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Rigidity Theorems in Kähler Geometry and Fundamental Groups of Varieties

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Abstract. We review some developments in rigidity theory of compact Kähler manifolds and related developments on restrictions on their possible fundamental groups.

1. Introduction

This article surveys some developments, which started almost twenty years ago, on the applications of harmonic mappings to the study of topology and geometry of Kähler manifolds. The starting point of these developments was the strong rigidity theorem of Siu [1980], which is a generalization of a special case of the strong rigidity theorem of Mostow [1973] for locally symmetric manifolds.

Siu's theorem introduced for the first time an effective way of using, in a broad way, the theory of harmonic mappings to study mappings between manifolds. Many interesting applications of harmonic mappings to the study of mappings of Kähler manifolds to nonpositively curved spaces have been developed since then by various authors. More generally the linear representations (and other representations) of their fundamental groups have also been studied. Our purpose here is to give a general survey of this work.

One interesting by-product of this study is that it has produced new results on an old an challenging question: what groups can be fundamental groups of smooth projective varieties (or of compact Kähler manifolds)? These groups are called *Kähler groups* for short, and have been intensively studied in the last decade. New restrictions on Kähler groups have been obtained by these techniques. On the other hand new examples of Kähler groups have also shown the limitations of some of these methods. We do not discuss these developments in much detail because we have nothing to add to the recent book [Amorós et al. 1996] on this subject.

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Even though the motivation for much of what we cover here came from the general rigidity theory for lattices in Lie groups, we do not attempt to review this important subject. We begin our survey with the statement of Mostow's strong rigidity theorem for hyperbolic space forms, and refer the reader to [Pansu 1995] and the references therein for more information on both the history and the present state of rigidity theory. We also refer the reader to [Amorós et al. 1996; Arapura 1995; Corlette 1995; Katzarkov 1997; Kollár 1995; Simpson 1997] for surveys that have some overlap and give more information on some of the subjects specifically covered here.

2. The Theorems of Mostow and Siu

We begin by recalling the first strong rigidity theorem of all, Mostow's strong rigidity theorem for hyperbolic space forms. We have slightly restated the original formulation found in [Mostow 1968].

THEOREM 2.1. *Let M and N be compact manifolds of constant negative curvature and dimension at least three*, *and let $f : M \to N$ be a homotopy equivalence*. *Then f is homotopic to an isometry*.

This theorem says in particular that there are no continuous deformations of metrics of constant negative curvature in dimensions greater than two, in sharp contrast to the situation for Riemann surfaces, where there are deformations. All proofs of this theorem seem to involve the study of an extension to the boundary of hyperbolic n-space of the lift of f to the universal cover of M. Besides the original proof in [Mostow 1968] we mention the proof by Gromov and Thurston (explained in [Thurston 1978] in dimension 3 and now known to be valid in all dimensions). They prove actually more: if in the statement of Theorem 2.1 we assume that f is a map of degree equal to the ratio of the hyperbolic volumes, then f is homotopic to a covering isometry. This stronger statement is also proved in [Besson et al. 1995].

For the purposes of this survey, we note that a natural way to attempt to prove Theorem 2.1 would be the following. First, the basic existence theorem of Eells and Sampson [1964] implies that f is homotopic to a harmonic map (unique in this case because of the strict negativity of the curvature [Hartman 1967]). We can thus assume that the homotopy equivalence f is harmonic, and it is natural to expect that one could prove directly that f is an isometry, thus establishing Mostow's theorem 2.1.

It is very curious to note that this has not been done, and in some sense is one of the outstanding problems in the theory of harmonic maps. All the developments in harmonic map theory that we mention in this article, by the very nature of the methods employed, must leave this case untouched. Of course one knows *a fortiori*, from Mostow's theorem and the uniqueness of harmonic maps that f is an isometry. But it does not seem to be known even how to prove that

f is a diffeomorphism without appealing to Mostow's theorem. In this context it should be noted than in dimension two, where the rigidity theorem 2.1 fails, it is known that a harmonic homotopy equivalence between compact surfaces of constant negative curvature is a diffeomorphism [Sampson 1978; Schoen and Yau 1978].

It is the author's impression that during the 1960's and 1970's several mathematicians attempted to prove Theorem 2.1 by showing that the harmonic map is an isometry. The failure of all these attempts was taken at that time as an indication of the limited applicability of the theory of harmonic maps.

In the early 1970's Mostow proceeded to prove his general rigidity theorem [Mostow 1973], namely the same as Theorem 2.1 with M and N now irreducible compact locally symmetric manifolds, the statement otherwise unchanged. Since what was thought to be the simplest case, namely that of constant curvature manifolds, was not accessible by harmonic maps, no one expected the more general case to be approachable this method. It was thus surprising when Siu [1980] was able to prove, by harmonic maps, the following strengthening of Mostow's rigidity theorem for Hermitian symmetric manifolds:

THEOREM 2.2 (SIU'S RIGIDITY THEOREM). *Let M and N be compact Kähler manifolds. Assume that the universal cover of N is an irreducible bounded symmetric domain other that the unit disc in $\mathbb{C}$. Let $f : M \to N$ be a homotopy equivalence. Then f is homotopic to a holomorphic or anti-holomorphic map.*

This strengthens Mostow's rigidity theorem because only one of the two manifolds is assumed to be locally symmetric. The conclusion may seem weaker (biholomorphic map rather than isometric), but recall that if M is also locally symmetric, that is, its universal cover is a bounded symmetric domain, then f is indeed homotopic to an isometry because biholomorphic maps of bounded domains are isometric for their Bergmann metrics.

Siu proves his theorem by showing that the harmonic map homotopic to f is holomorphic or antiholomorphic. We explain the details, and some extensions, in the next two sections.

We close this section with the remark that this theorem was one of the first two substantial applications of harmonic maps to geometry. The other application, appearing about the same time, was the solution by Siu and Yau [1980] of Frankel's conjecture: a compact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to complex projective space. This theorem was somewhat overshadowed by Mori's proof [1979], at about the same time, of the more general Hartshorne conjecture in algebraic geometry: a smooth projective variety with ample tangent bundle is biholomorphic to complex projective space. This is another type of rigidity property, in the context of positive curvature rather than negative curvature. It concerns the rigidity properties of Hermitian symmetric spaces of compact, rather than noncompact type. We do not cover this interesting line of development here, but refer the reader to [Hwang and Mok 1998; 1999; Mok 1988; Siu 1989; Tsai 1993] and the references in these papers for more information.

It is worth noting that both the theorem of Siu and Yau and the theorem of Mori are based on producing suitable rational curves. Siu and Yau use harmonic two-spheres, Mori uses the action of Frobenius in positive characteristic to produce the rational curves. It has been remarked to the author by M. Gromov the philosophical similarity between elliptic theory and the action of Frobenius, and the fact that the latter should also be used to study rigidity problems in non-positively curved situations. This author would not be surprised to find that the solution to some of the open problems mentioned in this article will eventually depend on ideas from algebraic geometry in positive characteristic.

3. Harmonic Maps are Pluriharmonic

We explain briefly the proof of Siu's rigidity theorem and some of its extensions, following the exposition in [Amorós et al. 1996; Carlson and Toledo 1989]. Recall that a map $f : M \to N$ between Riemannian manifolds is called *harmonic* if it is an extremal for the energy functional

$$E(f) = \int_M \|df\|^2 \, dV,$$

where dV is the Riemannian volume element of M. Being an extremal is equivalent to the Euler–Lagrange equation

$$\Delta f := *d_\nabla * df = 0, \tag{3–1}$$

where the symbols have the following meaning. We write $A^k(M, f^*TN)$ to denote the space of smooth k-forms on M with coefficients in f^*TN, $d_\nabla : A^k(M, f^*TN) \to A^{k+1}(M, f^*TN)$ the exterior differentiation induced by the the Levi-Civita connection of N: $d_\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \otimes \nabla s$ for $\alpha \in A^k(M)$ and s a smooth section of f^*TN. Then $d_\nabla^2 = -R$, where R is the curvature tensor of N.

In a Hermitian manifold of complex dimension n one has an identity on one forms (up to a multiplicative constant) $*\alpha = \omega^{n-1} \wedge J\alpha$, where ω is the fundamental 2-form associated to the metric and J is the complex structure. Thus there is an identity (up to multiplicative constant) $*df = \omega^{n-1} \wedge Jdf = \omega^{n-1} \wedge d^c f$. Thus in a Hermitian manifold the harmonic equation (3–1) is equivalent to the equation

$$d_\nabla(\omega^{n-1} \wedge d^c f) = 0.$$

Thus in a Kähler manifold, since $d\omega = 0$, the harmonic equation is equivalent to the equation

$$\omega^{n-1} \wedge d_\nabla d^c f = 0. \tag{3–2}$$

Observe that if $n = 1$ then (3–2) is equivalent to $d_\nabla d^c f = 0$, which is independent of the Hermitian metric on M (depends just on the complex structure of M).

Thus if M is a complex manifold, N is a Riemannian manifold, and $f : M \to N$ is a smooth map, it makes sense to say that f is a *pluriharmonic* map if its restriction to every germ of a complex curve in M is a harmonic map. Clearly f is pluriharmonic if and only if it satisfies the equation

$$d_\nabla d^c f = 0. \tag{3–3}$$

The basic discovery of Siu was that harmonic maps of compact Kähler manifolds to Kähler manifolds with suitable curvature restrictions are pluriharmonic. This was later extended by Sampson to more general targets. The curvature condition on N is called *nonpositive Hermitian curvature* and is defined to be the condition:

$$R(X, Y, \bar{X}, \bar{Y}) \le 0$$

for all $X, Y \in TN \otimes \mathbb{C}$. Here R is the curvature tensor of N, extended by complex multilinearity to complex vectors. The theorem is then the following:

THEOREM 3.1 (SIU–SAMPSON). *Let M be a compact Kähler manifold, let N be a Riemannian manifold of nonpositive Hermitian curvature, and let $f : M \to N$ be a harmonic map. Then f is pluriharmonic.*

We now explain the proof of this theorem. If $n = 1$ there is nothing to prove, since there is no difference between harmonic and pluriharmonic. If $n \ge 2$ the proof proceeds by an integration by parts argument (or Bochner formula) as follows. First, by Stokes's theorem and the compactness of M we have

$$\int_M d(\langle d^c f \wedge d_\nabla d^c f\rangle \wedge \omega^{n-2}) = 0. \tag{3–4}$$

Here, and it what follows, we use the symbol $< \alpha >$ to denote the scalar-valued form obtained from a form α with values in $f^*(TN \otimes TN)$ by composing with the inner product $\langle\ ,\ \rangle : TN \otimes TN \to \mathbb{R}$. Expanding the integrand using the Leibniz rule and $d\omega = 0$, we get a sum of two terms:

$$\langle d_\nabla d^c f \wedge d_\nabla d^c f\rangle \wedge \omega^{n-2} - \langle d^c f \wedge d_\nabla^2 d^c f\rangle \wedge \omega^{n-2}.$$

Now the first term is pointwise negative definite on harmonic maps by the so-called Hodge signature theorem: $\alpha \wedge \alpha \wedge \omega^{n-1} \le 0$ on the space of $(1,1)$-forms α such that $\alpha \wedge \omega^{n-1} = 0$, with equality if and only if $\alpha = 0$. Now the harmonic equation on Kähler manifolds we have just seen is equivalent to (3–2), thus the asserted negativity on harmonic maps.

The second term, when rewritten using the definition of curvature $d_\nabla^2 = -R$, turns out to be the average value of $R(df(X), df(Y), df(\bar{X}), df(\bar{Y}))$ over all unit length decomposable vectors $X \wedge Y \in \bigwedge^2 T^{1,0}M$ (that is, over all two-dimensional subspaces of $T^{1,0}M$). This computation can be found in [Amorós et al. 1996] (in the notation used here), or in equivalent forms in [Siu 1980; Sampson 1986].

Thus if N has nonpositive Hermitian curvature the two terms have the same sign and add to zero, thus each is zero. The vanishing of the first term is the pluriharmonic equation (3–3):

$$d_\nabla d^c f = 0$$

and the vanishing of the second term gives the following equations, which are also directly a consequence (by differentiation) of the pluriharmonic equation:

$$R(df(X), df(Y), df(\bar{X}), df(\bar{Y})) = 0 \; for \; all \; X, Y \in T^{1,0}M. \tag{3–5}$$

This concludes the proof of the Siu–Sampson theorem 3.1.

Before proceedings to applications, we point out three generalizations of this theorem that will be needed in the sequel:

Generalization 1. Theorem 3.1 holds for *twisted harmonic maps*. This means the following. Let X be a Riemannian manifold of nonpositive Hermitian curvature, let G be its group of isometries, and let $\rho : \pi_1(M) \to G$ be a representation. A twisted harmonic map (twisted by ρ) means a ρ-equivariant harmonic map $f : \tilde{M} \to X$, where $\tilde{M}$ denotes the universal cover of M and $\pi_1(M)$ acts on $\tilde{M}$ by covering transformations. Equivariant means as usual that $f(\gamma x) = \rho(\gamma) f(x)$ holds for all $\gamma \in \pi_1(M)$ and all $x \in \tilde{M}$. Equivariant maps are in one to one correspondence with sections of the flat bundle over M with fiber X associated to ρ, and equivariant harmonic maps correspond to harmonic sections of this bundle.

Since the integrand in (3–4) is an invariant form on $\tilde{M}$ (and thus descends to a form on M) for f a ρ-equivariant map, it is clear that the proof of Theorem 3.1 still holds in this context. Thus twisted harmonic maps of compact Kähler manifolds to Riemannian manifolds of nonpositive Hermitian curvature are pluriharmonic (and (3–5) holds).

Generalization 2. Theorem 3.1 holds under the following variation of its hypotheses: M, rather than a Kähler manifold, is a hermitian manifold whose fundamental form ω satisfies $dd^c(\omega^{n-2}) = 0$, and f, rather than a harmonic map, is a map that satisfies the equation (3–2). This was observed by Jost and Yau [1993b] where they call such manifolds M *astheno-Kählerian* and such maps f *Hermitian harmonic*. The proof of this extension is that the condition $dd^c(\omega^{n-2}) = 0$ is exactly what is needed to carry through the above integration by parts argument, provided of course that f satisfies the equation (3–2) (which differs from the harmonic equation by a lower order term if $d\omega \neq 0$).

Generalization 3. Theorem 3.1 holds for harmonic maps (or twisted harmonic maps) of compact Kähler manifolds to suitable singular spaces of nonpositive curvature (for example trees, or Bruhat–Tits buildings). This has been proved by Gromov and Schoen [1992]. The main points are, first, to define what is meant by a harmonic map, and then to prove that such a map has sufficient regularity for the integrand in (3–4) to make sense and the argument to go through.

4. Applications of Pluriharmonic Maps

We specialize the considerations of the last section to the case where N is a locally symmetric space of noncompact type. This means that the universal covering manifold of N is a symmetric space G/K, where G is a connected semisimple linear Lie group without compact factors and K is its maximal compact subgroup, and G/K is given the invariant metric determined by the Killing form $\langle\ ,\ \rangle$ on $\mathfrak{g}$. All computations can be reduced to Lie algebra computations: We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively and $\mathfrak{p}$ is a K-invariant complement to $\mathfrak{k}$. The Killing form is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. We have the equations

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

expressing the invariance of $\mathfrak{p}$ and the fact that $\mathfrak{k}$ is the fixed point set of an involution of $\mathfrak{g}$. For our purposes it will be harmless to make the identification $T_xN \cong \mathfrak{p}$ of the tangent space to N at any fixed point $x \in N$ with $\mathfrak{p}$. (Strictly speaking, we should have a varying isotropy subalgebra $\mathfrak{k}$ and thus varying complement $\mathfrak{p}$.)

Under this identification the curvature tensor is given (up to multiplicative constant) by

$$R(X, Y) = [X, Y],$$

and the Hermitian curvature on $TN \otimes \mathbb{C}$ is given by

$$R(X, Y, \bar{X}, \bar{Y}) = \langle [X, Y], [\bar{X}, \bar{Y}] \rangle$$

which is nonpositive, and zero if and only if $[X, Y] = 0$, because the Killing form is negative definite on $\mathfrak{k}$.

Thus if N is a locally symmetric manifold of noncompact type the Siu–Sampson theorem 3.1 applies, the map f is pluriharmonic and satisfies the further equations (also consequence of the pluriharmonic equation):

$$R(df(X), df(Y)) = [df(X), df(Y)] = 0 \textit{ for all } X, Y \in T^{1,0}M. \tag{4–1}$$

This vanishing of curvature has the following interpretation (compare [Amorós et al. 1996; Carlson and Toledo 1989]). Let

$$d''_\nabla : A^{0,k}(M, f^*TN \otimes \mathbb{C}) \to A^{0,k+1}(M, f^*TN \otimes \mathbb{C})$$

denote the Cauchy–Riemann operator induced by the Levi-Civita connection of N. Then $(d''_\nabla)^2 = 0$, thus $f^*TN \otimes \mathbb{C}$ is a holomorphic vector bundle over M. Then, if $d'f$ denotes the restriction of df to $T^{1,0}M$, the pluriharmonic equation (3–3) reads

$$d''_\nabla d'f = 0, \tag{4–2}$$

which means that $d'f$ is a holomorphic section of $Hom(T^{1,0}M, f^*TN \otimes \mathbb{C})$. Moreover, the Lie bracket form of (4–1) means that, if we identify T_xN with $\mathfrak{p}$ as above, then $df(T^{1,0}M)$ *is an abelian subalgebra of* $\mathfrak{p} \otimes \mathbb{C}$.

This last statement, which was observed by Sampson in [Sampson 1986], represents a nontrivial set of equations that must be satisfied by pluriharmonic maps. These equations extend, to targets which are not hermitian symmetric, the equations that Siu used in [Siu 1980] to prove his rigidity theorem. Namely, observe that if G/K is a Hermitian symmetric space, then correspoding to any invariant complex structure on G/K (there are only two if G/K is irreducible) we have the decomposition

$$\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}^{1,0} \oplus \mathfrak{p}^{0,1},$$

and the integrability condition $[\mathfrak{p}^{1,0}, \mathfrak{p}^{1,0}] \subset \mathfrak{p}^{1,0}$ is equivalent, in view of $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, to $[\mathfrak{p}^{1,0}, \mathfrak{p}^{1,0}] = 0$, thus $\mathfrak{p}^{1,0}$ *is an abelian subalgebra of* $\mathfrak{p} \otimes \mathbb{C}$. The idea of rigidity can thus be explained by saying that the Cauchy–Riemann equations

$$df(T^{1,0}M) \subset \mathfrak{p}^{1,0} = T^{1,0}N$$

can be forced on a pluriharmonic map f if one knows that abelian subalgebras of large dimension are rare. To this end the following algebraic theorem was proved in [Carlson and Toledo 1989].

THEOREM 4.1. *Let G/K be a symmetric space of noncompact type that does not contain the hyperbolic plane as a factor. Let $\mathfrak{a} \subset \mathfrak{p} \otimes \mathbb{C}$ be an abelian subalgebra. Then* $\dim(\mathfrak{a}) \leq 1/2 \ \dim(\mathfrak{p} \otimes \mathbb{C})$. *Equality holds in this inequality if and only if G/K is hermitian symmetric and $\mathfrak{a} = \mathfrak{p}^{1,0}$ for an invariant complex structure on G/K.*

This theorem gives a simple proof of the geometric version of Siu's rigidity theorem, namely the following statement:

THEOREM 4.2. *Let M be a compact Kähler manifold, let N be a manifold whose universal cover is an irreducible bounded symmetric domain other than the unit disk in $\mathbb{C}$, let $f : M \to N$ be a harmonic map, and suppose there is a point $x \in M$ such that $df(T_xM) = T_{f(x)}N$. Then f is either holomorphic or antiholomorphic.*

The proof of this theorem is now very simple. By the Siu–Sampson theorem 3.1, f is pluriharmonic. Since, by (4–2), $d'f$ is a holomorphic section of $Hom(T^{1,0}M, f^*TN \otimes \mathbb{C})$, the subset U of M on which df is surjective is the complement of an analytic subvariety. Since, by assumption, U is not empty, it is a dense connected open subset of M. By (4–1) and Theorem 4.1, at each $x \in U$ f satisfies the Cauchy–Riemann equations with respect to one of the two invariant complex structures on N. This complex structure is independent of x by the connectedness of U, hence f is holomorphic on a dense open set, hence holomorphic, with respect to this structure. In other words, f is holomorphic or antiholomorphic with respect to a preassigned complex structure on N and the proof of Theorem 4.2 is complete.

This proof of Theorem 4.2, taken from [Carlson and Toledo 1989] contains two simplifications of Siu's original proof [1980]. The first is the simple way in which the Cauchy–Riemann eauations follow from Theorem 4.1 at the points of maximum rank. The second is the observation (4–2) implies that these points form a dense connected open set, thus obviating one difficult (although interesting) result needed by Siu [1980], namely his unique continuation theorem to the effect that a harmonic map which is holomorphic on a nonempty open set is everywhere holomorphic.

The rigidity theorem 2.2 follows immediately from 4.2 and the existence theorem for harmonic maps of Eells and Sampson [1964]. Namely, since M and N are compact and N has nonpositive curvature, the main theorem of [Eells and Sampson 1964] asserts that any continuous map is homotopic to a harmonic one. Thus one may assume that the homotopy equivalence in Theorem 2.2 is harmonic. Since a smooth homotopy equivalence must have maximal rank at at least one point, Theorem 4.2 implies that it is holomorphic or anti-holomorphic, thus proving Theorem 2.2.

Now it is clear from this proof that knowledge of the abelian subalgebras of $\mathfrak{p}^{\mathbb{C}}$ should place restrictions on the harmonic maps of compact Kähler manifolds to locally symmetric spaces for G/K, and consequently, by the Eells–Sampson theorem, on the possible homotopy classes of maps. This has been done in the following cases:

Large Abelian Subalgebras of Hermitian Symmetric Spaces

THEOREM 4.3 [Siu 1982]. *Let G/K be a Hermitian symmetric space. Then there is an integer $\nu(G/K)$ with the property that if $\mathfrak{a} \subset \mathfrak{p} \otimes \mathbb{C}$ is an abelian subalgebra of dimension larger than $\nu(G/K)$, then $\mathfrak{a} \subset \mathfrak{p}^{1,0}$ for an invariant complex structure in G/K. Thus if M is compact Kähler and $f : M \to \Gamma\backslash\Gamma/K$ is a harmonic map of rank larger than $2\nu(G/K)$, then f is holomorphic with respect to an invariant complex structure on G/K.*

The numbers $\nu(G/K)$ are computed in [Siu 1982] for the irreducible Hermitian symmetric spaces.

These numbers $\nu(G/K)$ turn out to be sharp, because they happen to coincide with the largest (complex) dimension of a totally geodesic complex subspace of G/K that contains the hyperbolic plane as a factor. Thus using the nonrigidity of Riemann surfaces one can readily construct examples of nonholomorphic harmonic maps up to this rank. In this connection the most elementary and interesting case is perhaps that of the unit ball (complex hyperbolic space) where $\nu = 1$ and harmonic maps of real rank larger than two are holomorphic or antiholomorphic. An immediate topological consequence of 4.3 that any continous map that for some topological reason forces any smooth map in its homotopy class to have rank larger than $2\nu(G/K)$ (for instance, being nontrivial in homology above that dimension) is homotopic to a holomorphic map.

When the target G/K is not Hermitian symmetric the Siu–Sampson theorem still has interesting consequences. For instance an immediate consequence of Theorem 4.1 is the following theorem [Carlson and Toledo 1989]:

THEOREM 4.4. *Let M be a compact Kähler manifold, let N be a locally symmetric space whose universal cover is not Hermitian symmetric, and let $f : M \to N$ be a harmonic map. Then* $\operatorname{rank} f < \dim(N)$.

Now one would like to improve this theorem by giving a sharp upper bound for the rank of harmonic maps. Also one may want to know more about the structure of the harmonic maps of maximum rank:

Maximum-Dimensional Abelian Subalgebras

THEOREM 4.5 [Carlson and Toledo 1993]. *Let G/K be a symmetric space of noncompact type which is not Hermitian symmetric, and let $\mu(G/K)$ be the maximum dimension of an abelian subalgebra of $\mathfrak{p}\otimes\mathbb{C}$. If M is a compact Kähler manifold and $f : M \to \Gamma\backslash G/K$ is a harmonic map, then* $\operatorname{rank} f \le 2\mu(G/K)$.

The numbers $\mu(G/K)$ are computed in [Carlson and Toledo 1993] for all classical groups G.

The earliest, simplest, and most dramatic computation of μ was for real hyperbolic space by Sampson [1986], where he shows that $\mu = 1$ in this case, thus proving the following theorem:

THEOREM 4.6. *Let M be a compact Kähler manifold, let N be a manifold of constant negative curvature, and let $f : M \to N$ be a harmonic map. Then* $\operatorname{rank} f \le 2$.

This theorem implies that any continuous map of a compact Kähler manifold to a compact constant curvature manifold has image deformable to a two-dimensional subspace. Thus Kähler geometry and constant negative curvature geometry are incompatible in a very strong sense.

The computations of the numbers $\mu(G/K)$ in [Carlson and Toledo 1993] show that they are typically about $1/4\ \dim(G/K)$, thus giving an upper bound of about $\frac{1}{2}\dim(G/K)$ for the rank of harmonic maps. In some cases the bounds coincide with the largest dimension of a totally geodesic Hermitian subspace of G/K, thus they are sharp for suitable choice of discrete group Γ. In other cases the bound is one more than this number. In some of the cases when the two numbers coincide there is a further rigidity phenomenom: any harmonic map of this maximum rank of a compact Kähler manifold must have image contained in a totally geodesic Hermitian symmetric subspace. This is the case, for example, when $G = \mathrm{SO}(2p, 2q)$ for $p, q \ge 4$.

There are other numbers, less understood than the numbers in 4.5, which are the analogues of the numbers in 4.3 for the non-Hermitian G/K: any harmonic map of rank larger than twice this number must arise from a variation of Hodge structure (see Section 6). This number is shown to be one for quaternionic hyperbolic space in [Carlson and Toledo 1989], in analogy to the results of Siu and Sampson just discussed: $\nu = 1$ for complex hyperbolic space and $\mu = 1$ for real hyperbolic space. For classical G these numbers are estimated in [Carlson and Toledo 1993]. In contrast with the situation of the numbers in Theorem 4.3 and most of the numbers in Theorem 4.5, these estimates are not always sharp. In some cases they can be improved by more global methods [Jost and Zuo 1996; Zuo 1994] discussed in Section 5. The general picture still has to be worked out.

Finally, we would like to mention the fact that all these results on harmonic maps can be immediately extended to *twisted harmonic maps* as in the last section. Then, thanks to the existence theorem for twisted harmonic maps, all the analogous topological applications hold.

First, the existence theorem asserts that if M is a compact Riemannian manifold, X is a complete manifold of nonpositive curvature with group of isometries G, and if $\rho : \pi_1(M) :\rightarrow G$ is a suitable representation, then a ρ-equivariant harmonic map $f : \tilde{M} \rightarrow X$ exists. The first theorem of this nature was proved by Diederich and Ohsawa [1985] for X the hyperbolic plane, then in different contexts by other authors [Donaldson 1987; Corlette 1988; Labourie 1991; Jost and Yau 1991]. In more general contexts for X not a manifold it is proved in [Gromov and Schoen 1992; Korevaar and Schoen 1993]. We state here Corlette's theorem because it is the one most relevant to this survey. It was also the first fairly general statement of the existence theorem, and the first that was stated with a broad range of applicability in mind:

THEOREM 4.7. *Let M be a compact Riemannian manifold, let G be a semisimple algebraic group, and let $\rho : \pi_1(M) \rightarrow G$ be a representation. Then a ρ-equivariant harmonic map $f : \tilde{M} :\rightarrow G/K$ exists if and only if the Zariski closure of the image of ρ is a reductive group.*

This theorem is proved in [Corlette 1988] where an application to rigidity is also given, namely the rigidity in $\mathrm{PSU}(1, n+1)$ of representations of $\pi_1(M)$ in $\mathrm{PSU}(1, n)$, $n \geq 2$, with nonvanishing volume invariant, thus solving a conjecture of Goldman and Millson. (The corresponding statement for $n = 1$ is stated and proved in [Toledo 1989].) Another application to rigidity in presence of nonvanishing volume invariant is given in [Corlette 1991]. An application in a similar spirit, proving that certain $\mathrm{SO}(2p, 2q)$ representations must factor through $\mathrm{SU}(p, q)$ is given in [Carlson and Toledo 1993, Theorem 9.1].

5. Further Applications of Pluriharmonic Maps

Pluriharmonic maps are very special even when they are not holomorphic. For instance their fibers and their fibration structure are special. This seems to have been first exploited by Jost and Yau [1983], who proved that the fibers of a pluriharmonic map of constant maximum rank of a compact Kähler manifold to a Riemann surface are complex manifolds which vary holomorphically. Thus the harmonic map can be made holomorphic by changing the complex structure of the target surface. In this way they prove that all deformations of Kodaira surfaces arise from deformations of the base curve.

More generally, even it the rank of the pluriharmonic map $f : M \to N$ is not constant, or if N is not a Riemann surface, one can still try to form the quotient of M by the equivalence relation: two points are equivalent if and only if they lie in the same connected component of a maximal complex subvariety of a fiber. In some cases one can show that the quotient of M by this equivalence relation is a complex space V and that the pluriharmonic map factors as

$$M \to V \to N, \tag{5–1}$$

where the first map is holomorphic and the second is pluriharmonic. This works very well in case that the generic fiber of f is a divisor, and V is then a Riemann surface: see [Siu 1987; Carlson and Toledo 1989; Jost and Yau 1991]. Thus one can prove that harmonic maps $f : M \to N$, where N is a hyperbolic Riemann surface factor as in (5–1), where V is a Riemann surface of possibly higher genus than N, the first map in (5–1) is holomorphic and the second is harmonic [Siu 1987]. Similar factorization theorems hold for maps to real hyperbolic space (consequently strengthening Sampson's theorem 4), for nonholomorphic maps to complex hyperbolic space, and for maps to quaternionic hyperbolic space that do not arise from variations of Hodge structure [Carlson and Toledo 1989; Jost and Yau 1991]. Finally, an analog of this factorization theorem has been proved by Gromov and Schoen for maps to trees (thus N is a tree rather than a manifold). See [Gromov and Schoen 1992, § 9].

These factorizations theorems have interesting applications to the study of Kähler groups. The first is that the property of a compact Kähler manifold of fibering over a Riemann surface is purely a property of its fundamental group [Beauville 1991; Catanese 1991; Siu 1987]; compare the general discussion in [Amorós et al. 1996, Chapter 2]:

Theorem 5.1. *Let M be a compact Kähler manifold. Then there exists a surjective holomorphic map $f : M \to N$, where N is a compact Riemann surface of genus $g \geq 2$ if and only if there exists a surjection $\pi_1(M) \to \Gamma_h$, where Γ_h is the fundamental group of a compact surface of genus $h \geq 2$ and $h \leq g$.*

The second application is the following restriction on fundamental groups of compact Kähler manifolds [Carlson and Toledo 1989]:

Theorem 5.2. *Let Γ be the fundamental group of a compact manifold of constant negative curvature and dimension at least* 3. *Then Γ is not a Kähler group.*

The interest of this theorem is that it provided the first application of pluriharmonic maps to the study of Kähler groups. Namely, the results of the last section can be used to restrict the possible homotopy types of compact Kähler manifolds (see [Carlson and Toledo 1989, Theorem; Amorós et al. 1996, Theorem 6.17] for a concrete restriction of this type), but it is hard to give restrictions on the fundamental group by these methods.

This theorem has been extended in two directions. In [Hernández 1991], Hernández proves the same statement for Γ the fundamental group of a compact pointwise $\frac{1}{4}$-pinched negatively curved manifold. In [Carlson and Toledo 1997] the authors apply an existence theorem of Jost and Yau for Hermitian harmonic maps and Generalization 2 of section 4 to prove that such groups are not fundamental groups of compact complex surfaces.

The third application of these factorization theorems to fundamental groups of compact Kähler manifolds is the Theorem of Gromov and Schoen on amalgamated products [Gromov and Schoen 1992]:

THEOREM 5.3. *Let M be a compact Kähler manifold with $\pi_1(M) = \Gamma_1 *_\Delta \Gamma_2$, where the index of Δ in Γ_1 is at least 2 and its index in Γ_2 is at least 3. Then there exists a representation $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{R})$ with discrete, cocompact image, and a holomorphic equivariant map $f : \tilde{M} \to D$, where D is the Poincaré disc.*

The interest of this theorem is that it provides restrictions on fundamental groups of compact Kähler manifolds that do not assume (as, for instance, theorem 5.2 does), that the group is linear. We will see in section 7 that there is good reason for doing this. One consequence of this theorem is that it excludes amalgamated products that are not residually finite from being Kähler groups. See [Amorós et al. 1996, § 6.5, 6.6] for further discussion of this point.

So far we have used the existence of factorizations (5–1) in situations where the generic fiber of f is a divisor. For fibers of higher codimension the situation is much more subtle. The (singular) foliation of M by the maximal complex subvarieties of the fibers of f may not have compact leaves. In cases where it can be proved to have compact leaves, the factorization (5–1) need only hold after blowing up M. It is technically much more difficult to obtain factorization theorems. A very careful discussion of such a theorem is given in [Mok 1992], where Mok proves a factorization theorem for discrete $\mathrm{SL}(k,\mathbb{R})$ representations of the fundamental group. This general philosophy makes it plausible that representations of fundamental groups of compact Kähler manifolds should factor through lower dimensional varieties. These ideas are further pursued by several authors, see [Jost and Zuo 1996; Katzarkov and Pantev 1994; Zuo 1994].

A certain picture emerges from these works, and from the work of Simpson [1991] where many of these considerations started: If a representation is not rigid, then it factors through a representation of the fundamental group of a lower-dimensional variety, whose dimension is bounded by the rank of the group. On the other hand there are rigid representations that cannot factor. It is not yet known how to combine these pictures into a picture of the general representation.

We refer to [Simpson 1993] for many examples and for formulation of specific problems that may help in seeing this general picture.

Another subject related to these ideas is the Shafarevich conjecture. This is the name given to the statement that the universal cover of a smooth projective variety is holomorphically convex, and which is posed as a question in the last section of [Shafarevich 1974]. To relate this question to pluriharmonic maps, we first make the following stricly heuristic remark. Suppose that M is compact Kähler and $\rho : \pi_1(M) \to G$ is a discrete, faithful and reductive representation with image Γ, where $G = \mathrm{SL}(k, \mathbb{R})$. Then by Theorem 4.7 there is a harmonic map $f : M \to \Gamma\backslash G/K$. It is easy to see that a pluriharmonic map pulls back convex functions to plurisubharmonic functions. Thus if $\phi : G/K \to \mathbb{R}$ denotes distance from a point, then ϕ is convex and consequently $f^*\phi$ is a plurisubharmonic exhaustion function on $\tilde{M}$. If it were *strictly* plurisubharmonic one would of course prove that $\tilde{M}$ is Stein, hence holomorphically convex. It is however well known that the existence of (weakly) plurisubharmonic exhaustion functions does not imply holomorphic convexity, so this approach does not prove the Shafarevich conjecture for M. But I hope that it makes it plausible that there could be a connection between pluriharmonic maps and the Shafarevich conjecture for (discrete, reductive) linear groups.

In fact there are such connections, of course in more subtle and involved ways. It is now known, thanks to work of Napier, Ramachandran, Lassell, Katzarkov, Pantev that the Shafarevich conjecture holds for *surfaces with linear fundamental group*; see [Katzarkov 1997; Katzarkov and Ramachandran 1998; Lasell and Ramachandran 1996; Napier 1990; Napier and Ramachandran 1995]. Very briefly, pluriharmonic maps to symmetric spaces and to buildings are used to prove the Shafarevich conjecture for linear reductive groups in [Katzarkov and Ramachandran 1998], where a reduction to a criterion of Napier [1990] (no infinite connected chain of compact curves in the universal cover) is used. The nonreductive case case is described in [Katzarkov 1997] by combining the reductive ideas with relative nilpotent completion ideas.

On the other hand, there is a tantalizing idea, due to Bogomolov and further developed in [Bogomolov and Katzarkov 1998; Katzarkov 1997], for possibly giving counterexamples to the Shafarevich conjecture. Part of the idea is to find relations with the free Burnside groups. The groups in question will of course be far from linear. Even though this work has not yet produced the desired counterexamples, this author feels that this type of construction will eventually prove fruitful in producing examples of nontrivial behavior of fundamental groups.

Much of what has been said in this section concerns the factorization of a manifold by a suitable equivalence relation. We point out the paper [Kollár 1993] where Kollár proves that the natural equivalence relation related to the Shafarevich conjecture is generically well-behaved. (See also [Campana 1994].) The book [Kollár 1995] contains many interesting examples and information on this equivalence relation, and in relations of the fundamental group with algebraic geometric properties of varieties.

Finally, we mention another recent application of pluriharmonic maps, namely the solution of Bloch's conjecture by Reznikov [1995]. This is the statement that the higher Chern–Simons classes of a flat vector bundle over a smooth projective variety are torsion classes.

6. Nonabelian Hodge Theory

Closely related to the theory of harmonic maps is the nonabelian Hodge theory of Corlette and Simpson. Since this theory is amply described in [Amorós et al. 1996; Simpson 1992; Simpson 1997], we limit ourselves to a few comments most closely related to this survey.

For simplicity we let $G = GL(m, \mathbb{C})$ and observe that if $\rho : \pi_1(M) :\to G$ is a reductive representation, then by Theorem 4.7 a twisted harmonic map exists, which is pluriharmonic by Theorem 3.1. If we let $\theta = d'f$, then equation (4–2) can be interpreted as saying that the flat $\mathbb{C}^n$-bundle undelying the flat $GL(m, \mathbb{C})$-bundle has a holomorphic structure, so that the induced holomophic structure on $\mathrm{End}(E)$ is the holomorphic structure given by d''_∇ (using the identification $\mathrm{End}(\mathbb{C}^n) = \mathfrak{p} \otimes \mathbb{C}$). Thus

$$\theta \in H^0(M, \Omega^1 \otimes \mathrm{End}(E)), \tag{6–1}$$

and the abelian equations (4–1) are equivalent to

$$[\theta, \theta] = 0 \in H^0(M, \Omega^2 \otimes \mathrm{End}(E)). \tag{6–2}$$

Now the data: a holomorphic vector bundle E over M and a holomorphic one-form θ as in (6–1) satisfying (6–2) is by definition a Higgs bundle over M. This notion was introduced by Hitchin [1987] for M a Riemann surface, where (6–2) is vacuous, and for higher-dimensional M by Simpson [1992].

We have just seen that a reductive representation of $\pi_1(M)$ gives rise to a Higgs bundle, which must satisfy, as a consequence of reductivity, a suitable stability condition (in the sense of geometric invariant theory). Conversely, Simpson proves that a stable Higgs bundle arises from a representation of $\pi_1(M)$. The end result is that the subset $H^1_{\mathrm{red}}(M, G)$ of the first cohomology set $H^1(M, G)$ given by reductive representations is in one to one correspondence with the set of isomorphism classes of stable Higgs bundles.

Simpson uses this correspondence to define a $\mathbb{C}^*$-action on $H^1_{\mathrm{red}}(M, G)$, namely the action such that $t \in \mathbb{C}^*$ sends a Higgs bundle E, θ to the Higgs bundle $E, t\theta$. This action (which is interpreted as the nonabelian analogue of the Hodge filtration on abelian cohomology) has for fixed points the *variations of Hodge structure*. For our purposes these can be defined as the representations of $\pi_1(M)$ to $GL(m, \mathbb{C})$ that have image in a subgroup of type $U(p, q)$ and whose harmonic section, with values in the symmetric space of $U(p,q)$, lifts to a horizontal holomorphic map of a suitable homogeneous complex manifold fibering over this symmetric space.

Simpson proceeds to prove that every reductive representation of $\pi_1(M)$ can be deformed to a variation of Hodge structure. In particular, rigid representations must be variations of Hodge structure. He then uses the fact that the Zarisiki closure of the monodromy of a variation of Hodge structure is what he calls a group of Hodge type (equivalently, a group with a compact Cartan subgroup, equivalent a group where the geodesic symmetry in its symmetric space is in the connected component of the identity) and the infinitesimal rigidity of most lattices to derive the following theorem:

THEOREM 6.1. *Let Γ be a lattice in a simple Lie group G and suppose that Γ is a Kähler group. Then G has a compact Cartan subgroup.*

As an application, we see that lattices in simple complex Lie groups, in $\mathrm{SL}(n,\mathbb{R})$, in $\mathrm{SO}(2p{+}1,\ 2q{+}1)$ are not Kähler groups.

Now if G has a compact Cartan subgroup but is not the group of automorphisms of a bounded symmetic domain (for example, the group $\mathrm{SO}(2p,2q)$ where $p,q>1$) it is not known if lattices in G can be Kähler groups. It is conjectured in [Carlson and Toledo 1989] that they are not, but except for the cases $\mathrm{SO}(1,2n)$, solved in the same paper, and the automorphism group of the Cayley hyperbolic plane, solved in [Carlson and Hernández 1991], this question remains open.

Another open question is the following, Suppose G is the group of automorphisms of an irreducible bounded symmetric domain of dimension at least two, $\Gamma \subset G$ is a lattice, and M is a compact Kähler manifold with fundamental group Γ. Does there exist a *holomorphic* map $f : M \to \Gamma\backslash G/K$ inducing an isomorphism on fundamental group? If G/K is complex hyperbolic space (of dimension at least 2) the answer is affirmative, but in other cases it remains open. One needs to know whether the harmonic map is holomorphic, equivalently whether the variation of Hodge structure given by the proof of Simpson's theorem 6.1 is the standard one.

Finally, we mention that perhaps the first geometric application of nonabelian Hodge theory was the computation by Hitchin of the components of the space of $\mathrm{SL}(2,\mathbb{R})$ (or $\mathrm{PSL}(2,\mathbb{R})$)-representations of a surface group. Recall that the space of representations of the fundamental group of a surface of genus $g>1$ in $\mathrm{PSL}(2,\mathbb{R})$ has $4g{-}3$ components, indexed by the value k of the Euler class, which can take any value k such that $|k| \le 2g-2$ [Goldman 1985]. Let $r = 2g-2-|k|$. It follows from [Hitchin 1987] that the component with Euler class k is the total space of a vector bundle over the r^{th} symmetric power of the base surface. This identification has the draw-back that it requires a fixed complex structure on the surface and does not allow one to draw any conclusions as to the action of the mapping class group. Knowledge of this action on the components of Euler class k, for $|k| < 2g-2$, is an interesting open problem [Goldman 1985].

7. Nonlinear Kähler Groups

We have given a number of examples of how harmonic map techniques, as well as the nonabelian Hodge theory, can be used to study Kähler groups. These techniques are a natural extension of the classical ones of linear Hodge theory (compare [Amorós et al. 1996, Chapters 1 and 3]). We have seen that the nonlinear harmonic equation and nonabelian Hodge theory can be used effectively to study linear representations of Kähler groups. We have have seen one example where a more general harmonic theory applies to possibly nonlinear groups, namely Theorem 5.3. Other restrictions on Kähler groups that do not assume linearity of the group arise from L_2 harmonic theory. This development started in [Gromov 1989] and we refer to [Amorós et al. 1996, Chapter 4] for discussion of the present state of this particular subject.

Now nonlinear Kähler groups do exist. This means that the nonabelian Hodge theory can only capture part of the fundamental group, and there is indeed good reason for developing methods that apply to nonlinear groups, as the methods just mentioned.

The first example of a non-residually finite, and hence nonlinear, Kähler group was given in [Toledo 1993]. The construction is briefly the following. Let M be a compact locally symmetric variety for the symmetric space of $\mathrm{SO}(2,4)$ such that M contains a smooth totally geodesic divisor D corresponding to a standard embedding of $\mathrm{SO}(2,3)$ in $\mathrm{SO}(2,4)$. It is proved in [Toledo 1993] that there is a smooth projective variety $X \subset M - D$ so that the inclusion induces an isomorphism $\pi_1(X) \cong \pi_1(M - D)$. Now there is an exact sequence

$$1 \to K \to \pi_1(M - D) \to \pi_1(M) \to 1, \tag{7–1}$$

where K is a free group of infinite rank, namely $K = \pi_1(\tilde{M} - \pi^{-1}(D))$, where $\tilde{M}$, the universal cover of M, is the symmetric space for $\mathrm{SO}(2,4)$ and $\pi^{-1}(D)$ is the disjoint union of countably many copies of the symmetric space of $\mathrm{SO}(2,3)$, each totally geodesically embedded in $\tilde{M}$.

Let N denote a tubular neighborhood of D in M, and let ∂N denote its boundary, which is a circle bundle over D. Then there is an exact sequence

$$1 \to \mathbb{Z} \to \pi_1(\partial N) \to \pi_1(D) \to 1, \tag{7–2}$$

and it is easily seen that the maps induced by inclusion map each element of (7–2) *injectively* to the corresponding element of (7–1). In particular $\pi_1(\partial N)$ is a subgroup of $\pi_1(M - D) \cong \pi_1(X)$. Now since ∂N is a locally homogeneous circle bundle over D, it is easy to identify this bundle and to show that its fundamental group is a lattice in an infinite cyclic covering group of $\mathrm{SO}(2,3)$. Now this covering group is a nonlinear Lie group, and a remarkable theorem of Raghunathan [1984] implies that this lattice is not residually finite. Thus $\pi_1(X)$ contains the non-residually finite subgroup $\pi_1(\partial N)$, thus it is itself not residually finite. From this it is easy to see that the intersection of all subgroups of finite index of $\pi_1(X)$ is a free group of infinite rank.

It is possible to prove, essentially as a consequence of the Margulis superrigidity theorem (see Section 8 for the Margulis theorem), that $\pi_1(\partial N)$ is not a linear group. This is a weaker, but less subtle, result than Raghunathan's theorem. It gives immediately the weaker result that $\pi_1(X)$ is not a linear group. We leave the details of this simpler result to the interested reader.

There have been other constructions of non-residually finite Kähler groups. A construction by Nori and independently by Catanese and Kollár [1992] gives Kähler groups such that the intersection of all subgroups of finite index is a finite cylic group. The present author then constructed examples where this intersection is any finitely generated abelian group. See [Amorós et al. 1996, Chapter 8] for a detailed discussion of all these examples.

It is interesting to note that to date all known examples of non-residually finite Kähler groups are based on Raghunathan's theorem (or similar theorems for lattices in covering groups of automorphism groups of other symmetric domains [Prasad and Rapinchuk 1996]. There is an interesting proposal arising from the work of Bogomolov and Katzarkov and a suggestion of Nori's that may give a different kind of example, where the interesection of all subgroups of finite index is itself not residually finite. However the verification of the proposed examples is still conjectural and depends on the solution of difficult problems in group theory [Bogomolov and Katzarkov 1998].

8. Other rigidity Theorems

Even though this article is concerned mostly with applications of harmonic maps to complex analysis, there are closely related applications of harmonic maps to rigidity theorems that should be mentioned here. We refer to the surveys [Corlette 1995; Pansu 1995], and to the original references [Corlette 1992; Jost and Yau 1993a; Mok et al. 1993] for more details.

In retrospect, one can say that the reason that the Siu–Sampson theorem works is that the holonomy group of a Kähler manifold is contained in the unitary group $U(n)$ which is a proper subgroup of the holonomy group $\mathrm{SO}(2n)$ of the general oriented Riemannian manifold of dimension $2n$. On the general Riemannian manifold the only Bochner formula that is available is the original formula of Eells and Sampson [1964] which involves both the curvature of the target *and* the Ricci curvature of the domain. One of the achievements of [Siu 1980] was to find a Bochner identity that did not involve the curvature tensor of the domain. One can now say that the reason Siu was successful was that Kähler manifolds of complex dimension at least two admit a parallel form distinct from the volume form, namely the Kähler form. And this is equivalent to the fact that the holonomy group of a Kähler manifold is contained in $U(n)$.

Once it was realized that the Siu–Sampson theorem was probably related to special holonomy groups, the search began for other Bochner formulas for other holonomy groups. The interest in this search was to complete the superrigidity theorems of Margulis [1975; 1991]. Namely, Margulis had proved his celebrated generalization of the Mostow rigidity theorem for irreducible lattices in a real algebraic groups G *of real rank at least two*, and which says essentially that a homomorphism of such a lattice to a simple algebraic group H over a local field either extends to a homomorphism of algebraic groups or it has relatively compact image. (See [Margulis 1975; 1991; Zimmer 1984] for the precise statement of the theorem and for proofs and applications.) The methods of Margulis used in an essential way the hypothesis of the real rank of G (i.e., the rank of the symmetric space G/K) being at least two. It was known that the theorem failed for the groups $\mathrm{SO}(1,n)$ — [Gromov and Piatetski-Shapiro 1988] and the references therein — and (at least for small n) for the groups $\mathrm{SU}(1,n)$ of real rank one, but it was possible that Margulis's theorem was still true for lattices in the remaining simple groups of real rank one: $\mathrm{Sp}(1,n)$ and the automorphism group of the Cayley hyperbolic plane.

The local field in the statement of Margulis's theorem may be Archimedean or nonarchimedean. In the Archimedean case for the target group, and assuming also that the lattice is cocompact, the existence theorem for equivariant harmonic maps [Corlette 1988], reduces the Margulis theorem to the following statement (where K, K' denote the maximal compact subgroups of G, H respectively):

THEOREM 8.1. *Let $\Gamma \subset G$ be a torsion-free cocompact lattice*, *let $\rho : \Gamma \to H$ be a representation*, *let $f : G/K \to H/K'$ be a ρ-equivariant harmonic map. Then f is totally geodesic.*

In [Corlette 1992] Corlette succeeded in this search by proving a Bochner identity for harmonic maps with domain a manifold with a parallel form which implies, in the case that the domain has holonomy $\mathrm{Sp}(1)\cdot\mathrm{Sp}(n)$, for $n \geq 2$, as in quaternionic hyperbolic space, where there is a parallel 4-form, that harmonic maps are totally geodesic as in 8.1. He also proves 8.1 for G/K the noncompact dual of the Cayley plane, which has a parallel 8-form, thus proving the Archimedean superrigidity for cocompact lattices in these real rank one groups.

If the local field in the statement of Margulis's theorem is nonarchimedean, then then the symmetic space H/K' of the Archimedean case is replaced by the Tits building X, which is a nonpositively curved simplicial complex which plays the analogous role, for p-adic Lie groups, that the symmetric spaces play for real Lie groups. In this case the existence theorem for harmonic maps was developed by Gromov and Schoen [1992] where they reduce the Margulis theorem to the analogous statement to 8.1, where one must note that a totally geodesic map from a symmetric space of noncompact type to a building must be constant. They also prove that Corlette's Bochner formula also applies in this case to give the nonarchimedean version of 8.1 for lattices acting on quaternionic hyperbolic space (of quaternionic dimension at least two) and the hyperbolic Cayley plane. The main interest in the nonarchimedean superrigidity is that it implies the *arithmeticity* of lattices; see [Margulis 1991; Zimmer 1984].

Finally, both these results can be extended to noncocompact lattices. For the existence theorem of equivariant harmonic maps one needs an initial condition of finite energy, and one knows how to do this in the case that the target manifold has negative curvature bounded away from zero, as is the case in finite volume quotients of the rank one symmetric spaces. This requires some understanding of the nature of the cusps, as does the integration by parts argument required for the Bochner formula. All this is understood and explained in [Corlette 1992; Gromov and Schoen 1992], thus completing the Margulis superridity theorem for these rank one groups. I consider the results of these two papers the best applications of harmonic maps to rigidity questions since Siu's original rigidity theorem.

There has been another important development, namely a new proof, by harmonic maps, of most cases of the Margulis theorem. The statement of Theorem 8.1 has now been proved for G any simple noncompact group other than $\mathrm{SO}(1,n)$ and $\mathrm{SU}(1,n)$ (and H/K' replaced by manifolds with suitable curvature assumptions) in [Jost and Yau 1993a; Mok et al. 1993]. Instead of the Bochner formula these authors use a suitable version of Matsushima's formula, which also exploits the fact that the holonomy of the domain manifold is special.

We mention again (compare Section 2) that these methods cannot prove the original Mostow rigidity theorem for hyperbolic space forms, Theorem 2.1, because the holonomy group of a constant negative curvature manifold is the full orthogonal group, so it does not allow any of the integration by parts formulas that have been used to derive rigidity from harmonic maps. Similarly these methods, even though they easily prove Mostow rigidity for lattices in $\mathrm{SU}(1,n)$, $n \geq 2$, by their very nature they cannot shed any light on the open question of the possibility of geometric superrigidity theorem 8.1 for lattices in $\mathrm{SU}(1,n)$ for n large. Geometric super-rigidity fails for $n = 2$ and $n = 3$ because of the existence of non-arithmetic lattices; see [Mostow 1980; Deligne and Mostow 1986]. For $n = 2$ there is a further more dramatic failure of super-rigidity due to the existence of "non-standard homomorphisms"; [Mostow 1980, § 22]. What happens for large n seems to be wide open. Since the holonomy of a constant negative holomorphic sectional manifold is the full unitary group, it does not allow any of the additional formulas used to prove superrigidity for other Hermitian symmetric spaces. It seems that the question of geometric superrigidity 8.1 for lattices in $\mathrm{SU}(1,n)$, n large, is the main open question in this subject.

From the point of view of the theory of harmonic mappings, two aesthetic problems that one would like to solve are the following: First, the proofs of superrigidity in [Jost and Yau 1993a; Mok et al. 1993] require an intense amount of case by case verification, which would be nice to replace by more conceptual and general arguments. Second, the harmonic map techniques have not yet been successful in proving a known and important part of Margulis's theorem, namely the superrigidity for noncocompact lattices in Lie groups of real rank at least two. The existence theorem for an equivariant harmonic map is not known here because it is not known in all generality how to find an initial condition of finite energy in the heat equation method.

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