at multiplicity n so that multiplicity greater than n is counted only as n. The refinement is as follows. From Equations (0.7.2) and (0.7.3) it follows that

$$N\left(r,\tilde{\varphi}^*\left(\frac{P}{\theta^q}\right),\infty\right)\le qN_n\left(r,D,0\right).$$

and

$$T\left(r, \tilde{\varphi}^*\left(\frac{P}{\theta^q}\right)\right) = \mathcal{A}_r\left(\log^+\left|\varphi^*\left(\frac{P}{\theta^q}\right)\right|\right) + N\left(r, \varphi^*\left(\frac{P}{\theta^q}\right), \infty\right).$$

Moreover, it follows from (0.7.4) that

$$T\left(r, \tilde{\varphi}^*\left(\frac{P}{\theta^q}\right)\right) \leq pT(r,\varphi) + qN\left(r, D, 0\right) + O\left(\log r + \log T(r,\varphi)\right).$$

and

$$\begin{aligned} q\, m\left(r, \varphi, D\right) + qN\left(r, \varphi, D\right) &= T\left(r, \tilde{\varphi}^*\left(\frac{1}{\theta^q}\right)\right) + O(1) \\ &= T\left(r, \tilde{\varphi}^*\left(\frac{P}{\theta^q}\frac{1}{P}\right)\right) + O(1) \\ &\leq T\left(r, \tilde{\varphi}^*\left(\frac{P}{\theta^q}\right)\right) + T\left(r, \tilde{\varphi}^*\left(\frac{1}{P}\right)\right) + O(1) \\ &\leq T\left(r, \tilde{\varphi}^*\left(\frac{P}{\theta^q}\right)\right) + T\left(r, \tilde{\varphi}^* P\right) + O(1) \\ &\leq 2pT(r,\varphi) + qN_n\left(r, \varphi, D\right) + O\left(\log r + \log T(r,\varphi)\right). \end{aligned}$$

Dividing both sides by q and using p/q yields the stronger Second Main Theorem.

0.8. Perturbation of Holomorphic Maps. By the second approach of using techniques motivated by diophantine approximation, McQuillan [1996] gives an alternative proof of Bloch's theorem and obtains [1997] the zero defect of an ample divisor D of A. He uses different rescalings of variables of $\mathbb{C}$ to handle the problem of the identical vanishing of the pullback of a section constructed for an appropriate line bundle. It comes as a great surprise that his method of perturbation by rescaling of variables works, but in fact it does. Since in Chapter 1 of this paper we will apply the rescaling method to the complex projective space to get a proof of Theorem 0.0.1, we will not elaborate further on that method here.

We make a remark about the difficulty of using perturbation for the approach by jet differentials. For hyperbolicity problems Pit-Mann Wong with his collaborators introduces the method of perturbing the map $d^k\varphi : \mathbb{C} \to J_k(M)$ into another map $\Phi_k : \mathbb{C} \to J_k(M)$ so that the composite of Φ_k and the natural projection $J_k(M) \to M$ is φ. The main difficulty with such a perturbation is that, unlike the case of using the product of a number of copies of the target manifold, there is yet no known good way of perturbation which could control the change of the proximity term, even when the perturbation is done by rescaling. The problem can be illustrated by the simple case of $k = 1$ and M being an abelian variety A whose universal cover has coordinates $z_1, \ldots, z_n$. Suppose

$$\varphi(\zeta) = (\varphi_1(\zeta), \ldots, \varphi_n(\zeta))$$

in terms of $z_1, \dots, z_n$ and we perturb $d\varphi$ to

$$(d\varphi)(\zeta) = \left(\varphi(\zeta), \left(\frac{\partial \varphi_1}{\partial \zeta}\right)(\xi_1\zeta), \dots, \left(\frac{\partial \varphi_n}{\partial \zeta}\right)(\xi_n\zeta)\right) \in A \times \mathbb{C}^n$$

with some rescaling factors $\xi_1, \dots, \xi_n$. When we estimate the effect of the perturbation on the proximity function for some theta function s_D defining an ample divisor D, even with the possible use of another rescaling factor ξ' there is no way to handle the difficulty coming from the discrepancy between

$$\left(\frac{\partial s_D}{\partial \zeta}\right)(\xi'\zeta) \qquad \text{and} \qquad \sum_{\nu=1}^{n}\left(\frac{\partial s_D}{\partial z_\nu}\right)(\varphi(\zeta))\left(\frac{\partial \varphi_\nu}{\partial \zeta}\right)(\xi_\nu\zeta).$$

0.9. Since the main ideas of the streamlined version of the proof of Theorem 0.1.4 will be discussed in the overview in Chapter 2, here in the Introduction we will confine ourselves to only a couple of comments on the relation between number theory and the easier first step of finding meromorphic 1-jet differentials whose pullback on the entire holomorphic curve vanishes.

The construction of 2-jet differentials of certain explicit forms given in Chapter 2 is accomplished by using polynomials whose terms contain the factors $f, df, d^2 f$ to a certain order, where f is the polynomial defining the plane curve C of degree δ (see 2.1.2). This means that the constructed jet differential vanishes to that order along $J_2(C)$. This requirement is related to the techniques discussed above.

On the branched cover X over $\mathbb{P}_2$ with branching along C, the construction of holomorphic 2-jet differentials is possible because there are more divisors on $J_k(X)$ and some factors from the additional ways of factorization become holomorphic jet differentials; see Section 2.3. This is analogous to the following observation due to Vojta in number theory. The finiteness of rational points for a subvariety of abelian varieties not containing the translate of an abelian variety is the consequence of the fact that in the product space of many copies of the subvariety there are more line bundles or divisors than constructed from the factors which are copies of the subvariety [Faltings 1991; Vojta 1992].

On the other hand, the existence of more divisors in $J_k(X)$ and more ways of factorization mean that it is easier for two jet differentials to share a common factor and as a result it is more difficult to conclude that the zero-sets of two jet differentials do not have a branch in common.

0.10. Overview of the Proofs. We conclude this introduction with a brief discussion of the proofs of the main results. The proof of Theorem 0.0.1 is parallel to that of Roth's Theorem [Roth 1955; Schmidt 1980]. It provides more tangible evidence to support the formal analogy between Nevanlinna theory and diophantine approximation pointed out by Osgood [1985] and Vojta [1987]. It also introduces a new approach to the hyperbolicity problem of the complement of a generic hypersurface of high degree in a complex projective space, which might hold a better promise than other approaches for an eventual solution to the full conjecture with optimal bounds involving such complements of hypersurfaces. There is no attempt to get the optimal bound from the proof of Theorem 0.0.1. Some small improvements in the bounds may be possible from that argument.

Theorem 0.0.1 is not a new result. The case of $\hat{m} = 1$ of Theorem 0.0.1 is contained in the defect relation of Cartan [1933] and Ahlfors [1941] and the following result of Eremenko and Sodin. The case of general $\hat{m}$ of Theorem 0.0.1 follows from the standard process of averaging over the complex lines in the complex vector space $\mathbb{C}^{\hat{m}}$.

THEOREM 0.10.1 [Erëmenko and Sodin 1991, p. 111, Theorem 1]. *If Q_ν $(1 \leq \nu \leq q)$ are homogeneous polynomials of degree d_ν in $n+1$ variables so that no more than n of them have a common zero in $\mathbb{C}^{n+1} - 0$ and if $\varphi : \mathbb{C} \to \mathbb{P}_n$ so that $\varphi^* Q_\nu$ is not identically zero for $1 \leq \nu \leq q$, then*

$$(q - 2n) T(r, \varphi) \leq \sum_{\nu=1}^{q} \frac{1}{d_\nu} N(r, Q_k, 0) + o(T(r, \varphi))$$

where $T(r, \varphi)$ is the characteristic function, $N(r, Q_k, 0)$ is the counting function, and the inequality holds outside a subset of the real line with finite measure with respect to dr/r.

Chapter 2 is devoted to the proof of Theorem 0.0.2, which contains two main steps. The first is to produce a meromorphic 1-jet differential h whose pullback to the entire holomorphic curve is zero; see Sections 2.2 to 2.5. When the degree of h in the affine variables is at least 4 times its degree in the differentials of those variables, the proof is rather easily finished by using arguments of Riemann–Roch to construct some holomorphic 1-jet differential defined only on a branched cover of the zero-set of h which vanishes on an ample divisor of $\mathbb{P}_2$; see Section 2.6. The second step is to deal with the most difficult remaining case. When the curve C is defined by a polynomial f of the affine coordinates, the main idea is to use an appropriate meromorphic 1 form η of low degree and consider the restriction of $\frac{\eta}{f}$ to the zero-set of h. When there is a good upper bound for the touching order of the "integral curves" of h and C, the argument for the Ahlfors–Schwarz lemma for log-pole jet differentials finishes the proof; see Section 2.8. The main streamlining is some new ingredients in the touching order argument in the difficult last step; see Section 2.7. A less important streamlining is that we employ more the cleaner language of cohomology theory, instead of the direct arguments of using polynomials, in the first step of constructing the meromorphic 1-jet differential h whose pullback to the entire holomorphic curve is zero. The method of proof is chosen and presented in a way which facilitates possible generalizations to the higher dimensional case.

1. Sum of Defects of Hypersurfaces in the Projective Space

We prove in this chapter the following theorem, which is the case of $\hat{m} = 1$ in Theorem 0.0.1. All the principal difficulties of the proof of Theorem 0.0.1 already occur in the special case of $\hat{m} = 1$. So for notational simplicity we give only the details for the case of $\hat{m} = 1$ and then present the minor modifications needed for the case of a general $\hat{m}$ after the proof of Theorem 1.0.1.

THEOREM 1.0.1. *Let V_λ $(1 \leq \lambda \leq \Lambda)$ be regular complex hypersurface in $\mathbb{P}_n$ of degree δ in normal crossing. Let $\varphi : \mathbb{C} \to \mathbb{P}_n$ be a holomorphic map whose image is not contained in any hypersurface of $\mathbb{P}_n$. Then the sum of the defects $\sum_{\lambda=1}^{\Lambda} \mathrm{Defect}(\varphi, V_\lambda)$ is no more than ne for any $\delta \geq 1$ and is no more than $n+1$ for $\delta = 1$.*

The method of proof uses techniques motivated by diophantine approximation. We construct holomorphic section s of low degree on the product $\mathbb{P}_n^{\times m}$ of m copies of $\mathbb{P}_n$ which vanishes to high order at points of $\bigcup_{\lambda \in \Lambda} V_\lambda^{\times m}$. Then we use McQuillan's estimate [1997] for the proximity function with a rescaling of the variable of $\mathbb{C}$. The m different rescalings on $\mathbb{C}$ for the map from $\mathbb{C}$ to $\mathbb{P}_n^{\times m}$ induced by φ guarantee the non identical vanishing of the pullback to $\mathbb{C}$ of s by the perturbed map. The defect relation then follows from the standard argument of the Poisson–Jensen formula or the First Main Theorem. The normal crossing condition is required to make sure that the product of the multi-order ideal sheaves for $V_\lambda^{\times m}$ is equal to their intersection.

1.1. Preliminaries on Combinatorics and Integrals

LEMMA 1.1.1. *Let n be a positive integer. For any positive number $\tau > 1$ let $\Theta_n(\tau)$ be*

$$\overline{\lim}_{m\to\infty} \left(\int_{\left\{ \substack{x_1+\cdots+x_m < \frac{m}{\tau(n+1)} \\ 0<x_1<1,\ldots,0<x_m<1} \right\}} (1-x_1)^{n-1} \cdots (1-x_m)^{n-1}\, dx_1 \cdots dx_m \right)^{1/m}.$$

Then

$$\Theta_n(\tau) \leq \min\left(\frac{e}{\tau(n+1)}, \frac{1}{n} e^{-\frac{1}{4(n+1)^2}\left(1-\frac{1}{\tau}\right)^2} \right).$$

PROOF. First we show that

$$\Theta_n(\tau) \leq \frac{1}{n} e^{-\frac{1}{4(n+1)^2}\left(1-\frac{1}{\tau}\right)^2}.$$

We need the following combinatorial lemma, which follows from [Schmidt 1980, p. 122, Lemma 4C] and the fact that the number of n-tuples of nonnegative integers $i_1, \ldots, i_n$ with $i_1 + \cdots + i_n = r$ is equal to $\binom{r+n-1}{r}$.

Lemma 1.1.2. *Let $d_1, \ldots, d_m$ be positive integers, $0 < \varepsilon < 1$, and n be a positive integer. Then*

$$\sum_{\left|\left(\frac{j_1}{d_1}+\cdots+\frac{j_m}{d_m}\right)-\frac{m}{n+1}\right|\geq\varepsilon m} \binom{d_1-j_1+n-1}{n-1}\cdots\binom{d_m-j_m+n-1}{n-1} \leq \binom{d_1+n}{n}\cdots\binom{d_m+n}{n}\cdot 2e^{-\frac{\varepsilon^2 m}{4}}.$$

Setting $\varepsilon = \frac{1}{n+1}\left(1-\frac{1}{\tau}\right)$ and $d_1 = \cdots = d_m = d$, we get

$$\sum_{j_1+\cdots+j_m<\frac{md}{\tau(n+1)}} \binom{d-j_1+n-1}{n-1}\cdots\binom{d-j_m+n-1}{n-1} \leq \binom{d+n}{n}^m 2e^{-\frac{m}{4(n+1)^2}\left(1-\frac{1}{\tau}\right)^2}.$$

Forming the Riemann sum by choosing $1/d$ as the size of an increment for each variable and choosing the points $x_\nu = j_\nu/d$ for $1 \leq j_\nu \leq d$ from each rectangular parallelpiped of size $1/d$ and passing to limit as $d \to \infty$, we get

$$\begin{aligned}\lim_{d\to\infty}\frac{1}{d^{nm}} &\sum_{j_1+\cdots+j_m<\frac{md}{\tau(n+1)}} \binom{d-j_1+n-1}{n-1}\cdots\binom{d-j_m+n-1}{n-1} \\ &= \frac{1}{((n-1)!)^m}\int_{\left\{\substack{x_1+\cdots+x_m<\frac{m}{\tau(n+1)} \\ 0<x_1<1,\ldots,0<x_m<1}\right\}} (1-x_1)^{n-1}\cdots(1-x_m)^{n-1}\,dx_1\cdots dx_m.\end{aligned}$$

On the other hand,

$$\lim_{d\to\infty}\frac{1}{d^{nm}}\binom{d+n}{n}^m 2e^{-\frac{m}{4(n+1)^2}\left(1-\frac{1}{\tau}\right)^2} = \frac{1}{(n!)^m}2e^{-\frac{m}{4(n+1)^2}\left(1-\frac{1}{\tau}\right)^2}.$$

After taking the m-th root in the above two limits and using 1.1.2 and letting $m \to \infty$, we get

$$\Theta_n(\tau) \leq \frac{1}{n}e^{-\frac{1}{4(n+1)^2}\left(1-\frac{1}{\tau}\right)^2}.$$

For the other inequality, $\Theta_n(\tau) \leq \frac{e}{\tau(n+1)}$, we make the substitution $x_\nu = \frac{y_\nu}{\tau}$ and get

$$\begin{aligned}&\int_{\left\{\substack{\frac{y_1+\cdots+y_m<m}{\tau(n+1)} \\ 0<x_1<1,\ldots,0<x_m<1}\right\}} (1-x_1)^{n-1}\cdots(1-x_m)^{n-1}\,dx_1\cdots dx_m \\ &\qquad= \frac{1}{\tau^m}\int_{\left\{\substack{y_1+\cdots+y_m<\frac{m}{n+1} \\ 0<y_1<\tau,\ldots,0<y_m<\tau}\right\}} \left(1-\frac{y_1}{\tau}\right)^{n-1}\cdots\left(1-\frac{y_m}{\tau}\right)^{n-1} dy_1\cdots dy_m \\ &\qquad\leq \frac{1}{\tau^m}\text{ Volume of }\left\{y_1+\cdots+y_m < \frac{m}{n+1} : y_1 > 0,\ldots,y_m > 0\right\} \\ &\qquad\leq \frac{m^m}{m!}\left(\frac{1}{\tau(n+1)}\right)^m.\end{aligned}$$

Taking the m-th root and letting $m \to \infty$ and using

$$\lim_{m\to\infty} \frac{m!\, e^m}{m^m \sqrt{2\pi m}} = 1,$$

from Stirling's formula, we get $\Theta_n(\tau) \leq \frac{e}{\tau(n+1)}$. □

LEMMA 1.1.3. *Let δ, Λ be positive integers and τ be a number > 1 such that $\delta n \Theta_n(\tau) < 1$. Then there exists m_0 such that for $m \geq m_0$ there exists d_0 depending on m with the property that for $d \geq d_0$ one has*

$$\Lambda \delta^m \sum_{j_1+\cdots+j_m<\frac{md}{\tau(n+1)}} \binom{d-j_1+n-1}{n-1} \cdots \binom{d-j_m+n-1}{n-1} < \binom{d+n}{n}^m .$$

PROOF. Let $0 < \eta < 1$ such that $\delta n \Theta_n(\tau) < 1 - \eta$. There exists m_0 such that $\Lambda\left(1-\eta\right)^m < 1$ for $m \geq m_0$ and such that for $m \geq m_0$ we have

$$(\delta n)^m \int_{\left\{\substack{x_1+\cdots+x_m<\frac{m}{\tau(n+1)} \\ 0<x_1<1,\ldots,0<x_m<1}\right\}} \left(1-x_1\right)^{n-1} \cdots \left(1-x_m\right)^{n-1} dx_1 \cdots dx_m < \left(1-\eta\right)^m .$$

Choose any $m \geq m_0$. Forming the Riemann sum by choosing $1/d$ as the size of an increment for each variable and choosing the points $x_\nu = j_\nu/d$ for $1 \leq j_\nu \leq d$ from each rectangular parallelpiped of size $1/d$ and passing to limit as $d \to \infty$, we get

$$\begin{aligned}\lim_{d\to\infty} \frac{1}{d^{nm}} &\sum_{j_1+\cdots+j_m<\frac{md}{\tau(n+1)}} \binom{d-j_1+n-1}{n-1} \cdots \binom{d-j_m+n-1}{n-1} \\ &= \frac{1}{((n-1)!)^m} \int_{\left\{\substack{x_1+\cdots+x_m<\frac{m}{\tau(n+1)} \\ 0<x_1<1,\ldots,0<x_m<1}\right\}} (1-x_1)^{n-1} \cdots (1-x_m)^{n-1} dx_1 \cdots dx_m.\end{aligned}$$

Since

$$\lim_{d\to\infty} \frac{1}{d^{nm}} \binom{d+n}{n}^m = \frac{1}{(n!)^m},$$

it follows that there exists d_0 depends on m such that for $d \geq d_0$ one has

$$\delta^m \sum_{j_1+\cdots+j_m<\frac{md}{\tau(n+1)}} \binom{d-j_1+n-1}{n-1} \cdots \binom{d-j_m+n-1}{n-1} < \binom{d+n}{n}^m (1-\eta)^m$$

and

$$\Lambda \delta^m \sum_{j_1+\cdots+j_m<\frac{md}{\tau(n+1)}} \binom{d-j_1+n-1}{n-1} \cdots \binom{d-j_m+n-1}{n-1} < \binom{d+n}{n}^m . \qquad \square$$

1.2. Construction of Sections of Low Degree and High Vanishing Order

PROPOSITION 1.2.1. *Let V_λ $(1 \leq \lambda \leq \Lambda)$ be nonsingular hypersurfaces of degree δ in $\mathbb{P}_n$ in normal crossing. Let $\tau > 1$ satisfy $\delta n \Theta_n(\tau) < 1$. There exists m_0 and for $m \geq m_0$ there exists d_0 depending on m such that for $d \geq d_0$ there exists an element*

$$F \in H^0\big(\mathbb{P}_n^{\times m}, \mathcal{O}_{\mathbb{P}_n^{\times m}}(d, \ldots, d)\big)$$

which vanishes at each $V_\lambda^{\times m}$ to every multi-order $(j_1, \ldots, j_m)$ which satisfies

$$j_1 + \cdots + j_m < \frac{dm}{\tau(n+1)}.$$

PROOF. The space of all homogeneous polynomials of degree r on V_λ is equal to the space of all polynomials of degree r on $\mathbb{P}_n$ quotiented by the ideal generated by the defining polynomial for V_λ. Thus

$$\dim_{\mathbb{C}} H^0(V_\lambda, \mathcal{O}_{V_\lambda}(r)) = \binom{r+n}{n} - \binom{r-\delta+n}{n}.$$

It follows from the following identity for binomial coefficients

$$\binom{b+1}{c+1} - \binom{b}{c+1} = \binom{b}{c}$$

that

$$\begin{aligned}\dim_{\mathbb{C}} H^0(V_\lambda, \mathcal{O}_{V_\lambda}(r)) &= \sum_{\nu=1}^{\delta} \binom{r-\delta+\nu+n}{n} - \binom{r-\delta+\nu+n-1}{n} \\ &= \sum_{\nu=1}^{\delta} \binom{r-\delta+\nu+n-1}{n-1} \leq \delta \binom{r+n-1}{n-1},\end{aligned}$$

where we use the definition

$$\binom{a}{b} = \frac{\prod_{\nu=1}^{b}(a-b+\nu)}{b!}$$

so that $\binom{a}{b} = 0$ for $a < b$ and we use the inequality

$$\binom{a}{b} < \binom{c}{b}$$

for integers $b \leq a < c$. By Künneth's formula we have

$$\dim_{\mathbb{C}} H^0(V_\lambda^{\times m}, \mathcal{O}_{V_\lambda^{\times m}}(d_1, \ldots, d_m)) \leq \delta^m \prod_{\nu=1}^{m} \binom{d_\nu + n - 1}{n-1}.$$

Let $z_{\nu,0}, \ldots, z_{\nu,n}$ be the homogeneous coordinates for the ν-th factor of $\mathbb{P}_n^{\times m}$. An element

$$F \in H^0\big(\mathbb{P}_n^{\times m}, \mathcal{O}_{\mathbb{P}_n^{\times m}}(d, \ldots, d)\big)$$

is represented by a polynomial in the $m(n+1)$ variables

$$z_{1,0}, \ldots, z_{1,n}, \ldots, z_{m,0}, \ldots, z_{m,n},$$

which is homogeneous of degree d_ν in the variables $z_{\nu,0}, \ldots, z_{\nu,n}$ for $1 \leq \nu \leq m$. Assume that the complex line $z_{\nu,1} = \cdots = z_{\nu,n} = 0$ is not contained in any V_λ. Then the vanishing of F on V_λ to every multi-order $(j_1, \ldots, j_m)$ with

$$j_1 + \cdots + j_m < \frac{md}{\tau(n+1)}$$

means that

$$\frac{\partial^{j_1+\cdots+j_m}}{\partial z_{1,0}^{j_1} \cdots \partial z_{m,0}^{j_m}} F$$

as an element of

$$H^0\big(\mathbb{P}_n^{\times m}, \mathcal{O}_{\mathbb{P}_n^{\times m}}(d-j_1, \ldots, d-j_m)\big)$$

vanishes identically on V_λ for every multi-order $(j_1, \ldots, j_m)$ satisfying

$$j_1 + \cdots + j_m \leq \frac{md}{\tau(n+1)}.$$

There exists

$$F \in H^0\big(\mathbb{P}_n^{\times m}, \mathcal{O}_{\mathbb{P}_n^{\times m}}(d, \ldots, d)\big)$$

which vanishes at each $V_\lambda^{\times m}$ to every multi-order $(j_1, \ldots, j_m)$ which satisfies

$$j_1 + \cdots + j_m < \frac{md}{\tau(n+1)}$$

if

$$\Lambda \delta^m \sum_{j_1+\cdots+j_m < \frac{md}{\tau(n+1)}} \binom{d-j_1+n-1}{n-1} \cdots \binom{d-j_m+n-1}{n-1} < \binom{d+n}{n}^m,$$

which is the case by Lemma 1.1.3 and the assumption $\delta n \Theta_n(\tau) < 1$. □

1.3. Effect of Rescaling on Proximity Term. For the estimate of the effect of rescaling on the proximity term we follow the method of [McQuillan 1997]. Let

$$G_{R,a}(\zeta) = \frac{R^2 - \overline{a}\zeta}{R(\zeta - a)},$$

so that

$$\frac{1}{G_{R,a}(\rho\zeta)} = \frac{R(\rho\zeta - a)}{R^2 - \overline{a}\rho\zeta}.$$

We have

$$\begin{aligned}\frac{1}{G_{R,a}(\rho_1\zeta)}-\frac{1}{G_{R,a}(\rho_2\zeta)}&=\frac{R(\rho_1\zeta-a)}{R^2-\overline{a}\rho_1\zeta}-\frac{R(\rho_2\zeta-a)}{R^2-\overline{a}\rho_2\zeta}\\&=R\left\{\frac{(\rho_1-\rho_2)R^2\zeta+(\rho_2-\rho_1)a\overline{a}\zeta}{(R^2-\overline{a}\rho_1\zeta)(R^2-\overline{a}\rho_2\zeta)}\right\}\\&=R(\rho_1-\rho_2)\zeta\frac{R^2-\overline{a}a}{(R^2-\overline{a}\rho_1\zeta)(R^2-\overline{a}\rho_2\zeta)}.\end{aligned}$$

Now we impose the conditions

$$|\rho_1|<R,\quad |\rho_2|<R,\quad |a|\le R.$$

Let

$$\gamma_{R,\rho_1,\rho_2}=\frac{R|\rho_1-\rho_2|}{(R-|\rho_1|)(R-|\rho_2|)}.$$

For $|\zeta|=1$ we have

$$\begin{aligned}\left|\frac{1}{G_{R,a}(\rho_1\zeta)}-\frac{1}{G_{R,a}(\rho_2\zeta)}\right|&\le R|\rho_1-\rho_2|\cdot\frac{R^2}{(R^2-|\rho_1a|)(R^2-|\rho_2a|)}\\&\le\frac{R|\rho_1-\rho_2|}{(R-|\rho_1|)(R-|\rho_2|)}=\gamma_{R,\rho_1,\rho_2}\end{aligned}$$

and

$$\left|\frac{G_{R,a}(\rho_2\zeta)}{G_{R,a}(\rho_1\zeta)}-1\right|\le\gamma_{R,\rho_1,\rho_2}\left|G_{R,a}(\rho_2\zeta)\right|,\qquad\left|\frac{G_{R,a}(\rho_2\zeta)}{G_{R,a}(\rho_1\zeta)}\right|\le1+\gamma_{R,\rho_1,\rho_2}\left|G_{R,a}(\rho_2\zeta)\right|.$$

Poisson's integral formula states that for $h(\zeta)$ meromorphic on $\{|\zeta|\le R\}$ we have

$$\log|h(\zeta)|=\int_{\theta=0}^{2\pi}\log|h(Re^{i\theta})|\operatorname{Re}\left(\frac{Re^{i\theta}+\zeta}{Re^{i\theta}-\zeta}\right)\frac{d\theta}{2\pi}-\log\prod_{|a|\le R}\left|\frac{R^2-\overline{a}\zeta}{R(\zeta-a)}\right|^{\operatorname{ord}_a h}.$$

In particular, when $\zeta=0$ we have

$$\log|h(0)|=\int_{\theta=0}^{2\pi}\log|h(Re^{i\theta})|\frac{d\theta}{2\pi}-\log\prod_{|a|\le R}\left|\frac{R}{a}\right|^{\operatorname{ord}_a h}.$$

Apply the last equation to the special case

$$h(\zeta)=\prod_{|a|\le R}\left(\frac{R^2-\overline{a}\zeta}{R(\zeta-a)}\right)^{\operatorname{ord}_a h}$$

with R replaced by $r < R$ in the formula. Then

$$\log \prod_{|a|\leq R} \left|\frac{R}{a}\right|^{\operatorname{ord}_a h} = \frac{1}{2\pi}\int_{|\zeta|=r} \log \prod_{|a|\leq R} \left|\frac{R^2 - \overline{a}\zeta}{R(\zeta - a)}\right|^{\operatorname{ord}_a h} - \log \prod_{|a|\leq r} \left|\frac{r}{a}\right|^{\operatorname{ord}_a h}.$$

If Z is a divisor on $\mathbb{C}$ and $Z \cap \{|\zeta| < t\} = \{a_1, \ldots, a_N\}$ with multiplicity, then

$$N(R, Z) = \int_{t=0}^{R} n(t, Z)\frac{dt}{t}\sum_{\nu=1}^{N} \log \left|\frac{R}{a_\nu}\right|.$$

Hence

$$\frac{1}{2\pi}\int_{|\zeta|=r} \log \prod_{|a|\leq R} \left|\frac{R^2 - \overline{a}\zeta}{R(\zeta - a)}\right|^{\operatorname{ord}_a h}$$
$$= \left(N(R, \{h = 0\}) - N(r, \{h = 0\})\right) - \left(N(R, \{h = \infty\}) - N(r, \{h = \infty\})\right).$$

Now for $|\rho_1| < R, |\rho_2| < R$ we have

$$\log \left|\frac{h(\rho_1\zeta)}{h(\rho_2\zeta)}\right| = \int_{\theta=0}^{2\pi} \log \left|h(R\, e^{i\theta})\right| \operatorname{Re}\left(\frac{R\, e^{i\theta} + \rho_1\zeta}{R\, e^{i\theta} - \rho_1\zeta} - \frac{R\, e^{i\theta} + \rho_2\zeta}{R\, e^{i\theta} - \rho_2\zeta}\right)\frac{d\theta}{2\pi}$$
$$- \log \prod_{|a|\leq R} \left|\frac{G_{R,a,\rho_1\zeta}}{G_{R,a}(\rho_2\zeta)}\right|^{\operatorname{ord}_a h}.$$

To estimate the right-hand side, we observe that

$$\frac{R\, e^{i\theta} + \rho_1\zeta}{R\, e^{i\theta} - \rho_1\zeta} - \frac{R\, e^{i\theta} + \rho_2\zeta}{R\, e^{i\theta} - \rho_2\zeta} = \frac{2(\rho_1 - \rho_2)\zeta R\, e^{i\theta}}{(R\, e^{i\theta} - \rho_1\zeta)(R\, e^{i\theta} - \rho_2\zeta)}.$$

Hence

$$\left|\operatorname{Re}\left(\frac{R\, e^{i\theta} + \rho_1\zeta}{R\, e^{i\theta} - \rho_1\zeta} - \frac{R\, e^{i\theta} + \rho_2\zeta}{R\, e^{i\theta} - \rho_2\zeta}\right)\right| \leq \frac{2|\rho_1 - \rho_2|R}{(R - |\rho_1|)(R - |\rho_2|)}.$$

So

$$\log^+ \left|\frac{h(\rho_1\zeta)}{h(\rho_2\zeta)}\right| \leq \int_{\theta=0}^{2\pi} \log^+ \left|h(R\, e^{i\theta})\right| \frac{2|\rho_1 - \rho_2|R}{(R - |\rho_1|)(R - |\rho_2|)}\frac{d\theta}{2\pi}$$
$$+ \log \prod_{|a|\leq R, \operatorname{ord}_a h > 0} (1 + \gamma_{R,\rho_1,\rho_2})\, |G_{R,a}(\rho_1\zeta)|^{\operatorname{ord}_a h}$$
$$+ \log \prod_{|a|\leq R, -\operatorname{ord}_a h > 0} (1 + \gamma_{R,\rho_1,\rho_2})\, |G_{R,a}(\rho_2\zeta)|^{-\operatorname{ord}_a h}.$$

Now averaging over $\{|\zeta|=1\}$ gives us

$$
\begin{aligned}
&\oint_{|\zeta|=1} \log^+ \left|\frac{h(\rho_1\zeta)}{h(\rho_2\zeta)}\right| \\
&\quad\le \frac{2|\rho_1-\rho_2|R}{(R-|\rho_1|)(R-|\rho_2|)} \int_{\theta=0}^{2\pi} \log^+ \left|h(R\,e^{i\theta})\right| \frac{d\theta}{2\pi} \\
&\qquad + \sum_{\substack{|a|\le R\\ \mathrm{ord}_a h>0}} \log(1+\gamma_{R,\rho_1,\rho_2}) + \sum_{\substack{|a|\le R\\ \mathrm{ord}_a h>0}} \oint_{|\zeta|=1} (\mathrm{ord}_a h)\log|G_{R,a}(\rho_1\zeta)| \\
&\qquad + \sum_{\substack{|a|\le R\\ -\mathrm{ord}_a h>0}} \log(1+\gamma_{R,\rho_1,\rho_2}) + \sum_{\substack{|a|\le R\\ -\mathrm{ord}_a h>0}} \oint_{|\zeta|=1} (-\mathrm{ord}_a h)\log|G_{R,a}(\rho_2\zeta)| \\
&\quad= \frac{2|\rho_1-\rho_2|R}{(R-|\rho_1|)(R-|\rho_2|)} \int_{\theta=0}^{2\pi} \log^+ \left|h(R\,e^{i\theta})\right| \frac{d\theta}{2\pi} \\
&\qquad + \sum_{\substack{|a|\le R\\ \mathrm{ord}_a h\ne 0}} \log(1+\gamma_{R,\rho_1,\rho_2}) + \left(N(R,\{h=0\}) - N(|\rho_1|,\{h=0\})\right) \\
&\qquad + \left(N(R,\{h=\infty\}) - N(|\rho_2|,\{h=\infty\})\right).
\end{aligned}
$$

Observe that if Z is a divisor in $\mathbb{C}$ whose support does not contain the origin, then

$$
\begin{aligned}
n(R,Z) = \sum_{\substack{a\in Z\\ 0<|a|<R}} \mathrm{ord}_a Z &\le \frac{1}{\log\frac{\tilde R}{R}} \sum_{\substack{a\in Z\\ 0<|a|<R}} (\mathrm{ord}_a Z)\log\frac{\tilde R}{|a|} \\
&\le \frac{1}{\log\frac{\tilde R}{R}} N(\tilde R, Z).
\end{aligned}
$$

Moreover, for $0<\rho<R$ we have

$$
\begin{aligned}
N(R,Z)-N(\rho,Z) &= \sum_{0<|a|<\rho} \mathrm{ord}_a Z \log\frac{R}{\rho} + \sum_{\rho\le|a|<R} \mathrm{ord}_a Z\log\frac{R}{|a|} \\
&\le \sum_{0<|a|<R} \mathrm{ord}_a Z\log\frac{R}{\rho} \\
&= \log\frac{R}{\rho}\, n(R,Z) \le \frac{\log\frac{R}{\rho}}{\log\frac{\tilde R}{R}} N(\tilde R, Z).
\end{aligned}
$$

Using $\log(1+x)\le x$ for $x\ge 0$, we now summarize our result in the following proposition.

Proposition 1.3.1. *Let $h(\zeta)$ be a holomorphic function on $\{\zeta \in \mathbb{C} : |\zeta| \leq R\}$ and let ρ_1, ρ_2 be complex numbers such that $|\rho_1| < R, |\rho_2| < R$. Let $\tilde{R} > R$. Then*

$$
\begin{aligned}
&\oint_{|\zeta|=1} \log^+ \left|\frac{h(\rho_1\zeta)}{h(\rho_2\zeta)}\right| \\
&\quad \leq \frac{|\rho_1 - \rho_2| R}{(R - |\rho_1|)(R - |\rho_2|)} \\
&\qquad\qquad \times \left(2\int_{\theta=0}^{2\pi} \log^+ \left|h(R\,e^{i\theta})\right| \frac{d\theta}{2\pi} + n(\tilde{R}, \{h=0\}) + n(\tilde{R}, \{h=\infty\})\right) \\
&\qquad + (N(R, \{h=0\}) - N(|\rho_1|, \{h=0\})) \\
&\qquad + (N(R, \{h=\infty\}) - N(|\rho_2|, \{h=\infty\})) \\
&\quad \leq \frac{|\rho_1 - \rho_2| R}{(R - |\rho_1|)(R - |\rho_2|)} \\
&\qquad\qquad \times \left(2\int_{\theta=0}^{2\pi} \log^+ \left|h(R\,e^{i\theta})\right| \frac{d\theta}{2\pi} + \frac{N(\tilde{R}, \{h=0\})}{\log \frac{\tilde{R}}{R}} + \frac{N(\tilde{R}, \{h=\infty\})}{\log \frac{\tilde{R}}{R}}\right) \\
&\qquad + (N(R, \{h=0\}) - N(|\rho_1|, \{h=0\})) \\
&\qquad + (N(R, \{h=\infty\}) - N(|\rho_2|, \{h=\infty\}))\,.
\end{aligned}
$$

1.4. Lower Bound of Some Derivative at One Point. Now we make precise what rescaling is required for the perturbation of the holomorphic map to make sure that the pullback of the constructed section to $\mathbb{C}$ is not identically zero. Let

$$\tilde{\varphi}_m : \mathbb{C}^{\times m} \to \mathbb{P}_n^{\times m}$$

be defined by

$$\tilde{\varphi}_m(\zeta_1, \ldots, \zeta_m) = (\varphi(\zeta_1), \ldots, \varphi(\zeta_m))\,.$$

We expand $\tilde{\varphi}_m^* F$ into homogeneous components $\tilde{\varphi}_m^* F = \sum_{\mu=0}^{\infty} G_\mu$ in the m variables $(\zeta_1, \ldots, \zeta_n)$. Since the image of φ is not contained in any hypersurface of $\mathbb{P}_n$, it follows that there exists the smallest l such that G_l is not identically zero. We now consider the worst case where $F(\varphi(\zeta), \ldots, \varphi(\zeta))$ is identically zero. In particular, $G_l(1, \ldots, 1) = 0$. Choose positive numbers $\tau_1, \ldots, \tau_m$ less than $\frac{1}{2}$ such that $G_l(1+\tau_1, \ldots, 1+\tau_m)$ is nonzero. Since $G_l(\zeta_1, \ldots, \zeta_m)$ is homogeneous in the m variables $\zeta_1, \ldots, \zeta_m$, we can write

$$G_l(1 + \tau_1\zeta, \ldots, 1 + \tau_m\zeta) = \chi_p\zeta^p + \chi_{p+1}\zeta^{p+1} + \cdots + \chi_l\zeta^l$$

with $0 \neq \chi_p \in \mathbb{C}$. Let η_0 be a positive number such that

$$\left|\chi_{p+1}\zeta + \cdots + \chi_l\zeta^{l-p}\right| \leq \tfrac{1}{2}|\chi_p|$$

for $|\zeta| \leq \eta_0$. Suppose $A > 1/\eta_0$. Let r be a positive number and let $\rho_\nu = r\,(1 + \tau_\nu/A)$. Let

$$\varphi_{\rho_1,\ldots,\rho_m}(\zeta) = (\varphi(\rho_1\zeta), \ldots, \varphi(\rho_m\zeta))\,.$$

Then

$$\left|\lim_{\zeta\to 0} \frac{1}{\zeta^l}\left(\varphi^*_{\rho_1,\ldots,\rho_m} F\right)(\zeta)\right| \geq \frac{r^l}{2}|\chi_p|\frac{1}{A^p},$$

because

$$\begin{aligned} G_l\left(\rho_1\zeta, \ldots, \rho_m\zeta\right) &= r^l\zeta^l F_l\left(1 + \frac{\tau_1}{A}, \ldots, 1 + \frac{\tau_m}{A}\right) \\ &= r^l\zeta^l\frac{1}{A^p}\left(\chi_p + \chi_{p+1}\left(\frac{1}{A}\right) + \cdots + \chi_l\left(\frac{1}{A}\right)^{l-p}\right). \end{aligned}$$

In our application we will use $A = \frac{1}{r^2 T(r,\varphi)^\kappa}$ with $\kappa > 4$.

1.5. Computation of Defect and the Proof of Theorem 0.0.1. Let $s_{V_\lambda} \in H^0(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\delta))$ $(1 \leq \lambda \leq \Lambda)$ define the smooth hypersurface V_λ in $\mathbb{P}_n$. By Lemma 1.1.3 we can choose $\tau > 1$ such that $\delta n \Theta_n(\tau) < 1$. Then we can choose m sufficiently large and then choose d sufficiently large such that there exists

$$F \in H^0\big(\mathbb{P}_n^{\times m}, \mathcal{O}_{\mathbb{P}_n^{\times m}}(d, \ldots, d)\big)$$

so that F vanishes to any multi-order $(j_1, \ldots, j_m)$ at $V_\lambda^{\times m}$ $(1 \leq \lambda \leq \Lambda)$ which satisfies

$$j_1 + \cdots + j_m < \frac{md}{\tau(n+1)}.$$

Let x be an $(n+1)$-tuple of functions which form the coordinate system of the affine part $\mathbb{C}^n$ of $\mathbb{P}_n$. When $\mathbb{P}_n$ is the j-th factor of $\mathbb{P}_n^{\times m}$ we relabel x as x_j so that $(x_1, \ldots, x_m)$ form the affine coordinate system of the affine part of $\mathbb{P}_n^{\times m}$. We rescale the coordinate ζ of $\mathbb{C}$ to $\rho_\nu\zeta$ to get from φ another map from $\mathbb{C}$ to $\mathbb{P}_n$ for $1 \leq \nu \leq m$, where $\rho_1, \ldots, \rho_m$ are from Section 1.4. We let $\tilde{x}_\nu = x_\nu\left(\varphi_\nu(\rho_\nu\zeta)\right)$ and $\hat{x} = x(\varphi(r\zeta))$. Let q be the largest integer less than $\frac{md}{\tau(n+1)}$.

We now make the following trivial observation. Let Λ, m, N be positive integers such that $\Lambda n \leq N$. Let $z_1, \ldots, z_N$ be the coordinates of $\mathbb{C}^N$. For $1 \leq \lambda \leq \Lambda$ let $I(\lambda, j_{\lambda,1}, \ldots, j_{\lambda,n})$ be the principal ideal generated by $\prod_{\nu=1}^n z_{\lambda n+\nu}^{j_{\lambda,\nu}}$ over the local ring $\mathcal{O}_{\mathbb{C}^N,0}$ of $\mathbb{C}^N$ at the origin. Then

$$\bigcap_{\lambda=1}^{\Lambda} I(\lambda, j_{\lambda,1}, \ldots, j_{\lambda,n}) = \prod_{\lambda=1}^{\Lambda} I(\lambda, j_{\lambda,1}, \ldots, j_{\lambda,n})$$

for any nonnegative integers $j_{\lambda,\nu}$ $(1 \leq \lambda \leq \Lambda, 1 \leq \nu \leq n)$, because both are equal to the principal ideal generated by the single element

$$\prod_{\substack{1\leq\lambda\leq\Lambda \\ 1\leq\nu\leq m}} z_{\lambda n+\nu}^{j_{\lambda,\nu}}.$$

Since the hypersurfaces V_λ $(1 \leq \lambda \leq \Lambda)$ of $\mathbb{P}_n$ are in normal crossing, the trivial observation implies that the ideal sheaf of germs of holomorphic functions on $\mathbb{P}_n^{\times m}$ which vanish to multi-order $(j_1, \ldots, j_m)$ on each $V_\lambda^{\times m}$ is generated by

$$\prod_{\substack{1 \leq \lambda \leq \Lambda \\ 1 \leq \nu \leq m}} \left(\pi_\nu^* s_{V_\lambda}\right)^{j_\nu},$$

where $\pi_\nu : \mathbb{P}_n^{\times m} \to \mathbb{P}_n$ is the projection onto the ν-th factor.

Let $l_\mu(x_1, \ldots, x_m)$ $(1 \leq \mu \leq k)$ be a product of m generic polynomials respectively of degree 1 in the affine coordinates $x_1, \ldots, x_m$ of $\mathbb{P}_n^{\times m}$. For N sufficiently large we can write

$$\begin{aligned} &F(x_1, \ldots, x_m) l_\mu(x_1, \ldots, x_m)^N \\ &\qquad = \sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q \\ \cdots\cdots\cdots \\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q}} \left(\prod_{\substack{1 \leq \lambda \leq \Lambda \\ 1 \leq \nu \leq m}} s_{V_\lambda}(x_\nu)^{j_{\lambda,\nu}} \right) G_{\mu, \{j_{\lambda,\nu}\}_{\substack{1 \leq \lambda \leq \Lambda \\ 1 \leq \nu \leq m}}} (x_1, \ldots, x_m). \end{aligned}$$

We have

$$\begin{aligned} &\frac{\prod_{\nu=1}^m \left(1 + |\tilde{x}_\nu|^2\right)^{\frac{d+N}{2}}}{|F(\tilde{x}_1, \ldots, \tilde{x}_m)| \sum_{\mu=1}^k |l_\mu(\tilde{x}_1, \ldots, \tilde{x}_m)|^N} \\ &\quad \geq \frac{\left(1 + |\hat{x}|^2\right)^{\frac{q\delta\Lambda}{2}}}{\prod_{\lambda=1}^\Lambda |s_{V_\lambda}(\hat{x})|^q} \frac{\prod_{\nu=1}^m \left(1 + |\tilde{x}_\nu|^2\right)^{\frac{d+N}{2}} \Big/ \left(1 + |\hat{x}|^2\right)^{\frac{q\delta\Lambda}{2}}}{\sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q \\ \cdots\cdots\cdots \\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q \\ 1 \leq \mu \leq k}} \left|\prod_{\substack{1 \leq \lambda \leq \Lambda \\ 1 \leq \nu \leq m}} s_{V_\lambda}(\tilde{x}_\nu)^{j_{\lambda,\nu}}\right| \left|G_{\mu, \{j_{\lambda,\nu}\}_{\substack{1 \leq \lambda \leq \Lambda \\ 1 \leq \nu \leq m}}}\right|}. \end{aligned}$$

Note that instead of using $l_\mu(x_1, \ldots, x_m)$ $(1 \leq \mu \leq k)$, one could also write $F(x_1, \ldots, x_m)$ as a linear combination of

$$\prod_{\substack{1 \leq \lambda \leq \Lambda \\ 1 \leq \nu \leq m}} s_{V_\lambda}(x_\nu)^{j_{\lambda,\nu}}$$

with smooth sections of

$$\mathcal{O}_{\mathbb{P}_n^{\times m}}(d - q, \ldots, d - q)$$

over $\mathbb{P}_n^{\times m}$ as coefficients as in Section 0.7. We consider the following long string of inequalities:

(1.5.1)

$$\log \frac{\left(1+|\hat{x}|^2\right)^{\frac{q\delta\Lambda}{2}}}{\prod_{\lambda=1}^\Lambda |s_{V_\lambda}(\hat{x})|^q}$$

$$\leq \log \frac{\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{d+N}{2}}}{|F(\tilde{x}_1,\ldots,\tilde{x}_m)|\sum_{\mu=1}^{k}|l_\mu(\tilde{x}_1,\ldots,\tilde{x}_m)|^N}$$

$$+\log \frac{\sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q\\ \cdots\cdots\cdots\\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q\\ 1\leq\mu\leq k}} \prod_{\substack{1\leq\lambda\leq\Lambda\\ 1\leq\nu\leq m}} \left|\frac{s_{V_\lambda}(\tilde{x}_\nu)}{s_{V_\lambda}(\hat{x})}\right|^{j_{\lambda,\nu}} \left|G_{\mu,\{j_{\lambda,\nu}\}_{\substack{1\leq\lambda\Lambda\\ 1\leq\nu\leq m}}}(\tilde{x}_1,\ldots,\tilde{x}_m)\right|}{\prod_{\nu=1}^{m}(1+|\tilde{x}_\nu|^2)^{\frac{d+N}{2}} \big/ (1+|\hat{x}|^2)^{\frac{q\delta\Lambda}{2}}}$$

$$\leq \log \frac{\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{d+N}{2}}}{|F(\tilde{x}_1,\ldots,\tilde{x}_m)|\sum_{\mu=1}^{k}|l_\mu(\tilde{x}_1,\ldots,\tilde{x}_m)|^N}$$

$$+\log \sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q\\ \cdots\cdots\cdots\\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q}} \prod_{\substack{1\leq\lambda\leq\Lambda\\ 1\leq\nu\leq m}} \left|\frac{s_{V_\lambda}(\tilde{x}_\nu)}{s_{V_\lambda}(\hat{x})}\right|^{j_{\lambda,\nu}}$$

$$\times\sum_{\mu=1}^{k}\left(\frac{\left|G_{\mu,\{j_{\lambda,\nu}\}_{\substack{1\leq\lambda\leq\Lambda\\ 1\leq\nu\leq m}}}\right|}{\prod_{\nu=1}^{m}(1+|\tilde{x}_\nu|^2)^{\frac{d+N-(j_{1,\nu}+\cdots+j_{\Lambda,\nu})\delta}{2}}}\prod_{\nu=1}^{m}\left(\frac{1+|\hat{x}|^2}{1+|\tilde{x}_\nu|^2}\right)^{\frac{(j_{1,\nu}+\cdots+j_{\Lambda,\nu})\delta}{2}}\right)$$

$$\leq \log \frac{\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{d+N}{2}}}{|F(\tilde{x}_1,\ldots,\tilde{x}_m)|\sum_{\mu=1}^{k}|l_\mu(\tilde{x}_1,\ldots,\tilde{x}_m)|^N} + \sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q\\ \cdots\cdots\cdots\\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q}} \log^+ \prod_{\substack{1\leq\lambda\leq\Lambda\\ 1\leq\nu\leq m}} \left|\frac{s_{V_\lambda}(\tilde{x}_\nu)}{s_{V_\lambda}(\hat{x})}\right|^{j_{\lambda,\nu}}$$

$$+\sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q\\ \cdots\cdots\cdots\\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q\\ 1\leq\mu\leq k}} \log^+ \frac{\left|G_{\mu,\{j_{\lambda,\nu}\}_{\substack{1\leq\lambda\leq\Lambda\\ 1\leq\nu\leq m}}}\right|}{\prod_{\nu=1}^{m}(1+|\tilde{x}_\nu|^2)^{\frac{d+N-(j_{1,\nu}+\cdots+j_{\Lambda,\nu})\delta}{2}}}$$

$$+\sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q\\ \cdots\cdots\cdots\\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q\\ 1\leq\mu\leq k,1\leq\nu\leq m}} \log^+ \left(\frac{1+|\hat{x}|^2}{1+|\tilde{x}_\nu|^2}\right)^{\frac{(j_{1,\nu}+\cdots+j_{\Lambda,\nu})\delta}{2}} + C_{m,q}$$

$$= \log \frac{A\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{d}{2}}}{|F(\tilde{x}_1,\ldots,\tilde{x}_m)|} + \log \frac{\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{N}{2}}}{\sum_{\mu=1}^{k}|l_\mu(\tilde{x}_1,\ldots,\tilde{x}_m)|^N}$$

$$+\sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q\\ \cdots\cdots\cdots\\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q}} (j_{1,\nu}+\cdots+j_{\Lambda,\nu}) \log^+ \left|\frac{s_{V_\lambda}(\tilde{x}_\nu)}{s_{V_\lambda}(\hat{x})}\right|$$

$$+\sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q\\ \cdots\cdots\cdots\\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q\\ 1\leq\mu\leq k}} \log^+ \frac{\left|G_{\mu,\{j_{\lambda,\nu}\}_{\substack{1\leq\lambda\leq\Lambda\\ 1\leq\nu\leq m}}}\right|}{\prod_{\nu=1}^{m}(1+|\tilde{x}_\nu|^2)^{\frac{d+N-(j_{1,\nu}+\cdots+j_{\Lambda,\nu})}{2}}}$$

$$+\sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q\\ \cdots\cdots\cdots\\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q\\ 1\leq\mu\leq k,1\leq\nu\leq m}} \tfrac{1}{2}(j_{1,\nu}+\cdots+j_{\Lambda,\nu})\delta \log^+ \left(\frac{1+|\hat{x}|^2}{1+|\tilde{x}_\nu|^2}\right) + C_{m,q} - \log A,$$

where $C_{m,q}$ is a constant depending only on m and q and A is a positive constant chosen so large that

$$\log \frac{A\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{d}{2}}}{|F(\tilde{x}_1,\ldots,\tilde{x}_m)|}>0$$

at every point of $\mathbb{P}_n^{\times m}$. We will average the left-hand side and the right-hand side of 1.5.1 over the unit circle $\{|\zeta|=1\}$. We will consider a lower bound for the averaged left-hand side of 1.5.1 and also conisder the an upper bound for each of the averaged term on the right-hand side of 1.5.1, in order to get the defect relation stated in Theorem 1.0.1.

First we look at an upper bound for each of the averaged term on the right-hand side of 1.5.1. Both terms

$$\log \frac{\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{N}{2}}}{\sum_{\mu=1}^{k}|l_\mu(\tilde{x}_1,\ldots,\tilde{x}_m)|^N}$$

and

$$\log^+ \frac{\left|G_{\mu,\{j_{\lambda,\nu}\}_{\substack{1\leq\lambda\leq\Lambda\\ 1\leq\nu\leq m}}}(\tilde{x}_1,\ldots,\tilde{x}_m)\right|}{\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{md+mN-(j_{1,\nu}+\cdots+j_{\Lambda,\nu})\delta}{2}}}$$

are uniformly bounded on $\mathbb{P}_n^{\times m}$.

To get an upper bound of the average of

$$\log \frac{A\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{d}{2}}}{|F(\tilde{x}_1,\ldots,\tilde{x}_m)|}$$

over the circle $\{|\zeta|=1\}$, we apply the standard First Main Theorem argument of two integrations to

$$\partial\bar{\partial}\log\left|\frac{F(\tilde{x}_1,\ldots,\tilde{x}_m)}{\zeta^l}\right|$$

which is nonzero at $\zeta=0$, we get

$$\begin{aligned}
\oint_{|\zeta|=1}\log \frac{A\prod_{\nu=1}^{m}\left(1+|\tilde{x}_\nu|^2\right)^{\frac{d}{2}}}{|F(\tilde{x}_1,\ldots,\tilde{x}_m)|} &\leq d\sum_{\nu=1}^{m}T(\rho_\nu,\varphi)+\lim_{\zeta\to 0}\log\frac{|\zeta^l|}{|F(\tilde{x}_1,\ldots,\tilde{x}_m)|}+O(1)\\
&\leq d\sum_{\nu=1}^{m}T(\rho_\nu,\varphi)+\log\left(\frac{r^l}{2}|\chi_p|r^2T(r,\varphi)^\kappa\right)+O(1)\\
&\leq d\sum_{\nu=1}^{m}T(\rho_\nu,\varphi)+O\left(\log r+\log T(r,\varphi)\right),
\end{aligned}$$

where l,p,χ_p come from Section 1.4.

To get an upper bound for

$$\log^+\left(\frac{1+|\hat{x}|^2}{1+|\tilde{x}_\nu|^2}\right)$$

we use the following trivial inequality

$$\frac{1+a_1+\cdots+a_n}{1+b_1+\cdots+b_n} \le 1+\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}.$$

for positive numbers $a_1,\dots,a_n,b_1,\dots,b_n$. Let $x=(z_1,\dots,z_n)$. Then $\hat{x}(\zeta)=(z_1(\varphi(r\zeta)),\dots,z_n(\varphi(r\zeta)))$ and $\tilde{x}_\nu(\zeta)=(z_1(\varphi(\rho_\nu\zeta)),\dots,z_n(\varphi(\rho_\nu\zeta)))$. We have

$$\begin{aligned}\log^+\left(\frac{1+|\hat{x}|^2}{1+|\tilde{x}_\nu|^2}\right) &= \log^+\left(\frac{1+\sum_{\lambda=1}^n |z_\lambda(\varphi(r\zeta))|^2}{1+\sum_{\lambda=1}^n |z_\lambda(\varphi(\rho_\nu\zeta))|^2}\right)\\ &\le \log^+\left(1+\sum_{\lambda=1}^n \left|\frac{z_\lambda(\varphi(r\zeta))}{z_\lambda(\varphi(\rho_\nu\zeta))}\right|^2\right)\\ &\le \sum_{\lambda=1}^n \log^+\left(\left|\frac{z_\lambda(\varphi(r\zeta))}{z_\lambda(\varphi(\rho_\nu\zeta))}\right|^2\right)+\log(n+1).\end{aligned}$$

To estimate the discrepancy from rescaling of the coordinate of $\mathbb{C}$, we need to compare at the same time both the characteristic function and the counting function at a pair of points whose distance is of the order of the reciprocal of the characteristic function. For that we need the following simple lemma on real functions, which is modified from [Hayman 1964, p. 14] so that the conclusion is valid at the same time for several functions.

LEMMA 1.5.2 (REAL FUNCTIONS [Hayman 1964, p. 14]). *Suppose that $S_1(r)$, $\dots$, $S_k(r)$ are positive nondecreasing functions for $r_0 \le r < \infty$ which are bounded in every interval $[r_0,r_1]$ for $r_0 \le r_1 < \infty$. Then given $K>1$, $B_1>1$, and $B_2>1$ with $B_2\sum_{\nu=1}^k S_\nu(r_0)>1$ there exists a sequence $r_\mu\to\infty$ such that*

$$S_\nu(r) < K\,S_\nu(r_\mu) \qquad \textit{for } r_\mu < r < r_\mu + \frac{B_1}{\left(\log\left(B_2\sum_{\nu=1}^k S_\nu(r_\mu)\right)\right)^K}.$$

PROOF. Assume that our conclusion is false. Then for all sufficiently large r we can find ρ such that

$$r<\rho<r+\frac{B_1}{\left(\log\left(B_2\sum_{\nu=1}^k S_\nu(r)\right)\right)^K}$$

and $S_\nu(\rho)\ge K\,S_\nu(r)$ for some ν with $1\le\nu\le k$.

Choose r_1 so that this holds for $r\ge r_1$. Then if r_μ has already been defined we define $r_{\mu+1}$ so that

$$r_\mu<r_{\mu+1}<r_\mu+\frac{B_1}{\left(\log\left(B_2\sum_{\nu=1}^k S_\nu(r_\mu)\right)\right)^K}$$

and $S_{\nu_\mu}(r_{\mu+1})\ge K\,S_{\nu_\mu}(r_\mu)$ for some ν_μ with $1\le\nu_\mu\le k$.

Let $p_{\nu,\mu}=1$ if $\nu=\nu_\mu$ and $p_{\nu,\mu}=0$ for $\nu\ne\nu_\mu$ and $1\le\nu\le k$. Then

$$S_\nu(r_{\mu+1})\ge K^{p_{\nu,\mu}}S_\nu(r_\mu) \quad \text{for} \quad 1\le\nu\le k.$$

We have

$$\sum_{\nu=1}^{k} S_\nu(r_{\mu+1}) \geq \sum_{\nu=1}^{k} K^{p_{\nu,1}+\cdots+p_{\nu,\mu}} S_\nu(r_0) \geq K^\mu \min\left(S_1(r_0),\ldots,S_k(r_0)\right).$$

Thus

$$r_{\mu+1} - r_\mu \leq \frac{B_1}{\left(\mu \log K + \log\left(B_2 \min\left(S_1(r_0),\ldots,S_k(r_0)\right)\right)\right)^K}$$

and $\sum_{\mu=1}^{\infty}(r_{\mu+1}-r_\mu)$ converges so that $\sup_\mu r_\mu$ is finite. On the other hand, there exists some ν_0 such that there are infinitely many $p_{\nu_0,m_l} = 1$ with $1 \leq m_l < \infty$ and from the nondecreasing property of $S_{\nu_0}(r)$ we have

$$S_{\nu_0}(r_{\mu+1}) \geq K^{q_\mu} S_{\nu_0}(r_1),$$

where q_μ is the number of m_l less than μ. Since $q_\mu \to \infty$ as $n \to \infty$, we conclude that $S_{\nu_0}(r)$ is unbounded on the finite interval $[r_0, \sup_\mu r_\mu]$, which is a contradiction. □

Corollary 1.5.3. *Given any $K > 1$ and $B > 1$ there exists a sequence $r_\mu \to \infty$ such that*

$$T(r_\mu + \frac{B}{T(r_\mu,\varphi)}, \varphi) \leq KT(r_\mu), \qquad N(r_\mu + \frac{B}{T(r_\mu,\varphi)}, \varphi) \leq KN(r_\mu).$$

Proof. If $T(r,\varphi)$ is bounded, the statement is trivial. If $T(r,\varphi)$ is unbounded, we have

$$\frac{B}{T(r,\varphi)} < \frac{B}{\log\left(2T(r,\varphi)\right)} < \frac{B+1}{\log\left(T(r,\varphi)+N(r,\varphi)\right)}$$

for r sufficiently large. □

Let η be an arbitrary positive number and we choose $\kappa > 4$. Now choose a sequence $\{r_\mu\}_{1\leq\mu<\infty}$ going to infinity such that

$$r_\mu \geq 2, \qquad T(r_\mu,\varphi) \geq 2, \qquad T\Big(r_\mu + \frac{1}{T(r_\mu,\varphi)},\, \varphi\Big) \leq (1+\eta)T(r_\mu,\varphi),$$
$$N\Big(r_\mu + \frac{1}{T(r_\mu,\varphi)},\, \varphi\Big) \leq (1+\eta)N(r_\mu,\varphi), \qquad R = r_\mu + \frac{1}{2T(r_\mu,\varphi)},$$
$$\tilde{R} = r_\mu + \frac{1}{T(r_\mu,\varphi)}, \qquad \rho_\nu = r_\mu + \frac{\tau_\nu}{r_\mu T(r_\mu,\varphi)^\kappa},$$

where $\tau_1,\ldots,\tau_m$ are from Section 1.4. From here to the end of the section r will be a member of the sequence $\{r_\mu\}_{1\leq\mu<\infty}$ though for notational simplicity we suppress the subscript μ of r_μ. Since $\log(1+\eta) \geq \eta - \frac{\eta^2}{2}$ for $\eta < 1$, it follows that both $\log\frac{\tilde{R}}{R}$ and $\log\frac{R}{\rho_\nu}$ are at most

$$\frac{1}{rT(r,\varphi)} - \frac{1}{2(rT(r,\varphi))^2} - \frac{1}{2rT(r,\varphi)} \geq \frac{1}{4rT(r,\varphi)}.$$

Moreover, both

$$\frac{(\rho_\nu - \rho_\mu)R}{(R-\rho_\nu)(R-\rho_\mu)} \qquad \text{and} \qquad \frac{(\rho_\nu - r)R}{(R-\rho_\nu)(R-\rho_\mu)}$$

are no less than

$$\frac{\dfrac{\tau_\nu - \tau_\mu}{r^2 T(r,\varphi)^\kappa}(r + \dfrac{1}{4})}{\left(\frac{1}{4}T(r,\varphi)\right)^2} \leq \frac{32}{r\,T(r,\varphi)^{\kappa-2}}.$$

By Proposition 1.3.1,

$$\begin{aligned}\oint_{|\zeta|=1} \log^+ \left|\frac{z_\lambda \circ \varphi(r\zeta)}{z_\lambda \circ \varphi(\rho_\nu \zeta)}\right|^2 & \\ &\leq \frac{32}{r\,T(r,\varphi)^{\kappa-2}}\left(2(1+\eta)T(r,\varphi) + 8r\,T(r,\varphi)^2\right) + 2\eta T(r,\varphi) + O(1) \\ &\leq 2\eta T(r,\varphi) + O(1),\end{aligned}$$

because $\kappa > 4$. Hence

$$\oint_{|\zeta|=1} \log^+ \left(\frac{1+|\hat{x}|^2}{1+|\tilde{x}_\nu|^2}\right) \leq 4n\eta T(r,\varphi) + O(1)$$

and

$$\oint_{|\zeta|=1} \log^+ \left|\frac{s_{V_\lambda}(\tilde{x}_\nu)}{s_{V_\lambda}(\hat{x})}\right| \leq 2\delta\eta T(r,\varphi) + O(1).$$

Thus we have the upper bounds

$$\begin{aligned}\oint_{|\zeta|=1} \sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q \\ \cdots\cdots\cdots \\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q \\ 1\leq\mu\leq k, 1\leq\nu\leq m}} \tfrac{1}{2}\left(j_{1,\nu}+\cdots+j_{\Lambda,\nu}\right)\delta \log^+\left(\frac{1+|\hat{x}|^2}{1+|\tilde{x}_\nu|^2}\right) & \\ \leq 2n\delta km\Lambda q \binom{q+m-1}{m-1}^{\Lambda} \eta T(r,\varphi) + O(1).\end{aligned}$$

and

$$\begin{aligned}\oint_{|\zeta|=1} \sum_{\substack{j_{1,1}+\cdots+j_{1,m}=q \\ \cdots\cdots\cdots \\ j_{\Lambda,1}+\cdots+j_{\Lambda,m}=q}} \left(j_{1,\nu}+\cdots+j_{\Lambda,\nu}\right) \log^+\left|\frac{s_{V_\lambda}(\tilde{x}_\nu)}{s_{V_\lambda}(\hat{x})}\right| & \\ \leq 2\delta\Lambda q \binom{q+m-1}{m-1}^{\Lambda} \eta T(r,\varphi) + O(1).\end{aligned}$$

To get a lower bound for the left-hand side of 1.5.1, we use the definition of defect and get

$$\oint_{|\zeta|=1} \log \frac{\left(1+|\hat{x}|^2\right)^{\frac{q\delta\Lambda}{2}}}{\prod_{\lambda=1}^{\Lambda} |s_{V_\lambda}(\hat{x})|^q} \geq q\delta\left(\sum_{\lambda=1}^{\Lambda} \operatorname{Defect}(\varphi, s_{V_\lambda}) - \eta\right) T(r,\varphi)$$

for r sufficiently large.

We now put together the lower bound for the averaged left-hand side of 1.5.1 and the upper bounds for the averaged terms on the right-hand side of 1.5.1. We get

$$q\delta\left(\sum_{\lambda=1}^{\Lambda}\mathrm{Defect}(\varphi, s_{V_\lambda})-\eta\right)T(r,\varphi)$$
$$\leq d(1+\eta)\sum_{\nu=1}^{m}T(r,\varphi_\nu)+2\delta(nmk+1)\Lambda q\binom{q+m-1}{m-1}^{\Lambda}\eta T(r,\varphi)+O(1).$$

Since η is an arbitrary positive number, it follows that

$$\sum_{\lambda=1}^{\Lambda}\mathrm{Defect}(\varphi, s_{V_\lambda})\leq\frac{md}{q\delta}.$$

The number q is chosen so that q is the largest integer less than $\frac{md}{\tau(n+1)}$ with $\delta n\Theta_n(\tau)<1$. Hence

$$\sum_{\lambda\in\Lambda}\mathrm{Defect}(\varphi, V_\lambda)\leq\frac{n+1}{\delta}\Theta_n^{-1}\left(\frac{1}{n\delta}\right).$$

This gives Theorem 1.5.4 below for the case $q=1$. It now follows from

$$\Theta_n(\tau)\leq\min\left(\frac{e}{\tau(n+1)},\ \frac{1}{n}e^{-\frac{1}{4(n+1)^2}\left(1-\frac{1}{\tau}\right)^2}\right)$$

that $\sum_{\lambda\in\Lambda}\mathrm{Defect}(\varphi, V_\lambda)$ is no more than ne for any $\delta\geq 1$ and is no more than $n+1$ for $\delta=1$. This proves the Theorem 1.0.1. The modification needed to prove Theorem 0.0.1 and Theorem 1.5.4 is standard. The modification is to restrict φ to a complex line in the complex vector space $\mathbb{C}^{\hat{m}}$ and then compute the proximity term by restricting and average over the complex line with respect to the Fubini–Study volume form of $\mathbb{P}_{\hat{m}-1}$.

THEOREM 1.5.4. *For $\tau>0$ let $\Theta_n(\tau)$ be*

$$\overline{\lim}_{m\to\infty}\left(\int_{\left\{\substack{x_1+\cdots+x_m<\frac{m}{\tau(n+1)}\\ 0<x_1<1,\ldots,0<x_m<1}\right\}}(1-x_1)^{n-1}\cdots(1-x_m)^{n-1}\,dx_1\cdots dx_m\right)^{1/m},$$

which is bounded by the minimum of $\frac{e}{\tau(n+1)}$ and $\frac{1}{n}e^{-\frac{1}{4(n+1)^2}\left(1-\frac{1}{\tau}\right)^2}$. Let V_λ $(1\leq\lambda\leq\Lambda)$ be regular complex hypersurfaces in $\mathbb{P}_n$ of degree δ in normal crossing. Let $\varphi:\mathbb{C}\to\mathbb{P}_n$ is a holomorphic map whose image is not contained in any hypersurface of $\mathbb{P}_n$. Then

$$\sum_{\lambda=1}^{\Lambda}\mathrm{Defect}(\varphi, V_\lambda)\leq\frac{n+1}{\delta}\Theta_n^{-1}\left(\frac{1}{n\delta}\right).$$

2. Hyperbolicity of the Complement of a Generic High Degree Plane Curve

2.1. Overview of the Method of Proof. In this Chapter we will give a streamlined version of the proof of Theorem 0.1.4 [Siu and Yeung 1996a]. As explained in the introduction of this paper for the approach of jet differentials, the main difficulty of proving hyperbolicity is how to construct enough holomorphic jet differentials vanishing on an ample divisor which are independent in an appropriate sense.

As discussed in Section 0.10, there are two main steps in the proof. Though the first step is easier, we will spend more time in explaining the techniques in it, because these techniques may be generalizable to the higher dimensional case. In this overview the techniques of the first step are explained from here to the end of 2.1.4 and the techniques of the second step are explained in 2.1.5.

For the first step of constructing a meromorphic 1-jet differential whose pullback to the entire holomorphic curve vanishes, we use the following three ingredients to construct holomorphic 2-jet differentials vanishing on an ample divisor on a branched cover of $\mathbb{P}_2$ (see 2.1.1 and 2.1.2):

(i) meromorphic *nonlinear* connections of low pole order for the tangent bundle,
(ii) the Wronskian, and
(iii) the positivity of the canonical line bundle.

This particular way of constructing holomorphic 2-jet differentials gives us some control over their explicit forms so that by comparing degrees with respect to suitable distinct polarizations we can get the independence of two 2-jet differentials to obtain our desired meromorphic 1-jet differential as their resultant (see 2.1.4). A polarization here means a collection of affine variables and their differentials with respect to which degrees are measured. So far our method works only in the 2-dimensional case. The difficulty of extending it to the case of general dimension is that the algebraic procedure of concluding independence by comparing degrees with respect to suitable distinct polarizations is not yet developed for the case of general dimension. Such an algebraic procedure used in the dimension two case is done in a very *ad hoc* way by brute force.

2.1.1. Use of Linear Connections. To put our construction in the proper context, we first consider the the use of meromorphic *linear* connection of low pole order in some special cases. Let us look at the situation of lifting a connection for the tangent bundle of the base manifold to a branched cover. We assume that the branching is cyclic and the branching locus is smooth. In addition we assume that the second fundamental form of the branching locus with respect to the connection is zero in the sense that with respect to the connection the derivative of a local vector field with another local vector field always vanishes when both vector fields are tangential to the branching. Let $z^1, \ldots, z^n$ be local coordinates for the base manifold and $w^1, \ldots, w^n$ be local coordinates for the branched cover so that $w^n = (z^n)^{\frac{1}{\delta}}$ and $w^\alpha = z^\alpha$ for $1 \leq \alpha \leq n-1$. Let $\mathcal{D}$ denote a connection for the tangent bundle of the base manifold and let $\Gamma^\gamma_{\alpha\,\beta}$ be its Christoffel symbol so that

$$\mathcal{D}_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial z^\beta} = \Gamma^\gamma_{\alpha\,\beta} \frac{\partial}{\partial z^\gamma}.$$

Here we use the summation convention of summing over an index appearing both in the superscript and subscript positions. Let the lifting of $\mathcal{D}$ to the branched cover be $\tilde{\mathcal{D}}$ with Christoffel symbol $\tilde{\Gamma}^\nu_{\lambda\,\mu}$ so that

$$\mathcal{D}_{\frac{\partial}{\partial w^\lambda}} \frac{\partial}{\partial w^\mu} = \tilde{\Gamma}^\nu_{\lambda\,\mu} \frac{\partial}{\partial w^\nu}.$$

From

$$\mathcal{D}_{\frac{\partial}{\partial w^\lambda}} \frac{\partial}{\partial w^\mu} = \mathcal{D}_{\frac{\partial}{\partial w^\lambda}} \left(\frac{\partial z^\beta}{\partial w^\mu} \frac{\partial}{\partial z^\beta} \right) = \frac{\partial^2 z^\beta}{\partial w^\lambda \partial w^\mu} \frac{\partial}{\partial z^\beta} + \frac{\partial z^\alpha}{\partial w^\lambda} \frac{\partial z^\beta}{\partial w^\mu} \Gamma^\gamma_{\alpha\,\beta} \frac{\partial}{\partial z^\gamma}$$

it follows that

$$\tilde{\Gamma}^\nu_{\lambda\,\mu} = \frac{\partial^2 z^\beta}{\partial w^\lambda \partial w^\mu} \frac{\partial w^\nu}{\partial z^\beta} + \frac{\partial z^\alpha}{\partial w^\lambda} \frac{\partial w^\nu}{\partial z^\gamma} \frac{\partial z^\beta}{\partial w^\mu} \Gamma^\gamma_{\alpha\,\beta}. \tag{2.1.1.1}$$

Suppose $\mathcal{D}$ is locally holomorphic. We would like to compute the pole-order of $\tilde{\mathcal{D}}$ by using the condition that the second fundamental form of the branching locus is zero with respect to $\mathcal{D}$. From (2.1.1.1) the only pole contribution comes from

$$\frac{\partial w^n}{\partial z^n} = \frac{1}{\delta} \frac{1}{(w^n)^{\delta - 1}}.$$

The pole could occur only in $\tilde{\Gamma}^n_{\lambda\,\mu}$, which has the two terms

$$T_1 := \frac{\partial^2 z^n}{\partial w^\lambda \partial w^\mu} \frac{\partial w^n}{\partial z^n}, \qquad T_2 := \frac{\partial z^\alpha}{\partial w^\lambda} \frac{\partial w^n}{\partial z^n} \frac{\partial z^\beta}{\partial w^\mu} \Gamma^n_{\alpha\,\beta}.$$

Since the only term in T_1 that is nonzero is for the case $\lambda = \mu = n$, it follows that

$$T_1 = \frac{\partial^2 z^n}{(\partial w^n)^2} \frac{\partial w^n}{\partial z^n} = (\delta - 1) \frac{1}{w^n}.$$

For the term T_2 the only pole contribution comes from the case $1 \leq \lambda, \mu \leq n-1$. In that case from the vanishing of the second fundamental form of the branching locus with respect to $\mathcal{D}$ we know that

$$\Gamma^n_{\lambda\,\mu} = O(z^n) = O((w^n)^\delta)$$

which more than makes up for the pole contribution from $\frac{\partial w^n}{\partial z^n}$. Thus we conclude that the pole of $\tilde{\mathcal{D}}$ is at most order one along the branching locus $\{w^n = 0\}$ of the branched cover.

Let the branching locus be defined locally by a function $f = 0$. We would like to see what the vanishing of the second fundamental form of the branching locus means in terms of the defining function f of the branching locus. Let ξ, η be arbitrary local vector fields tangential to the branching locus. Let $df, \omega_1, \ldots, \omega_{n-1}$ be a local basis of 1-forms. From the vanishing of $\langle df, \xi\rangle$ and $\langle df, \mathcal{D}_\eta \xi\rangle$ the equation

$$d_\eta \langle df, \xi\rangle = \langle \mathcal{D}_\eta df, \xi\rangle + \langle df, \mathcal{D}_\eta \xi\rangle$$

imples that $\langle \mathcal{D}df, \xi \otimes \eta\rangle = 0$. By writing

$$\mathcal{D}df = df \otimes df + df \otimes \sum_{j=1}^{n-1} a_j \omega_j + \left(\sum_{j=1}^{n-1} b_j \omega_j\right) \otimes df + \sum_{j,k=1}^{n-1} c_{j\,k} \omega_j \otimes \omega_k$$

for some local scalar functions $a_j, b_j, c_{j\,k}$, we conclude that the term

$$\sum_{j,k=1}^{n-1} c_{j\,k} \omega_j \otimes \omega_k$$

must vanish on $\{f = 0\}$. Thus on $\{f = 0\}$ we have

$$\mathcal{D}df = df \otimes df + df \otimes \sum_{j=1}^{n-1} a_j \omega_j + \left(\sum_{j=1}^{n-1} b_j \omega_j\right) \otimes df,$$

or in terms of the first-order derivative f_α and the second-order derivatives $f_{\alpha\,\beta}$ we have scalar functions $A_\beta, B_\alpha, C_{\alpha\,\beta}$ such that

$$f_{\alpha\,\beta} - \Gamma^\gamma_{\alpha\,\beta} f_\gamma = f_\alpha A_\beta + B_\alpha f_\beta + C_{\alpha\,\beta} f.$$

We now start our construction of holomorphic jet differentials from meromorphic connections. Let $z = \varphi(\zeta)$ represent a local holomorphic curve in an n-dimensional complex manifold X and let $\tilde{\mathcal{D}}$ be the meromorphic connection for the tangent bundle of X. Then

$$\varphi \mapsto d\varphi \wedge \tilde{\mathcal{D}} d\varphi \wedge \cdots \wedge \tilde{\mathcal{D}}^{n-1} d\varphi$$

defines a K_X-valued n-jet differential. Let $\omega \in \Gamma(X, mK_X)$. Then

$$\varphi \mapsto \left\langle \omega,\, (d\varphi \wedge \tilde{\mathcal{D}} d\varphi \wedge \cdots \wedge \tilde{\mathcal{D}}^{n-1} d\varphi)^{\otimes m}\right\rangle$$

defines an n-jet differential. If the pole order of $\tilde{\mathcal{D}}$ is small and the vanishing order of ω is high, then the n-jet differential

$$\varphi \mapsto \left\langle \omega,\, (d\varphi \wedge \tilde{\mathcal{D}} d\varphi \wedge \cdots \wedge \tilde{\mathcal{D}}^{n-1} d\varphi)^{\otimes m}\right\rangle$$

is holomorphic.

Suppose C is a smooth curve in $\mathbb{P}_2$ defined by the polynomial $f(x, y) = 0$ in the affine coordinates x, y of degree δ and X is the branched cover over $\mathbb{P}_2$ with cyclic branching of order δ along C. Suppose $\mathcal{D}$ is a meromorphic connection of low pole order for the tangent bundle of $\mathbb{P}_2$ such that the second fundamental form of C with respect to $\mathcal{D}$ is zero in the sense that the covariant derivatives of tangent vector fields of C in the direction of C with respect to $\mathcal{D}$ are zero. Then the connection $\mathcal{D}$ for the tangent bundle of $\mathbb{P}_2$ can be lifted to a connection $\tilde{\mathcal{D}}$ for the tangent bundle of X. We could define such a connection $\mathcal{D}$ if

$$\begin{aligned} f_{xx} &= a_0 f + a_1 f_x + a_2 f_y, \\ f_{xy} &= b_0 f + b_1 f_x + b_2 f_y, \\ f_{yy} &= c_0 f + c_1 f_x + c_2 f_y, \end{aligned}$$

by using $z^1 = x$, $z^2 = y$ and defining the Christoffel symbol

$$\Gamma^l_{jk} \otimes \frac{\partial}{\partial z^l} \otimes dz^j \otimes dz^k$$

for the connection $\mathcal{D}$ by

$$\begin{aligned} &\Gamma^1_{1\,1} = a_1, \quad \Gamma^1_{1\,2} = b_1, \quad \Gamma^1_{1\,1} = c_1, \\ &\Gamma^2_{1\,1} = a_2, \quad \Gamma^2_{1\,2} = b_2, \quad \Gamma^2_{2\,2} = c_2. \end{aligned}$$

For a local holomorphic curve $\varphi : U \to X$ parametrized by an open subset U of $\mathbb{C}$, we form

$$\Phi = (\mathcal{D}_\zeta \varphi^\alpha_\zeta)\varphi^\beta_\zeta \left(\frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial z^\beta}\right) = \tfrac{1}{2}\big((\mathcal{D}_\zeta \varphi^\alpha_\zeta)\varphi^\beta_\zeta - (\mathcal{D}_\zeta \varphi^\beta_\zeta)\varphi^\alpha_\zeta\big) \left(\frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial z^\beta}\right).$$

Let $s = s_{\alpha\,\beta} dz^\alpha \wedge dz^\beta$ be a 2-form. Then the evaluation of s at Φ gives

$$\langle s, \Phi \rangle = \tfrac{1}{2}\big((\mathcal{D}_\zeta \varphi^\alpha_\zeta)\varphi^\beta_\zeta - (\mathcal{D}_\zeta \varphi^\beta_\zeta)\varphi^\alpha_\zeta\big) s_{\alpha\,\beta}.$$

From

$$\mathcal{D}_\zeta \varphi^\alpha_\zeta = \varphi^\alpha_{\zeta\,\zeta} + \Gamma^\alpha_{\lambda\,\mu} \varphi^\lambda_\zeta \varphi^\mu_\zeta$$

it follows that

$$(\mathcal{D}_\zeta \varphi^\alpha_\zeta)\varphi^\beta_\zeta - (\mathcal{D}_\zeta \varphi^\beta_\zeta)\varphi^\alpha_\zeta = \big(\varphi^\alpha_{\zeta\,\zeta}\varphi^\beta_\zeta - \varphi^\beta_{\zeta\,\zeta}\varphi^\alpha_\zeta\big) + \varphi^\lambda_\zeta \varphi^\mu_\zeta \big(\Gamma^\alpha_{\lambda\,\mu}\varphi^\beta_\zeta - \Gamma^\beta_{\lambda\,\mu}\varphi^\alpha_\zeta\big).$$

For our special case, when we set $s = dx \wedge dy$ and $z^1 = x, z^2 = y$, we get

$$\begin{aligned} \langle s, \Phi \rangle &= \varphi^* \big\{ (d^2x\, dy - dx\, d^2y) \\ &\qquad + dx^2 (a_1\, dy - a_2\, dx) + 2\, dx\, dy (b_1\, dy - b_2\, dx) + dy^2 (c_1\, dy - c_2\, dx) \big\} \\ &= \varphi^* \big\{ (d^2x\, dy - \, dx\, d^2y) \\ &\qquad + (a_1\, dx^2 + 2b_1\, dx\, dy + c_1\, dy^2)\, dy - (a_2\, dx^2 + 2b_2\, dx\, dy + c_2\, dy^2)\, dx \big\} \end{aligned}$$

Let $t^\delta = f(x, y)$. On X the pullback of the 2-form $dx \wedge dy$ yields a holomorphic 2-jet differential after we divide it by an appropriate power of t, because its vanishing order in t along the branching locus more than offsets its pole order along the infinity line of $\mathbb{P}_2$. An analytic way of seeing it is that

$$dx \wedge dy = \frac{dx \wedge df}{f_y} = \frac{\delta t^{\delta - 1}}{f_y}\,(dx \wedge dt),$$

which says that

$$\frac{1}{t^{\delta-1}}\,(dx \wedge dy)$$

is a holomorphic 2-jet differential on X. The key point is that $dx \wedge df$ is divisible by f_y as well as by $t^{\delta-1}$.

Now we come back to $\langle s, \Phi \rangle$. Geometrically we know that

$$\frac{g(x,y)}{t^{\delta-2}} \langle s, \Phi \rangle$$

is a holomorphic 2-jet differential if the pole divisor of meromorphic connection $\tilde{\mathfrak{D}}$ is contained in the zero divisor of $g(x,y)$ in the affine part, because the pullback of a holomorphic 2-jet differential to the branched cover has at most a simple pole along the branching locus. We would like to see analytically why we the 2-jet differential

$$\frac{g(x,y)}{t^{\delta-2}} \langle s, \Phi \rangle$$

is holomorphic on X. We do it in a way analogous to the analytic proof of the holomorphicity of

$$\frac{1}{t^{\delta-1}} (dx \wedge dy)$$

by the divisibility of $dx \wedge df$ by f_y as well as by $t^{\delta-1}$. Just as in the case of the analytic proof of the holomorphicity of

$$\frac{1}{t^{\delta-1}} (dx \wedge dy),$$

we first convert $d^j y$ to $d^j f$ for $j = 1, 2$. We use

$$d^2 f\, dx - d^2 x\, df = f_y (d^2 y\, dx - d^2 x\, dy) + \mathrm{II}\, dx,$$

where

$$\mathrm{II} = f_{xx}\, dx^2 + 2 f_{xy}\, dx\, dy + f_{yy}\, dy^2.$$

Write

$$\begin{aligned}\mathrm{II} = {} & (a_0\, dx^2 + 2b_0\, dx\, dy + c_0\, dy^2) f \\ & + (a_1\, dx^2 + 2b_1\, dx\, dy + c_1\, dy^2) f_x + (a_2\, dx^2 + 2b_2\, dx\, dy + c_2\, dy^2) f_y\end{aligned}$$

and use

$$f_x\, dx = df - f_y\, dy$$

to get

$$
\begin{aligned}
&d^2 f\,dx - d^2 x\,df \\
&\quad = f_y(d^2 y\,dx - d^2 x\,dy) + (a_0\,dx^2 + 2b_0\,dx\,dy + c_0\,dy^2) f\,dx \\
&\qquad + (a_1\,dx^2 + 2b_1\,dx\,dy + c_1\,dy^2) f_x\,dx + (a_2\,dx^2 + 2b_2\,dx\,dy + c_2\,dy^2) f_y\,dx \\
&\quad = f_y(d^2 y\,dx - d^2 x\,dy) + (a_0\,dx^2 + 2b_0\,dx\,dy + c_0\,dy^2) f\,dx \\
&\qquad + (a_1\,dx^2 + 2b_1\,dx\,dy + c_1\,dy^2)(df - f_y\,dy) + (a_2\,dx^2 + 2b_2\,dx\,dy + c_2\,dy^2) f_y\,dx \\
&\quad = f_y\big\{(d^2 y\,dx - d^2 x\,dy) + (a_2\,dx^2 + 2b_2\,dx\,dy + c_2\,dy^2)\,dx \\
&\qquad\qquad - (a_1\,dx^2 + 2b_1\,dx\,dy + c_1\,dy^2)\,dy\big\} \\
&\qquad + (a_0\,dx^2 + 2b_0\,dx\,dy + c_0\,dy^2) f\,dx + (a_1\,dx^2 + 2b_1\,dx\,dy + c_1\,dy^2)\,df.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\langle s, \Phi\rangle &= \varphi^*\big\{(d^2 x\,dy - dx\,d^2 y) \\
&\qquad + (a_1\,dx^2 + 2b_1\,dx\,dy + c_1\,dy^2)\,dy - (a_2\,dx^2 + 2b_2\,dx\,dy + c_2\,dy^2)\,dx\big\} \\
&= \frac{1}{f_y}\big\{d^2 f\,dx - d^2 x\,df \\
&\qquad -(a_0\,dx^2 + 2b_0\,dx\,dy + c_0\,dy^2) f\,dx - (a_1\,dx^2 + 2b_1\,dx\,dy + c_1\,dy^2)\,df\big\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{g(x,y)}{t^{\delta-2}}\,\langle s, \Phi\rangle &= \frac{g(x,y)}{t^{\delta-2} f_y}\big\{d^2 f\,dx - d^2 x\,df \\
&\qquad -(a_0\,dx^2 + 2b_0\,dx\,dy + c_0\,dy^2) f\,dx - (a_1\,dx^2 + 2b_1\,dx\,dy + c_1\,dy^2)\,df\big\}
\end{aligned}
$$

is holomorphic, because $f = t^\delta$ implies that

$$
\begin{aligned}
df &= \delta t^{\delta-1} dt, \\
d^2 f &= \delta(\delta-1) t^{\delta-2} dt^2 + \delta t^{\delta-2} d^2 t = \delta t^{\delta-2}((\delta-1)(dt)^2 + t d^2 t).
\end{aligned}
$$

2.1.2. Use of Nonlinear Connections. In general, we do not have

$$
\begin{aligned}
f_{xx} &= a_0 f + a_1 f_x + a_2 f_y, \\
f_{xy} &= b_0 f + b_1 f_x + b_2 f_y, \\
f_{yy} &= c_0 f + c_1 f_x + c_2 f_y,
\end{aligned}
$$

with low pole order for a_j, b_i, c_j $(j = 0, 1, 2)$. On the other hand, we know that the theorem of Riemann–Roch guarantees the existence of holomorphic 2-jet differentials in general. The theorem of Riemann–Roch is just a more refined form of counting the number of unknowns and the number of equations. The disadvantage of the use of the theorem of Riemann–Roch is that we do not have any explicit form of holomorphic 2-jet differentials to obtain any conclusion about independence. For the general case we need to modify our approach of using connections to get holomorphic 2-jet differentials in an explicit form. The connections constructed above for the special cases are linear connections.

When we differentiate a tangent vector field without a connection, we end up with a field of 2-jets. A connection is a way of converting such a field of 2-jets back to a tangent vector field. For the purpose of constructing a holomorphic 2-jet differential we do not have to confine ourselves to a linear connection. The conversion of a field of 2-jets back to a tangent vector field can involve a conversion function which is not linear. For example, the conversion function can be an algebraic function which is a root of a polynomial equation. Geometrically there is no existing interpretation for a connection which is an algebraic function. If we just carry out in a purely analytic way the analog of the argument for a linear connection, we should consider, in the case of a connection which is an algebraic function, a polynomial of the form

$$\Phi = \sum_{k=0}^{m} \omega_{s+3k} f^{2(m-k)} (d^2 f\, dx - d^2 x\, df)^{m-k}$$

which is divisible by f_y, where

$$\omega_\mu = \sum_{\nu_0+\nu_1+\nu_2=\mu} a_{\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2}$$

and $a_{\nu_0\nu_1\nu_2}(x,y)$ is a polynomial in x and y of degree $\leq p$. The integers s, p, m are chosen so that the counting of the number of coefficients and the number of equations yields the existence of a function Φ which is not identically zero. The powers of f in the above expressions are used so that

$$\frac{1}{f^{s+3m}}\Phi = \sum_{k=0}^{m} \Big(\frac{1}{f^{s+3k}}\omega_{s+3k}\Big)\Big(\frac{d^2 f}{f}\, dx - d^2 x\, \frac{df}{f}\Big)^{m-k}$$

is divisible by f_y, where

$$\frac{1}{f^\mu}\omega_\mu = \sum_{\nu_0+\nu_1+\nu_2=\mu} a_{\nu_0\nu_1\nu_2}(x,y)\Big(\frac{df}{f}\Big)^{\nu_0}(dx)^{\nu_1}(dy)^{\nu_2}.$$

With

$$\begin{aligned}
\frac{df}{f} &= \delta\frac{dt}{t},\\
\frac{d^2 f}{f} &= \delta\Big(\frac{d^2 t}{t} + (\delta-1)\Big(\frac{dt}{t}\Big)^2\Big),\\
\frac{d^2 f}{f}\, dx - d^2 x \frac{df}{f} &= \delta\Big(\frac{d^2 t}{t}\, dx - \frac{dt}{t}\, d^2 x\Big) + \delta(\delta-1)\Big(\frac{dt}{t}\Big)^2 dx,
\end{aligned}$$

it means that, when we set

$$\tilde\Phi = \frac{1}{f^{s+3m}}\Phi, \qquad \tilde\omega_\mu = \frac{1}{f^\mu}\omega_\mu,$$

we are looking for

$$\tilde{\Phi} = \sum_{k=0}^{m} \tilde{\omega}_{s+3k} \left(\delta\Big(\frac{d^2t}{t}\,dx - \frac{dt}{t}\,d^2x\Big) + \delta(\delta-1)\Big(\frac{dt}{t}\Big)^2 dx \right)^{m-k}$$

to be divisible by f_y, where

$$\tilde{\omega}_\mu = \sum_{\nu_0+\nu_1+\nu_2=\mu} a_{\nu_0\nu_1\nu_2}(x,y)\Big(\delta\frac{dt}{t}\Big)^{\nu_0}(dx)^{\nu_1}(dy)^{\nu_2}.$$

Thus we can construct a 2-jet differential which has small degrees with respect to

$$dx,\ dy,\ \frac{dt}{t},\ \frac{d^2t}{t}\,dx - \frac{dt}{t}\,d^2x.$$

2.1.3. Independence from Degree Considerations for Different Polarizations. By interchanging the rôles of x and y, we can also construct a 2-jet differential which has small degrees with respect to

$$dx,\ dy,\ \frac{dt}{t},\ \frac{d^2t}{t}\,dy - \frac{dt}{t}\,d^2y.$$

The expressions $dx, dy, \frac{dt}{t}$ used in the two sets of polarizations above are not completely independent. They are related by

$$\frac{dt}{t} = \frac{1}{\delta}\Big(\frac{f_x}{f}dx + \frac{f_y}{f}dy\Big).$$

The difference between the two sets of polarizations

$$dx,\ dy,\ \frac{dt}{t},\ \frac{d^2t}{t}\,dx - \frac{dt}{t}\,d^2x$$

and

$$dx,\ dy,\ \frac{dt}{t},\ \frac{d^2t}{t}\,dy - \frac{dt}{t}\,d^2y$$

is the last component in each, namely

$$\frac{d^2t}{t}\,dx - \frac{dt}{t}d^2x \quad \text{and} \quad \frac{d^2t}{t}\,dy - \frac{dt}{t}d^2y.$$

They are related by

$$\frac{1}{f_y}\Big(\frac{d^2t}{t}dx - \frac{dt}{t}d^2x\Big) + \frac{1}{f_x}\Big(\frac{d^2t}{t}dy - \frac{dt}{t}d^2y\Big) = \left(\frac{\mathrm{II}}{\delta f} - (\delta-1)\Big(\frac{dt}{t}\Big)^2\right)\Big(\frac{1}{f_y}dx - \frac{1}{f_x}dy\Big)$$

which has large degree in x, y. From this, for generic affine coordinates x, y and for generic f of sufficiently high degree we get the following statement which will later be given and proved in detail in Section 2.8.

Claim 2.1.4. *There exist two affine coordinate systems so that the irreducible branch of the zero-set of one 2-jet differential containing the entire holomorphic curve constructed from one affine coordinate system is different from the one constructed from the other affine coordinate system.*

This statement is actually obtained by using the set of holomorphic 2-jet differentials ω_γ from the action of $\gamma \in \mathrm{SU}(2,\mathbb{C})$ on the affine coordinates x, y and by using the restriction placed on the coefficients of f by the differential equation on f which f is forced to satisfy when the set of holomorphic 2-jet differentials have a common irreducible branch containing the 2-jets of the entire holomorphic curve. We use more than just the high degree in x, y of the relation of the two different sets of polarizations, but we also use the fact that the polarizations involve differentials so that dependence in our sense implies that f satisfies a differential equation which imposes conditions on the coefficients of f, thereby making f not generic.

Since the 2-jet differential is of homogeneous weight in dx, dy, $d^2x\,dy - dx\,d^2y$, its zero-set is of complex dimension 3. The common zero-set of the two irreducible branches is of complex dimension 2.

Because dx, dy, $\frac{dt}{t}$ have the relation

$$\frac{dt}{t} = \frac{1}{\delta}\Big(\frac{f_x}{f}\,dx + \frac{f_y}{f}\,dy\Big),$$

when we factor any of the two 2-jet differentials we have to worry about losing the property of having small degree with respect to either

$$\Big(dx,\ dy,\ \frac{dt}{t},\ \frac{d^2t}{t}\,dx - \frac{dt}{t}\,d^2x\Big)$$

and

$$\Big(dx,\ dy,\ \frac{dt}{t},\ \frac{d^2t}{t}\,dy - \frac{dt}{t}\,d^2y\Big).$$

For that we need the following irreducibility criterion, which is given as Proposition 2.3.2 below.

Suppose $P(x, y, dx, dy, \frac{df}{f}, Z)$ is irreducible as a polynomial of the 6 variables with degree p in x and y and homogeneous degree m in $dx, dy, \frac{df}{f}, Z$. If $p+m+1 \le \delta$, then $P\big(x, y, dx, dy, \frac{df}{f}, Z\big)$ is irreducible as a polynomial in dx, dy, Z over the field $\mathbb{C}(x, y)$.

In the application the weight of Z is 3 while the weight of each of $dx, dy, \frac{df}{f}$ is 1. To handle that, we rewrite $P\big(x, y, dx, dy, \frac{df}{f}, Z\big)$ as $P_1\big(x, y, dx, dy, \frac{df}{f}, \frac{Z}{dx^2}\big)$ so that the weight of $\frac{Z}{dx^2}$ is 1 and can be regarded as a new variable $\tilde{Z} = \frac{Z}{dx^2}$.

2.1.5. Touching Order with 1-Jet Differential of Low Pole Order. When we take the resultant of the two irreducible factors of the two 2-jet differentials, we have to use either $\frac{d^2t}{t}\,dx - \frac{dt}{t}\,d^2x$ or $\frac{d^2t}{t}\,dy - \frac{dt}{t}\,d^2y$ at the same for both factors and we end up with a relation among x, y, dx, dy, $\frac{dt}{t}$ which is of small degree with respect to $\big(dx, dy, \frac{dt}{t}\big)$. We use $\delta\frac{dt}{t} = \frac{df}{f}$ to write the relation as a polynomial in x, y, dx, dy which is homogeneous in dx, dy. The pullback of this relation to the entire holomorphic curve is identically zero. So the pullback of one of its factors to the entire holomorphic curve is identically zero. Let $h = h(x, y, dx, dy)$ be that factor. Let q be its degree in x, y and m be its degree as a homogeneous polynomial in dx, dy.

Let $\tilde{h}$ be the pullback of h to the δ-sheeted branched cover X over $\mathbb{P}_2$. Let $V_{\tilde{h}}$ be the zero-set of $\tilde{h}$ as a function on the projectivization of $\mathbb{P}(T_X)$ of the tangent bundle T_X of X. Let L_X be the line bundle over $\mathbb{P}(T_X)$ so that the global sections of rL_X correspond to 1-jet differentials over X of degree r.

We use, for sufficiently large r, the existence of a nontrivial global holomorphic section of $r(F-G)$ over Y if F, G are two ample line bundles over a compact complex variety Y of complex dimension n with $F^n > nF^{n-1}G$. We apply it to the case $rL_X = F - G$ with $F = (r+1)(L_X + 3H_{\mathbb{P}_2})$ and $G = L_X + 3(r+1)H_{\mathbb{P}_2}$ over a branch of $V_{\tilde{h}}$. We use the branch of $V_{\tilde{h}}$ which contains a lifting of the entire holomorphic curve. The cyclic group of order δ which is the Galois group of $X \to \mathbb{P}_2$ acts on the set of all branches of $V_{\tilde{h}}$. When $q > 4m$, by using the Galois group of $X \to \mathbb{P}_2$, for sufficiently large δ we obtain a nontrivial global section s of rL_X over that branch of $V_{\tilde{h}}$ for r sufficiently large. The zero-set of s projects down to an algebraic curve in $\mathbb{P}_2$ which contains the entire holomorphic curve. So the case that remains is $q \leq 4m$.

The number m can be chosen to be independent of δ. There are integers N, δ_0 depending only on q, m such that a generic curve of degree $\delta \geq \delta_0$ cannot be tangential, to order N at any point, to any irreducible 1-jet differential $\theta(x, y, dx, dy)$ of degree q in x, y and of homogeneous degree m in dx, dy. We can choose δ sufficiently large relative to N. We choose a polynomial $S(x, y)$ with degree small relative to δ so that S vanishes to order N at all the points on the zero-set of $f(x, y)$ where the discriminant of $h(x, y, dx, dy)$ as a homogeneous polynomial of dx, dy vanishes. Let η be any meromorphic 1-jet differential of low pole order (for example, a suitable linear combination of dx and dy) whose pullback to the entire holomorphic curve is not identically zero. We then prove an inequality of Schwarz lemma type:

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\frac{\left\|f^{\frac{N-1}{N}}S(x,y)\eta\right\|^2}{\|f\|^2\left(\log\|f\|^2\right)^2}\right) \geq \varepsilon\frac{\left\|f^{\frac{N-1}{N}}S(x,y)\eta\right\|^2}{\|f\|^2\left(\log\|f\|^2\right)^2}$$

for some positive number ε when pulled back to $\mathbb{C}$ by the entire holomorphic curve. The Schwarz lemma type inequality implies the nonexistence of the entire holomorphic curve. This concludes the overview of our proof. Now we give the details.

2.2. Construction of Holomorphic 2-Jet Differentials. Let p be a positive integer and s be a nonnegative integer. We are going to construct a 2-jet differential Φ of degree m on X of the form

$$\Phi = \sum_{k=0}^{m} \omega_{s+3k} f^{2(m-k)} (d^2 f dx - d^2 x df)^{m-k},$$

where

$$\omega_\mu = \sum_{\nu_0+\nu_1+\nu_2=\mu} a_{\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2}$$

and $a_{\nu_0\nu_1\nu_2}(x,y)$ is a polynomial in x and y of degree $\leq p$. We are going to choose the polynomials $a_{\nu_0\nu_1\nu_2}(x,y)$ so that Φ is divisible by f_y. Then we will conclude that $t^{-N}f_y^{-1}\Phi$ is a holomorphic 2-jet differential on X when certain inequalities involving p, s, δ, m, and N are satisfied. This is done by regarding the coefficients of the polynomials $a_{\nu_0\nu_1\nu_2}(x,y)$ as unknowns and counting the number of linear equations corresponding to divisibility of Φ by f_y and solving the linear equations when the number of unknowns exceeds the number of equations. In order to guarantee that the 2-jet differential Φ obtained by solving the linear equations is not identically zero, we need the following lemma involving the independence of the coefficients of the polynomials $a_{\nu_0\nu_1\nu_2}(x,y)$.

Lemma 2.2.1. *Let q be a positive integer $< \delta$. Let l be any positive integer. For $\nu_0+\nu_1+\nu_2 = l$ let $b_{\nu_0\nu_1\nu_2}(x,y)$ be a polynomial in x and y of degree at most q. If $\sum_{\nu_0+\nu_1+\nu_2=l} b_{\nu_0\nu_1\nu_2}(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2}$ is identically zero, then $b_{\nu_0\nu_1\nu_2}(x,y)$ is identically zero for $\nu_0+\nu_1+\nu_2 = l$.*

Proof. Regard (x,y) as the affine coordinate for $\mathbb{P}_2$ and introduce the homogeneous coordinates $[\xi,\eta,\zeta]$ for anther $\mathbb{P}_2$. On the product $\mathbb{P}_2\times\mathbb{P}_2$ consider the hypersurface M of bidegree $(\delta,1)$ defined by

$$f(x,y)\zeta = f_x(x,y)\xi + f_y(x,y)\eta.$$

Let

$$s \in \Gamma\left(\mathbb{P}_2\times\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(q,l)\right)$$

be defined by

$$\sum_{\nu_0+\nu_1+\nu_2=l} b_{\nu_0\nu_1\nu_2}(x,y)\zeta^{\nu_0}\xi^{\nu_1}\eta^{\nu_2}.$$

The assumption of the Lemma means that the restriction of s to M is identically zero. Since $q<\delta$, from the exact sequence

$$0 = H^0\left(\mathbb{P}_2\times\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(q-\delta,l-1)\right) \to \\ H^0\left(\mathbb{P}_2\times\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(q,l)\right) \to H^0\left(M, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2}(q,l)|M\right)$$

it follows that s is identically zero. □

2.2.2. Computation of the numbers of equations and unknowns. On $\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1$ we use the affine coordinate (x,y) for the first factor and use the affine coordinate (dx,dy) for the second factor and then use the affine coordinate $d^2x\,dy - d^2y\,dx$ for the third factor. Then consider

$$\Phi = \sum_{k=0}^{m} \omega_{f,s+3k} f^{2(m-k)}(d^2f\,dx - d^2x\,df)^{m-k},$$

as a holomorphic section of $\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1}(a,b,c)$ over $\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1$ for suitable integers a,b,c and then restrict to the hypersurface defined by $f_y(x,y)=0$. We do the counting of the dimensions of the section modules to show that there exists Φ not identically zero whose restriction to $\{f_y(x,y)=0\}$ is identically zero. Here $\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1$ is regarded as birationally equivalent to the space of special 2-jet differentials over $\mathbb{P}_2$.

We now compute the number of equations involved in setting

$$\Phi=\sum_{k=0}^{m}\omega_{s+3k}f^{2(m-k)}(f_{xx}\,dx^2+2f_{xy}\,dx\,dy+f_{yy}\,dy^2)^{m-k}dx^{m-k}$$

equal to zero modulo f_y. Using

$$d^2f\,dx-d^2x\,df=(f_{xx}\,dx^2+2f_{xy}\,dx\,dy+f_{yy}\,dy^2)\,dx-f_y(d^2x\,dy-d^2y\,dx)$$

and expanding Φ, we end up with an expression of the form

$$\sum_{j=0}^{s+3m}b_j(x,y)\,dx^j\,dy^{s+3m-j}$$

modulo f_y, where $b_j=b_j(x,y)$ is a polynomial in x and y of degree at most $p+(s+3m)\delta$. The number of coefficients in each b_j is at most

$$\tfrac{1}{2}(p+(s+3m)\delta+2)(p+(s+3m)\delta+1).$$

For each $b_j(x,y)$ we have to rule out expressions of the form $q_j(x,y)f_y(x,y)$ with the degree of $q_j(x,y)$ in x and y no more than $p+(s+3m)\delta-(\delta-1)$. So the number of possible constraints for each b_j is at most

$$\tfrac{1}{2}(p+(s{+}3m)\delta+2)(p+(s{+}3m)\delta+1)\\ -\tfrac{1}{2}(p+(s{+}3m)\delta+2-(\delta{-}1))(p+(s{+}3m)\delta+1-(\delta{-}1)),$$

which is to say

$$(\delta-1)(p+(s{+}3m)\delta)-\tfrac{1}{2}(\delta^2-5\delta+4).$$

There are altogether $s+3m+1$ such functions $b_j(x,y)$. Thus the total number of equations is at most

$$(s+3m+1)((\delta-1)(p+(s+3m)\delta)-\tfrac{1}{2}(\delta^2-5\delta+4)).$$

Now we would like to compute the number of unknowns. The number of unknowns is the sum of the number of unknowns from each ω_μ. For

$$\omega_\mu=\sum_{\nu_0+\nu_1+\nu_2=\mu}a_{\nu_0\nu_1\nu_2}(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2},$$

the number of unknowns from ω_μ is equal to the sum of the number of coefficients in each of the polynomials $a_{\nu_0\nu_1\nu_2}$ with $\nu_0{+}\nu_1{+}\nu_2=\mu$. There are $\frac{1}{2}(\mu{+}2)(\mu{+}1)$

such $a_{\nu_0\nu_1\nu}$ and each $a_{\nu_0\nu_1\nu}$ has $\frac{1}{2}(p+2)(p+1)$ coefficients. Hence the number of unknowns in ω_μ is $\frac{1}{4}(\mu+2)(\mu+1)(p+2)(p+1)$. The total number of unknowns is

$$\sum_{k=0}^{m} \tfrac{1}{4}(s+3k+2)(s+3k+1)(p+2)(p+1).$$

When the number of unknowns exceeds the number of equations, for a generic f we can solve the linear equations and the solutions will be rational functions of the coefficients of f. We summarize the result in the following lemma.

LEMMA 2.2.3. *To be able to construct a 2-jet differential Φ which is divisible by f_y and which is of the form*

$$\sum_{k=0}^{m} \omega_{s+3k} f^{2(m-k)}(d^2 f\, dx - d^2 x\, df)^{m-k},$$

where

$$\omega_\mu = \sum_{\nu_0+\nu_1+\nu_2=\mu} a_{\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2}$$

and $a_{\nu_0\nu_1\nu_2}(x,y)$ is a polynomial in x and y of degree $\leq p$, it suffices to have the following inequalities $p < \delta - 1$ and

$$\sum_{k=0}^{m} \frac{1}{4}(s+3k+2)(s+3k+1)(p+2)(p+1) > (s+3m+1)((\delta-1)(p+(s+3m)\delta) - \tfrac{1}{2}(\delta^2 - 5\delta + 4)).$$

Moreover, for a generic f the coefficients of $a_{\nu_0\nu_1\nu_2}(x,y)$ are rational functions of the coefficients of f.

The reason for the last statement of Lemma 2.2.3 is as follows. When we solve the system of homogeneous linear equations for the coefficients of $a_{\nu_0\nu_1\nu_2}(x,y)$, we choose a square submatrix A with nonzero determinant in the matrix of the coefficients of the system of homogeneous linear equations so that A has maximum size among all square submatrices with nonzero determinants and then we apply Cramer's rule to those equations whose coefficients are involved in A to solve for the the coefficients of $a_{\nu_0\nu_1\nu_2}(x,y)$. When we do this process, we can regard the coefficients of the system of homogeneous linear equations as functions of the coefficients of f. The square submatrix A has maximum size among all square submatrices whose determinants are not identically zero as functions of the coefficients of f. A sufficient condition for the genericity of f involved in this process is that the point represented by the coefficients of f is outside the zero set of A when A is regarded as a function of the coefficients of f.

2.2.4. Condition for the holomorphicity of the 2-jet differential. We would like to determine under what condition the constructed 2-jet differential $t^{-N}f_y^{-1}\Phi$ is holomorphic on X and vanishes on some ample curve of X.

First we consider the pole order at infinity of various factors. Recall that $[\zeta^0,\zeta^1,\zeta^2]$ is the homogeneous coordinates of $\mathbb{P}_2$ with $x=\zeta^1/\zeta^0$ and $y=\zeta^2/\zeta^0$. At a point at the infinity line we assume without loss of generality that $\zeta^1\neq 0$. At that point of the infinity line we use the affine coordinates $u=\zeta^0/\zeta^1=1/x$ and $v=\zeta^2/\zeta^1=y/x$. Thus $x=1/u$ and $y=xv=v/u$. We have

$$dx=-\frac{du}{u^2},\quad dy=\frac{dv}{u}-\frac{v\,du}{u^2},\quad d^2x\,dy-d^2y\,dx=-\frac{1}{u^3}(d^2u\,dv-d^2v\,du).$$

Thus we conclude that the pole order of $d^2x\,dy-d^2y\,dx$ at infinity is 3. From

$$\begin{aligned}d^2xdf-d^2f\,dx=-f_{xx}\Big(-\frac{du}{u^2}\Big)^3-2f_{xy}\Big(-\frac{du}{u^2}\Big)^2\Big(\frac{dv}{u}-\frac{v\,du}{u^2}\Big)\\ -f_{yy}\Big(-\frac{du}{u^2}\Big)\Big(\frac{dv}{u}-\frac{v\,du}{u^2}\Big)^2-\frac{f_y}{u^3}(d^2u\,dv-d^2v\,du)\end{aligned}$$

we conclude that the pole order of d^2xdf-d^2fdx at infinity is $\delta+4$. From

$$df=f_x\Big(-\frac{du}{u^2}\Big)+f_y\Big(\frac{dv}{u}-\frac{v\,du}{u^2}\Big)$$

we have the pole order $\delta+1$ for df at infinity.

Since $f=t^\delta$ and $df=\delta t^{\delta-1}dt$ and $d^2f=\delta t^{\delta-1}d^2t+\delta(\delta-1)t^{\delta-2}dt^2$, it follows that from ω_μ we can factor out $t^{\mu(\delta-1)}$. From d^2fdx-d^2xdf we can factor out $t^{\delta-2}$. Hence from the term $\omega_{s+3k}f^{2(m-k)}(d^2fdx-d^2xdf)^{m-k}$ we can factor out t to the power $(s+3k)(\delta-1)+2(m-k)\delta+(m-k)(\delta-2)$ which is the same as $(s+3m)\delta-(s+2m+k)$ for $0\le k\le m$. We can only factor out the minimum power of t, namely $(s+3m)(\delta-1)$. When we can divide by f_y, we factor out a pole order of $\delta-1$ which corresponds to the power $\delta-1$ of t. On the other hand, the pole order at infinity for ω_μ is $p+\mu(\delta+2)$ and as a result the pole order of the term $\omega_{s+3k}f^{2(m-k)}(d^2fdx-d^2xdf)^{m-k}$ of Φ at infinity is

$$p+(s+3k)(\delta+2)+2(m-k)\delta+(m-k)(\delta+4)=p+(s+3m)\delta+2s+4m+2k$$

for $0\le k\le m$. We have to take in this case the maximum of the expression for $0\le k\le m$ and we get $p+(s+3m)(\delta+2)$. Take a positive integer q. To end up with a holomorphic jet differential $t^{-(s+3m)(\delta-1)}f_y^{-1}\Phi$ on X with at least q zero order at infinity, we can impose the condition

$$\delta-1+(s+3m)(\delta-1)\ge q+p+(s+3m)(\delta+2)$$

which is the same as $p\le\delta-q-1-3s-9m$.

2.3. Two Kinds of Irreducibility. In number theory it was first pointed out by Vojta that the finiteness of rational points for a subvariety of abelian varieties not containing the translate of an abelian variety is the consequence of the fact that in the product space of many copies of the subvariety there are more line bundles or divisors than constructed from the factors which are copies of the subvariety [Faltings 1991; Vojta 1992]. In hyperbolicity problems the analog of taking the product of copies of a manifold is to use the space of jets. The analog of the existence of more divisors or line bundles is the existence of more ways of factorization for meromorphic jet differentials. Some factors from the additional ways of factorization become holomorphic jet differentials. In our construction we pullback

$$\Phi = \sum_{k=0}^{m} \omega_{s+3k} f^{2(m-k)} (d^2 f dx - d^2 x df)^{m-k},$$

to the space of 2-jets of the branched cover and obtain a new factor t^N so that one of the other factors becomes a holomorphic 2-jet differntial on the branched cover.

On the other hand, the many more different ways of factorization makes it more difficult to control the factors to get the independence of holomorphic jet differentials. Two meromorphic jet differentials on the complex projective plane constructed in different ways may share a common factor when pulled back to the branched cover, because there are more ways of factorization in the space of jets of the branched cover. We have to strike a balance between having many ways of factorization to get holomorphic jet differentials and having not too many ways of factorization to get the independence of holomorphic jet differentials. The way we handle it is to construct an appropriate intermediate manifold between the space of jets of the complex projective space and the space of jets of the branched cover. On this intermediate manifold we introduce a certain class of meromorphic functions with the following property. Every memomorphic function in that class can be pulled back to the space of jets of the branched cover to give a factor which is a holomorphic jet differential. On the other hand, for that particular class of meromorphic functions the number of ways of factorization is not too numerous that we could construct two meromorphic functions in that class having no common factors before being pulled back to the jet space of the branched cover.

PROPOSITION 2.3.1. *Let $g_j(z_0, z_1, z_2)$ $(0 \leq j \leq 2)$ be homogeneous polynomials of degree δ whose common zero-set consists only of the single point*

$$(z_0, z_1, z_2) = 0.$$

Let $P(x, y, w_0, w_1, w_2, Y)$ be a polynomial of the 6 variables x, y, w_0, w_1, w_2, Y with degree p in x, y and homogeneous degree m in w_1, w_2, w_3 and of degree q in Y. Let Q be obtained from P by replacing w_0 by a function in w_1, w_2, x, y

satisfying $\sum_{j=0}^{2} g_j(1,x,y)w_j = 0$, *in other words,*

$$Q(x,y,w_1,w_2,Y) = P(x,y,-\frac{g_1(1,x,y)}{g_0(1,x,y)}w_1 - \frac{g_2(1,x,y)}{g_0(1,x,y)}w_2, w_1, w_2, Y).$$

Suppose $P(x,y,w_0,w_1,w_2,Y)$ *is irreducible as a polynomial of the* 6 *variables* x,y,w_0,w_1,w_2,Y. *If* $p < \delta$, *then* $Q(x,y,w_1,w_2,Y)$ *is irreducible as a polynomial of the* 3 *variables* w_1,w_2,Y *over the field* $\mathbb{C}(x,y)$.

PROOF. Introduce the homogeneous variables z_0, z_1, z_0 of $\mathbb{P}_2$ so that $x = \frac{z_1}{z_0}$ and $y = \frac{z_2}{z_0}$. Introduce the homogeneous variables Z_0, Z_1 of $\mathbb{P}_1$ so that $Y = \frac{Z_1}{Z_0}$. We use the coordinates $\big([z_0,z_1,z_2],\,[w_0,w_1,w_2],\,[Z_0,Z_1]\big)$ for the product $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1$. Let M be the subvariety in $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1$ defined by

$$\sum_{j=0}^{2} g_j(z_0,z_1,z_2)w_j = 0.$$

Since $g_j(z_0,z_1,z_2)$ $(0 \le j \le 2)$ have no common zeroes except the single point $(z_0,z_1,z_2) = 0$, it follows that M is a submanifold of $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1$. Let $\tilde{\pi}_j$ be the projection of $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1$ onto its j-th factor $(1 \le j \le 3)$. Let π_j be the restriction of $\tilde{\pi}$ to M. Let

$$\tilde{\pi} : \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1 \to \mathbb{P}_2 \times \mathbb{P}_1$$

be the projection

$$\big([z_0,z_1,z_2],\,[w_0,w_1,w_2],\,[Z_0,Z_1]\big) \mapsto \big([z_0,z_1,z_2],\,[Z_0,Z_1]\big);$$

in other words, $\tilde{\pi} = \tilde{\pi}_1 \times \tilde{\pi}_3$. Let $\pi : M \to \mathbb{P}_2 \times \mathbb{P}_1$ be the restriction of $\tilde{\pi}$ to M. Then $\pi : M \to \mathbb{P}_2 \times \mathbb{P}_1$ is a $\mathbb{P}_1$-bundle over $\mathbb{P}_2 \times \mathbb{P}_1$ whose fiber over the point $\big([z_0,z_1,z_2],\,[Z_0,Z_1]\big)$ is the complex line

$$\sum_{j=0}^{2} g_j(z_0,z_1,z_2)w_j = 0$$

in the projective plane $\mathbb{P}_2$ with homogeneous coordinates $[w_0,w_1,w_2]$.

Clearly the inclusion map $M \subset \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1$ induces the isomorphisms

$$\begin{aligned} R\tilde{\pi}^j_*\mathbb{Z} &\xrightarrow{\approx} R\pi^j_*\mathbb{Z} && (0 \le j \le 2),\\ R\tilde{\pi}^j_*\mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1} &\xrightarrow{\approx} R\pi^j_*\mathcal{O}_M && (0 \le j \le 2). \end{aligned}$$

From these isomorphisms and the standard spectral sequence arguments the following isomorphisms follow.

$$\begin{aligned} H^j\left(\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1, \mathbb{Z}\right) &\xrightarrow{\approx} H^j\left(M, \mathbb{Z}\right) && (0 \le j \le 2),\\ H^j\left(\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1}\right) &\xrightarrow{\approx} H^j\left(M, \mathcal{O}_M\right) && (0 \le j \le 2). \end{aligned}$$

In particular, we have the isomorphisms between the group of holomorphic line bundles over $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1$ and the group of holomorphic line bundles over M, namely,

$$H^1\big(\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}^*_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1}\big) \xrightarrow{\approx} H^1\left(M, \mathcal{O}^*_M\right). \tag{2.3.1.1}$$

Then a holomorphic line bundle over M is of the form

$$\mathcal{O}_M\left(k_1, k_2, k_3\right) := (\pi_1)^*\left(\mathcal{O}_{\mathbb{P}_2}(k_1)\right) \otimes (\pi_2)^*\left(\mathcal{O}_{\mathbb{P}_2}(k_2)\right) \otimes (\pi_3)^*\left(\mathcal{O}_{\mathbb{P}_1}(k_3)\right).$$

By Künneth's formula we have

$$H^1\left(\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1}(k_1, k_2, k_3)\right) = 0 \quad \text{for } k_3 \geq 1, \tag{2.3.1.2}$$

because $H^1(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(k)) = 0$ for every integer k and $H^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(k)) = 0$ for every integer $k \geq -1$. Let

$$\psi \in \Gamma(\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1}(\delta, 1, 0))$$

be defined by multiplication by $\sum_{j=0}^{2} g_j(z_0, z_1, z_2)w_j = 0$, From (2.3.1.2) and the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1}(k_1 - \delta, k_2 - 1, k_3) \xrightarrow{\theta} \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1}(k_1, k_2, k_3) \\ \to \mathcal{O}_M(k_1, k_2, k_3) \to 0$$

with θ defined by multiplication by ψ it follows that

$$\Theta_{k_1,k_2,k_3} : \Gamma(\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2\times\mathbb{P}_2\times\mathbb{P}_1}(k_1, k_2, k_3)) \to H^0(M, \mathcal{O}_M(k_1, k_2, k_3))$$

is surjective for $k_3 \geq -1$ and that Θ_{k_1,k_2,k_3} is injective for $k_1 < \delta$.

Let s be the meromorphic function on $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1$ be defined by P/w_0^m. Let $\tilde{H}_1$(respectively $\tilde{H}_2$, $\tilde{H}_3$)be the hypersurface in $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1$ defined by $z_0 = 0$ (respectively $w_0 = 0$, $Z_0 = 0$). Let $H_l = M \cap \tilde{H}_l$ for $1 \leq l \leq 3$. The pole divisor of s is $pH_1 + mH_2 + qH_3$.

Suppose $Q(x, y, w_1, w_2, Y)$ is not irreducible as a polynomial of the 3 variables w_1, w_2, Y over the field $\mathbb{C}(x, y)$. Then we can write $Q(x, y, w_1, w_2, Y)$ as a product of two factors $Q_j(x, y, w_1, w_2, Y)$ $(j = 1, 2)$ each of which is a polynomial of positive degree in the 3 variables w_1, w_2, Y over the field $\mathbb{C}(x, y)$. Thus the restriction $s|M$ of s to M can be written as the product of two meromorphic functions $s_1 s_2$ on M with s_j defined by $Q_j(x, y, w_1, w_2, Y)$ $(j = 1, 2)$. Let

$$W_j - V_j - \sum_{l=1}^{3} r'_{j,l} H_l$$

be the divisor of s_j $(j = 1, 2)$, where W_j, V_j are effective divisors with support not contained in $\bigcup_{l=1}^{3} H_l$. We know that $\pi_1(V_j)$ is a proper subvariety of $\mathbb{P}_2$ for $j = 1, 2$, because of the factorization of Q into the product of Q_1 and Q_2 over the field $C(x, y)$. We also know that for $j = 1, 2$ both $r'_{j,2}, r'_{j,3}$ are nonnegative and one of them is positive. The key point is that by (2.3.1.1) there exists a meromorphic function σ_j on M such that the divisor of σ_j is equal to

$$V_j - \sum_{l=1}^{3} r''_{j,l} H_l$$

for some integers $r''_{j,l}$.

The integers $r''_{j,l}$ $(1 \le l \le 3)$ are all nonnegative, because of the following fact.

Claim 2.3.1.3. *If u is a non-identically-zero meromorphic function on M whose divisor is $E - \sum_{l=1}^{3} \kappa_l H_l$, where E is an effective divisor of M, then the integers κ_l $(1 \le l \le 3)$ are all nonnegative.*

Proof. Suppose the contrary. Let $b = \max(-1, \kappa_3)$. Then one of κ_1, κ_2, b is negative. Let

$$\tau \in \Gamma\left(M, \mathcal{O}_M(r_{j,1}, r_{j,2}, b)\right)$$

be defined by $u(z_0)^{\kappa_1}(w_0)^{\kappa_2}(Z_0)^b$. Since $b \ge -1$, it follows from the surjectivity of $\Theta_{\kappa_1,\kappa_2,b}$ that τ can be lifted to an element

$$\tilde{\tau} \in \Gamma\big(\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_1}(\kappa_1, \kappa_2, b)\big).$$

Since one of κ_1, κ_2, b is negative, it follows that $\tilde{\tau}$ is identically zero, which is a contradiction and concludes the proof of Claim 2.3.1.3. □

Since $s|M = s_1\, s_2$ on M, it follows that the support of the divisor of the meromorphic function $(s|M)(\sigma_1\, s_1 \sigma_2\, s_2)^{-1}$ on M is contained in $\bigcup_{l=1}^{3} H_l$. By (2.3.1.1) we know that the meromorphic function $(s|M)(\sigma_1\, s_1 \sigma_2\, s_2)^{-1}$ on M must be a constant.

The divisor of $s_j\, \sigma_j$ is equal to

$$W_j - \sum_{l=1}^{3} (r'_{j,l} + r''_{j,l}) H_l.$$

At least one of the two integers $r'_{j,2} + r''_{j,2}$ and $r'_{j,3} + r''_{j,3}$ is positive. Both are nonnegative. By Claim (2.3.1.3) the integer $r'_{j,1} + r''_{j,1}$ is nonnegative for $j = 1, 2$. From $s|M = c(s_1\, \sigma_1)(s_2\, \sigma_2)$ on M for some nonzero constant c it follows that for $j = 1, 2$ we have

$$0 \le r'_{j,1} + r''_{j,1} \le p,$$
$$0 \le r'_{j,2} + r''_{j,2} \le m,$$
$$0 \le r'_{j,3} + r''_{j,3} \le q$$

and one of $r'_{j,2} + r''_{j,2}, r'_{j,3} + r''_{j,3}$ is positive. From $p < \delta$ it follows that

$$\Theta_{r_{j,1}+r'_{j,1}, r_{j,2}+r'_{j,2}, r_{j,3}+r'_{j,3}}$$

is an isomorphism and $s_j\, \sigma_j$ is induced by a polynomial $R_j(x, y, w_0, w_1, w_2, Y)$ of degree $r'_{j,1} + r''_{j,1} \le p$ in x, y and of degree $r'_{j,2} + r''_{j,2} \le m$ in w_0, w_1, w_2 and of degree $r'_{j,3} + r''_{j,3} \le q$ in Z. From $P = cR_1R_2$ and one of $r'_{j,2} + r''_{j,2}, r'_{j,3} + r''_{j,3}$

being positive for $j = 1, 2$, we have a contradiction to the irreducibility of P in the six variables x, y, w_0, w_1, w_2, Z. $\square$

Proposition 2.3.2. *Suppose $P\big(x, y, dx, dy, \frac{df}{f}, Z\big)$ is irreducible as a polynomial of the 6 variables with degree p in x and y and homogeneous weight m in dx, dy, $\frac{df}{f}$, Z when each of dx, dy, $\frac{df}{f}$ has weight 1 and Z has weight 3. If $p + m < \delta$, then $P\big(x, y, dx, dy, \frac{df}{f}, Z\big)$ is irreducible as a polynomial in dx, dy, Z over the field $\mathbb{C}(x, y)$ for generic f.*

Proof. We rewrite $P\big(x,\, y,\, dx,\, dy,\, \frac{df}{f}\big)$ as

$$P_1\left(x, y, \frac{dx}{x}, \frac{dy}{y}, \frac{df}{f}\right)$$

and introduce the symbols

$$w_0 = \frac{df}{f}, \quad w_1 = \frac{dx}{x}, \quad w_2 = \frac{dy}{y}.$$

The degree p' of $P_1\big(x,\, y,\, \frac{dx}{x},\, \frac{dy}{y},\, \frac{df}{f}\big)$ in x, y can be as high as $p + m$ when $P_1\big(x,\, y,\, \frac{dx}{x},\, \frac{dy}{y},\, \frac{df}{f}\big)$ is regarded as a polynomial of the 5 variables x, y, w_0, w_1, w_2. Let

$$\begin{aligned}
g_0(z_0, z_1, z_2) &= -z_0^\delta f\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right),\\
g_1(z_0, z_1, z_2) &= z_0^{\delta-1} z_1 f_x\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right),\\
g_2(z_0, z_1, z_2) &= z_0^{\delta-1} z_2 f_y\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right),
\end{aligned}$$

so that

$$\frac{g_1(z_0, z_1, z_2)}{g_0(z_0, z_1, z_2)} = -\frac{x f_x(x, y)}{f(x, y)}, \qquad \frac{g_2(z_0, z_1, z_2)}{g_0(z_0, z_1, z_2)} = -\frac{y f_y(x, y)}{f(x, y)},$$

with $x = z_1/z_0$ and $y = z_2/z_0$. For a generic f the three polynomials g_0, g_1, g_2 have no common zeroes other than the point $(z_0, z_1, z_2) = 0$, because it is the case for the special $f(x, y) = 1 + x^\delta + y^\delta$, where

$$\begin{aligned}
g_0(z_0, z_1, z_0) &= -(z_0^\delta + z_1^\delta + z_2^\delta),\\
g_1(z_0, z_1, z_2) &= \delta z_1^\delta,\\
g_2(z_0, z_1, z_2) &= \delta z_2^\delta.
\end{aligned}$$

The result now follows from Proposition 2.3.1. $\square$

2.4. Degree of Second Order Differential Greater Than One. We factor

$$\Phi = \sum_{k=0}^{m} \omega_{f,s+3k} f^{2(m-k)} (d^2 f\, dx - d^2 x\, df)^{m-k}$$

into irreducible factors

$$\Phi = \Phi_1 \Phi_2 \cdots \Phi_k$$

as polynomials in the independent variables

$$\frac{dx}{x}, \quad \frac{dy}{y}, \quad \frac{df}{f}, \quad \frac{d^2f\,dx - d^2x\,df}{f}$$

with coefficients in the field $\mathbb{C}(x,y)$ and then clear the denominators. The polynomial Φ satisfies the following three properties:

(1) Φ has homogeneous total weight $\leq s+3m$ when dx, dy, df are assigned weight 1 and $d^2f\,dx - df\,d^2x$ is assigned weight 3.
(2) The degree of Φ as a polynomial in $d^2f\,dx - df\,d^2x$ is at most m.
(3) When Φ is written as a polynomial in

$$x,\ y,\ \frac{dx}{x}, \ \frac{dy}{y}, \ \frac{df}{f}, \ \frac{d^2f\,dx - d^2x\,df}{f},$$

the degree of Φ in x, y is $\leq p + 3m + s$.

Hence each of the factors Φ_j $(1 \leq j \leq k)$ satisfies the same three properties. The third property means that, when Φ_j is written

$$\Phi_j = \sum_{k=0}^{m_j} \omega^{(j)}_{f,s_j+3k} f^{2(m_j-k)} (d^2f\,dx - d^2x\,df)^{m_j-k},$$

with

$$\omega^{(j)}_{f,\mu} = \sum_{\nu_0+\nu_1+\nu_2=\mu} a^{(j)}_{f,\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2},$$

the degree of the polynomial $a^{(j)}_{f,\nu_0\nu_1\nu_2}(x,y)$ in x, y is at most $p+3m+s$. Since Φ is divisible by f_y, at least one of the factors Φ_j divisible by f_y. We can now replace Φ by that factor Φ_j and assume that Φ is irreducible. One difference is that after this replacement the degree of the polynomial $a_{\nu_0\nu_1\nu_2}(x,y)$ in x, y is now at most $p+3m+s$ instead of at most p.

The degree m of the irreducible new Φ in $f^2(d^2f\,dx - d^2x\,df)$ may be equal to 1 or even 0. If m is zero, then we can get a holomorphic 1-jet differential on X which according to Sakai's result [1979] is impossible. We now would like to rule out the case of $m = 1$ for a generic f of sufficiently large degree δ relative to m, p, s. Assume $m = 1$ and we are going to derive a contradiction. The case of $m = 1$ means that we have the divisibility of $\omega_s \mathrm{II} + \omega_{s+3}$ by f_y. We use the following terminology. For a polynomial $g(x,y)$ of degree $\leq k$, by the element of $H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(k))$ defined by g we mean the element defined by the element of $H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(k))$ defined by the homogeneous polynomial $G(z_0, z_1, z_2)$ given by

$$G(z_0, z_1, z_2) = z_0^k\, g\Big(\frac{z_1}{z_0}, \frac{z_2}{z_0}\Big).$$

Lemma 2.4.1. *Suppose $g(x,y), g_1(x,y), g_2(x,y)$ are polynomials of degree δ in x,y. Let G, G_1, G_2 be elements of $H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(\delta))$ defined respectively by g, g_1, g_2. Assume that G, G_1, G_2 have no common zeroes on $\mathbb{P}_2$. Let $k \geq \delta$. If $a(x,y), a_1(x,y), a_2(x,y)$ are polynomials of degree $\leq k$ so that $ga = a_1 g_1 + a_2 g_2$, then there exist polynomials b_1, b_2 of degree $\leq k-\delta$ such that $a = b_1 g_1 + b_2 g_2$.*

Proof. Let E be the element in $H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(e))$ (with $0 \leq e \leq \delta$ whose zero-set is the union of all the common branches of the zero-set of G_1 and the zero-set of G_2. Let $\tilde{G}_j = \frac{G_j}{E} \in H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(\delta - e))$ for $j = 1, 2$. Let $\mathcal{I}$ be the ideal sheaf on $\mathbb{P}_2$ generated by G_1, G_2. Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_2}(k - 2\delta + e) \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}_2}(k-\delta)^{\oplus 2} \xrightarrow{\tau} \mathcal{I}(k) \to 0$$

with σ defined by the 2×1 matrix $\begin{pmatrix} -\tilde{G}_2 \\ \tilde{G}_1 \end{pmatrix}$ and with τ defined by the 1×2 matrix (G_1, G_2). Since $H^1(\mathbb{P}_2, \mathcal{I}(k - 2\delta + e)) = 0$, it follows that the map

$$\tilde{\sigma} : H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(k-\delta))^{\oplus 2})) \to H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(k))$$

is surjective. Let A, A_1, A_2 be elements of $H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(k))$ defined by a, a_1, a_2. It follows from $ga = a_1 g_1 + a_2 g_2$ that $GA = A_1 G_1 + A_2 G_2$. Since G, G_1, G_2 have no common zeroes in $\mathbb{P}_2$, it follows that $A \in H^0(\mathbb{P}_2, \mathcal{I}(k))$. Hence there exist $B_1, B_2 \in H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(k-\delta))$ such that $A = \tilde{\sigma}(B_1, B_2)$. Let $b_1(x,y), b_2(x,y)$ be polynomials of degree $\leq k - \delta$ corresponding respectively to B_1, B_2. Then $a = b_1 g_1 + b_2 g_2$. □

Lemma 2.4.2. *Suppose $g(x,y), g_1(x,y), g_2(x,y)$ are polynomials of degree δ in x,y. Let G, G_1, G_2 be elements of $H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(\delta))$ defined respectively by g, g_1, g_2. Assume that G, G_1, G_2 have no common zeroes on $\mathbb{P}_2$. Let $a_\mu(x,y)$ $(0 \leq \mu \leq s)$ be polynomials of degree at most p so that $a_\mu(x,y)$ $(0 \leq \mu \leq s)$ are not all identically zero. Let $h(x,y)$ be a polynomial of degree k. Let $b_\mu(x,y)$ $(0 \leq \mu \leq s+1)$ be polynomials of degree at most $p + k - \delta$. Suppose*

$$\left(\sum_{\mu=0}^{q} a_\mu(x,y) g_1(x,y)^{q-\mu} g_2(x,y)^\mu\right) h(x,y) + \left(\sum_{\nu=0}^{q+1} b_\nu(x,y) g_1(x,y)^{q+1-\nu} g_2(x,y)^\nu\right)$$

is divisible by $g(x,y)$. Then there exist non identically zero polynomials $a(x,y)$, $c_1(x,y)$, $c_2(x,y)$, $c(x,y)$ of degree at most p such that

$$a(x,y)h(x,y) = c_1(x,y)g_1(x,y) + c_2(x,y)g_2(x,y) + c(x,y)g(x,y).$$

Proof. By replacing $g_1(x,y), g_2(x,y)$ by

$$\tilde{g}_1(x,y) = \alpha_1 g_1(x,y) + \alpha_2 g_2(x,y),$$
$$\tilde{g}_2(x,y) = \beta_1 g_1(x,y) + \beta_2 g_2(x,y),$$

for some suitable constants α_j, β_j $(j = 1, 2)$, we can assume without loss of generality that $a_0(x, y)$ is not identically zero. Then

$$g_1(x,y)^q \left(a_0(x,y)h(x,y) + b_0(x,y)g_1(x,y)\right) = \psi_2(x,y)g_2(x,y) + \psi(x,y)g(x,y),$$

where

$$\psi_2(x,y) = -\sum_{\mu=1}^{q} a_\mu(x,y) g_1^{q-\mu} g_2(x,y)^{\mu-1} h(x,y) - \sum_{\mu=1}^{q+1} b_\mu(x,y) g_1^{q+1-\mu} g_2(x,y)^{\mu-1}$$

and $\psi(x, y)$ are polynomials in x, y of degree at most $q\delta + p + k$.

Applying q times Lemma 2.4.1 gives us polynomials $c_2(x, y), c(x, y)$ of degree at most $p + k - \delta$ such that

$$a_0(x,y)h(x,y) + b_0(x,y)g_1(x,y) = c_2(x,y)g_2(x,y) + c(x,y)g(x,y).$$

It suffices to set $a(x, y) = a_0(x, y)$ and $c_1(x, y) = -b_0(x, y)$. □

2.4.3. The case $m = 1$ means that there exist polynomials $a_{\nu_0\nu_1\nu_2}$ of degree at most p such that

$$\sum_{\nu_0+\nu_1+\nu_2=s} a_{\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2} f^2 (d^2 f\,dx - d^2 x\,df) + \sum_{\nu_0+\nu_1+\nu_2=s+3} a_{\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2}$$

is divisible by f_y. This means that

$$\sum_{\nu_0+\nu_1+\nu_2=s} a_{\nu_0\nu_1\nu_2}(x,y)(f_x)^{\nu_0} f^{\nu_1+\nu_2+2}(dx)^{\nu_0+\nu_1+1}(dy)^{\nu_2}\mathrm{II} + \sum_{\nu_0+\nu_1+\nu_2=s+3} a_{\nu_0\nu_1\nu_2}(x,y)(f_x)^{\nu_0} f^{\nu_1+\nu_2}(dx)^{\nu_0+\nu_1}(dy)^{\nu_2}$$

is divisible by f_y. Let

$$\xi_l = \sum_{\nu=0}^{l} a_{\nu,l-\nu,s-l}(x,y)(f_x)^\nu f^{s+2-\nu}, \qquad \eta_l = \sum_{\nu=0}^{l} a_{\nu,l-\nu,s+3-l}(x,y)(f_x)^\nu f^{s+3-\nu}.$$

Then

$$\sum_{l=0}^{s} \xi_l (dx)^{l+1}(dy)^{s-l}\left(f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2\right) - \sum_{l=0}^{s+3} \eta_l (dx)^l (dy)^{s+3-l}$$

is divisible by f_y.

Let l_0 be the largest l such that the polynomial $\xi_l(x, y)$ is not identically zero. Let l_1 be the smallest l such that the polynomial ξ_l is not identically zero. Then from the coefficient of $(dx)^{l_0+3}(dy)^{s-l_0}$ we conclude that

$$\xi_{l_0} f_{xx} - \eta_{l_0+3}$$

is divisible by f_y. From the coefficient of $(dx)^{l_1+1}(dy)^{s+2-l_1}$ we conclude that

$$\xi_{l_1} f_{yy} - \eta_{l_1+1}$$

is divisible by f_y. From the coefficient of $(dx)^{l_0+2}(dy)^{s-l_0+1}$ we conclude that

$$\xi_{l_0+1} f_{xx} + 2\xi_{l_0} f_{xy} - \eta_{l_0+2}$$

is divisible by f_y. Hence

$$2\xi_{l_0}^2 f_{xy} - \xi_{l_0}\eta_{l_0+2} + \xi_{l_0+1}\eta_{l_0+3}$$

is divisible by f_y.

Choose two polynomials $\lambda_1(x,y), \lambda_2(x,y)$ of degree 1 in x, y such that the elements in $H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(\delta))$ defined by $\lambda_1(x,y)f_x(x,y)$, $\lambda_2(x,y)f_y(x,y)$, and $f(x,y)$ have no common zeroes on $\mathbb{P}_2$. Let $g_1(x,y) = \lambda_1(x,y)f_x(x,y)$ and $g_2(x,y) = \lambda_2(x,y)f_y(x,y)$, and

$$\begin{aligned}
\tilde{\xi}_l = \lambda_1^l \xi_l &= \sum_{\nu=0}^{l} a_{\nu,l-\nu,s-l}(x,y)\lambda_1^{l-\nu}(g_1)^\nu f^{s+2-\nu}, \\
\tilde{\eta}_l = \lambda_1^l \eta_l &= \sum_{\nu=0}^{l} a_{\nu,l-\nu,s+3-l}(x,y)\lambda_1^{l-\nu}(g_1)^\nu f^{s+3-\nu}.
\end{aligned}$$

Then the three polynomials

$$\begin{gathered}
\lambda_2\lambda_1^3\tilde{\xi}_{l_0} f_{xx} - \lambda_2\tilde{\eta}_{l_0+3}, \\
\lambda_2\lambda_1\tilde{\xi}_{l_1} f_{yy} - \lambda_2\tilde{\eta}_{l_1+1}, \\
2\lambda_2\lambda_1^4\tilde{\xi}_{l_0}^2 f_{xy} - \lambda_2\lambda_1^2\tilde{\xi}_{l_0}\tilde{\eta}_{l_0+2} + \lambda_2\tilde{\xi}_{l_0+1}\tilde{\eta}_{l_0+3}
\end{gathered}$$

are all divisible by g_2.

By Lemma 2.4.2 there exist polynomials $c_{i,j}(x,y)$ such that

$$\begin{aligned}
c_{1,0} f_{xx} &= c_{1,1}\lambda_1 f_x + c_{1,2}\lambda_2 f_y + c_{1,3} f, \\
c_{2,0} f_{xy} &= c_{2,1}\lambda_1 f_x + c_{2,2}\lambda_2 f_y + c_{2,3} f, \\
c_{3,0} f_{xx} &= c_{3,1}\lambda_1 f_x + c_{3,2}\lambda_2 f_y + c_{3,3} f,
\end{aligned}$$

with

$$\begin{aligned}
&\deg\, c_{1,j} \le p + l_0 + 4, \\
&\deg\, c_{2,j} \le p + l_1 + 2, \\
&\deg\, c_{3,j} \le p + 2l_0 + 5
\end{aligned}$$

for $0 \le j \le 3$. Consider the above system of linear equations in

$$f_{xx},\ f_{xy},\ f_{yy},\ f_x,\ f_y$$

as a system of linear differential equations for the unknown functions f_x, f_y, f. Counting the degree of freedom for all the polynomials $c_{i,j}$, we conclude from the uniqueness property of the system of differential equations that the degree of freedom for f is no more than

$$3+4\left(\binom{p+s+4}{2}+\binom{p+s+2}{2}+\binom{p+2s+5}{2}\right).$$

So when

$$\binom{\delta+2}{2}>3+4\left(\binom{p+s+4}{2}+\binom{p+s+2}{2}+\binom{p+2s+5}{2}\right),$$

the case of $m=1$ cannot occur for a generic f of degree δ.

2.5. Independence of Special 2-Jet Differentials by Invariant Theory. Let p be a positive integer and s be a nonnegative integer. By solving linear equations we can generically construct a special 2-jet differential Φ of total weight $s+3m$ $(m\geq 1)$ on X of the form

$$\Phi=\sum_{k=0}^{m}\omega_{f,s+3k}f^{2(m-k)}(d^2f\,dx-d^2x\,df)^{m-k},$$

where

$$\omega_{f,\mu}=\sum_{\nu_0+\nu_1+\nu_2=\mu}a_{f,\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2}$$

and $a_{f,\nu_0\nu_1\nu_2}(x,y)$ is a polynomial in x and y of degree $\leq p$ so that Φ is divisible by f_y and as a consequence $t^{-N}f_y^{-1}\Phi$ is a holomorphic 2-jet differential on X defined by $t^\delta=f(x,y)$, when certain inequalities involving p, s, δ, m, and N are satisfied.

We can assume that Φ, as a polynomial in

$$x,\ y,\ dx,\ dy,\ \frac{df}{f},\ \frac{d^2f\,dx-dx\,d^2f}{f^2},$$

is irreducible and the coefficients of a_{ν_0,ν_1,ν_2} are rational functions of the coefficients of $f(x,y)$. This assumption is possible because we can replace Φ by the corresponding irreducible factor which is divisible by f_y. This means that we can assume without loss of generality that Φ as a polynomial in x, y, dx, dy, d^2x, $dy-dx$, d^2y is irreducible.

Consider the space $\mathcal{F}$ of polynomials f. Let $G=SL(2,\mathbb{C})$. Let C be the curve defined by f. For $\gamma\in G$, the defining function for $\gamma(C)$ is $(\gamma^{-1})^*f$. Let $(x_\gamma,y_\gamma)=\gamma(x,y)$. We have a procedure which gives us a special 2-jet differential Ψ_f for $f\in\mathcal{F}$ generically. We can use $\gamma\in SL(2,\mathbb{C})$ to get another $\gamma^*\Psi_{(\gamma^{-1})^*f}$. Suppose this procedure with the use of $\gamma\in SL(2,\mathbb{C})$ does not give us at least two independent special 2-jet differentials. By Proposition 3.3.1 each $\gamma^*\Psi_{(\gamma^{-1})^*f}$ is irreducible over $\mathbb{C}(x,y)$ as a polynomial of dx, dy, $d^2x\,dy-dx\,d^2y$. Then we have

$$\gamma^*\Psi_{(\gamma^{-1})^*f}=R_{\gamma,f}(x,y)\Psi_f$$

for some rational function $R_{\gamma,f}(x,y)$ in x, y. To take away $R_{\gamma,f}(x,y)$ we define for every γ the following. Let Z_f be the union of all algebraic complex curves Z'_f in $\mathbb{C}^2$ such that the inverse image of Z'_f in the space of 2-jets is contained in the zero-set of Ψ_f. In other words, Ψ_f is divisible by the polynomial in x, y which defines Z'_f. Let $g_f(x,y)$ be a polynomial in x, y which defines Z_f. In other words, $g_f(x,y)$ is the polynomial (defined up to a nonzero constant) which divides Ψ_f. Then we conclude that

$$\gamma^*\left(\frac{1}{g_{(\gamma^{-1})^*f}}\Psi_{(\gamma^{-1})^*f}\right) = c_{\gamma,f}\frac{1}{g_f}R_{\gamma,f}(x,y)\Psi_f$$

for some nonzero constant $c_{\gamma,f}$. Let $g_{\gamma,f} = \gamma^* g_{(\gamma^{-1})^*f}$.

2.5.1. For $\gamma \in SL(2,\mathbb{C})$ let f_{y_γ} be the partial derivative with respect to y_γ in the coordinate system (x_γ, y_γ). We have

$$\frac{1}{g_{\gamma,f}}f_{y_\gamma}^{-1}\sum_{k=0}^{m}\gamma^*(\omega_{(\gamma^{-1})^*f,s+3k})f^{2(m-k)}(d^2f\,d(x_\gamma) - d^2(x_\gamma)\,df)^{m-k}$$
$$= c_{\gamma,f}\frac{1}{g_{1,f}}f_y^{-1}\sum_{k=0}^{m}\omega_{f,s+3k}f^{2(m-k)}(d^2f\,dx - d^2x\,df)^{m-k}$$

and

$$\gamma^*\omega_{(\gamma^{-1})^*f,\mu} = \sum_{\nu_0+\nu_1+\nu_2=\mu} a_{(\gamma^{-1})^*f,\nu_0\nu_1\nu_2}(x_\gamma,y_\gamma)(df)^{\nu_0}(f\,d(x_\gamma))^{\nu_1}(f\,d(y_\gamma))^{\nu_2}.$$

We use

$$d^2f\,dx - d^2x\,df = f_y(d^2y\,dx - d^2x\,dy) + \mathrm{II}\,dx,$$

where

$$\mathrm{II} = f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2.$$

Since $d^2y\,dx - d^2x\,dy$ and II are both invariant under $SL(2,\mathbb{C})$, it follows that

$$d^2f\,d(x_\gamma) - d^2(x_\gamma)\,df = \gamma^*\left(d^2((\gamma^{-1})^*f)\,dx - d^2x\,d(\gamma^{-1})^*f)\right)$$
$$= f_{y_\gamma}(d^2y\,dx - d^2x\,dy) + \mathrm{II}\,d(x_\gamma).$$

Thus

(2.5.1.1)
$$\frac{1}{g_{\gamma,f}}f_{y_\gamma}^{-1}\sum_{k=0}^{m}\gamma^*(\omega_{(\gamma^{-1})^*f,s+3k})f^{2(m-k)}\left(f_{y_\gamma}(d^2y\,dx - d^2x\,dy) + \mathrm{II}\,d(x_\gamma)\right)^{m-k}$$
$$= c_{\gamma,f}\frac{1}{g_{1,f}}f_y^{-1}\sum_{k=0}^{m}\omega_{f,s+3k}f^{2(m-k)}(d^2f\,dx - d^2x\,df)^{m-k}.$$

We consider the terms in (2.5.1.1) with the highest power for the factor $d^2y\,dx - d^2x\,dy$ and conclude that

$$\frac{1}{g_{\gamma,f}}\gamma^*(\omega_{(\gamma^{-1})^*f,s})(f_{y_\gamma})^{m-1} = c_{\gamma,f}\frac{1}{g_{1,f}}\omega_{f,s}(f_y)^{m-1}. \tag{2.5.1.2}$$

2.5.2. From Section 2.4 we know that $m > 1$. Let q be the largest integer such that $g_{\gamma,f}$ is divisible by $(f_{y_\gamma})^q$ for a generic γ.

Again we differentiate between two cases. The first case is that $q < m-1$. Since for any integer $l \geq 2$ the distinct generic elements

$$\gamma_j = \begin{pmatrix} \alpha_j & \beta_j \\ \sigma_j & \tau_j \end{pmatrix} \in SL(2,\mathbb{C}) \qquad (1 \leq j \leq l)$$

the l polynomials $f_{\gamma_j y} = -\beta_j f_x + \alpha_j f_y$ are relatively prime. it follows from (2.5.1.2) that $\omega_{f,s}$ contains the factor $\prod_{j=1}^{l}(-\beta_j f_x + \alpha_j f_y)$ for arbitrarily large l and we have a contradiction.

Now consider the second case of $q \geq m-1$. Then $(f_{y_\gamma})^{m-1}$ divides $g_{\gamma,f}$ for a generic γ. As a consequence

$$\sum_{k=0}^{m} \gamma^*(\omega_{(\gamma^{-1})^* f, s+3k}) f^{2(m-k)} \left(f_{y_\gamma}(d^2y\,dx - d^2x\,dy) + \mathrm{II}\, d(x_\gamma) \right)^{m-k}$$

is divisible by $(f_{y_\gamma})^m$. Since we consider a generic f, we can assume that γ being equal to the identity element is the generic case. By considering the coefficient of $\left(d^2y\,dx - d^2x\,dy\right)^{m-1}$, we conclude that f_y divides $m\omega_s \mathrm{II} + \omega_{s+3}$ and the 2-jet differential

$$f_y^{-1}\left(m\omega_s f^2(d^2f\,dx - df\,d^2x) + \omega_{s+3}\right)$$

gives rise to a holomorphic 2-jet differential, which means that we have the case of $m = 1$, contradicting the earlier conclusion that the case of $m = 1$ cannot occur.

2.6. Construction of Sections of Multiples of Differences of Ample Line Bundles. We now take the resultant for the two independent holomorphic 2-jet differentials and get a meromorphic 1-jet differential h whose pullback by the entire holomorphic curve is identically zero. After replacing h by one of its factors, we can also assume without loss of generality that h is and its homogeneous degree q in x, y be m.

LEMMA 2.6.1 (AMPLE LINE BUNDLE DIFFERENCE [Siu 1993]). *Let F and G be ample line bundles over a reduced compact complex space X of complex dimension n. If $F^n > nF^{n-1}G$, then for k sufficiently large there exists a nontrivial holomorphic section of $k(F-G)$ which vanishes on some ample divisor of X.*

PROOF. By replacing F and G by their sufficiently high powers, we can assume without loss of generality that both F and G are very ample. Let k be any positive integer. We select $k+1$ reduced members G_j, $1 \leq j \leq k+1$ in the linear system $|G|$ and consider the exact sequence

$$0 \to H^0(X, kF - \sum_j G_j) \to H^0(X, kF) \to \textstyle\bigoplus_{j=1}^{k+1} H^0(G_j, kF|G_j).$$

By Kodaira's vanishing theorem and the theorem of Riemann–Roch

$$\begin{aligned}\dim_{\mathbb{C}} H^0(X, kF-(k+1)G) &\geq \frac{k^n}{n!}F^n - \sum_{j=1}^{k+1}\frac{k^{n-1}}{(n-1)!}F^{n-1}G_j - o(k^{n-1})\\ &\geq \frac{k^n}{n!}(F^n - nF^{n-1}G) - o(k^n).\end{aligned}$$

So for k sufficiently large there exists a nontrivial global holomorphic section s of $kF-(k+1)G$ over X. We multiply s by a nontrivial global holomorphic section of G on X to get a nontrivial holomorphic section of $k(F-G)$ over X which vanishes on an ample divisor of X. □

LEMMA 2.6.2. *Let $h(x, y, dx, dy)$ be an irreducible polynomial in x, y, dx, dy which is of degree q in x, y and is of homogeneous degree $m \geq 1$ in dx, dy. Suppose $q \geq 4m$ and $\delta \geq 1$. Let $f(x, y)$ be a polynomial of degree δ such that the curve C in $\mathbb{P}_2$ defined by f is smooth. Then there exists no holomorphic map $\varphi : \mathbb{C} \to \mathbb{P}_2 - C$ such that the image of φ is Zariski dense in $\mathbb{P}_2$.*

PROOF. Assume that there is a holomorphic map $\varphi : \mathbb{C} \to \mathbb{P}_2 - C$ such that the image of φ is Zariski dense in $\mathbb{P}_2$. We are going to derive a contradiction.

Let X be the surface in $\mathbb{P}_3$ which has affine coordinates x, y, t with $t^\delta = f(x, y)$ so that X is a cyclic branched cover over $\mathbb{P}_2$ with branching along C with projection map $\pi : X \to \mathbb{P}_2$. Let $\tilde{C} = \pi^{-1}(C)$ and $\tilde{\varphi} : \mathbb{C} \to X - \tilde{C}$ be the lifting of φ so that $\pi \circ \tilde{\varphi} = \varphi$.

We use the following notations. For a vector space E over $\mathbb{C}$, we let $\mathbb{P}(E)$ denote the space of all 1-dimensional $\mathbb{C}$-linear subspaces of E. For a vector bundle $\sigma : B \to Y$ we let $\mathbb{P}(B)$ denote the bundle of projective spaces over Y so that the fiber of $\mathbb{P}(B)$ over a point $y \in Y$ is $\mathbb{P}(\sigma^{-1}(y))$. We let L_X denote the line bundle over $\mathbb{P}(T_X)$ whose restriction to the fiber of $\mathbb{P}(T_X) \to X$ over $x \in X$ is the hyperplane section line bundle of $\mathbb{P}(T_{X,x})$, where $T_{X,x}$ is the tangent space of X at x. We regard h as a holomorphic section of $mL_X + qH_{\mathbb{P}_2}$. For the proof we will produce a non identically zero holomorphic section of L_X over the Zariski closure of the image of $\tilde{\varphi}$ which vanishes on ample divisor, which then yields a contradiction by the usual Schwarz lemma argument.

We will compute the Chern classes of L_X and use the following well-known formula of Grothendieck [Fulton 1976; Grothendieck 1958] to do the computation to produce such a holomorphic section of L_X.

FORMULA 2.6.3 (GROTHENDIECK). *Let E be a vector bundle of rank r over X and $p : \mathbb{P}(E^*) \to X$ be the projection from the projectivization of the dual of E. Let L_E be the hyperplane section line bundle over $\mathbb{P}(E^*)$. Then*

$$\sum_{j=0}^{r}(-1)^j p^*(c_j(E^*))(c_1(L_E))^j = 0,$$

where $c_0(E^*)$ means 1. □

To compute the Chern classes of T_X we use the exact sequence

$$0 \to T_X \to T_{\mathbb{P}_3}|X \to \delta H_{\mathbb{P}_3}|X \to 0.$$

From the Euler sequence

$$0 \to 1 \to H_{\mathbb{P}_3}^{\oplus 4} \to T_{\mathbb{P}_3} \to 0$$

we conclude that the total Chern class of $T_{\mathbb{P}_3}$ is $(1 + H_{\mathbb{P}_3})^4$. Thus the total Chern class of T_X is $(1 + H_{\mathbb{P}_3})^4(1 + \delta H_{\mathbb{P}_3})^{-1}|X$ and the total Chern class of T_X^* is $(1 - H_{\mathbb{P}_3})^4(1 - \delta H_{\mathbb{P}_3})^{-1}|X$. We conclude that $c_1(T_X^*) = (\delta - 4)H_{\mathbb{P}_3}|X$ and $c_2(T_X^*) = (\delta^2 - 4\delta + 6)H_{\mathbb{P}_3}^2|X$. Grothendieck's formula yields $L_X^2 - (\delta - 4)H_{\mathbb{P}_3}L_X + (\delta^2 - 4\delta + 6)H_{\mathbb{P}_3}^2 = 0$ on $\mathbb{P}(T_X)$. Since $H_{\mathbb{P}_3}$ is lifted up from X via the projction map $\mathbb{P}(T_X) \to X$, we have $H_{\mathbb{P}_3}^3|X = 0$. Hence $L_X^2 H_{\mathbb{P}_3} = (\delta - 4)H_{\mathbb{P}_3}^2 L_X$ and

$$\begin{aligned} L_X^3 &= (\delta - 4)H_{\mathbb{P}_3}L_X^2 - (\delta^2 - 4\delta + 6)H_{\mathbb{P}_3}^2 L_X \\ &= (\delta - 4)^2 H_{\mathbb{P}_3}^2 L_X - (\delta^2 - 4\delta + 6)H_{\mathbb{P}_3}^2 L_X \\ &= (-4\delta + 10)H_{\mathbb{P}_3}^2 L_X. \end{aligned}$$

Note that $H_{\mathbb{P}_3}|X = \pi^*(H_{\mathbb{P}_2})$ so that we simply write $H_{\mathbb{P}_3}|X = H_{\mathbb{P}_2}$. It follows from $H_{\mathbb{P}_2}^2|\mathbb{P}_2 = 1$ that $H_X^2|X = \delta$ and $L_X H_{\mathbb{P}_2}^2|\mathbb{P}(T_X) = \delta$. Hence $L_X^2 H_{\mathbb{P}_2} = \delta(\delta - 4)$ and $L_X^3 = \delta(-4\delta + 10)$.

We know that $L_X + 3H_{\mathbb{P}_2}$ is positive on $\mathbb{P}(T_X)$ as we can easily see by using dx, dy, dt and considering the order of their poles at infinity. Take a large positive integer r. Now to apply Lemma 2.6.1, we let $F = (r + 1)(L_X + 3H_{\mathbb{P}_2})$ and $G = L_X + 3(r + 1)H_{\mathbb{P}_2}$ so that $rL_X = F - G$. We have to verify $F^2 > 2FG$ on V_h. In other words,

$$\begin{aligned} &((r + 1)(L_X + 3H_{\mathbb{P}_2}))^2 (mL_X + qH_{\mathbb{P}_2}) \\ &\qquad > 2\left((r + 1)(L_X + 3H_{\mathbb{P}_2})\right)(L_X + 3(r + 1)H_{\mathbb{P}_2})(mL_X + qH_{\mathbb{P}_2}), \end{aligned}$$

because V_h as a hypersurface in $\mathbb{P}(T_X)$ is defined by $h = 0$. We rewrite this inequality as

$$\begin{aligned} &(r{+}1)^2 \left(mL_X^3 + (6m{+}q)L_X^2 H_{\mathbb{P}_2} + (6q{+}9m)L_X H_{\mathbb{P}_2}^2\right) \\ &\qquad > 2(r{+}1)\left(mL_X^3 + (3m(r{+}2){+}q)L_X^2 H_{\mathbb{P}_2} + (9m(r{+}1){+}3q(r{+}2))L_X H_{\mathbb{P}_2}^2\right). \end{aligned}$$

Dividing both sides of the inequality by $(r + 1)\delta$, we get

$$\begin{aligned} &(r + 1)\left(m(-4\delta + 10) + (6m + q)(\delta - 4) + (6q + 9m)\right) \\ &\qquad > 2\left(m(-4\delta + 10) + (3m(r + 2) + q)(\delta - 4) + (9m(r + 1) + 3q(r + 2))\right). \end{aligned}$$

Since we are free to choose arbitrarily large r, it suffices to consider the coefficients of r on both sides. The coefficient of r on the left-hand side is

$$(2m + q)\delta + 2q - 5m = (2m + q)(\delta - 1) + 3q - 3m$$

and the coefficient of r on the right-hand side is

$$6m\delta + 3q - 15m = 6m(\delta - 1) + 3q - 9m.$$

If $q \geq 4m$, we get the inequality we want for r sufficiently large.

Since the 1-jet differential h on $\mathbb{P}_2$ is irreducible, its zero-set in $\mathbb{P}\left(T_{\mathbb{P}_2}\right)$ is again irreducible. However, the pullback $\tilde{h}$ of h to the branched cover X over $\mathbb{P}_2$ may not be irreducible. The holomorphic section of rL_X over $V_{\tilde{h}}$ we get may be identically zero on the branch of $V_{\tilde{h}}$ which contains the lifting of the entire holomorphic curve. To deal with this case, we will use the observation that the subvariety $V_{\tilde{h}}$ of $\mathbb{P}(T_X)$ is branched over the subvariety V_h of $\mathbb{P}\left(T_{\mathbb{P}_2}\right)$ and the branching is cylic. The action of the cyclic group of order δ acting on $V_{\tilde{h}}$ will in the following way help us get a non identically zero section on the branch we want.

We lift $\varphi : \mathbb{C} \to \mathbb{P}_2 - C$ to $\tilde{\varphi} : \mathbb{C} \to X - \tilde{C}$. We consider the projectivization $\mathbb{P}(T_X)$ of the tangent bundle T_X of X and let $p_X : \mathbb{P}(T_X) \to X$ be the projection map. We also consider the projectivization $\mathbb{P}(T_{\mathbb{P}_2})$ of the tangent bundle $T_{\mathbb{P}_2}$ of $\mathbb{P}_2$ and let $p_{\mathbb{P}_2} : \mathbb{P}(T_{\mathbb{P}_2}) \to X$ be the projection map. The projection map $\pi : X \to \mathbb{P}_2$ induces a meromorphic map $\mathbb{P}(\pi) : \mathbb{P}(T_X) \to \mathbb{P}(T_{\mathbb{P}_2})$ whose restriction to $\mathbb{P}(T_{X-\tilde{C}})$ is holomorphic. We have a holomorphic map $\mathbb{P}(d\varphi) : \mathbb{C} \to \mathbb{P}(T_{\mathbb{P}_2})$ which we define first at points of $\mathbb{C}$ where $d\varphi$ is nonzero and then extend by holomorphicity to all of $\mathbb{C}$. Likewise we have a holomorphic map $\mathbb{P}(d\tilde{\varphi}) : \mathbb{C} \to \mathbb{P}(T_X)$ which we define first at points of $\mathbb{C}$ where $d\varphi$ is nonzero and then extend by holomorphicity to all of $\mathbb{C}$. Let W be the Zariski closure in $\mathbb{P}(T_{\mathbb{P}_2})$ of the image of $\mathbb{P}(d\varphi)$. Let $\tilde{W}$ be the Zariski closure in $\mathbb{P}(T_X)$ of the image of $\mathbb{P}(d\tilde{\varphi})$. We let $(\mathbb{P}(\pi))(\tilde{W})$ denote the proper image of $\tilde{W}$ under $\mathbb{P}(\pi)$ in the sense that it is Zariski closure in $\mathbb{P}(T_{\mathbb{P}_2})$ of the image of $\mathbb{P}(\pi)$ of $W \cap \mathbb{P}(T_{X-\tilde{C}})$. Then $W = (\mathbb{P}(\pi))(\tilde{W})$. We know that $W = V_h$. Also we know that $\tilde{W}$ is a branch of $V_{\tilde{h}}$. Let $\hat{W}$ be a branch of $V_{\tilde{h}}$ where the 1-jet differential ω constructed as a section of the difference of ample line bundles is not identically zero. There is a proper subvariety $\tilde{E}$ of X such that the projection under p_X of the intersection of any two distinct branches of $V_{\tilde{h}}$ onto X is contained in $\tilde{Z}$. Let Z be the projection of $\tilde{Z}$ to $\mathbb{P}_2$. Take a point $P_0 \in \mathbb{P}_2 - (C \cap Z)$ such that $V_h \cap \pi_{\mathbb{P}_2}^{-1}(P_0)$ consists of precisely m distinct points $Q_1, \ldots, Q_m$. The inverse image of P_0 under π consists of δ distinct points P_0^ν $(1 \leq \nu \leq \delta)$. The inverse image of Q_j under $\mathbb{P}(\pi)$ consists of δ distinct points $Q_j^{(\nu)}$ $(1 \leq \nu \leq m)$ so that $Q_j^{(\nu)} \in \pi_X^{-1}(P_0^{(\nu)})$. Some $Q_{j_0}^{(\nu_0)} \in \hat{W}$. Then there exists some ν_1 such that $Q_{j_0}^{(\nu_1)} \in \tilde{W}$. There exists an element γ in the Galois group of automorphisms of X over $\mathbb{P}_2$ such that γ maps $P_0^{(\nu_1)}$ to $P_0^{(\nu_0)}$. Then the induced automorphism of $\tilde{\gamma}$ $\mathbb{P}(T_X)$ over $\mathbb{P}(T_{\mathbb{P}_2})$ maps $Q_{j_0}^{(\nu_1)}$ to $Q_{j_0}^{(\nu_0)}$. As a consequence $\tilde{\gamma}^*(\omega)$ is not identically zero on the branch $\tilde{W}$ of $V_{\tilde{h}}$ which is the Zariski closure of the image of $\mathbb{P}(d\tilde{\varphi})$. This forces the pullback by $d\tilde{\varphi}$ of $\tilde{\gamma}^*(\omega)$ to vanish identically on $\mathbb{C}$, which is a contradiction. $\square$

2.7. An Algebraic Geometric Lemma on Touching Order

LEMMA 2.7.1. *Let $F(x,y)=\sum_{\nu=0}^{m} a_\nu(x)y^\nu$ be an irreducible polynomial in x,y, where the degree of $a_\nu(x)$ in x is no more than q. Let $y_0(x)$ be a polynomial in x such that the vanishing order N of $F(x,y_0(x))$ in x at $x=0$ is greater than $(2m-1)q$. Let e be the vanishing order of $\frac{\partial F}{\partial y}(x,y_0(x))$ in x at $x=0$. Then $e\le(2m-1)q$.*

PROOF. Consider the system of $2m-1$ linear equations

$$\sum_{\nu=0}^{m} a_\nu(x)y_0^{\nu+j}=x^N g(x)y_0^j \qquad (0\le j\le m-2),$$

$$\sum_{\nu=0}^{m-1}(\nu+1)a_{\nu+1}(x)y_0^{\nu+j}=x^e h(x)y_0^j \qquad (0\le j\le m-1).$$

Let $D(x)$ be the resultant of $F(x,y)$ and $\frac{\partial F}{\partial y}$ as polynomials in y. We can solve for the unknowns $1,y(x),\dots,y(x)^{2m-2}$ in the above system of $2m-1$ linear equations and get $D(x)y(x)^k\equiv 0 \bmod x^{\min(N,e)}$ for $0\le k\le 2m-2$. The degree of the $(2m-1)\times(2m-1)$ determinant $D(x)$ in x is at most $(2m-1)q$. Since $D(x)$ is not identically zero due to the irreducibility of $F(x)$, the vanishing order of $D(x)$ in x is at most $(2m-1)q$. Since $D(x)\equiv 0 \bmod x^{\min(N,e)}$, it follows from the case $k=0$ in $D(x)y(x)^k\equiv 0 \bmod x^{\min(N,e)}$ and from $N>(2m-1)q$ that $e\le(2m-1)q$. □

LEMMA 2.7.2. *Let $F(x,y)=\sum_{\nu=0}^{m} a_\nu(x)y^\nu$ be a polyomial in x,y, where the degree of $a_\nu(x)$ in x is no more than q. Let e be the vanishing order of $\Delta(x)=\frac{\partial F}{\partial y}(x,y_0(x))$. Let l be a positive integer $>2e$. Let $y_0(x)$ be a polynomial in x such that*

$$F(x,y_0(x))\equiv 0 \ \mathit{mod}\ x^l.$$

Then there exists a convergent power series $\tilde y(x)$ in x such that $F(x,\tilde y(x))=0$ and $\tilde y(x)\equiv y_0(x)$ mod x^{l-e}. In particular, if $l>2(2m-1)q$ and the polynomial $F(x,y)$ is irreducible, then there exists a convergent power series $\tilde y(x)$ in x such that $F(x,\tilde y(x))=0$ and $\tilde y(x)\equiv y_0(x)$ mod $x^{l-(2m-1)q}$.

PROOF (adapted from the proof of [Artin 1968, Lemma 2.8]). Let $\Delta(x)=\frac{\partial F}{\partial y}(x,y_0(x))$. We now apply Taylor's formula and consider the equation

$$\begin{aligned}0&=F(x,y_0(x)+x^{l-2e}\Delta(x)h(x))\\&=F(x,y_0(x))+\Delta(x)^2x^{l-2e}h(x)+P(x)\Delta(x)^2x^{2(l-2e)}h(x)^2.\end{aligned}$$

It follows from

$$F(x,y_0(x))\equiv 0 \bmod x^l.$$

that $F(x,y_0(x))=x^{l-2e}\Delta(x)^2\psi(x)$ for some convergent power series $\psi(x)$. We have

$$0=x^{l-2e}\Delta(x)^2\psi(x)+x^{l-2e}\Delta(x)^2h(x)+P(x)\Delta(x)^2x^{2(l-2e)}h(x)^2$$

for some polynomial $P(x)$. Division by $x^{l-2e}\Delta(x)^2$ yields

$$0 = \psi(x) + h(x) + P(x)x^{l-2e}h(x)^2.$$

From $l > 2e$ it follows that

$$\frac{\partial}{\partial Y}\left(\psi(x) + Y + P(x)x^{l-2e}Y^2\right) = 1 + 2P(x)x^{l-2e}Y^2 = 1$$

at $x = 0$. The implicit function theorem yields a convergent power series $h(x)$ so that

$$0 = F(x, y_0(x) + x^{l-2e}\Delta(x)h(x))$$

It suffices to set $y(x) = y_0(x) + x^{l-2e}\Delta(x)h(x)$. When $F(x,y)$ is irreducible, it follows from $l > (2m-1)q$ and Lemma 2.7.1 that $e \leq (2m-1)q$. □

For the rest of this paper, for any real number u we use $\lfloor u \rfloor$ to denote the round-down of u, which means the largest integer not exceeding u.

Lemma 2.7.3. *Let $F(x,y) = \sum_{\nu=0}^{m} a_\nu(x)y^\nu$ be a non identically zero polyomial in x, y, where the degree of $a_\nu(x)$ in x is no more than q. Let l be a positive integer $> 2m(2m-1)q$. Let $y_0(x)$ be a polynomial in x such that*

$$F(x, y_0(x)) \equiv 0 \mod x^l.$$

Then there exists a convergent power series $\tilde{y}(x)$ in x such that $F(x, \tilde{y}(x)) = 0$ and $\tilde{y}(x) \equiv y_0(x)$ mod $x^{\lfloor l/m \rfloor - (2m-1)q}$.

Proof. Let

$$F(x,y) = \prod_{\lambda=1}^{\tilde{m}} F_\lambda(x,y)$$

be the decomposition into irreducible factors. Then $1 \leq \tilde{m} \leq m$ and the degree of each $F_\lambda(x,y)$ in x is no more than q and its degree in y is no more than m. It follows from

$$F(x, y_0(x)) \equiv 0 \bmod x^l$$

that there exists some $1 \leq \lambda \leq \tilde{m}$ such that

$$F_\lambda(x, y_0(x)) \equiv 0 \bmod x^{\lfloor l/m \rfloor}.$$

By Lemma 2.7.2 there exists a convergent power series $\tilde{y}(x)$ in x such that $F_\lambda(x, \tilde{y}(x)) = 0$ and $\tilde{y}(x) \equiv y_0(x) \bmod x^{\lfloor l/m \rfloor - (2m-1)q}$. Hence $F(x, \tilde{y}(x)) = 0$ and $\tilde{y}(x) \equiv y_0(x) \bmod x^{\lfloor l/m \rfloor - (2m-1)q}$. □

Lemma 2.7.4. *Let $a_\nu(x)$ be polynomials of degree at most q in x $(0 \leq \nu \leq m)$ not all identically zero. Let N be an integer $> 2m(2m-1)q$. Then in the space of all polynomials $y(x)$ of degree at most N in x the subset defined by*

$$\sum_{\nu=0}^{m} a_\nu(x)y(x)^\nu \equiv 0 \mod x^N$$

is of codimension at least $\lfloor N/m \rfloor - (2m-1)q$.

Proof. Let $F(x,y)=\sum_{\nu=0}^m a_\nu(x)y^\nu$ and let $y_0(x)$ be an arbitrary polynomial of degree at most N which satisfies

$$\sum_{\nu=0}^m a_\nu(x)y_0(x)^\nu \equiv 0 \bmod x^N.$$

By Lemma 2.7.3, there exists a convergent power series $\tilde{y}(x)$ such that

$$\sum_{\nu=0}^m a_\nu(x)\tilde{y}(x)^\nu = 0$$

and

$$\tilde{y}(x) \equiv y_0(x) \bmod x^{\lfloor N/m\rfloor-(2m-1)q}.$$

On the other hand, there are only a finite number of convergent power series $\tilde{y}(x)$ which could satisfy the equation

$$\sum_{\nu=0}^m a_\nu(x)\tilde{y}(x)^\nu = 0.$$

This means that there are only a finite number of possibilities for the first $\lfloor N/m\rfloor-(2m-1)q$ terms of $y_0(x)$ if $y_0(x)$ is an arbitrary polynomial of degree at most N in x satisfying

$$\sum_{\nu=0}^m a_\nu(x)y(x)^\nu \equiv 0 \bmod x^N. \qquad \square$$

Proposition 2.7.5. *Suppose m,q,N,δ are positive integers such that*

$$\binom{\delta+2}{2} > N \geq \frac{3}{2}(2m+q)(m+1)\left((2m-1)(q+m)+\binom{q+2}{2}(m+1)+2\right).$$

Then a generic polynomial $f(x,y)$ of degree δ in x,y cannot be tangential at any point to order at least N to any 1-jet differential h of the form

$$\sum_{\nu=0}^m a_\nu(x,y)(dx)^{m-\nu}(dy)^\nu$$

where $a_\nu(x,y)$ $(0\leq\nu\leq m)$ is a polynomial in x,y of degree at most q with $a_0(x,y),\dots,a_m(x,y)$ not all identically zero. Here tangential to order N at a point P means that the restriction, to the zero-set of $f(x,y)$, of

$$\sum_{\nu=0}^m a_\nu(x,y)(-f_y)^{m-\nu}(f_x)^\nu$$

vanishes to order at least N at P.

Proof. Let Ω be the set of all polynomials $f(x,y)$ of degree δ such that the homogeneous polynomial $z_0^\delta f\left(\frac{z_1}{z_0},\frac{z_1}{z_0}\right)$ in the homogeneous coordinates $[z_0,z_1,z_2]$ defines a nonsingular complex curve in $\mathbb{P}_2$. For any nonnegative integer l, any point $P_0\in\mathbb{C}^2$, and any non identically zero 1-jet differential

$$h:=\sum_{\nu=0}^{m}a_\nu(x,y)(dx)^{m-\nu}(dy)^\nu,$$

we let $\mathcal{A}_{h,l,P_0}$ be the set of all $f\in\Omega$ such that $f(P_0)=0$ and

$$\left(f_y\frac{\partial}{\partial x}-f_x\frac{\partial}{\partial y}\right)^j\left(\sum_{\nu=0}^{m}a_\nu(x,y)(-f_y)^{m-\nu}(f_x)^\nu\right)$$

vanishes at P_0 for all $0\le j<l$. In other words, $\mathcal{A}_{h,l,P_0}$ consists of all $f\in\Omega$ such that h is tangential to the zero-set of f at P_0 to order at least N. The definition of $\mathcal{A}_{h,l,P_0}$ shows how the algebraic set $\mathcal{A}_{h,l,P_0}$ depends algebraically on the coefficients of h and on the coordinates of P_0.

Let $\mathcal{H}_{m,q}$ be the set of all non identically zero polynomials

$$h(x,y,dx,dy)=\sum_{\nu=0}^{m}a_\nu(x,y)(dx)^{m-\nu}(dy)^\nu,$$

in x,y,dx,dy of degree no more than q in x,y and of homogeneous degree no more than m in dx,dy. The complex dimension of $\mathcal{H}_{m,q}$ is $(m+1)\binom{q+2}{2}$. The degree of freedom of the point P_0 is 2 as it varies in $\mathbb{C}^2$. Since the complex dimension of Ω is $\binom{\delta+2}{2}$, to finish the proof of the Proposition it suffices to show that for any fixed $h\in\mathcal{H}_{m,q}$ and $P_0\in\mathbb{C}^2$, the complex codimension of $\mathcal{A}_{h,N,P_0}$ is greater than $2+(m+1)\binom{q+2}{2}$, because then

$$\bigcup\left\{\mathcal{A}_{h,N,P_0}\;\middle|\;h\in\mathcal{H}_{m,q},P_0\in\mathbb{C}^2\right\}$$

is not Zariski dense in Ω. We will prove

$$\operatorname{codim}\mathcal{A}_{h,N,P_0}>2+(m+1)\binom{q+2}{2}$$

at a point $f\in\Omega$ by showing that

$$\operatorname{codim}\mathcal{A}_{h,N,P_0}\cap\mathcal{Z}>k+2+(m+1)\binom{q+2}{2}$$

for some subvariety germ $\mathcal{Z}$ of Ω at the point f defined by k local holomorphic functions on Ω at the point f.

Fix $P_0\in\mathbb{C}^2$. By an affine coordinate change in $\mathbb{C}^2$ we can assume without loss of generality that P_0 is the origin of $\mathbb{C}^2$. For a nonnegative integer l we define $\mathcal{Z}_l$ as the set of all $f\in\Omega$ such that

(1) $f(0,0)=f_x(0,0)=0$, and

(2) the convergent power series $y_f(x)$ defined by $f(x, y_f(x))=0$ satisfies $y_f(x)\equiv 0 \mod x^l$.

For the rest of the proof of this proposition we will use $y_f(x)$ to denote such a convergent power series. The subvariety $\mathcal{Z}_l$ of Ω is locally defined by l functions and its codimension in Ω is l when l does not exceed the dimension of Ω. Let $\kappa = \lfloor 2N/(3(2m+q))\rfloor$. Then

$$\binom{\delta+2}{2} > N \geq (2m+q)\tfrac{3}{2}\kappa$$

and

$$\min\left(\frac{\kappa}{2}, \frac{\kappa}{m} - (2m-1)(q+m)\right) > \binom{q+2}{2}(m+1)+2.$$

The subvariety germ $\mathcal{Z}$ mentioned above will be $\mathcal{Z}_\kappa$ and the number k mentioned above will be κ.

Fix an element

$$h(x,y,dx,dy) = \sum_{\substack{0\leq\lambda,\mu\leq q\\ 0\leq\nu\leq m}} c_{\lambda\mu\nu} x^\lambda y^\mu (dx)^{m-\nu}(dy)^\nu$$

of $\mathcal{H}_{m,q}$. Choose (μ_0,ν_0) so that $\mu_0+\nu_0$ is the minimum among all $\mu+\nu$ with $c_{\lambda\mu\nu}\neq 0$ for some λ. Let

$$P_\nu(x) = \sum_{\lambda=0}^{q} \sum_{\mu_0+\nu_0=\mu+\nu} c_{\lambda\mu\nu} x^{m-\nu+\lambda}.$$

When $P_\nu(x)$ is not identically zero for some $\nu>0$, we let $G_1(x),\dots,G_{\tilde m}(x)$ be the set of all convergent power series such that

$$\sum_{\nu=0}^{m} P_\nu(x) G_j(x)^\nu = 0.$$

We know that $\tilde m \leq m$. For a given nonnegative integer l we let $\mathcal{W}_l$ be the set of all $f\in\mathcal{Z}_0$ such that

(1) $y_f(x)\equiv 0 \mod x^l$, and
(2) when we write $y_f(x) = x^l \tilde y_f(x)$, we have

$$\tilde y_f(x) = \tilde y_f(0)\exp\left(\int_{\xi=0}^{x} \frac{G_j(\xi)-G_j(0)}{\xi} d\xi\right) \mod x^{\lfloor l/m\rfloor-(2m-1)(q+m)}$$

for some $1\leq j\leq \tilde m$.

The codimension of $\mathcal{W}_l$ in Ω is at least $l+\lfloor l/m\rfloor-(2m-1)(q+m)$ if the dimension of Ω is at least $l+\lfloor l/m\rfloor-(2m-1)(q+m)$, because each choice of the $\tilde m$ set of conditions means $\lfloor l/m\rfloor-(2m-1)(q+m)$ independent conditions on the coefficients of $\tilde y_f(x)$, which translates to $l+\lfloor l/m\rfloor-(2m-1)(q+m)$ independent conditions on the coefficients of $y_f(x) = x^l\tilde y_f(x)$. When $P_\nu(x)$ is identically zero for all $\nu>0$, we do not define $\mathcal{W}_l$.

CLAIM 2.7.5.1. *If $\mathcal{A}_{h,N,P_0} \cap \mathcal{Z}_\kappa$ is not contained in $\mathcal{Z}_{\lfloor 3\kappa/2 \rfloor}$, then $P_\nu(x)$ is not identically zero for some $\nu > 0$ and*

$$\mathcal{A}_{h,N,P_0} \cap \mathcal{Z}_\kappa \subset \mathcal{Z}_{\lfloor 3\kappa/2 \rfloor} \cup \left(\bigcup_{l=\kappa}^{\lfloor 3\kappa/2 \rfloor} \mathcal{W}_l \right).$$

PROOF. Take $f \in \mathcal{A}_{h,N,P_0} \cap \mathcal{Z}_\kappa$ such that f does not belong to $\mathcal{Z}_{\lfloor 3\kappa/2 \rfloor}$. Let l be the vanishing order at $x = 0$ of the convergent power series $y_f(x)$. Then $l < \frac{3}{2}\kappa$. Write $y_f(x) = x^l \tilde{y}_f(x)$. Then

$$\frac{xy_f'}{y_f} = l + \frac{x\tilde{y}_f'}{\tilde{y}_f}$$

which is equal to l at $x = 0$. We have

$$\begin{aligned} \sum_{\substack{0 \le \lambda, \mu \le q \\ 0 \le \nu \le m}} c_{\lambda\mu\nu} x^{m-\nu+\lambda} y_f^{\mu+\nu} \left(\frac{xy_f'}{y_f} \right)^\nu &= \sum_{\substack{0 \le \lambda, \mu \le q \\ 0 \le \nu \le m}} c_{\lambda\mu\nu} x^{m-\nu+\lambda} y_f^{\mu} (xy_f')^\nu \\ &= x^m \sum_{\substack{0 \le \lambda, \mu \le q \\ 0 \le \nu \le m}} c_{\lambda\mu\nu} x^{\lambda} y_f^{\mu} (y_f')^\nu \equiv 0 \mod x^{m+N} \end{aligned}$$

(which is from the definition of $\mathcal{A}_{h,N,P_0}$). It follows from $l < \frac{3}{2}\kappa$ that $N > (2m+q)l$. Since $\mu_0 \le q$ and $\nu_0 \le m$, we have $N > (\mu_0 + \nu_0 + 1)l$. Hence

$$\sum_{\substack{0 \le \lambda, \mu \le q \\ 0 \le \nu \le m}} c_{\lambda\mu\nu} x^{m-\nu+\lambda} y_f^{\mu+\nu} \left(\frac{xy_f'}{y_f} \right)^\nu \equiv 0 \mod x^{(\mu_0+\nu_0+1)l}.$$

Since $c_{\lambda\mu\nu} = 0$ for $\mu + \nu < \mu_0 + \nu_0$, it follows that we can divide the above congruence relation by $x^{(\mu_0+\nu_0)l}$ and get

$$\sum_{\substack{0 \le \lambda \le q \\ \mu+\nu=\mu_0+\nu_0}} c_{\lambda\mu\nu} x^{m-\nu+\lambda} \left(\frac{xy_f'}{y_f} \right)^\nu \equiv 0 \mod x^l.$$

We cannot have $c_{\lambda\mu\nu} = 0$ zero for all $\mu + \nu = \mu_0 + \nu_0$ and $\nu > 0$, otherwise

$$\sum_{\substack{0 \le \lambda \le q \\ \mu+\nu=\mu_0+\nu_0}} c_{\lambda\mu\nu} x^{m-\nu+\lambda} \equiv 0 \mod x^l,$$

contradicting $l \ge \kappa > m + q$ and $c_{\lambda\mu\nu} \ne 0$ for some $\mu + \nu = \mu_0 + \nu_0$. Thus

$$\sum_{\nu=0}^{m} P_\nu(x) \left(\frac{xy_f'}{y_f} \right)^\nu \equiv 0 \mod x^l$$

with $P_\nu(x)$ not identically zero for some $\nu > 0$. By Lemma 2.7.3 we know that

$$\frac{xy_f'}{y_f} \equiv G_j(x) \mod x^{\lfloor l/m \rfloor - (2m-1)(m+q)}$$

for some $1 \leq j \leq \tilde{m}$. It follows that

$$\tilde{y}_f(x) = \tilde{y}_f(0) \exp\left(\int_{\xi=0}^{x} \frac{G_j(\xi) - l}{\xi} d\xi\right) \mod x^{\lfloor l/m \rfloor - (2m-1)(q+m)}.$$

Thus $f \in \mathcal{W}_l$ and Claim (2.7.5.1) is proved. □

The codimension of $\mathcal{W}_l$ in Ω is at least $\kappa + \lfloor \kappa/m \rfloor - (2m-1)(q+m)$ and the codimension of $\mathcal{Z}_{\lfloor 3\kappa/2 \rfloor}$ in Ω is $\lfloor 3\kappa/2 \rfloor$. Hence the codimension of $\mathcal{A}_{h,N,P_0} \cap \mathcal{Z}_\kappa$ in Ω is at least

$$\min\left(\kappa + \lfloor \kappa/m \rfloor - (2m-1)(q+m), \lfloor 3\kappa/2 \rfloor\right).$$

Since $\mathcal{Z}_\kappa$ is locally defined by κ holomorphic functions, it follows that the codimension of $\mathcal{A}_{h,N,P_0}$ in Ω is at least

$$\min\left(\lfloor \kappa/m \rfloor - (2m-1)(q+m), \lfloor \kappa/2 \rfloor\right) > 2 + (m+1)\binom{q+2}{2}.$$

This concludes the proof of Proposition 2.7.5. □

2.8. A Schwarz Lemma Using Low Touching Order. We now resume our argument of the hyperbolicity of the complement of a generic plane curve of sufficiently high degree. We can assume that we have an irreducible meromorphic 1-jet differential $h(x, y, dx, dy)$ whose pullback by the entire holomorphic curve is identically zero. Moreover, the degree of $h(x, y, dx, dy)$ in x, y is q and the homogeneous degree of $h(x, y, dx, dy)$ in dx, dy is m with $q \leq 4m$. We consider the resultant $R(x, y)$ of

$$\frac{h(x, y, dx, dy)}{dx^m} = \sum_{\nu=0}^{m} h_\nu(x, y) \left(\frac{dy}{dx}\right)^\nu$$

and its derivative with respect to $\frac{dy}{dx}$

$$\sum_{\nu=0}^{m-1} (\nu+1) h_{\nu+1}(x, y) \left(\frac{dy}{dx}\right)^\nu$$

as polynomials in $\frac{dy}{dx}$. Since $h(x, y)$ is irreducible, the resultant $R(x, y)$ is not identically zero and its degree is no more than $(2m-1)q$. Let Z be the common zero-set of $R(x, y)$ and $f(x, y)$. The number of points in Z is no more than $(2m-1)q\delta$. When a point of $\mathbb{P}_2$ is not a zero of $R(x, y)$ we can have a finite number of families of local integral curves going through that point and the entire holomorphic curve is locally contained in such a local integral curve.

Let N be the smallest integer satisfying

$$N \geq \frac{3}{2}(2m+q)(m+1)\left((2m-1)(q+m) + \binom{q+2}{2}(m+1) + 2\right).$$

Assume that $\binom{\delta+2}{2} > N$. Then by Proposition 2.7.5 for our generic f, the touching order of f with $h(x,y,dx,dy)$ is no more than N. Let $S(x,y)$ be a non identically zero polynomial of degree r with

$$\binom{r+2}{2} > (2m-1)q\delta\binom{N+2}{2}$$

such that it vanishes to order at least N at each point of the common zero-set Z of $R(x,y)$ and $f(x,y)$. Let $e^{-\psi_0}$ be a smooth metric for the hyperplane section line bundle $H_{\mathbb{P}_2}$ of $\mathbb{P}_2$ with strictly positive curvature. Let A be a positive number and let locally $\psi = \psi_0 + A$ so that $e^{-\psi}$ is a metric for $H_{\mathbb{P}_2}$. We will later choose A to be sufficiently large for our purpose. Let $\theta_\psi = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$ be the curvature form of the metric $e^{-\psi}$.

For a holomorphic section u of a line bundle with a metric, we use $\|u\|$ to denote its pointwise norm with respect to the metric and we use $|u|$ to denote the absolute value of a function which represents u in a local trivialization of the line bundle. The pointwise norm $\|u\|$ is used to give a globally well defined expression. In proving results involving estimates of the norm, we will use local trivialization of the line bundle and it does not matter which local trivialization of the line bundle is used.

Consider f as a section of the δ-th power of the hyperplane section line bundle so that the pointwise norm of f is given by $\|f\|^2 = |f|^2 e^{-\delta\psi}$. We assume that A is chosen so large that $\|f\| < 1$ on all of $\mathbb{P}_2$. Let (x_j, y_j) $(1 \leq j \leq J)$ be a finite number of affine coordinates of affine open subsets of $\mathbb{P}_2$ so that $dx_1, dy_1, \ldots, dx_J, dy_J$ generate at every point of $\mathbb{P}_2$ the cotangent bundle of $\mathbb{P}_2$ tensored by $2H_{\mathbb{P}_2}$. Let $\{\eta_j\}_j$ denote the set $\{dx_1, dy_1, \ldots, dx_J, dy_J\}$. We use $\|\eta_j\|^2$ to denote $|\eta_j|^2 e^{-2\psi}$, which is a function on the tangent bundle of $\mathbb{P}_2$. Let

$$\left\|f^{\frac{N-1}{N}}S\right\|^2 = \left|f^{\frac{N-1}{N}}S\right|^2 e^{-\left(\frac{(N-1)\delta}{N}+r\right)\psi},$$

which can be geometrically interpreted as the N-th root of the pointwise square norm of the section of

$$N\left(\frac{(N-1)\delta}{N} + r\right)H_{\mathbb{P}_2}$$

over $\mathbb{P}_2$ defined by $\left(f^{\frac{N-1}{N}}S\right)^N$.

Proposition 2.8.1. *Assume $\delta > (r+2)N$. Let*

$$\Psi = \frac{\left\|f^{\frac{N-1}{N}}S\right\|^2 \sum_j \|\eta_j\|^2}{\|f\|^2\left(\log\frac{1}{\|f\|^2}\right)^2}.$$

Then, when A is sufficiently large, there exists a positive constant ε such that the pullback of

$$\sqrt{-1}\,\partial\bar{\partial}\log\Psi \geq \varepsilon\Psi$$

to any local holomorphic curve Γ in $\mathbb{P}_2 - \{f=0\}$ holds if Γ satisfies $h=0$.

PROOF. From standard direct computation we have the following Poincaré–Lelong formula on $\mathbb{P}_2$ in the sense of currents.

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \frac{\left\|f^{\frac{N-1}{N}}S\right\|^2 \sum_j \|\eta_j\|^2}{\|f\|^2\left(\log \frac{1}{\|f\|^2}\right)^2} = \left(\frac{\delta}{N} - (r+2) - \frac{2\delta}{\log \frac{1}{\|f\|^2}}\right)\theta_\psi - \frac{1}{N}Z_f$$
$$+Z_S + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \sum_j |\eta_j|^2 + \frac{2}{\|f\|^2\left(\log\frac{1}{\|f\|^2}\right)^2}\frac{\sqrt{-1}}{2\pi} Df \wedge \overline{Df}.$$

Here Z_f (respectively Z_S) is the $(1,1)$-current defined by the zero-set of f (respectively S) and Df is the smooth $\delta H_{\mathbb{P}_2}$-valued 1-form on $\mathbb{P}_2$ which is the covariant differentiation of the section of $\delta H_{\mathbb{P}_2}$ defined by f with respect to the metric $e^{-\psi}$ of $H_{\mathbb{P}_2}$.

Since we could change affine coordinates, we need only verify the inequality on any compact subset of the affine plane $\mathbb{C}^2$ with affine coordinates x, y. Fix a point P in the common zero-set Z of $R(x,y)$ and $f(x,y)$ and take a compact neighborhood U_P of P in $\mathbb{C}^2$ disjoint from $Z - \{P\}$. We are going to derive an inequality on U_P (which we may have to shrink to get the inequaltiy). Without loss of generality we can assume that $f_x \neq 0$ on U_P (after shrinking U_P and making an affine coordinate transformation if necessary). We write

$$h = \sum_{\nu=0}^{m} \hat h_\nu (df)^{m-\nu}(dy)^\nu$$

with

$$\hat h_m = \sum_{\nu=0}^{m} h_\nu \left(\frac{-f_y}{f_x}\right)^{m-\nu}.$$

We use the following two trivial inequalities for positive numbers a, b and α, β with $\alpha+\beta=1$.

$$a^\alpha b^\beta \leq \alpha a + \beta b,$$
$$a^m + b^m \leq (a+b)^m \leq \left(2\max(a,b)\right)^m \leq 2^m(a^m+b^m).$$

We use C_j to denote positive constants. We consider separately the case of $m>1$ and the case of $m=1$. We first look at the case of $m>1$. For a nonnegative bounded continuous function ρ we have

$$\rho|\hat h_m (dy)^m|^2 \leq C_1\left(\rho|h|^2 + \sum_{\nu=0}^{m-1}\left(|df|^{\frac{2(m-\nu)}{m}}\left(\rho^{\frac{1}{\nu}}|dy|^2\right)^{\frac{\nu}{m}}\right)^m |\hat h_\nu|^2\right)$$
$$\leq C_2\left(\rho|h|^2 + |df|^{2m} + \rho^{\frac{m}{m-1}}|dy|^{2m}\right).$$

Hence

$$\rho^{\frac{1}{m}}|\hat h_m|^{\frac{2}{m}}|dy|^2 \leq C_2^{\frac{1}{m}}\left(\rho^{\frac{1}{m}}|h|^{\frac{2}{m}} + |df|^2 + \rho^{\frac{1}{m-1}}|dy|^2\right),$$
$$\rho^{\frac{1}{m}}\left(|f|^2 + |\hat h_m|^{\frac{2}{m}}\right)|dy|^2 \leq C_3\left(\rho^{\frac{1}{m}}|h|^{\frac{2}{m}} + |f|^2|dy|^2 + |df|^2 + \rho^{\frac{1}{m-1}}|dy|^2\right).$$

For $m > 1$ we set $\rho = \big(2C_3\big)^{-m(m-1)}\big(|f|^2 + |\hat{h}_m|^{\frac{2}{m}}\big)^{m(m-1)}$. Then

$$C_3\rho^{\frac{1}{m-1}}|dy|^2 = \tfrac{1}{2}\rho^{\frac{1}{m}}\big(|f|^2 + |\hat{h}_m|^{\frac{2}{m}}\big)|dy|^2$$

and

$$\big(|f|^2 + |\hat{h}_m|^2\big)|dy|^2 \le C_4\big(|h|^{\frac{2}{m}} + |f|^2|dy|^2 + |df|^2\big).$$

For $m = 1$ the inequality is obviously true. Since the vanishing order of $\hat{h}_m$ on $\{f = 0\}$ at P is at most N and the vanishing order of $S(x, y)$ at P is at least N, it follows that on U_P (after shrinking U_P if necessary)

$$|S|^2|dy|^2 \le C_5\big(|f|^2 + |\hat{h}_m|^2\big)|dy|^2 \le C_6\big(|h|^{\frac{2}{m}} + |f|^2|dy|^2 + |df|^2\big).$$

Using the inequalities

$$|df|^2 \le C_7\big(|f|^2 + |Df|^2\big)$$

and

$$\frac{|f|^2|dy|^2}{\|f\|^2\big(\log\frac{1}{\|f\|^2}\big)^2} \le \varepsilon_0\theta_\psi$$

for any positive number ε_0 when A is sufficiently large, we conclude from the Poincaré–Lelong formula that

$$\frac{|S|^2|dy|^2}{\|f\|^2\big(\log\frac{1}{\|f\|^2}\big)^2} \le \varepsilon_0\theta_\psi + C_8\frac{|Df|^2}{\|f\|^2\big(\log\frac{1}{\|f\|^2}\big)^2} \le C_9\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\frac{\big\|f^{\frac{N-1}{N}}S\big\|^2\sum_j|\eta_j|^2}{\|f\|^2\big(\log\frac{1}{\|f\|^2}\big)^2}$$

when pulled back to any local holomorphic curve in U_P which is disjoint from the zero-set of f and which satisfies $h = 0$. We repeat the same argument for a finite number of other affine coordinates instead of (x, y) and sum up to get the inequality we want to prove on local holomorphic curves in U_P which are disjoint from the zero-set of f and which satisfies $h = 0$.

We can find an open neighborhood W of the zero-set of f so that $W - \bigcup_{P\in Z} U_P$ is disjoint from the zero-set of R. At every point Q of W where R is not zero, we can find an open neighborhood Ω_Q of Q in W so that the equation $h = 0$ gives rise to a finite number of families of integral curves. The vanishing order of f on each such integral curve Γ is at most N. With respect to a local holomorphic coordinate ζ, the function $f(\zeta) = \zeta^l g(\zeta)$ with $g(0) \neq 0$ for some $l \le N$. Since $\frac{\delta}{N} > r + 2$, by choosing A sufficiently large we have

$$\frac{\delta}{N} > r + 2 + \frac{2\delta}{\log\frac{1}{\|f\|^2}}.$$

Hence when pulled back to Γ, at points not on the zero-set of f we have

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log\frac{\big\|f^{\frac{N-1}{N}}\big\|^2\sum_j\|\eta_j\|^2}{\|f\|^2\big(\log\frac{1}{\|f\|^2}\big)^2} &\ge C_{10}\frac{|df|^2}{\|f\|^2\big(\log\frac{1}{\|f\|^2}\big)^2}\\ &\ge C_{11}\frac{|d\zeta|^2}{\|\zeta|^2\big(\log\frac{1}{\|f\|^2}\big)^2} \ge C_{12}\frac{\big\|f^{\frac{N-1}{N}}\big\|^2\sum_j\|\eta_j\|^2}{\|f\|^2\big(\log\frac{1}{\|f\|^2}\big)^2}.\end{aligned}$$

After shrinking W if necessary, the positive constant C_{12} can be made independent of the integral curve Γ of $h = 0$ as long as it is inside W. This gives us on $W - \bigcup_{P \in Z} U_P$ the inequality stated in the Proposition. On $\mathbb{P}_2 - W$ the inequality stated in the Proposition is clear, because there

$$\frac{\left\| f^{\frac{N-1}{N}} \right\|^2 \sum_j \|\eta_j\|^2}{\|f\|^2 \left(\log \frac{1}{\|f\|^2}\right)^2} \leq C_{13} \theta_\psi$$

and the Poincaré–Lelong formula gives

$$\theta_\psi \leq C_{14} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \frac{\left\| f^{\frac{N-1}{N}} \right\|^2 \sum_j \|\eta_j\|^2}{\|f\|^2 \left(\log \frac{1}{\|f\|^2}\right)^2}$$

when pulled back to local holomorphic curves in $\mathbb{P}_2 - \{f = 0\}$ which satisfy $h = 0$. □

COROLLARY 2.8.2. *If $\delta > (r+2)N$, then there is no entire holomorphic curve in $\mathbb{P}_2$ which is disjoint from the curve in $\mathbb{P}_2$ defined by $f = 0$ for a generic f.*

PROOF. The inequality

$$\sqrt{-1}\, \partial\bar{\partial} \log \Psi \geq \varepsilon \Psi$$

from Proposition 2.8.1 implies that the pullback of Ψ to any such entire holomorphic curve must be identically zero. This means that the entire holomorphic curve must be contained in the zero-set of S, which is not possible for a generic f. □

2.9. The Final Step. We now combine all the preceding steps together and formulate our theorem.

THEOREM 2.9.1. *Let δ, p, m, N, r be positive integers and s a nonnegative integer, and set $\tilde{m} = (s+3m)(2m-1)$. Assume that the following inequalities are satisfied*:

(*a*) $\sum_{k=0}^{m} \frac{1}{4}(s+3k+2)(s+3k+1)(p+2)(p+1) > (s+3m+1)$
$\times((\delta-1)(p+(s+3m)\delta) - \frac{1}{2}(\delta^2 - 5\delta + 4))$.

(b) $p \leq \delta - 2 - 4(s+3m)$.

(c) $\binom{\delta+2}{2} > 3 + 4\left(\binom{p+s+4}{2} + \binom{p+s+2}{2} + \binom{p+2s+5}{2}\right)$.

(d) $N \geq \frac{3}{2}(6\tilde{m}+1)(\tilde{m}+1)\left((2\tilde{m}-1)(5\tilde{m}+1) + \binom{4\tilde{m}+3}{2}(\tilde{m}+1) + 2\right)$.

(e) $\binom{r+2}{2} > (2\tilde{m}-1)(4\tilde{m}+1)\delta\binom{N+2}{2}$.

(f) $\delta > (r+2)N$.

Let $f(x,y)$ be any generic polynomial of degree δ in x, y and C be the complex curve in $\mathbb{P}_2$ defined by $f = 0$. Then $\mathbb{P}_2 - C$ is hyperbolic in the sense that there is no nonconstant holomorphic map from $\mathbb{C}$ to $\mathbb{P}_2 - C$.

Proof. Because of the inequality

$$\sum_{k=0}^{m} \tfrac{1}{4}(s+3k+2)(s+3k+1)(p+2)(p+1) \\ > (s+3m+1)((\delta-1)(p+(s+3m)\delta) - \tfrac{1}{2}(\delta^2-5\delta+4)),$$

by Lemma 2.2.1 we can construct a 2-jet differential Φ which is divisible by f_y and which is of the form

$$\sum_{k=0}^{m} \omega_{s+3k} f^{2(m-k)} (d^2 f\, dx - d^2 x\, df)^{m-k},$$

where

$$\omega_\mu = \sum_{\nu_0+\nu_1+\nu_2=\mu} a_{\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2}$$

and $a_{\nu_0\nu_1\nu_2}(x,y)$ is a polynomial in x and y of degree $\le p$. By Proposition 2.3.1 and the paragraphs before Lemma 2.4.1, we can factor Φ and get Φ_1 which is divisible by f_y and which is of the form

$$\Phi_1 \sum_{k=0}^{m_1} \omega^{(1)}_{s_1+3k} f^{2(m_1-k)} (d^2 f\, dx - d^2 x\, df)^{m_1-k}$$

which is irreducible as a polynomial in $dx, dy, d^2x\, dy - dx\, d^2y$, where

$$\omega^{(1)}_\mu = \sum_{\nu_0+\nu_1+\nu_2=\mu} a^{(1)}_{\nu_0\nu_1\nu_2}(x,y)(df)^{\nu_0}(f\,dx)^{\nu_1}(f\,dy)^{\nu_2}$$

and $a^{(1)}_{\nu_0\nu_1\nu_2}(x,y)$ is a polynomial in x and y of degree $\le p+3m_1+s_1$. We know that $s_1+3m_1 \le s+3m$.

By 2.2.4 it follows from the inequality

$$p+3m_1+s_1 \le \delta - 2 - 3(s_1+3m_1)$$

that $t^{-(s_1+3m_1)(\delta-1)} f_h^{-1} \Phi_1$ defines a holomorphic 2-jet differential on X which vanishes on an ample divisor. Thus the pullback of Φ_1 to the entire holomorphic curve in $\mathbb{P}_2 - C$ is identically zero. To emphasize the dependence of Φ_1 on f we denote Φ_1 also by $\Phi_{1,f}$. By Section 2.4 we know that $m_1 > 1$. By Section 2.5 we can choose an element $\gamma \in SL(2,\mathbb{C})$ such that

$$\tilde{\Phi}_1 := \gamma^*\big(\Phi_{1,(\gamma^{-1})^* f}\big)$$

and Φ_1 are independent in the sense that the resultant $h(x,y,dx,dy)$ of Φ_1 and $\tilde{\Phi}_1$ as polynomials in the variable $d^2x\, dy - dx\, d^2y$ is not identically zero. Since $t^{-(s_1+3m_1)(\delta-1)} f_h^{-1} \tilde{\Phi}_1$ also defines a holomorphic 2-jet differential on X which vanishes on an ample divisor, the pullback of $\tilde{\Phi}_1$ to the entire holomorphic curve in $\mathbb{P}_2 - C$ is also identically zero. It follows that the pullback of h to the entire holomorphic curve in $\mathbb{P}_2 - C$ is again identically zero. We factor the polynomial $h(x,y,dx,dy)$ into irreducible factors. Then one of the factors $h_1(x,y,dx,dy)$ satisfies the property that its pullback to the entire holomorphic curve in $\mathbb{P}_2 - C$ is identically zero. Since the homogeneous degree of $h(x, y, dx, dy)$ in the variables dx, dy is at most $(s_1 + 3m_1)(2m_1 - 1)$, the homogeneous degree of $h_1(x, y, dx, dy)$ in the variables dx, dy is at most $(s_1 + 3m_1)(2m_1 - 1)$ which is no more than $(s + 3m)(2m - 1)$ which is $\tilde{m}$. Let q be the degree of $h_1(x, y, dx, dy)$ in x, y. If $q \geq 4\tilde{m}$, then by § 6 we know that the entire holomorphic curve in $\mathbb{P}_2 - C$ must be contained in an algebraic curve in $\mathbb{P}_2$. This means that for a generic C there is no entire holomorphic curve in $\mathbb{P}_2 - C$. So we now assume that $q < 4\tilde{m}$. By Proposition 2.7.5 and Corollary 2.8.2 we know that there cannot be any entire holomorphic curve in $\mathbb{P}_2 - C$. □

2.9.1. Example of the Degree and a Set of Parameters. We could choose $s = 0$ and $m = 145$. Then $\tilde{m} = 3m(2m - 1) = 125715$ and we choose N to be the smallest integer satisfying

$$N \geq \frac{3}{2}(6\tilde{m} + 1)(\tilde{m} + 1)\left((2\tilde{m} - 1)(5\tilde{m} + 1) + \binom{4\tilde{m} + 3}{2}(\tilde{m} + 1) + 2\right).$$

and choose r to be the smallest integer satisfying

$$r \geq (2\tilde{m} - 1)(4\tilde{m} + 1)(N + 2)(N + 1)$$

and finally choose δ as the smallest integer satisfying

$$\delta > ((2\tilde{m} - 1)(4\tilde{m} + 1)(N + 2)(N + 1) + 3)\, N.$$

The number p is set to be the largest integer not exceeding $\frac{12}{145}\delta$. Such values of $s, p, m, \tilde{m}, N, r, \delta$ satisfy all the inequalities in the statement of Theorem 2.9.1. Note that the dominant term in

$$\sum_{k=0}^{m} \frac{1}{4}(s + 3k + 2)(s + 3k + 1)(p + 2)(p + 1)$$

is $\frac{3}{4}m^3p^2$ and the dominant term in $(s+3m+1)((\delta - 1)(p + (s+3m)\delta) - \frac{1}{2}(\delta^2 - 5\delta + 4))$ is $9m^2\delta^2$. To make sure that the condition

$$\binom{\delta + 2}{2} > 3 + 4\left(\binom{p + s + 4}{2} + \binom{p + s + 2}{2} + \binom{p + 2s + 5}{2}\right)$$

is satisfied for sufficiently large δ we have to require that $\delta^2 > 12p^2$. Hence the smallest m one should use to get a sufficiently large δ to satisfy the inequality

$$\sum_{k=0}^{m} \tfrac{1}{4}(s + 3k + 2)(s + 3k + 1)(p + 2)(p + 1) > (s + 3m + 1)((\delta - 1)(p + (s + 3m)\delta) - \tfrac{1}{2}(\delta^2 - 5\delta + 4)),$$

is $m = 145$.

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Yum-Tong Siu
Department of Mathematics
Harvard University
Cambridge, MA 02138
United States
siu@math.harvard.edu